Z_N equivariant Virasoro algebra via alternative Sugawara constructions

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In this paper, we study the $u(1)^2$ Kac-Moody algebra and generalize the standard Sugawara construction of the Virasoro algebra to an infinite family of new realizations. In this case, in addition to the standard invariant tensor δ^{ij}, there exists another invariant tensor ϵ^{ij}, which enables the construction of genuinely new realizations beyond the conventional one. We show that these new realizations arise from a $\mathbb{Z}_N$ grading of the mode index n of the Virasoro generators L_n and the space of such realizations corresponds to points of a possibly singular algebraic variety. For the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cases, the space of all such constructions is topologically equivalent to a cylinder, while for $\mathbb{Z}_4$ it forms a noncompact real four-dimensional manifold. We show that the spaces of constructions for Z_{2N} and Z_{2N+1} are closely similar. Furthermore, we reformulate the problem within an action-principle framework by introducing $\mathbb{Z}_N$-equivariant maps, which provide a systematic method for constructing conformal field theories endowed with these generalized Virasoro symmetries. This formulation reproduces the $\mathbb{Z}_2$ case and supports the idea that $\mathbb{Z}_N$ equivariance offers a consistent and unified approach to generating extended conformal algebras. Finally, we analyze the corresponding Virasoro-Kac-Moody-like algebras associated with these constructions and show that they represent nontrivial (up to local field transformations) of the well-known Virasoro-Kac-Moody algebra.

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I. INTRODUCTION

Conformal field theory (CFT) in two dimensions has been a cornerstone of theoretical physics for the past several decades. Since the seminal work of Belavin, Polyakov, and Zamolodchikov [1], two-dimensional CFT has provided a powerful and elegant framework for the study of critical phenomena and exactly solvable models [2–7]. Its applications range from statistical mechanics [8–10] and condensed matter physics from the Ising model to the fractional quantum Hall effect [11–14] to string theory [15–17]. It also continues to play an important role in holography. The seminal work of [18] showed that the asymptotic symmetry algebra of AdS_3 is formed by two copies of the Virasoro algebra. Needless to say, two-dimensional CFTs have a significant and broad contribution to the field of mathematical physics in the past three decades [19–23]. Moreover, 2D CFTs also have many applications in pure mathematics, through intercorrelated concepts. A few examples are vertex operator algebras [24–26], representation theory of affine Lie algebras [27–32], and even using the Wess–Zumino–Witten/Chern-Simon correspondence to knot theory [33]. Many families of 2D CFTs have been studied by different methods. The minimal models, which are built out of finitely many irreducible representations of the Virasoro algebra [34,35] and orbifold CFTs [36–38], are examples of such families.

Furthermore, there exist many conformal field theories in which the Virasoro algebra is not only generated by the energy-momentum tensor, but also supplemented by additional symmetries encoded in other fields of the theory. These fields furnish operator product expansions that close under a larger algebraic structure beyond the Virasoro algebra itself. The most prominent and well-studied examples of such extended symmetries are provided by affine Kac-Moody algebras [31,32,34], which arise as current algebras associated with conserved holomorphic and antiholomorphic currents [39]. The Kac-Moody algebra $\hat{g}_k$ for Lie algebra g and level k is given by

$$[J^a_m, J^b_n] = i f^{ab}_c J^c_{m+n} + km\delta^{ab}\delta_{m+n,0}. \tag{1}$$

Such symmetry algebras appear in a wide range of contexts. For instance, the affine algebra $\hat{su}(2)_1$ corresponds to a free boson, while, more generally, they emerge in non-Abelian bosonization [40], in the compactifications of string theory on group manifolds [41], and in the study of various statistical models. Notably, interesting models such as the Potts and Ising models can be realized through coset

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constructions of the Kac-Moody algebras [4,21]. The symmetry algebra generated by conserved currents, the Kac-Moody algebra, is not independent of the Virasoro algebra. Instead, consistency between these two structures is ensured through the Sugawara construction [42], where the energy-momentum tensor is expressed as a quadratic combination of the Kac-Moody currents.

The standard Sugawara construction has been extended in several directions. The general Virasoro affine construction on an affine Lie algebra $\mathfrak{g}$ was studied by Halpern and Kiritsis [43] and further explored (and extended) in subsequent works [44–47]. Such generalizations have also been shown to be connected to problems in quantum mechanics [48]. In this paper, we show that there are many more new realizations for special Kac-Moody algebras. For the group $g = U(1)^D$, the Kac-Moody algebra is simplified due to the Abelian nature of the group. In this case, the algebra reduces to a set of commuting current algebras (up to the central term), which is essentially identical to the oscillator algebra satisfied by the α_n^i modes in the free string theory. These α_n^i are the Fourier coefficients of the currents $J^i(z) = \partial X^i(z)$ arising from the mode expansion of free bosons, thereby providing a direct realization of an Abelian Kac-Moody algebra within string theory, and the Virasoro generators take the usual representations in terms of α_n^i as follows:

$$L_n = \frac{1}{2}\sum_q \alpha^i_{n-q}\alpha^i_q. \tag{2}$$

This is consistent with the Sugawara construction. This paper aims to show that such a construction is not unique for $U(1)^2$, and there are infinitely many alternative constructions of L_n in terms of α_n^i that reproduce the Virasoro algebra. Our construction relies on two main ingredients. First, the Virasoro generators can be decomposed into N classes by considering the integer index n modulo N. In this framework, the operators are denoted as L_n^k,[1] where the index takes values of the form $n = N\mathbb{Z} + k$ with $k = 0, 1, \ldots, N-1$, which is the $\mathbb{Z}_N$ partitioning of the integer numbers module N. Consequently, the usual Virasoro generator L_n is naturally decomposed into N distinct classes. Furthermore, since the Hamiltonian of a two-dimensional conformal field theory is given by $H = L_0 + \tilde{L}_0$, we fix the $k = 0$ class to correspond to the standard Virasoro construction (2), where $n \in N\mathbb{Z}$, which is Hermitian and positive definite.[2] Second, in two dimensions, in addition to the invariant tensor δ^{ij}, there also exists the antisymmetric tensor ϵ^{ij}, which is invariant under rotations. The combination of δ^{ij} and ϵ^{ij} is sufficient to generate infinitely many additional constructions of Virasoro-like operators.

The $\mathbb{Z}_N$-equivariant Virasoro algebra and its constructions in terms of current modes also have an interesting relation to the deformation theory of $u(1)^2$ Kac-Moody-Virasoro algebras. Deformation theory plays a significant role in modern physics, and many examples can be understood from the viewpoint of deformation theory. In quantum physics, for example, deformations are central to the deformation quantization program, where quantum mechanics is viewed as a deformed version of classical Poisson algebra which originated from physical motivations [49,50] (review from a physical point of view [51]) and is one of the cornerstone of mathematical physics with work of many mathematicians, especially Kontsevich [52] (review from a mathematical point of view [53]). Another example is the deformation of Galilean algebra to Poincaré algebra, where the speed of light is the parameter of the deformation [54] and Poincaré algebra itself can be deformed to de Sitter algebra [55]. We should note that, although such constructions in terms of α_n^i reproduce the Virasoro algebra, they lead to the deformation of the standard $u(1)^2$ Kac-Moody-Virasoro algebra, which in its own right is interesting.

The organization of this paper is as follows: In Sec. II, we present a motivating example that illustrates a new construction of the Virasoro algebra in terms of the α_n^i modes, thereby setting the stage for a more detailed investigation of the $\mathbb{Z}_N$ case. In Sec. III, we carry out a systematic study of the new construction and demonstrate that it exhibits a natural $\mathbb{Z}_2$ grading of the integers labeling the Virasoro generators L_n with $n \in \mathbb{Z}$. We then extend the analysis to the general $\mathbb{Z}_N$ case and derive the conditions that these new constructions must satisfy to be consistent with the Virasoro algebra. It turns out that the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cases admit infinitely many new constructions. The corresponding space of solutions is topologically equivalent to a cylinder, which can be compactified to smooth $\mathbb{CP}^1$ by adjoining two points at infinity, which is a Riemann surface of genus $g = 0$. The algebras corresponding to the points at infinity are explored in [56], and it is shown that they are related to the generalization of the BMS_3 algebra. We then investigate the structure of the $\mathbb{Z}_4$ case, where the space of new Virasoro generators expressed in terms of α_n^i is four-dimensional. In Sec. IV, we approached the new constructions from the perspective of an action principle rather than purely algebraic considerations. In particular, we developed the concept of $\mathbb{Z}_N$-equivariant maps as a systematic method to construct new conformal field theories. Using this framework, one can successfully generate the $\mathbb{Z}_2$ case, providing strong evidence that the notion of $\mathbb{Z}_N$ equivariance is sufficient to produce these algebras in a controlled and consistent manner. In Sec. V, we have explored the Virasoro-Kac-Moody-like

[1]In the rest of this paper, we denote $L_n^k = S_n^k$ for $k > 0$ and $L_n^0 = L_n$.

[2]From a more algebraic perspective, this condition can be relaxed. For instance, in the case $N = 2$, the most general form of the operator S_n^0 can be written as $L_n = \frac{1}{2}\sum_q (A + B(-1)^q)\alpha^i_{n-q}\alpha^i_q$. However, in the remainder of the paper, we fix S_n^0 to be the standard Virasoro operator, postponing the discussion of these more general variants to Sec. IV B, where such modifications are considered.

algebras associated to such constructions, and, interestingly, they exhibit interesting deformations of such algebras (up to local field transformations).

