

Amplitudes and partial wave unitarity bounds

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We develop a formalism, based on spinor-helicity techniques, to generalize the formulation of partial wave unitarity bounds. We discuss unitarity bounds for $N \rightarrow M$ (with $N, M \geq 2$) scattering processes—relevant for high-energy future colliders—and spin-2 or higher-spin theories—relevant for effective field theories of gravity—that are not approachable by standard methods. Moreover, we emphasize the power and complementarity of positivity and partial wave unitarity bounds to constrain the parameter space of effective field theories.

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Introduction. New interactions of nature can be probed through a wide variety of experimental tests, encompassing direct searches of new particles at high-energy colliders, indirect signals in low-energy precision measurements, as well as cosmological observations. On the theoretical side, perturbative unitarity—relying on the properties of the scattering S matrix—provides an independent and complementary way to infer upper limits on the strength of such interactions, irrespectively of any experimental information. Generally, unitarity violation signals the breakdown of the low-energy description of a theory that can be restored by introducing new degrees of freedom or a new strong dynamics at the scale where unitarity is broken. In the past, unitarity bounds for the WW scattering process led to the famous no-lose Higgs theorem, which implied an upper bound on the Higgs boson mass below the TeV scale [1,2]. Such a theoretical prediction was of paramount importance in motivating the construction of the LHC at CERN.

More recently, unitarity bounds have been extensively discussed in the literature to assess the range of validity of effective field theories (EFTs) [3–6]. Indeed, nonrenormalizable interactions generate scattering amplitudes growing with energy, thus leading to unitarity violation at some large energy scale. As a result, a correct interpretation of

experimental data, such as those inherent to tails of kinematical distributions, has to be necessarily supplemented by unitarity bounds.

The standard way of implementing unitarity bounds relies on $2 \rightarrow 2$ scatterings of particles with helicities h_i and proceeds in two steps: i) expansion of the helicity amplitudes into partial waves $a_{h_i}^J$ with fixed total angular momentum J by means of the Wigner rotation matrix [7] and ii) diagonalization of the partial wave scattering matrix. In this coupled channel analysis, the most stringent limit is then obtained by imposing the partial wave unitarity bound $|a_{h_i}^J|_{\max} \lesssim 1$ on the largest eigenvalue.

Although very popular, the above method suffers from two serious shortcomings that significantly reduce its regime of applicability. On the one hand, partial wave decompositions are known only for $2 \rightarrow 2$ processes. However, since in many cases $2 \rightarrow N$ (with $N > 2$) amplitudes show a faster growth with energy than $2 \rightarrow 2$ amplitudes, unitarity bounds from $2 \rightarrow N$ processes are expected to dominate at high energies, which can be probed at future colliders such as FCC-ee [8] or a high-energy muon collider [9].

On the other hand, even in the $2 \rightarrow 2$ case, a standard partial wave decomposition for spin-2 or higher-spin theories—relevant for EFTs of gravity—is impracticable given the difficulty in evaluating the amplitudes via Feynman rules. In this respect, on-shell methods—which proved to be very efficient to capture ultraviolet quantum effects of EFTs [10–22]—seem to be the ideal tool also to probe possible unitarity violations of EFTs at high energies.

The primary goal of this Letter is to generalize the formulation of partial wave unitarity bounds in order to overcome the above shortcomings. In particular, we

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elaborate on a vectorial formalism [23], based on the spinor-helicity formalism [24], which will allow us to determine the amplitude basis thanks to the amplitude-operator correspondence [25–28]. As a result, we find the generalization of the Wigner rotation matrix entering the partial wave expansion of $N \rightarrow M$ scattering amplitudes. This will enable us to set the most stringent partial wave unitarity bounds on those EFT interactions that generate contact $2 \rightarrow 3$ scattering amplitudes.

Furthermore, working in a Minkowski background, we discuss partial wave unitarity bounds for EFTs of gravity and higher-spin theories, which have been conjectured to be connected to weakly coupled conformal field theories through the AdS/CFT correspondence [29]. The recent direct observation of gravitational waves from merging black holes opens the possibility of probing the theory of gravity in the strong regime at an unprecedented level. It is therefore very interesting to explore which EFT extensions of gravity could be detected. In this respect, our work serves as an important guide for exploring viable models that are consistent with first principles of quantum field theory.

As an interesting byproduct of our study, we analyze for the first time the complementarity and interplay of positivity bounds [30]—stemming from the analyticity and causality of scattering amplitudes—and partial wave unitarity bounds. The resulting constraints will be of utmost importance to connect experimental data at colliders to fundamental properties of EFTs. Examples include the possibility to use the above bounds as a prior in statistical interpretations and to design phenomenological studies to test positivity and partial wave unitarity bounds at colliders.