II. WARM-UP: A NEW REPRESENTATION OF THE VIRASORO ALGEBRA IN $D=2$

As mentioned in the Introduction, the connection between the Abelian $u(1)^D$ algebra and the Virasoro algebra is not limited to the well-known construction $L_n = \frac{1}{2}\sum_{q=-\infty}^{\infty} :\alpha^i_{n-q}\alpha^i_q:$, where $::$ denotes normal ordering. This section aims to explore this relationship in greater depth. We claim that, in $D=2$, there exist many additional constructions that also satisfy the Virasoro algebra. In two dimensions, besides δ^{ij}, there is also a rotationally invariant tensor ϵ^{ij}, which allows us to define a richer class of operators beyond the standard Virasoro generators.

As a warm-up, let us define the following operator:

$$S_n = \frac{1}{2}\sum_q (-1)^q \epsilon^{ij}\alpha^i_{n-q}\alpha^j_q. \tag{3}$$

This is a new operator, well defined for odd values of n (for even n, it simply vanishes). The commutators of S_n with α^i_m, namely, $[S_n, \alpha^i_m]$, can be computed straightforwardly. Since the commutators of α^i_n are c numbers, we may safely ignore normal ordering. Consequently, the above commutators simplify to the following:

$$[S_n, \alpha^i_m] = (-1)^m m\epsilon^{ij}\alpha^j_{n+m}. \tag{4}$$

It is straightforward to verify that the set consisting of the standard L_n with even n together with the new operators S_n with odd n satisfies the Virasoro algebra, thus providing an extended realization of the Virasoro algebra beyond the standard construction.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d}{12}n(n^2-1)\delta_{n+m,0}, \tag{5}$$

$$[S_n, L_m] = (n-m)S_{n+m}, \tag{6}$$

$$[S_n, S_m] = (n-m)L_{n+m} + \frac{d}{12}n(n^2-1)\delta_{n+m,0}. \tag{7}$$

Here $d=2$. A natural question that arises is, what are the most general operators constructed from the modes α^i_n that satisfy the Virasoro algebra?

III. THE GENERIC CASE: THE MOD N OPERATORS

As illustrated by the example in the previous section, the Virasoro operators naturally split into two sets: those with even indices, corresponding to the standard representation, and those with odd indices, corresponding to the new operators. This observation motivates the consideration of operators S^k_n defined modulo N, where each operator belongs to an equivalence class labeled by $[n]=k$ for $k=0,1,\ldots,N-1$. In analogy with the $N=2$ case discussed in the previous section, we assume that the class S^0_n corresponds to the standard Virasoro representation

$$S^0_n = L_n = \frac{1}{2}\sum_q \alpha^i_{n-q}\alpha^i_q, \qquad n = N\mathbb{Z}. \tag{8}$$

The most general operators in class k, which we denote by V^k_m and U^k_m, can be constructed from the tensors ϵ^{ij} and δ^{ij} as follows:

$$V^k_n = \frac{1}{2}\sum_q f^k(q)\epsilon^{ij}:\alpha^i_{n-q}\alpha^j_q:, \tag{9}$$

$$U^k_n = \frac{1}{2}\sum_q h^k(q):\alpha^i_{n-q}\alpha^i_q:, \tag{10}$$

where $k=[n]=1,\ldots,N-1$. Requiring these operators to be invariant under the transformation $q \to m-q$ imposes the following constraints:

$$f^k(n-q) = -f^k(q), \qquad h^k(n-q) = h^k(q), \tag{11}$$

for all q. The most general operator for S^k_n is then the sum of these two operators: $S^k_n = U^k_n + V^k_n$. Let us now compute the commutators of L_n with V^k_n and[3] U^k_n:

$$[L_n, V^k_m] = \frac{1}{2}\sum_q \epsilon^{ij}(-(m-q)f^k(q) - f^k(q-n)(q-n))\alpha^i_{n+m-q}\alpha^j_q. \tag{12}$$

To ensure the closure of the algebra under commutation, the results of the previous computation require that the following relation must hold:

$$-(m-q)f^k(q) - f^k(q-n)(q-n) = (n-m)f^k(q). \tag{13}$$

Since n takes values in $N\mathbb{Z}$ (the index of L_n), it is sufficient for $f^k(q)$ to be invariant modulo N to satisfy the above relation. The most general solution is then given by

$$f^k(q) = \sum_{w=0}^{N-1} c^k_w e^{\frac{2\pi w i}{N}q}. \tag{14}$$

We also need to impose the constraints (11), which consequently yield the following relation:

[3]Note that, since $k \neq 0$, normal ordering is not relevant.

$$\sum_{w=0}^{N-1} c_w^k e^{\frac{2\pi w i}{N} q} = -\sum_{w=0}^{N-1} c_w^k e^{\frac{2\pi w i}{N}(k-q))}. \tag{15}$$

This condition is satisfied when the coefficients take the following form:

$$c_{N-w}^k = -c_w^k e^{\frac{2\pi i w k}{N}}. \tag{16}$$

It should be noted that $c_N^k = c_0^k$. The same holds for U_n^k, with a different set of constraints arising from (11), given as follows:

$$d_{N-w}^k = d_w^k e^{\frac{2\pi i w k}{N}}. \tag{17}$$

Therefore, the most general form of the operator S_n^k that is consistent with L_n is given by

$$\begin{aligned} S_n^k &= U_n^k + V_n^k \\ &= \frac{1}{2}\sum_q \sum_{w=0}^{N-1} (d_w^k e^{\frac{2\pi w i}{N} q} \alpha_{n-q}^i \alpha_q^i + c_w^k e^{\frac{2\pi w i}{N} q} \epsilon^{ij} \alpha_{n-q}^i \alpha_q^j). \end{aligned} \tag{18}$$

This is subject to the conditions (16) and (17). As shown in Appendix A, the closedness of the Witt algebra, $[S_n^{k_1}, S_m^{k_1}] = (n-m)S_{n+m}^{k_1+k_2}$, leads to the following relations[4]:

$$\sum_{\substack{w_1,w_2 \\ w_1+w_2=s \mod N}} (d_{w_1}^{k_1} d_{w_2}^{k_2} - c_{w_1}^{k_1} c_{w_2}^{k_2}) \left(\frac{\zeta_{k_1}^{-w_2} + \zeta_{k_2}^{-w_1}}{2}\right) = d_s^{k_1+k_2}, \tag{19}$$

$$\sum_{\substack{w_1,w_2 \\ w_1+w_2=s \mod N}} (d_{w_1}^{k_1} c_{w_2}^{k_2} + c_{w_1}^{k_1} d_{w_2}^{k_2}) \left(\frac{\zeta_{k_1}^{-w_2} + \zeta_{k_2}^{-w_1}}{2}\right) = c_s^{k_1+k_2}, \tag{20}$$

where $\zeta_k = e^{\frac{2\pi i k}{N}}$. Furthermore, we require that the following constraints be satisfied by c_ω^k and d_ω^k:

$$\sum_{\substack{w_1,w_2 \\ w_1+w_2=s \mod N}} (d_{w_1}^{k_1} d_{w_2}^{k_2} - c_{w_1}^{k_1} c_{w_2}^{k_2}) \left(\frac{\zeta_{k_1}^{-w_2} - \zeta_{k_2}^{-w_1}}{2}\right) = 0, \tag{21}$$

$$\sum_{\substack{w_1,w_2 \\ w_1+w_2=s \mod N}} (d_{w_1}^{k_1} c_{w_2}^{k_2} + c_{w_1}^{k_1} d_{w_2}^{k_2}) \left(\frac{\zeta_{k_1}^{-w_2} - \zeta_{k_2}^{-w_1}}{2}\right) = 0, \tag{22}$$

for all $s = 0, 1, \ldots, N-1$.

Solving all of the above constraints simultaneously is not straightforward; however, in the next subsections, we present the complete set of solutions for $\mathbb{Z}_2$ and $\mathbb{Z}_3$, suggesting that solutions may exist for all $\mathbb{Z}_N$.

A. The $\mathbb{Z}_2$ case

The mod 2 case provides the simplest example. Here, the class $k = 1$ is the only additional class besides the standard class $k = 0$, which corresponds to the usual representation of L_n for even n. Imposing the conditions (16) and (17) for $N = 2$ leads to the following identities:

$$c_0^1 = 0, \qquad d_1^1 = 0, \tag{23}$$

so that d_0^1 and c_1^1 are the only nonzero coefficients in the construction of S_n^1. All the equations (19)–(22) are satisfied when the following relation between d_0^1 and c_1^1 holds:

$$(d_0^1)^2 + (c_1^1)^2 = 1. \tag{24}$$

The standard representation of S_n^1, which corresponds to the usual construction of the Virasoro operator L_n for odd n, is obtained for $d_0^1 = 1$ and $c_1^1 = 0$. The alternative construction discussed in Sec. II corresponds to $d_0^1 = 0$ and $c_1^1 = 1$. Allowing the coefficients c_1^1 and d_0^1 to be complex numbers, the solutions of (24) satisfy $(d_0^1)^2 + (c_1^1)^2 = 1$ defines an algebraic curve in $\mathbb{C}^2$. The space of the solutions in $\mathbb{C}^2$ is topologically equivalent to a cylinder or a sphere with two punctures.

However, let us consider the above equation as the affine equation of $X^2 + Y^2 = Z^2$ in the chart defined by $Z \neq 0$, $d_0^1 = \frac{X}{Z}$, and $c_1^1 = \frac{Y}{Z}$ for some projective space with appropriate dimension. This identifies the solution space with the complex projective line $\mathbb{C}P^1$, which is topologically a sphere. This is an interesting result: There exist infinitely many alternative constructions beyond the usual Virasoro operator L_n for odd n.

1. Equivalent solutions

It appears that we have identified the true moduli space of constructions for the $\mathbb{Z}_2$ case. However, there exist automorphisms of the current algebra

$$[\alpha_n^i, \alpha_m^j] = n\delta^{ij}\delta_{n+m,0} \tag{25}$$

which relate the two seemingly distinct solutions to each other. By introducing mod -2 operators α_n^{ki}, where $n \in 2\mathbb{Z} + k$ and $k = 0, 1$, the above current algebra decomposes into two commuting current algebras as follows:

$$\begin{aligned} [\alpha_n^{0i}, \alpha_m^{0j}] &= n\delta^{ij}\delta_{n+m,0}, \\ [\alpha_n^{1i}, \alpha_m^{1j}] &= n\delta^{ij}\delta_{n+m,0}, \\ [\alpha_n^{0i}, \alpha_m^{1j}] &= 0. \end{aligned} \tag{26}$$

[4]In the computations, we ignore normal ordering, which becomes important when $k_1 + k_2 \equiv 0 \mod N$. The central term of the Virasoro algebra can be determined consistently via the Jacobi identity.

Each sector admits a $\mathbb{Z}_2$ automorphism of the form $\alpha_n^{ki} \to -\alpha_n^{ki}$. Consequently, the total algebra possesses a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. This observation implies that $(d_0^1, c_1^1) \sim -(d_0^1, c_1^1)$, and, therefore, the space of solutions must be quotiented by an overall $\mathbb{Z}_2$ factor.

There is an interesting nonlocal equivalence between these algebras. The Abelian Kac-Moody algebra arises from free boson theory. Suppose we have two free bosons $\tilde{X}^i$ and X^i. Then the following nonlocal transformation:

$$\tilde{X}(\tau,\sigma) = \frac{1+e^{i\theta}}{2} X(\tau,\sigma) + \frac{1-e^{i\theta}}{2} X(\tau,\sigma+\pi), \quad (27)$$

where $X \equiv (X^1 + iX^2)/\sqrt{2}$, relates the above algebras to each other, with θ parametrizing the circle defined in (24). However, we do not consider such nonlocal transformations in this paper.

2. Unitarian subsector

The above analysis demonstrates that there exists a vast number of new constructions beyond the standard Virasoro realization. However, from the conventional point of view, not all of these constructions correspond to Hermitian operators or possess a natural relation to their adjoint counterparts. To maintain consistency with the standard framework of two-dimensional conformal field theory, we assume the usual adjoint action on the α_n^i modes:

$$\alpha_n^{i\dagger} = \alpha_{-n}^i, \quad (28)$$

which induces an adjoint action on the Virasoro operators S_n. A natural expectation is for the following relation to hold:

$$L_n^\dagger = L_{-n}. \quad (29)$$

This imposes a strong constraint on the space of solutions. The even sector, represented by $S_n^0 = L_n$, is by definition consistent with this adjoint action. However, the odd sector requires additional restrictions. Explicitly, one finds

$$(S_n^1)^\dagger = \frac{1}{2}\left(d_0^{1*} \sum_q \alpha_{-n-q}^i \alpha_q^i + c_1^{1*} \sum_q \epsilon^{ij} \alpha_{-n-q}^i \alpha_q^j \right), \quad (30)$$

which is consistent with Eq. (29), when $d_0^{1*} = d_0^1$ and $c_1^{1*} = c_1^1$. Therefore, the space of solutions consistent with the adjoint action is $\frac{S^1}{\mathbb{Z}_2}$, which is topologically a circle.

B. The $\mathbb{Z}_3$ case

Although we cannot solve Eqs. (19)–(22) in closed form, they can still be solved directly without much difficulty. Beyond the $\mathbb{Z}_2$ case, it is instructive to examine the structure of the $\mathbb{Z}_3$ case and understand how it differs from $\mathbb{Z}_2$.

The constraints (16) and (17) determine the following relations between c_ω^k and d_ω^k:

$$d_2^1 = d_1^1 e^{\frac{2\pi i}{3}}, \qquad d_2^2 = d_1^2 e^{\frac{4\pi i}{3}}, \quad (31)$$

$$c_2^1 = -c_1^1 e^{\frac{2\pi i}{3}}, \qquad c_2^2 = -c_1^2 e^{\frac{4\pi i}{3}}, \qquad c_0^k = 0. \quad (32)$$

Working with the independent coefficients, the consistency of Eqs. (19)–(22) yields the following constraints on the coefficients:

$$d_0^1 d_0^2 + 2(d_1^1 d_1^2 + c_1^1 c_1^2) = 1, \quad (33)$$

$$d_1^1 d_1^2 - c_1^1 c_1^2 + e^{\frac{4\pi i}{3}} d_1^1 d_0^2 + e^{\frac{2\pi i}{3}} d_0^1 d_1^2 = 0, \quad (34)$$

$$e^{\frac{2\pi i}{3}} d_0^1 c_1^2 + e^{\frac{4\pi i}{3}} d_0^2 c_1^1 - (d_1^1 c_1^2 + c_1^1 d_1^2) = 0, \quad (35)$$

$$(d_0^1)^2 - e^{\frac{2\pi i}{3}}((d_1^1)^2 + (c_1^1)^2) = d_0^2, \quad (36)$$

$$(d_1^1)^2 - (c_1^1)^2 - e^{\frac{2\pi i}{3}} d_0^1 d_1^1 = d_1^2, \quad (37)$$

$$-e^{\frac{2\pi i}{3}} d_0^1 c_1^1 - 2 d_1^1 c_1^1 = c_1^2, \quad (38)$$

$$(d_0^2)^2 - e^{\frac{4\pi i}{3}}((d_1^2)^2 + (c_1^2)^2) = d_0^1, \quad (39)$$

$$-e^{\frac{4\pi i}{3}} d_0^2 d_1^2 + (d_1^2)^2 - (c_1^2)^2 = d_1^1, \quad (40)$$

$$-e^{\frac{4\pi i}{3}} d_0^2 c_1^2 - 2 d_1^2 c_1^2 = c_1^1, \quad (41)$$

$$(d_0^1 d_1^2 + d_1^1 d_0^2)(e^{\frac{4\pi i}{3}} - 1) + (d_1^1 d_1^2 - c_1^1 c_1^2)(e^{\frac{2\pi i}{3}} - e^{\frac{4\pi i}{3}}) + d_1^1 d_0^2 (1 - e^{\frac{2\pi i}{3}}) = 0, \quad (42)$$

$$(d_1^1 c_1^2 + c_1^1 d_1^2)(e^{\frac{2\pi i}{3}} - e^{\frac{4\pi i}{3}}) - d_0^1 c_1^2 (e^{\frac{2\pi i}{3}} - 1) - c_1^1 d_0^2 (1 - e^{\frac{2\pi i}{3}}) = 0. \quad (43)$$

The above equations appear too complicated to be solved analytically. However, one can use Eqs. (36)–(38) to express d_0^2, d_1^2, and c_1^2 in terms of d_0^1, d_1^1, and c_1^1. To simplify the notation, let us define $d_0^1 \equiv x$, $d_1^1 \equiv w$, and $c_1^1 \equiv y$. Eliminating d_0^2, d_1^2, and c_1^2 from the equations then yields the following simplified system:

$$x^3 + 2w^3 - 3e^{\frac{2\pi i}{3}} x(w^2 + y^2) - 6wy^2 = 1, \quad (44)$$

$$z_i(x^3 + 2w^3 - 3e^{\frac{2\pi i}{3}} x(w^2 + y^2) - 6wy^2 = 1), \quad (45)$$

where z_i is equal to x, w, or y.

1. Equivalent solutions

In the $\mathbb{Z}_3$ case, the automorphisms of the current algebra are more intricate, and one must carefully account for equivalent solutions. The current algebra remains invariant under the transformations

$$\alpha_n^{0i} \to -\alpha_n^{0i}, \qquad \alpha_n^{1i} \to \lambda\alpha_n^{1i}, \qquad \alpha_n^{2i} \to \frac{1}{\lambda}\alpha_n^{2i}, \tag{46}$$

for any nonzero $\lambda \in \mathbb{C}^\times$. Since the coefficients of the operators in (18) depend only on the residue classes modulo N, we need only to consider automorphisms that act identically on all members of a given class. Consequently, the automorphism group compatible with the $\mathbb{Z}_3$ grading is isomorphic to $\mathbb{C}^\times \times \mathbb{Z}_2$. Hence, two solutions (x', w', y') and (x, w, y) are equivalent, if there exist $\lambda \in C^\times$ satisfying the following relations:

$$x' - \omega^2 w' = \lambda(x - \omega^2 w), \tag{47}$$

$$x' + 2\omega^2 w' = \frac{1}{\lambda^2}(x + 2\omega^2 w), \tag{48}$$

$$y' = \lambda y, \tag{49}$$

where $\omega = e^{\frac{2\pi i}{3}}$. Therefore, the space of inequivalent constructions is obtained by quotienting the solution space (44) by the above equivalence relation. Let us define

$$x_1 = x - \omega^2 w, \tag{50}$$

$$x_2 = x + 2\omega^2 w, \tag{51}$$

$$x_3 = \sqrt{3}\omega^2 y, \tag{52}$$

and, interestingly, Eq. (44) can be written as

$$(x_1 - x_3)(x_1 + x_3)x_2 = 1. \tag{53}$$

Since both x_2 and λ are nonzero, we can always fix the gauge by setting $x_2 = 1$ through an appropriate choice of λ. Under this gauge fixing, the defining equation simplifies to

$$x_1^2 - x_3^2 = 1, \tag{54}$$

which admits the following parametrization:

$$x_1 = \frac{t + t^{-1}}{2}, \qquad x_3 = \frac{t - t^{-1}}{2}. \tag{55}$$

This representation makes it clear that the space of solutions is topologically equivalent to a cylinder.