Generalized partial wave unitarity bounds. The partial wave analysis allows one to project a generic amplitude $|\mathcal{A}_{i \rightarrow f}\rangle$ onto a kinematic basis $|\mathcal{B}_{i \rightarrow f}^J\rangle$ with definite angular momentum J as follows

$$|\mathcal{A}_{i \rightarrow f}\rangle = \sum_J a_{i \rightarrow f}^J |\mathcal{B}_{i \rightarrow f}^J\rangle, \quad (1)$$

where $a_{i \rightarrow f}^J$ are the partial wave coefficients. The fundamental building blocks of such a decomposition are the Poincaré Clebsch-Gordan coefficients $\mathcal{C}_{\mathcal{I} \rightarrow *}^{J,h}$ defined as [23]

$$\langle P, J, h | \mathcal{I} \rangle = \mathcal{C}_{\mathcal{I} \rightarrow *}^{J,h} \delta^{(4)} \left(P - \sum_{i \in \mathcal{I}} p_i \right), \quad (2)$$

namely the overlap between the multiparticle state $|\mathcal{I}\rangle$ and the Poincaré irreducible multiparticle state $|P, J, h\rangle$, where h denotes the helicity. The coefficients $\mathcal{C}_{\mathcal{I} \rightarrow *}^{J,h}$ can be seen as elements of the complex vector space $V_{\mathcal{I} \rightarrow *}$, i.e., $|\mathcal{C}_{\mathcal{I} \rightarrow *}^{J,h}\rangle \in V_{\mathcal{I} \rightarrow *}$, and, correspondingly, the angular momentum basis elements are given by

$$|\mathcal{B}_{i \rightarrow f}^J\rangle = \sum_h |\mathcal{C}_{i \rightarrow *}^{J,h}\rangle \otimes |\mathcal{C}_{* \rightarrow f}^{J,h}\rangle \in V_{i \rightarrow f}, \quad (3)$$

where $|\mathcal{C}_{* \rightarrow \mathcal{I}}^{J,h}\rangle = |(\mathcal{C}_{\mathcal{I} \rightarrow *}^{J,h})^*\rangle \in V_{* \rightarrow \mathcal{I}}$ and $V_{i \rightarrow f} = V_{i \rightarrow *} \otimes V_{* \rightarrow f}$. The partial wave coefficients $a_{i \rightarrow f}^J$ can then be conveniently obtained by extending the inner product in $V_{\mathcal{I} \rightarrow *}$ [31] to the vector space $V_{i \rightarrow f}$ as follows

$$\begin{aligned} a_{i \rightarrow f}^J &= \frac{1}{2J+1} \langle \mathcal{B}_{i \rightarrow f}^J | \mathcal{A}_{i \rightarrow f} \rangle \\ &= \frac{1}{2J+1} \int d\Phi_{\mathcal{I}} d\Phi_f \mathcal{A}_{i \rightarrow f} (\mathcal{B}_{i \rightarrow f}^J)^*, \end{aligned} \quad (4)$$

where $d\Phi_{\mathcal{I}}$ refers to the Lorentz invariant phase space measure associated with $|\mathcal{I}\rangle$ [32]. Notice that we crucially chose the basis normalization $\langle \mathcal{B}_{i \rightarrow f}^J | \mathcal{B}_{i \rightarrow f}^{J'} \rangle = (2J+1) \delta^{JJ'}$ in order to obtain the partial wave unitarity bounds in the standard form. Indeed, this choice implies

$$\int d\Phi_X |\mathcal{B}_{i \rightarrow X}^J\rangle \otimes |\mathcal{B}_{X \rightarrow f}^{J'}\rangle = |\mathcal{B}_{i \rightarrow f}^J\rangle \delta^{JJ'}, \quad (5)$$

which, when applied to the generalized optical theorem

$$|\mathcal{A}_{i \rightarrow f}\rangle - |\mathcal{A}_{f \rightarrow i}^*\rangle = i \sum_X \int d\Phi_X |\mathcal{A}_{i \rightarrow X}\rangle \otimes |\mathcal{A}_{X \rightarrow f}^*\rangle \quad (6)$$

leads to

$$a_{i \rightarrow f}^J - (a_{f \rightarrow i}^J)^* = i \sum_X a_{i \rightarrow X}^J (a_{X \rightarrow f}^J)^* \quad (7)$$

and, in turn, to the partial wave unitarity bounds

$$|\Re a_{i \rightarrow i}^J| \leq 1, \quad 0 \leq \Im a_{i \rightarrow i}^J \leq 2, \quad |a_{i \rightarrow f}^J| \leq 1, \quad (8)$$

where $i \neq f$. In practice, the determination of the angular momentum basis elements $|\mathcal{B}_{i \rightarrow f}^J\rangle$, required by the partial wave decomposition, can be achieved without passing through the Poincaré Clebsch-Gordan coefficients $\mathcal{C}_{\mathcal{I} \rightarrow *}^{J,h}$, as one can proceed as follows:

- (1) Find a set of kinematic monomials in spinor-helicity variables that are consistent with the particle helicities, span $V_{i \rightarrow f}$, and are independent after accounting for momentum conservation and Schouten [37].
- (2) Apply the Pauli-Lubanski operator squared [39]

$$\begin{aligned} \mathbb{W}_{\mathcal{I}}^2 &= \frac{1}{8} \mathbb{P}_{\mathcal{I}}^2 (\epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \mathbb{M}_{\mathcal{I},\alpha\beta} \mathbb{M}_{\mathcal{I},\gamma\delta} + \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\dot{\beta}\dot{\delta}} \tilde{\mathbb{M}}_{\mathcal{I},\dot{\alpha}\dot{\beta}} \tilde{\mathbb{M}}_{\mathcal{I},\dot{\gamma}\dot{\delta}}) \\ &\quad + \frac{1}{4} \mathbb{P}_{\mathcal{I}}^{\alpha\dot{\alpha}} \mathbb{P}_{\mathcal{I}}^{\beta\dot{\beta}} \mathbb{M}_{\mathcal{I},\alpha\beta} \tilde{\mathbb{M}}_{\mathcal{I},\dot{\alpha}\dot{\beta}}, \end{aligned} \quad (9)$$

where

$$\mathbb{P}_{\mathcal{I}}^{\alpha\dot{\alpha}} = \sum_{i \in \mathcal{I}} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \quad (10)$$

$$\mathbb{M}_{\mathcal{I}}^{\alpha\beta} = \sum_{i \in \mathcal{I}} \left(\lambda_i^\alpha \frac{\partial}{\partial \lambda_{i,\beta}} + \lambda_i^\beta \frac{\partial}{\partial \lambda_{i,\alpha}} \right), \quad (11)$$

$$\tilde{\mathbb{M}}_{\mathcal{I}}^{\dot{\alpha}\dot{\beta}} = \sum_{i \in \mathcal{I}} \left(\tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{i,\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_{i,\dot{\alpha}}} \right), \quad (12)$$

and $\mathcal{I} = i, f$ [40], to these monomials and construct the associated matrix.

- (3) Find the eigenvectors of the above matrix and the associated values of angular momentum from the Casimir eigenvalues $-P_{\mathcal{I}}^2 J_{\mathcal{I}}(J_{\mathcal{I}} + 1)$.
- (4) Normalize the elements of the so obtained orthogonal basis so that their norm is equal to $\sqrt{2J_{\mathcal{I}} + 1}$.

As expected, for $2 \rightarrow 2$ scattering, the basis $|\mathcal{B}_{i \rightarrow f}^J\rangle$ is proportional to the Wigner d -matrix [23]. Moreover, we have explicitly verified that the partial wave coefficients for $N \rightarrow M$ scattering (with $N, M \geq 2$) exactly reproduce the elements of the reduced scattering matrix [41–44], which captures only the s -wave contribution.

As an interesting result of our analysis, in Supplemental Material [45] we report the angular momentum basis for $2 \rightarrow 3$ amplitudes which is valid for generic values of J .

EFT of gravity and light-by-light scattering. As a first application of the above method, we analyze the unitarity bounds for the EFT of gravity and light-by-light scattering involving operators with four gravitons and photons, respectively. The versatility of the spinor-helicity formalism in handling particles of arbitrary spin [46] allows us to approach the problem in a unified manner.