2. Special constructions

To obtain concrete examples of the new representations of $\mathbb{Z}_3$, we focus on the subset of solutions characterized by the vanishing of at least one of x, w, or y. In the zero-dimensional case, we impose two constraints on the solution space simultaneously. No solutions exist when $x = w = 0$, which implies that there are no representations for the classes $k = 1$ and $k = 2$ defined solely by the tensor ϵ^{ij} (without contributions from the δ^{ij} term). This is in contrast to the $\mathbb{Z}_2$ case, where a representation does exist for $k = 1$.

Suppose that $w = y = 0$. In this case, we obtain

$$x^3 = 1 \to d_0^1 = e^{\frac{2\pi i a}{3}}, \tag{56}$$

for $a = 0, 1, 2$. The only remaining nonzero coefficient is $d_0^2 = e^{\frac{4\pi i a}{3}}$. Substituting this solution into Eq. (18) reproduces the standard representations of the Virasoro algebra:

$$[S_n^{k_1}, S_m^{k_2}] = (n - m)S_{n+m}^{k_1+k_2} + \frac{d}{12}(n^3 - n)\delta^{k_1+k_2,0}\delta_{m+n,0}. \tag{57}$$

The Virasoro algebra (57) is invariant under the transformation $S_n^k \to e^{\frac{2\pi i k}{N}} S_n^k$, which is an element of the automorphism group of the Virasoro algebra. Consequently, the factors $e^{\frac{2\pi i a}{3}}$ and $e^{\frac{4\pi i a}{3}}$ can be omitted, yielding the standard representation of the Virasoro algebra.

For a more nontrivial solution, let us impose the constraints $x = 0$ and $y^2 + w^2 = 0$. In this case, we obtain

$$8w^3 = 1 \to d_1^1 = \frac{1}{2}e^{\frac{2\pi i a}{3}}, \tag{58}$$

for $a = 0, 1, 2$. The remaining nonzero independent coefficients are $d_1^2 = \frac{1}{2}e^{\frac{4\pi i a}{3}}$ and $c_1^2 = \mp\frac{1}{2}ie^{\frac{4\pi i a}{3}}$. As before, the phase factors $e^{\frac{2\pi i a}{3}}$ and $e^{\frac{4\pi i a}{3}}$ can be absorbed by rescaling the operators. Consequently, we are left with two independent solutions, corresponding to the two roots of $w^2 = -y^2$. The resulting algebras are given by

$$S_n^1 = \frac{1}{4}\sum_q [(e^{\frac{2\pi i}{3}q} + \omega e^{\frac{4\pi i}{3}q})\delta^{ij} + i(e^{\frac{2\pi i}{3}q} - \omega e^{\frac{4\pi i}{3}q})\epsilon^{ij}]\alpha_{n-q}^i \alpha_q^j, \tag{59}$$

$$S_n^2 = \frac{1}{4}\sum_q [(e^{\frac{2\pi i}{3}q} + \omega^2 e^{\frac{4\pi i}{3}q})\delta^{ij} - i(e^{\frac{2\pi i}{3}q} - \omega^2 e^{\frac{4\pi i}{3}q})\epsilon^{ij}]\alpha_{n-q}^i \alpha_q^j, \tag{60}$$

and the other solution is given by

$$S_n^1 = \frac{1}{4}\sum_q [(e^{\frac{2\pi i}{3}q} + \omega e^{\frac{4\pi i}{3}q})\delta^{ij} - i(e^{\frac{2\pi i}{3}q} - \omega e^{\frac{4\pi i}{3}q})\epsilon^{ij}]\alpha_{n-q}^i \alpha_q^j, \tag{61}$$

$$S_n^2=\frac{1}{4}\sum_q[(e^{\frac{2\pi i}{3}q}+\omega^2 e^{\frac{4\pi i}{3}q})\delta^{ij}+i(e^{\frac{2\pi i}{3}q}-\omega^2 e^{\frac{4\pi i}{3}q})\epsilon^{ij}]\alpha^i_{n-q}\alpha^j_q. \tag{62}$$

Together with $S_n^0 = L_n$ for $n \in 3\mathbb{Z}$, these operators define new representations of the Virasoro algebra expressed in terms of α_n^i. The above operators are of special interest, which are compatible with the adjoint action. More precisely, for the above two solutions, we have

$$S_n^{1\dagger} = S_{-n}^2. \tag{63}$$

We analyze the space of such solutions for the $\mathbb{Z}_3$ in Sec. III B 3. It is easy to show that there is no nontrivial solution that depends on the δ^{ij} solely. Such a space of solutions corresponds to $y = 0$:

$$x^3 + 2w^3 - 3\omega x w^2 = 1, \tag{64}$$

which admits a rational parametrization. Explicitly, a rational parametrization of the affine curve (64) is given by

$$x = \frac{2t^3+1}{3t^2}\omega^2, \qquad w = \frac{1-t^3}{3t^2}, \tag{65}$$

for $t \neq 0$. By choosing $\lambda = \frac{1}{\omega^2 t}$ and applying the equivalence relations (47)–(49), we find that the following identities hold:

$$\lambda(x-\omega^2 w) = 1, \qquad \frac{1}{\lambda^2}(x+2\omega^2 w) = 1. \tag{66}$$

This implies that there are no nontrivial solutions that depend solely on δ^{ij} and the only such solution is the trivial one. In Sec. V, we will demonstrate that this conclusion extends to the general $\mathbb{Z}_N$ case as well.

3. Unitarian subsector

As in the case of the $\mathbb{Z}_2$ construction, it is of particular interest to determine the subset of constructions that remain compatible with the condition (29):

$$S_{3n+1}^{1\dagger} = S^2_{-(3n+1)}. \tag{67}$$

Since the space of constructions is determined by Eq. (54), we can explicitly express L_n^1 and L_n^2 in terms of the coordinates x_1 and x_3 (after fixing the gauge $x_2 = 1$) as follows:

$$S^1_{3n+1} = x_1\alpha^i_{3n+1-3q}\alpha^i_{3q} + \frac{1}{2}\alpha^i_{3n+1-(3q+2)}\alpha^i_{3q+2} - \frac{1}{2\sqrt{3}}(\omega-\omega^2)x_3\epsilon^{ij}(\alpha^i_{3n+1-3q}\alpha^j_{3q} - \alpha^i_{3n+1-(3q+1)}\alpha^j_{3q+1}), \tag{68}$$

$$S^2_{3n+2} = x_1\alpha^i_{3n+2-3q}\alpha^i_{3q} + \frac{1}{2}\alpha^i_{3n+2-(3q+1)}\alpha^i_{3q+1} - \frac{1}{2\sqrt{3}}(\omega^2-\omega)x_3\epsilon^{ij}(\alpha^i_{3n+2-3q}\alpha^j_{3q} - \alpha^i_{3n+2-(3q+2)}\alpha^j_{3q+2}), \tag{69}$$

where the sum over q is implicit.

The unitarity condition (67) requires that

$$x_1^* = x_1, \qquad x_3^* = x_3, \tag{70}$$

which implies that the space of unitary constructions forms a hyperbola in the (x_1, x_3) plane.

C. The $\mathbb{Z}_4$ case

Both the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cases exhibit a similar pattern. The spaces of constructions given in Eqs. (24) and (54) are topologically cylinders over the complex numbers. However, the unitary subsector differs between cases: For $\mathbb{Z}_2$, it corresponds to a circle, while for $\mathbb{Z}_3$, it forms a hyperbola. This observation motivates us to investigate the $\mathbb{Z}_4$ case, where new layers of complexity emerge. The number of defining equations in the $\mathbb{Z}_4$ case is considerably larger, and, for clarity, we have summarized them in Appendix B. The constraints (16) and (17) indicate that several coefficients are not independent, leading to the following relations:

$$d_3^1 = id_1^1, \quad d_2^1 = 0, \quad d_3^2 = -d_1^2, \quad d_3^3 = -id_1^3, \quad d_2^3 = 0, \tag{71}$$

$$c_3^1 = -ic_1^1, \quad c_3^2 = c_1^2, \quad c_2^2 = 0, \quad c_3^3 = ic_1^3. \tag{72}$$

For convenience, we introduce the notations $d_0^1 \equiv x$, $d_1^1 \equiv w$, $c_1^1 \equiv y$, and an additional variable $c_2^1 \equiv u$. It is then straightforward that the space of all equations in Appendix B reduces to a single equation as follows:

$$\begin{aligned}&-4w^4 - 4iw^2x^2 + x^4 + 8w^2y^2 - 4ix^2y^2 - 4y^4\\&-16wxyu + 4iw^2u^2 + 2x^2u^2 + 4iy^2u^2 + u^4 = 1,\end{aligned} \tag{73}$$

which defines a six-dimensional hypersurface. However, as we discussed for the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ cases, this hypersurface must be quotiented by the automorphisms of the current

algebra that are consistent with the $\mathbb{Z}_4$ grading. The current algebra with $\mathbb{Z}_4$ grading is invariant under the following rescaling transformations:

$$\alpha_q^{0i} \to -\alpha_q^{0i}, \qquad \alpha_q^{2i} \to -\alpha_q^{2i}, \qquad \alpha_q^{1i} \to \lambda\alpha_q^{1i}, \qquad \alpha_q^{3i} \to \frac{1}{\lambda}\alpha_q^{3i}, \tag{74}$$

where λ is a nonzero complex parameter. Hence, the automorphism group of the current algebra consistent with the $\mathbb{Z}_4$ grading is given by $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{C}^\times$.

Analogous to the $\mathbb{Z}_3$ case, we introduce the following coordinate redefinitions:

$$\begin{aligned} x_1 &= x + (1+i)w, \qquad & x_2 &= x - (1+i)w, \\ x_3 &= (1-i)y + u, \qquad & x_4 &= (1-i)y - u. \end{aligned} \tag{75}$$

With these definitions, Eq. (73) simplifies to the compact form

$$(x_1 + ix_3)(x_1 - ix_3)(x_2 + ix_4)(x_2 - ix_4) = 1, \tag{76}$$

and the rescaling transformation (74) induces the following equivalence relations among the new coordinates:

$$x_1 \sim \pm\lambda x_1, \qquad x_2 \sim \frac{1}{\lambda}x_2, \qquad x_3 \sim \pm\lambda x_3, \qquad x_4 \sim \frac{1}{\lambda}x_4. \tag{77}$$

Therefore, to obtain the true moduli space of constructions, the six-dimensional hypersurface must be quotiented by the group $\mathbb{Z}_2 \times \mathbb{C}^\times$. Since $x_2 - ix_4$ cannot be zero, one can set $x_2 - ix_4 = 1$ by an appropriate $\lambda \in \mathbb{C}^\times$. Then, Eq. (76) simplifies as follows:

$$(2x_2 - 1)(x_1 + ix_3)(x_1 - ix_3) = 1, \tag{78}$$

which should also be quotiented by an additional $\mathbb{Z}_2$ symmetry acting as $x_1 \sim -x_1$ and $x_3 \sim -x_3$.

D. The general pattern

The analysis of the $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$ cases reveals the general pattern. For a general $\mathbb{Z}_N$ grading, the corresponding current algebra

$$[\alpha_n^{k_1 i}, \alpha_m^{k_2 j}] = n\delta^{k_1+k_2=N \bmod N}\delta_{n+m,0}\delta^{ij} \tag{79}$$

admits automorphisms that preserve the $\mathbb{Z}_N$ grading of the form

$$\alpha_n^{ki} \to \lambda_k\alpha_n^{ki}, \qquad \alpha_n^{(N-k)i} \to \lambda_k^{-1}\alpha_n^{(N-k)i}, \tag{80}$$

for arbitrary nonzero complex parameters λ_k. When N is even, the number of complex parameters λ_k is $\frac{N-2}{2}$ and we have an extra $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. For odd N, the number of complex parameters λ_k is $\frac{N-1}{2}$ together with an extra $\mathbb{Z}_2$ symmetry. The preceding analysis shows that the hypersurface determined by Eqs. (19)–(22) should be quotiented by the symmetry group $(\mathbb{C}^\times)^{\frac{N-2}{2}} \times \mathbb{Z}_2$ when N is even and by $(\mathbb{C}^\times)^{\frac{N-1}{2}}$ when N is odd. This demonstrates that the space of constructions for $\mathbb{Z}_{2N}$ and $\mathbb{Z}_{2N+1}$ is more or less the same over the complex numbers. However, (24) and (54) indicate that the unitarian subsectors, which are defined over the real numbers, may differ.

IV. NEW ALGEBRAS FROM THE ACTION PRINCIPLE

In the preceding sections, we have investigated the relationship between the $u(1)^2$ Kac-Moody algebra and the Virasoro algebra from a strictly algebraic standpoint, neglecting the question of how these algebras arise from symmetries of an underlying action. This perspective is often preferred in the physics literature. In this section, we turn to a more physically motivated problem: the construction of actions whose symmetry algebras reproduce the specific representations of the Virasoro algebra considered above.

As a first step, we focus on the representation defined in Eq. (3) and analyze the conditions under which it can be derived from an action principle. This example will then serve as a prototype for formulating a more general and systematic procedure for obtaining such representations. In this construction, the index n associated with the operator S_n^1 is restricted to odd integers:

$$S_n^1 = \frac{1}{2}\sum_q (-1)^q \epsilon^{ij} :\alpha_{n-q}^i \alpha_q^j:, \tag{81}$$

which couples the odd-index modes α_n^i to the even-index modes.

To analyze the origin of this representation, let us consider a set of scalar fields ϕ^i defined on a flat cylinder with coordinates (τ, σ), where $-\infty < \tau < \infty$ and $0 \le \sigma \le 2\pi$. We impose the periodic boundary condition

$$\phi^i(\tau, \sigma) = \phi^i(\tau, \sigma + 2\pi), \tag{82}$$

ensuring that ϕ^i is single valued on the spatial circle. Under this condition, the scalar field admits the Fourier mode expansion

$$\phi^i = \sum_n a_n^i(\tau)e^{in\sigma} \equiv \phi_0^i + \phi_1^i, \tag{83}$$

where ϕ_0^i contains only the modes with $n \in 2\mathbb{Z}$ while ϕ_1^i contains those with $n \in 2\mathbb{Z} + 1$. This decomposition naturally suggests that the operator defined in (81) originates from a schematic interaction terms of the form

$$\sim\epsilon_{ij}\phi^i_{k_1}\phi^j_{k_2}, \tag{84}$$

where $k_{1,2}=0,1$ label the two $\mathbb{Z}_2$ sectors corresponding to even and odd modes, respectively. In other words, the representation (81) reflects a mod-2 partition of the integer mode numbers.

While this interpretation appears natural, it suffers from a significant drawback: The decomposition into even and odd Fourier modes relies crucially on the flat metric of the cylinder. Fourier modes acquire an unambiguous meaning only once a particular representative of the conformal class has been chosen, effectively amounting to a gauge fixing of the underlying gauge symmetries. Ideally, one would like the index $k\in\mathbb{Z}_2$ to possess an intrinsic, coordinate-independent meaning that does not depend on the specific choice of representative within the conformal class of metrics on the cylinder.

In what follows, we develop a framework capable of addressing this difficulty in a systematic manner. Let M be the underlying manifold and let $\mathrm{Diff}(M)$ denote its diffeomorphism group. Given a diffeomorphism $f\in\mathrm{Diff}(M)$,

$$f:M\to M \tag{85}$$

induces a natural pullback map on the algebra of smooth complex-valued functions defined on M:

$$f^*:C^\infty(M,\mathbb{C})\to C^\infty(M,\mathbb{C}), \tag{86}$$

which acts on any $\phi\in C^\infty(M,\mathbb{C})$ according to

$$f^*\phi(p)=\phi(f(p)), \tag{87}$$

A further ingredient in this construction is the introduction of a group action (finite group or a finite Lie group) on M. More precisely, let G be a group and consider a homomorphism

$$\begin{aligned}\psi:\ G&\to\mathrm{Diff}(M),\\ g&\mapsto\psi_g,\end{aligned} \tag{88}$$

which assigns to each $g\in G$ a diffeomorphism $\psi_g\in\mathrm{Diff}(M)$. In this way, G acts on the space of smooth functions on M via the pullback induced by ψ_g. This viewpoint will allow us to reformulate the even or odd decomposition in terms of group representations, potentially giving the $\mathbb{Z}_2$ grading an intrinsic interpretation.

We now impose the requirement that the field $\phi:M\to\mathbb{C}$ satisfy the equivariance condition

$$\psi_g^*\phi(p)=\phi(\psi_g(p))=L(g)\phi(p), \tag{89}$$

where L denotes a (generally linear) representation of the group G.[5]

[5]For notational simplicity, we have suppressed all additional indices; the indexed case is straightforward.

We formalize the above property by saying that the field $\phi:M\to V$ (with $V=\mathbb{C}$) is *G-equivariant* with respect to the group action

$$\psi:\ G\to\mathrm{Diff}(M),\qquad g\mapsto\psi_g,$$

and a representation

$$L:\ G\to GL(V),$$

if the following diagram commutes for all $g\in G$:

$$\begin{array}{ccc} M & \xrightarrow{\psi_g} & M \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ V & \xrightarrow[L(g)]{} & V \end{array}$$

This expresses the condition $\phi(\psi_g(p))=L(g)\phi(p)$ for every $p\in M$. Moreover, if $f:N\to M$ is a diffeomorphism, the diagram extends as

$$\begin{array}{ccc} N & \xrightarrow{\tilde\psi_g} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\psi_g} & M \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ V & \xrightarrow[L(g)]{} & V \end{array}$$

where $\tilde\psi_g=f^{-1}\circ\psi_g\circ f$ is the conjugated action on N and $\tilde\phi=\phi\circ f$ is the pullback field. The commutativity of the diagram guarantees that the equivariance property is preserved under diffeomorphic changes of the base manifold.

More specifically, suppose that $\phi(p)$ satisfies the equivariance condition (89). Then its pullback under f, denoted by $\tilde\phi:=\phi\circ f$, satisfies the same relation with respect to the induced action $\tilde\psi_g$. Indeed, we have

$$\begin{aligned}\tilde\psi^*\tilde\phi(p)&=\tilde\phi(\tilde\psi_g(p))=(\phi\circ f\circ f^{-1}\circ\psi_g\circ f)(p)\\ &=\phi(\psi_g(f(p)))=\psi_g^*(\phi\circ f(p))\\ &=L(g)\phi\circ f(p)=L(p)\tilde\phi(p).\end{aligned} \tag{90}$$

Thus, the pullback $\tilde\phi$ inherits the same transformation property under the conjugated action $\tilde\psi_g$. Consequently, the condition (89) is invariant under diffeomorphisms between manifolds and, therefore, constitutes a well-defined notion.

Let us return to the case of the flat cylinder. The isometry group of the flat cylinder is $\mathbb{R}\rtimes O(2)$, where $\mathbb{R}$ corresponds to translations along the noncompact direction and $O(2)$ acts on the compact circle. The connected component of the identity is given by $\mathbb{R}\times S^1$. Consequently, one can always

construct a homomorphism from the finite cyclic subgroup Z_N into the diffeomorphism group of the flat cylinder. In fact, such a homomorphism can be defined for any manifold diffeomorphic to the flat cylinder, ensuring that the construction is robust under diffeomorphic changes of the background.