We focus on the P- and CP-violating Lagrangian,

$$\frac{\mathcal{L}^{(S)}}{\sqrt{-g}} = c_1^{(S)} (\mathcal{Q}^{(S)})^2 + c_2^{(S)} (\tilde{\mathcal{Q}}^{(S)})^2 + c_3^{(S)} \mathcal{Q}^{(S)} \tilde{\mathcal{Q}}^{(S)} \quad (13)$$

where

$$\mathcal{Q}^{(1)} = F_{\mu\nu} F^{\mu\nu}, \quad \tilde{\mathcal{Q}}^{(1)} = F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (14)$$

$$\mathcal{Q}^{(2)} = M_{\text{P}}^2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \quad \tilde{\mathcal{Q}}^{(2)} = M_{\text{P}}^2 R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}, \quad (15)$$

$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual photon field-strength tensor, $\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R^{\alpha\beta}_{\rho\sigma}$ is the dual Riemann tensor [47], $M_{\text{P}}^2 = (8\pi G_{\text{N}})^{-1}$ is the Planck mass squared, and $c_i^{(S)}$ are dimensionful Wilson coefficients with $[c_i^{(S)}] = -4S$. For $S = 1$, we recover the electromagnetic Euler-Heisenberg Lagrangian [48], while, for $S = 2$, we obtain the eight-derivative corrections to the gravitational Einstein-Hilbert action [49]. In both cases, the three operators appearing in Eq. (13) constitute a basis for such effective dimension- $(4S + 4)$ operators [50,51].

Denoting by $i = \{(1^{+S}, 2^{+S}), (1^{+S}, 2^{-S}), (1^{-S}, 2^{-S})\}$ all the possible initial two-particle states and by $f = \{(3^{+S}, 4^{+S}), (3^{+S}, 4^{-S}), (3^{-S}, 4^{-S})\}$ the final ones, the full four-particle scattering matrix reads

$$|\mathcal{A}_{i \rightarrow f}\rangle = \begin{pmatrix} 8c_+^{(S)} \langle 12 \rangle^{2S} [34]^{2S} & 0 & 8c_-^{(S)} (\langle 12 \rangle^{2S} \langle 34 \rangle^{2S} + \langle 13 \rangle^{2S} \langle 24 \rangle^{2S} + \langle 14 \rangle^{2S} \langle 23 \rangle^{2S}) \\ 0 & 8c_+^{(S)} \langle 14 \rangle^{2S} [23]^{2S} & 0 \\ 8(c_-^{(S)})^* ([12]^{2S} [34]^{2S} + [13]^{2S} [24]^{2S} + [14]^{2S} [23]^{2S}) & 0 & 8c_+^{(S)} \langle 34 \rangle^{2S} [12]^{2S} \end{pmatrix} \quad (16)$$

with $c_+^{(S)} = c_1^{(S)} + c_2^{(S)} \in \mathbb{R}$ and $c_-^{(S)} = c_1^{(S)} - c_2^{(S)} + ic_3^{(S)} \in \mathbb{C}$. A simple way to obtain it relies on writing the photon field strength in the spinor space as $F_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \sim \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{\dot{\gamma}} \tilde{\lambda}_{\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\gamma}} \lambda_{\dot{\delta}}$ and similarly for the Riemann tensor $R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}\dot{\epsilon}\dot{\zeta}} \sim \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{\gamma}\dot{\delta}} \tilde{\lambda}_{\dot{\epsilon}} \tilde{\lambda}_{\dot{\zeta}} + \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{\dot{\gamma}\dot{\delta}} \lambda_{\dot{\epsilon}} \lambda_{\dot{\zeta}}$ [52].

The kinematic vector space for the maximally helicity violating amplitude $\mathcal{A}_{1^{-S}, 2^{-S} \rightarrow 3^{+S}, 4^{+S}}$ in Eq. (16), has dimension $2S + 1$ and is spanned by $\{|m_k\rangle\}_{k=1}^{2S+1}$, with

$$|m_k\rangle = [12]^{2S-k+1} [14]^{k-1} [23]^{k-1} [34]^{2S-k+1}, \quad (17)$$

which constitutes a basis. Of course, the same is true for $\mathcal{A}_{1^{+S}, 2^{+S} \rightarrow 3^{-S}, 4^{-S}}$, provided that all the square inner products are substituted with angle ones. In this basis,

the Pauli-Lubanski operator squared is represented by an upper bidiagonal matrix,

$$\mathbb{W}_{12}^2 = -s \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 2 & 4 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 6 & 9 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 12 & 16 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 20 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 4S^2 \\ 0 & \dots & \dots & \dots & 0 & 0 & 2S(2S+1) \end{pmatrix} \quad (18)$$

where $s = (p_1 + p_2)^2$. In fact, its action on the basis monomials $|m_k\rangle$, with $k > 1$, is

$$\begin{aligned} \mathbb{W}_{12}^2 |m_k\rangle = & -s \frac{k-1}{2} [12]^{2S-k+1} [14]^{k-2} [23]^{k-2} [34]^{2S-k+1} \\ & \times ((k+1)[14][23] + (k-1)[13][24] \\ & + [12][34]), \end{aligned} \quad (19)$$

which, upon using the Schouten identity $[13][24] = [12][34] + [14][23]$, becomes

$$\mathbb{W}_{12}^2 |m_k\rangle = -s((k-1)^2 |m_{k-1}\rangle + k(k-1) |m_k\rangle). \quad (20)$$

The eigenvalues of this matrix are given by its diagonal entries and correspond to $J_{12} = 0, 1, \dots, 2S$.