Since $\mathbb{Z}_N$ is a finite Abelian group, all of its irreducible representations are one-dimensional. If g denotes the generator of the group, then each irreducible representation is of the form

$$\rho_k\colon \mathbb{Z}_N \to \mathbb{C}^\times, \qquad \rho_k(g) = e^{\frac{2\pi i k}{N}}, \tag{91}$$

where $k = 0, 1, \ldots, N-1$. We are now in a position to give a natural and intrinsic meaning to the $\mathbb{Z}_N$ index k labeling the fields ϕ_k. Let M be a smooth manifold with diffeomorphism group $\mathrm{Diff}(M)$, and let $Z_N = \langle g \rangle$ be the finite cyclic group of the order of N. Choose a group homomorphism as follows:

$$\psi\colon Z_N \to \mathrm{Diff}(M), \qquad g \mapsto \psi_g.$$

A field $\phi_k\colon M \to \mathbb{C}$ is said to be of *sector* k if it is Z_N equivariant with respect to the character

$$\rho_k\colon Z_N \to \mathbb{C}^\times, \qquad \rho_k(g) = e^{2\pi i k/N}.$$

In other words, ϕ_k satisfies

$$\phi_k(\psi_g(p)) = \rho_k(g)\phi_k(p), \quad \forall\ p \in M,$$

which may be summarized by the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi_g} & M \\ {\scriptstyle\phi_k}\downarrow & & \downarrow{\scriptstyle\phi_k} \\ \mathbb{C} & \xrightarrow[\rho_k(g)]{} & \mathbb{C} \end{array}$$

for every $g \in Z_N$.

This construction provides an intrinsic, representation-theoretic interpretation of the index k: It labels the irreducible characters of Z_N under which the field transforms.

A. Equivariant conformal field theories

In the previous section, we introduced the concept of G-equivariant functions on a manifold endowed with a group action. Here, our goal is to formulate actions constructed from such functions and to investigate whether they can give rise to conformal symmetry. Although G equivariance is naturally connected to the new representations introduced in Sec. III, one must be cautious: These representations do not necessarily preserve conformal invariance, and their compatibility with conformal structures requires a careful analysis. As a concrete illustration, we turn to the case of $U(1)$-equivariant functions defined on a flat cylinder.

1. Example: $U(1)$-equivariant field on $\mathbb{R} \times S^1$

Let $M = \mathbb{R} \times S^1$ with coordinate $\sigma \in [0, 2\pi)$ for the circle. Consider the group $G = U(1)$ acting by rotations

$$\psi_\alpha\colon S^1 \to S^1, \qquad \psi_\alpha(\sigma) = \sigma + \alpha \pmod{2\pi}.$$

Let $V = \mathbb{C}$ and take the representation

$$L\colon U(1) \to \mathbb{C}^\times, \qquad L(e^{i\alpha}) = e^{in\alpha},$$

for integer n. Then a $U(1)$-equivariant field $\phi\colon S^1 \to \mathbb{C}$ satisfies

$$\phi(\sigma + \alpha) = e^{in\alpha}\phi(\sigma),$$

for all σ.

Now, let us consider the Polyakov action on manifolds that are topologically equivalent to the flat cylinder:

$$S \sim \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha \bar{\phi} \partial_\beta \phi. \tag{92}$$

One can always construct a homomorphism from the $U(1)$ group into the diffeomorphism group of such manifolds. For definiteness, let us fix the metric to be that of the flat cylinder and take the homomorphism as in the preceding example. Suppose further that the phase space of the theory is restricted to those maps satisfying the equivariance condition

$$\phi(\tau, \sigma + \alpha) = e^{in\alpha}\phi(\tau, \sigma). \tag{93}$$

In this case, the only surviving mode is of the form $e^{in\sigma}$. Consequently, the full Virasoro algebra cannot be recovered from such a restricted spectrum, as the dynamics reduce essentially to a single mode.

However, if we allow all possible representations, the full Virasoro algebra can, in fact, be recovered. This suggests that the obstruction observed in the single-mode case arises merely from an overly restrictive choice of equivariance. To address this, we turn to the study of the entire space of equivariant functions associated with a given group action. A natural question is whether a generic field can be decomposed into a sum of equivariant components, each transforming according to a different representation of the group. Such a decomposition would restore the richness of the mode expansion and, consequently, the algebraic structures associated with conformal symmetry.

Suppose that $C^\infty(M, \mathbb{C})$ is the space of all smooth complex-valued functions on the manifold M. As in the previous section, we assume that there exists a homomorphism from the finite group G to the diffeomorphism group of the manifold M by $\psi_g\colon G \to \mathrm{Diff}(M)$. We show that the

space of smooth functions on the manifold M, namely, $C^\infty(M,\mathbb{C})$, can be decomposed into subspaces of ρ-equivariant functions. For an irreducible representation ρ of G, we define

$$C^\infty(M)_\rho := \{F: M \to \mathbb{C}\,\text{smooth} | F(\psi_g(x)) = \rho(g)F(x), \quad \forall\ g \in G\}. \tag{94}$$

The decomposition of $C^\infty(M,\mathbb{C})$ into these isotypic subspaces is obtained by means of natural projection operators. Specifically, the projection onto $C^\infty(M,\mathbb{C})_\rho$ can be expressed by averaging over the group G against the character χ_ρ of ρ:

$$(P_\rho f)(x) := \frac{d_\rho}{|G|}\sum_{g\in G}\chi_\rho(g^{-1})f(\psi_g(x)). \tag{95}$$

Here, $d_\rho = \dim\rho$ and $\chi_\rho = \mathrm{Tr}(\rho)$ denotes the character of ρ. Since G is finite and Abelian, every irreducible representation ρ is one-dimensional, so $d_\rho = 1$ and $\chi_\rho(g) = \rho(g)$ for all $g \in G$.

The standard orthogonality relations of characters imply the following properties:

$$P_\rho^2 = P_\rho, \qquad P_\rho P_\sigma = 0 \quad \text{if } \rho \not\simeq \sigma, \qquad \sum_{\rho\in\hat{G}} P_\rho = \mathrm{Id}_{C^\infty(M)}. \tag{96}$$

Here, $\hat{G}$ denotes the set of all inequivalent irreducible representations of G. Hence, the family $\{P_\rho\}_{\rho\in\hat{G}}$ forms a complete set of mutually orthogonal projections, and one obtains the direct sum decomposition:

$$C^\infty(M,\mathbb{C}) \cong \bigoplus_{\rho\in\hat{G}} C^\infty(M)_\rho. \tag{97}$$

We are now in a position to introduce the action associated with the new representation discussed in Sec. III. To this end, let us consider the Polyakov action (92) and expand the fields ϕ^i in terms of G-equivariant components. More precisely, using the decomposition (97), we write

$$\phi^i = \sum_{k=0}^{N-1} a_k \phi^i_k. \tag{98}$$

This expansion allows us to reformulate the Polyakov action as a sum of contributions from the various isotypic components of the field space, thereby making the role of the G-equivariant decomposition explicit. Then the Polyakov action takes the following form:

$$S \sim \int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha\bar{\phi}^i_{k_1}\partial_\beta\phi^i_{k_2}\bar{a}_{k_1}a_{k_2}, \tag{99}$$

where summation over the indices k_1 and k_2 is understood. Here, the coefficients a_k encode the expansion of the fields ϕ^i in terms of the G-equivariant components ϕ^i_k, while the barred quantities denote complex conjugates.

There is a natural generalization of the action described above, which incorporates both symmetric and antisymmetric couplings among the equivariant components of the fields. Explicitly, we propose

$$S = \frac{-1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha\bar{\phi}^i_{k_1}\partial_\beta\phi^j_{k_2}(\delta_{ij}A_{k_1k_2} + \epsilon_{ij}B_{k_1k_2}), \tag{100}$$

where $A_{k_1k_2}$ (respectively, $B_{k_1k_2}$) are symmetric (respectively, antisymmetric) complex coefficients. The first term, proportional to δ_{ij}, is the generalized usual structure of the Polyakov action, while the second term, proportional to the antisymmetric tensor ϵ_{ij}, provides a natural generalization of it. We claim that this generalized action reproduces the Virasoro algebra in a form consistent with the new algebraic structures introduced in Sec. III. In the next section, we illustrate this construction in detail by working out the explicit cases of the finite groups $\mathbb{Z}_2$, which already display the essential features of the general framework.

B. $\mathbb{Z}_2$ case

In this section, we quantize the proposed action (100) for the case $G = \mathbb{Z}_2$ and demonstrate that the resulting Virasoro algebra possesses the same structure in terms of the mode operators α^1_n. For concreteness, we restrict our attention to world-sheet manifolds that are topologically equivalent to a cylinder. Since the conformal class of metrics on the cylinder is unique, we may employ the $\mathrm{Diff}(M)\rtimes Weyl$ gauge symmetry to fix the metric to the standard flat form, as is customary in conformal gauge

$$ds^2 = d\tau^2 + d\sigma^2, \tag{101}$$

with $-\infty < \tau < \infty$ and $0 \le \sigma < 2\pi$. Therefore, in this gauge the action (100) takes the form

$$S = \frac{-1}{4\pi\alpha'}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha\bar{\phi}^i_{k_1}\partial_\beta\phi^j_{k_2}(\delta_{ij}A_{k_1k_2} + \epsilon_{ij}B_{k_1k_2}). \tag{102}$$

The corresponding energy-momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha\phi^i_{k_1}\partial_\beta\phi^j_{k_2}(\delta_{ij}A_{k_1k_2} + \epsilon_{ij}B_{k_1k_2}) - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}(\partial_\gamma\phi^i_{k_1}\partial_\delta\phi^j_{k_2}(\delta_{ij}A_{k_1k_2} + \epsilon_{ij}B_{k_1k_2})). \tag{103}$$

Now consider the $\mathbb{Z}_2$ symmetry. We take the homomorphism between $\mathbb{Z}_2 = \{e, g\}$ and $\mathrm{Diff}(M)$ to be

$$\psi:\ \mathbb{Z}_2 \to \mathrm{Diff}(M), \tag{104}$$

$$\psi_g(\sigma) \to \sigma + \pi, \quad \text{Mod } 2\pi, \tag{105}$$

which corresponds to a half-period shift along the spatial direction of the cylinder. The two representations of $\mathbb{Z}_2$ are $\rho(g) = 1$ and $\rho(g) = -1$. Accordingly, the $\mathbb{Z}_2$-equivariant functions on the cylinder decompose into two sectors. In the trivial representation, the fields are invariant under the half-period shift, while in the nontrivial representation, they pick up a minus sign. Explicitly, they satisfy

$$\phi_0^i(\sigma + \pi) = \phi_0^i(\sigma), \tag{106}$$

$$\phi_1^i(\sigma + \pi) = -\phi_1^i(\sigma), \tag{107}$$

which have the following mode decomposition:

$$\phi_k^i(\tau, \sigma) = \sum_{m=N\mathbb{Z}+k} a_m^i(\tau) e^{im\sigma}. \tag{108}$$

The action (102) can be further simplified by noting that the $\mathbb{Z}_2$-equivariant functions form orthogonal sectors with respect to the inner product defined by integration over the spatial coordinate σ. In particular, one finds

$$\int_0^{2\pi} d\sigma \bar{\phi}_{k_1}^i(\tau, \sigma) \phi_{k_2}^j(\tau, \sigma) \sim \delta_{k_1 k_2}, \tag{109}$$

which expresses the fact that fields belonging to inequivalent representations of $\mathbb{Z}_2$ are orthogonal under the cylinder integral. As a consequence, cross-terms with $k_1 \neq k_2$ vanish in the action, and the dynamics decomposes into independent $k = 0$ and $k = 1$ sectors. Therefore, as $B_{k_1 k_2}$ is antisymmetric and couples different sectors, its contribution is zero. Thus, the only contribution comes from the diagonal part of $A_{k_1 k_2}$. Plugging the mode expansion (108) into the action (102) and using the orthogonality of the $\mathbb{Z}_2$-equivariant modes, the action reduces to

$$S = \frac{1}{2\alpha'} \Bigg(\sum_{n=\text{even}} \int d\tau (\dot{\bar{a}}_n^i \dot{a}_n^i - n^2 \bar{a}_n^i a_n^i) A_{00} + \sum_{n=\text{odd}} \int d\tau (\dot{\bar{a}}_n^i \dot{a}_n^i - n^2 \bar{a}_n^i a_n^i) A_{11} \Bigg). \tag{110}$$

Thus, the theory decomposes into two independent oscillator towers: one associated with the even modes ($k = 0$ sector) and the other with the odd modes ($k = 1$ sector). The quantization is straightforward. The canonical momentum conjugate to a_n^i is defined as

$$p_n^i = \frac{\partial L}{\partial \dot{a}_n^i} = \frac{A_{k_1 k_2}}{2\alpha'} \dot{\bar{a}}_n^i, \tag{111}$$

where $A_{k_1 k_1}$ depends on the sector we have chosen. This leads to the standard canonical commutation relations

$$[a_n^i, p_m^j] = i\delta^{ij} \delta_{n,m}. \tag{112}$$

The equations of motion for the mode coefficients follow directly from the action and take the form

$$\ddot{a}_n^i + n^2 a_n^i = 0 \tag{113}$$

for both the $k = 0$ (even) and $k = 1$ (odd) sectors. The general solutions are given by

$$\begin{cases} a_n^i(\tau) = i\sqrt{\frac{\alpha'}{A_{00}}} \left(\frac{\alpha_n^i}{n} e^{-in\tau} - \frac{\tilde{\alpha}_{-n}^i}{n} e^{in\tau} \right), & n \neq 0 \\ a_0^i(\tau) = x_0^i + 2\sqrt{\frac{\alpha'}{A_{00}}} \alpha_0^i \tau, & n = 0, \end{cases}$$

for the $k = 0$ sector. For the $k = 1$ sector (odd modes), the same expression holds for $n \neq 0$, but by construction there is no $n = 0$ mode in this sector, since only odd Fourier modes appear. The quantization rules (112) induce the following commutators among the oscillator modes. For $n \neq 0$ one finds

$$[\alpha_n^i, \bar{\alpha}_m^j] = n\delta^{ij} \delta_{n,m}, \tag{114}$$

$$[\tilde{\alpha}_n^i, \bar{\tilde{\alpha}}_m^j] = n\delta^{ij} \delta_{n,m}, \tag{115}$$

valid in both the $k = 0$ (even) and $k = 1$ (odd) sectors. For the zero mode in the trivial sector ($k = 0$), the situation is different. Here, the constant mode x_0^i and its conjugate momentum α_0^i satisfy the canonical commutation relation

$$[x_0^i, \bar{\alpha}_0^j] = i\sqrt{\frac{\alpha'}{A_{00}}} \delta^{ij}, \tag{116}$$

which corresponds to the standard position-momentum algebra of the degrees of freedom of the center of mass. The mode expansions of the equivariant fields $\phi_k^i(\tau, \sigma)$ then take the form

$$\phi_0^i = x_0^i + 2\sqrt{\frac{\alpha'}{A_{00}}} \alpha_0^i \tau + i\sqrt{\frac{\alpha'}{A_{00}}} \sum_{n=2\mathbb{Z}-\{0\}} \left(\frac{\alpha_n^i}{n} e^{-\text{in}(\tau-\sigma)} + \frac{\bar{\tilde{a}}_n^i}{n} e^{-\text{in}(\tau+\sigma)} \right), \tag{117}$$

$$\phi_1^i = i\sqrt{\frac{\alpha'}{A_{11}}} \sum_{n=2\mathbb{Z}+1} \left(\frac{\alpha_n^i}{n} e^{-\text{in}(\tau-\sigma)} + \frac{\bar{\tilde{a}}_n^i}{n} e^{-\text{in}(\tau+\sigma)} \right). \tag{118}$$

It is instructive to examine the canonical structure. If we define the conjugate momentum of $\phi_k^i(\tau, \sigma)$ by

$$P_k^i(\tau, \sigma) = \frac{A_{kk}}{4\pi\alpha'} \dot{\bar{\phi}}_k^i(\tau, \sigma), \tag{119}$$

then the canonical commutation relations take the form

$$[\phi^i_{k_1}(\tau,\sigma), P^j_{k_2}(\tau,\sigma)] = \frac{1}{2} i\delta^{ij}\delta_{k_1 k_2}\delta(\sigma-\sigma'). \tag{120}$$

This shows that each $\mathbb{Z}_2$ sector contributes one-half of the usual quantization rule for a free boson. In other words, the total phase space of the theory naturally decomposes into two independent halves, corresponding to the $k=0$ (periodic) and $k=1$ (antiperiodic) sectors.

Let us now return to the Virasoro algebra. By definition, the Virasoro generators L_m are obtained as the Fourier coefficients of the energy-momentum tensor. Working in light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, the right-moving component of the stress tensor takes the form

$$\begin{aligned} T_{--} &= \partial_- \bar{\phi}^i_{k_1} \partial_- \phi^j_{k_2} (\delta_{ij} A_{k_1 k_2} + \epsilon_{ij} B_{k_1 k_2}) \\ &\equiv 2\sqrt{\frac{\alpha'^2}{A_{00}A_{11}}} \sum_{m\in\mathbb{Z}} L_m e^{-2im\sigma^-}. \end{aligned} \tag{121}$$

If we define the complex conjugate of the oscillator modes by $\bar{\alpha}^i_n = \alpha^i_{-n}$, then the Virasoro generators can be written in terms of the oscillator bilinears. Explicitly, they take the form

$$\begin{aligned} L_{2n} &= \frac{1}{2}\left(A_{00} \sum_q \alpha^i_{2n-2q}\alpha^i_{2q} + A_{11}\sum_q \alpha^i_{2n-(2q+1)}\alpha^i_{2q+1}\right), \\ L_{2n+1} &= \frac{1}{2}\left(A_{01}\sum_q \alpha^i_{2n+1-q}\alpha^i_q \right. \\ &\quad \left. + B_{01}\sum_q (-1)^q \alpha^i_q \alpha^j_{2n+1-q}\epsilon_{ij}\right). \end{aligned} \tag{122}$$

Focusing on the Witt (centerless Virasoro) part of the commutator $[L_{2n}, L_{2m}]$, closure of the algebra imposes algebraic constraints on the coupling coefficients

$$A^2_{00} = A_{00}, \qquad A^2_{11} = A_{11}. \tag{123}$$

The standard solution in Sec. III A corresponds to $A_{00} = A_{11} = 1$ with the following constraints:

$$A^2_{01} + B^2_{01} = 1, \tag{124}$$

which comes from the closedness of the odd part of the algebra.