Instead, for all the amplitudes in Eq. (16) that are proportional to $c_+^{(S)}$, the dimension of the kinematic vector space is 1, and thus such amplitudes have a definite angular momentum. In particular, $\mathcal{A}_{1\pm S, 2\pm S \rightarrow 3\pm S, 4\pm S}$ has $J_{12} = 0$, while $\mathcal{A}_{1\pm S, 2\pm S \rightarrow 3\pm S, 4\pm S}$ has $J_{12} = 2S$, since

$$\mathbb{W}_{12}^2 \langle 12 \rangle^{2S} [34]^{2S} = \mathbb{W}_{12}^2 \langle 34 \rangle^{2S} [12]^{2S} = 0 \quad (21)$$

and

$$\mathbb{W}_{12}^2 \langle 14 \rangle^{2S} [23]^{2S} = -s[2S(2S+1)] \langle 14 \rangle^{2S} [23]^{2S}. \quad (22)$$

The normalized eigenvectors corresponding to $J_{12} = 0$ are then

$$|\mathcal{B}_{i \rightarrow f}^0\rangle = \frac{16\pi}{s^{2S}} \begin{pmatrix} \langle 12 \rangle^{2S} [34]^{2S} & 0 & \langle 12 \rangle^{2S} \langle 34 \rangle^{2S} \\ 0 & 0 & 0 \\ [12]^{2S} [34]^{2S} & 0 & [12]^{2S} \langle 34 \rangle^{2S} \end{pmatrix}, \quad (23)$$

and, by projecting the amplitudes in Eq. (16) onto them, it follows that the corresponding partial waves read [53]

$$a_{i \rightarrow f}^0 = \frac{s^{2S}}{2\pi} \begin{pmatrix} c_+^{(S)} & 0 & \frac{2S+3}{2S+1} c_-^{(S)} \\ 0 & 0 & 0 \\ \frac{2S+3}{2S+1} (c_-^{(S)})^* & 0 & c_+^{(S)} \end{pmatrix}, \quad (24)$$

whose nonvanishing eigenvalues are

$$\frac{s^{2S}}{2\pi} \left(c_+^{(S)} \pm \frac{2S+3}{2S+1} |c_-^{(S)}| \right). \quad (25)$$

As a result, we find that the strongest unitarity bound is provided by the partial wave with $J_{12} = 0$,

$$\mathcal{U}: \frac{s^{2S}}{2\pi} \left(|c_+^{(S)}| + \frac{2S+3}{2S+1} |c_-^{(S)}| \right) \leq 1. \quad (26)$$

The regions of the parameter space satisfying the partial wave unitarity condition of Eq. (26) are shown in red in Fig. 1 for the cases of $S = 1, 2$, respectively [54].

Complementary bounds on our EFT parameter space can be obtained by imposing positivity bounds, stemming from

the analyticity and causality of scattering amplitudes [30]. In particular, they impose specific inequalities on the Wilson coefficients to ensure compatibility with a well-behaved UV completion. In our scenarios, they read $|c_-^{(S)}| \leq c_+^{(S)}$, or, equivalently,

$$\mathcal{P}: c_{1,2}^{(S)} \geq 0 \quad \text{and} \quad (c_3^{(S)})^2 \leq 4c_1^{(S)} c_2^{(S)}, \quad (27)$$

and are valid for both $S = 1$ [30, 52, 60–62] and $S = 2$ [49, 52, 63–66]. The blue regions in Fig. 1 satisfy positivity bounds. Interestingly, we emphasize that the viable parameter space is significantly reduced once both positivity and partial wave unitarity bounds are imposed.

To highlight the impact of positivity bounds, we compare the volume of the parameter space allowed by unitarity, $\text{Vol}(\mathcal{U})$, with the one obtained after imposing the positivity constraints, $\text{Vol}(\mathcal{U} \cap \mathcal{P})$. In particular, for fixed values of the center-of-mass energy, we find

$$\frac{\text{Vol}(\mathcal{U} \cap \mathcal{P})}{\text{Vol}(\mathcal{U})} = \frac{1}{32} \left(\frac{2S+3}{S+1} \right)^2 \approx \begin{cases} 0.20 & \text{for } S = 1, \\ 0.17 & \text{for } S = 2, \end{cases} \quad (28)$$

which is a monotonically decreasing function of S .

$N \rightarrow M$ scattering amplitudes. Another important advantage of our method over standard techniques is its capability of addressing unitarity bounds for $N \rightarrow M$ processes (with $N, M \geq 2$). As an illustrative example, we consider the dimension-six SMEFT interactions [67]

$$\begin{aligned} \mathcal{L}^{(6)} \supset & C_{eH}^{pr} (H^\dagger H) \bar{\ell}_p e_r H + C_{dH}^{pr} (H^\dagger H) \bar{q}_p d_r H \\ & + C_{uH}^{pr} (H^\dagger H) \bar{q}_p u_r \tilde{H} + \text{H.c.} \end{aligned} \quad (29)$$

where $\tilde{H} = i\sigma^2 H^*$ and p, r are flavor indices. As is well known, $\mathcal{L}^{(6)}$ induces modifications to the fermionic Yukawa couplings, which are a target of the High-Luminosity LHC program, and generates new contributions to multiboson interactions, which might be probed at future colliders.

To obtain the unitarity bounds, we performed a coupled channel analysis involving all possible contact amplitudes for $\phi_a \phi_b \rightarrow \phi_c \psi_i^\pm \bar{\psi}_j^\pm$ and $\psi_i^\pm \bar{\psi}_j^\pm \rightarrow \phi_a \phi_b \phi_c$, where $\{\phi_a\}_{a=1}^4$ denote the real components of $H = (\phi_1 + i\phi_3, \phi_2 + i\phi_4)^T / \sqrt{2}$ and ψ_i, ψ_j any Dirac fermions. We find that the strongest unitarity bound arises from the $J = 0$ partial waves and reads

$$\sqrt{\text{Tr}[3C_{uH} C_{uH}^\dagger + 3C_{dH} C_{dH}^\dagger + C_{eH} C_{eH}^\dagger]} \leq \frac{32\pi^2}{\sqrt{3}s}, \quad (30)$$

where the trace is understood over flavor indices. As for the $\phi_a \phi_b \rightarrow \phi_c \bar{\psi}_i^\pm \psi_j^\pm$ amplitude

$$|\mathcal{A}_{1\phi_1, 2\phi_1 \rightarrow 3\phi_1, 4\psi_p^+, 5\psi_r^+}\rangle = \frac{3\sqrt{2}}{2} (C_{eH}^{pr})^* [54], \quad (31)$$

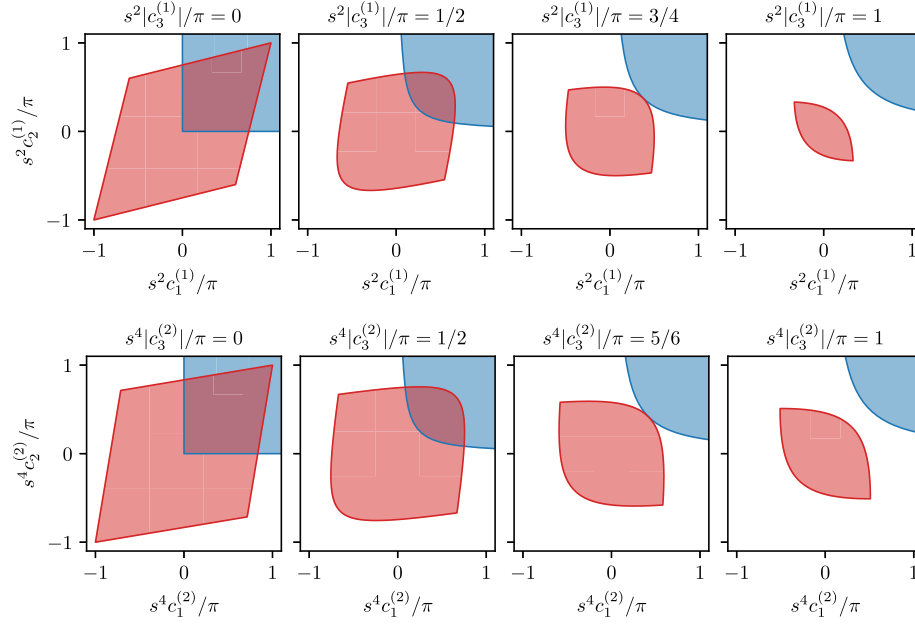


FIG. 1. Allowed regions for the Euler-Heisenberg EFT (upper plot) and the EFT of gravity (lower plot) by partial wave unitarity bounds (red) and positivity bounds (blue) in the $s^{2S} c_1^{(S)}/\pi$ and $s^{2S} c_2^{(S)}/\pi$ plane for different values of $c_3^{(S)}$, see Eq. (13).

the relevant vector basis element of Supplemental Material [45] is

$$|\mathcal{B}_{1^0, 2^0 \rightarrow 3^0, 4^1, 5^1}^0\rangle = \frac{32\sqrt{6}\pi^2}{s} [54], \quad (32)$$

where $s = (p_1 + p_2)^2$. Therefore, without performing any phase space integral, we get the $J = 0$ partial wave

$$a_{1_{\phi_1}, 2_{\phi_1} \rightarrow 3_{\phi_1}, 4_{\phi_1}^+, 5_{\phi_1}^+}^0 = \frac{\sqrt{3}s}{64\sqrt{2}\pi^2} (C_{eH}^{pr})^*, \quad (33)$$

where we included a $1/\sqrt{2}$ factor accounting for the identical particles in the initial state.

Instead, the unitarity bound from $2 \rightarrow 2$ amplitudes, arising from Eq. (29) picking one vacuum expectation value v out of a Higgs doublet, reads

$$\sqrt{\text{Tr}[3C_{uH}C_{uH}^\dagger + 3C_{dH}C_{dH}^\dagger + C_{eH}C_{eH}^\dagger]} \leq \frac{4\sqrt{2}\pi}{\sqrt{3sv^2}}, \quad (34)$$

and is weaker than that of Eq. (30) for $\sqrt{s} > 4\pi v\sqrt{2}$.

Next, we consider the following dimension-eight SMEFT interactions

$$\mathcal{L}^{(8)} \supset C_{X^3H^2}(H^\dagger H) f^{ABC} X_\mu^A X_\nu^B X_\rho^C \mu + \tilde{C}_{X^3H^2}(H^\dagger H) f^{ABC} X_\mu^A X_\nu^B \rho \tilde{X}_\rho^C \mu, \quad (35)$$

where $X_{\mu\nu}^A$ is the field strength associated with a generic non-Abelian gauge group G . The strongest unitarity constraint stems from the $J = 0$ partial waves and reads

$$\sqrt{C_{X^3H^2}^2 + \tilde{C}_{X^3H^2}^2} \leq \frac{32\sqrt{10}\pi^2}{s^2} \frac{1}{\sqrt{C_2(G)d(G)}}, \quad (36)$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of G and $d(G)$ is its dimension [68]. This bound is obtained via a coupled channel analysis which includes all the processes $\phi_a \phi_b \rightarrow X^{A\pm} X^{B\pm} X^{C\pm}$. Taking the related amplitudes

$$|\mathcal{A}_{1_{\phi_a}, 2_{\phi_b} \rightarrow 3_{X^A}^-, 4_{X^B}^-, 5_{X^C}^-}\rangle = 3\sqrt{2}(iC_{X^3H^2} - \tilde{C}_{X^3H^2})\delta_{ab} \times f^{ABC} \langle 34 \rangle \langle 45 \rangle \langle 35 \rangle, \quad (37)$$

$$|\mathcal{A}_{1_{\phi_a}, 2_{\phi_b} \rightarrow 3_{X^A}^+, 4_{X^B}^+, 5_{X^C}^+}\rangle = 3\sqrt{2}(iC_{X^3H^2} + \tilde{C}_{X^3H^2})\delta_{ab} \times f^{ABC} [43][54][53], \quad (38)$$

and the relevant vector basis element of Supplemental Material [45]

$$|\mathcal{B}_{1^0, 2^0 \rightarrow 3^1, 4^1, 5^1}\rangle = \frac{64\sqrt{30}\pi^2}{s^2} [43][53][54], \quad (39)$$

we can immediately obtain the partial wave coefficients by including the $1/\sqrt{2}$ symmetry factor,

$$a_{1_{\phi_a}, 2_{\phi_b} \rightarrow 3_{X^A}^\pm, 4_{X^B}^\pm, 5_{X^C}^\pm}^0 = \frac{3s^2 \delta_{ab} f^{ABC}}{64\sqrt{30}\pi^2} (iC_{X^3H^2} \pm \tilde{C}_{X^3H^2}). \quad (40)$$

Instead, $hX^{A\mp} \rightarrow X^{B\pm} X^{C\pm}$ (corresponding to $J = 1$) are the only available $2 \rightarrow 2$ channels and the related unitarity bound is

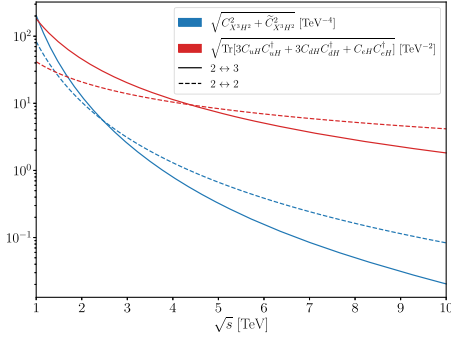


FIG. 2. Partial wave unitarity bounds on the Wilson coefficients of SMEFT operators, see Eqs. (29) and (35), vs $\sqrt{s}$ from $2 \rightarrow 2$ processes (dashed lines) and $2 \rightarrow 3$ processes (solid lines). $C_{X^3H^2}$ and $\tilde{C}_{X^3H^2}$ refer to $G = SU(3)$.

$$\sqrt{C_{X^3H^2}^2 + \tilde{C}_{X^3H^2}^2} \leq \frac{8\sqrt{2}\pi}{v s^{3/2}} \frac{1}{\sqrt{C_2(G)}}, \quad (41)$$

which is weaker than that of Eq. (36) for $\sqrt{s} > 4\pi v \sqrt{5/d(G)}$. In Fig. 2, we show the unitarity bounds arising from $2 \rightarrow 2$ and $2 \rightarrow 3$ processes as induced by $\mathcal{L}^{(6)}$ and $\mathcal{L}^{(8)}$, see Eqs. (29) and (35).

Conclusions. In this Letter, we elaborated on the application of on-shell methods for generalizing the formulation of partial wave unitarity bounds.

In particular, we provided the general angular momentum basis for $2 \rightarrow 3$ amplitudes, extending previous results

obtained by the reduced scattering matrix method in the limit of $J = 0$. As an illustration of our results, we discussed unitarity bounds for $2 \rightarrow 3$ scattering processes as induced by some dimension-six and -eight SMEFT operators, see Fig. 2, that might be probed at high-energy future colliders. Moreover, we discussed for the first time unitarity bounds for spin-2 or higher-spin theories—relevant for effective field theories of gravity—that are not approachable by standard methods, see Fig. 1.

Finally, we emphasized the complementarity of positivity and partial wave unitarity bounds to probe the parameter space of effective field theories, see Fig. 1.

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$$\begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix} = \frac{1}{\sqrt{1+z\bar{z}}} \begin{pmatrix} 1 & \bar{z} \\ -z & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

and writing the phase space measure as

$$\int d\Phi_{34} = -\frac{1}{2} \frac{1}{8\pi} \oint \frac{dz}{2\pi i} \int \frac{d\bar{z}}{(1+z\bar{z})^2},$$

where we included the $1/2$ symmetry factor due to indistinguishable particles. This yields the factors

$$-\text{Res}_{z=0} \int d\bar{z} \frac{1 + (z\bar{z})^{2S} + (1 + z\bar{z})^{2S}}{(1 + z\bar{z})^{2S+2}} = \frac{2S+3}{2S+1}$$

that appear in Eq. (24).

- [54] In deriving these bounds for the gravitational EFT ($S = 2$), we explicitly neglected the universal long-range contribution from single-graviton exchange

$$|\mathcal{A}_{1^{+2}, 2^{+2} \rightarrow 3^{+2}, 4^{+2}}\rangle = \frac{1}{M_{\text{p}}^2} \frac{\langle 12 \rangle^4 [34]^4}{stu},$$

responsible for the well-known forward and backward singularities that induce infrared divergences in the partial wave projections of the scattering amplitude. Such divergences are a well-understood feature of long-range interactions and can be systematically handled by factoring out a universal infrared-divergent phase [55] common to all partial waves. This universal phase, which arises from the resummation of iterated single-graviton exchange (the gravitational Born series), does not affect physical observables and can be stripped from the S matrix, as done for example in [56,57], yielding infrared-finite amplitudes suitable for unitarity and positivity analyses, and leaving the bounds presented here robust. The resolution scale $\mathcal{E}$ introduced by this procedure [57] admits a physical interpretation as the energy resolution of a macroscopic detector for soft gravitons. Alternative approaches to deal with the above singularity include the smearing method of Ref. [58] and the dimensional reduction technique of Ref. [59].

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