It is interesting that, by using the G-equivariance techniques, we uncover a richer structure than the purely algebraic constructions in Sec. III. Not only does the usual representation of L_{2n} (used as our starting point in Sec. III) remain a valid representation, but one can also express L_{2n} entirely in terms of the even or the odd oscillator towers. Explicitly,

$$L_{2n} = \frac{1}{2}\sum_{q\in\mathbb{Z}} \alpha^i_{2n-2q}\alpha^i_{2q} \text{ or } \frac{1}{2}\sum_{q\in\mathbb{Z}} \alpha^i_{2n-(2q+1)}\alpha^i_{2q+1}. \tag{125}$$

The above analysis reveals that the underlying mathematical structure governing these constructions is G equivariance. This concept can be naturally generalized to more intricate settings, such as non-Abelian Kac-Moody algebras, thereby offering alternative realizations of the Virasoro algebra beyond the conventional Sugawara construction [42].

V. DEFORMATIONS OF THE VIRASORO-KAC-MOODY-LIKE ALGEBRAS

We analyzed the degree of flexibility in the relation between the Abelian $U(1)^D$ Kac-Moody algebra and the Virasoro algebra. This analysis reveals additional possible realizations beyond the standard Sugawara construction for $D=2$, thereby suggesting a broader class of Virasoro-Kac-Moody-type algebras. As a concrete illustration, in the $\mathbb{Z}_2$ case we have the following Lie algebra[6]:

$$[j^i_n, j^j_n] = n\delta^{ij}\delta_{n+m}, \tag{126}$$

$$[S^1_n, j^i_m] = -maj^i_{m+n} + b\epsilon^{ij} m(-1)^m j^j_{m+n}, \tag{127}$$

$$[L_n, j^i_m] = -mj^i_{m+n}, \tag{128}$$

$$[S^{k_1}_n, S^{k_1}_m] = (n-m)S^{k_1+k_2}_{n+m} + \frac{c}{12}\delta^{k_1+k_2,0} n(n^2-1)\delta_{n+m,0}, \tag{129}$$

where the condition $k_1 + k_2 = 0$ is understood modulo N and as usual $L_n = S^0_n$.[7] The parameters a and b satisfy the algebraic relation (24). To the best of the author's knowledge, such a two-parameter generalization of the standard $U(1)^2$ algebra has not been mentioned in the literature.

One can show that the algebra introduced above cannot be transformed into the standard Virasoro-Kac-Moody algebra by any redefinition of the generators. This observation motivates the need for a precise notion of when two such algebras should be regarded as equivalent. In order to formalize this, let us define the exact meaning of an *isomorphism* between such structured algebras.

Definition. Let V be a vector space over a field $\mathbb{C}$, equipped with a basis $\{T_n\}_{n\in\mathbb{Z}}$. Although this basis is not unique for the vector space, we require that it has a special property. We assume that the vector space carries additional structure in the form of an equivalence relation on the index

[6]For coherence with the notation used in the literature, we denote $j^i_n \equiv \alpha^i_n$.

[7]It should be emphasized that the indices k labeling S^k_n correspond to the $\mathbb{Z}_N$ grading and must not be confused with the indices i, j associated with the $U(1)^2$ Kac-Moody algebra.

set $\mathbb{Z}$ defined by

$$n_1 \sim n_2 \quad \Leftrightarrow \quad n_1 \equiv n_2 \pmod{N},$$

for some fixed positive integer N. The equivalence relation on the index set $\mathbb{Z}$ partitions the basis $\{T_n\}_{n\in\mathbb{Z}}$ into N disjoint families. Each family spans a subspace V_k, and the vector space decomposes as a direct sum

$$V = \bigoplus_{k=0}^{N-1} V_k,$$

where

$$V_k = \text{span}\{T_{k+mN} | m \in \mathbb{Z}\}.$$

Similarly, for another vector space W with basis $\{S_n\}_{n\in\mathbb{Z}}$, we write

$$S_{n_1} \sim S_{n_2} \text{ if and only if } n_1 \sim n_2 \pmod{N}.$$

Let W be another Lie algebra with basis $\{S_n\}_{n\in\mathbb{Z}}$ and an equivalence relation $\sim'$ on its index set. A pair (ϕ, σ) is said to define an *equivariant Lie algebra isomorphism*

$$(\phi, \sigma)\colon (V, \{T_n\}, \sim) \to (W, \{S_n\}, \sim')$$

if the following conditions hold.

(1) $\phi\colon V \to W$ is a Lie algebra homomorphism, i.e.,

$$\phi([T_n, T_m]) = [\phi(T_n), \phi(T_m)] \quad \forall \ n, m \in \mathbb{Z}.$$

(2) $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ is a group homomorphism, typically of the form $\sigma(n) = an$ for some integer a, encoding the relabeling of the basis indices.

(3) The map ϕ acts on the basis by

$$\phi(T_n) = S_{\sigma(n)}.$$

(4) The equivalence relations are compatible in the sense that

$$n_1 \sim n_2 \pmod{N} \quad \Rightarrow \quad \sigma(n_1) \sim' \sigma(n_2).$$

If, in addition, ϕ is bijective and σ is invertible (as a group automorphism of $\mathbb{Z}$), then (ϕ, σ) defines an isomorphism of structured Lie algebras:

$$(V, \{T_n\}, \sim) \cong (W, \{S_n\}, \sim').$$

Such a definition imposes a strong constraint on the homomorphism σ. Suppose that the above algebra is isomorphic to the standard $U(1)^2$ Kac-Moody algebra, generated by $(\tilde{j}^i_n, \tilde{S}^k_n)$. Then, there exists a pair (ϕ, σ) characterizing this isomorphism, defined as follows:

$$\begin{aligned}\phi([j^i_n, j^j_m]) &= [\phi(j^i_n), \phi(j^i_m)] = [\tilde{j}^i_{\sigma(n)}, \tilde{j}^j_{\sigma(m)}] \\ &= \sigma(n)\delta^{ij}\delta_{\sigma(n)+\sigma(m),0}\delta^{\tilde{k}_1+\tilde{k}_2,0},\end{aligned} \tag{130}$$

where $\tilde{k}_1$ ($\tilde{k}_2$) denote the equivalence classes of $\sigma(n)$ and $\sigma(m)$, respectively. The condition (130) then leads to the following relation:

$$\sigma(n) = n. \tag{131}$$

Therefore, the $\mathbb{Z}_N$ grading is invariant under the map σ. The most general admissible transformation compatible with the $\mathbb{Z}_N$ grading takes the form[8]

$$\tilde{j}^i_n = \lambda_{[n]} A^{ij} j^j_n, \qquad A^{ik}A^{jl}\delta^{kl} = \delta^{ij}, \tag{132}$$

where A^{ij} is an orthonormal matrix and $\lambda_{[n]}$ is an automorphism of the current algebra that depends only on the equivalence class $[n]$.

For the $\mathbb{Z}_2$ case, we focus on the odd sector, where $m \in 2\mathbb{Z}+1$:

$$[\tilde{j}^i_n, \tilde{j}^j_m] = n\delta^{ij}\delta_{n+m}, \tag{133}$$

$$[S^k_n, \tilde{j}^i_m] = -m\tilde{j}^i_{m+n}. \tag{134}$$

The first relation is just $AA^T = 1$, while the second leads to $(a-1)A = bA\epsilon$. Taking the determinant of the latter relation gives

$$(a-1)^2 = b^2, \tag{135}$$

which is consistent with $a^2 + b^2 = 1$ only for $a = 1$ and $b = 0$. Therefore, for any nontrivial choice of (a, b), the resulting algebra is distinct from the standard Virasoro-Kac-Moody algebra (133) and (134).

It is natural to ask whether there exist nontrivial constructions beyond the trivial construction that depend solely on δ^{ij}. In the $\mathbb{Z}_3$ case, we showed in Sec. III B 2 that no such additional constructions exist. Here, we provide a sketch of the argument indicating that the same conclusion holds for general $\mathbb{Z}_N$. Let us define

$$j^{k\mathbf{i}}_n \equiv j^{\mathbf{i}}_n, \qquad [n] = k = 0, 1, \ldots, N-1 \quad \text{Mod } \mathbb{Z}_N. \tag{136}$$

Then, the S^1_{Nn+1} can be written as

[8]It is important to emphasize that, because we have restricted our attention to local field transformations (see Sec. III A 1), we apply the same rotation tensor A^{ij} to all Z_N classes.

$$S^1_{Nn+1} = \sum_{k=0}^{P} a_k j^{(N+1-k)\mathbf{i}}_{Nn+1-(Nq+k)} j^{(k)\mathbf{i}}_{Nq+k} \equiv (x_1 j^0_{Nn+1-(nq+1)} j^1_{Nq+1} + x_2 j^{N-1}_{Nn+1-(Nq+2)} j^2_{Nq+2} + \cdots), \quad (137)$$

where the sum over q is implicit and the relation between x_k and a_k can be easily derived. In the above expansion, we have $P = \frac{N-2}{2}$ when N is even and $P = \frac{N-1}{2}$ when N is odd. Using the automorphism group of the current algebra, we obtain the following equivalence relation for the variables x_k when N is even:

$$x_a \sim \frac{\epsilon_a \lambda_a}{\lambda_{a-1}} x_a, \quad a = 1, \ldots, \frac{N}{2}, \qquad \lambda_0 = \lambda_{\frac{N}{2}} = 1, \quad (138)$$

where $\epsilon_a = \pm 1$ for $a = 1$ and $a = \frac{N}{2}$ and is equal to 1 otherwise. The only invariant combination of the above coefficients is $x_1 x_2 \ldots x_{\frac{N}{2}}$. Therefore, the hypersurface resulting from Eqs. (19)–(22) is given by

$$(x_1 x_2 \ldots x_{\frac{N}{2}})^2 = 1. \quad (139)$$

Since the equivalence relations (138) introduce $\frac{N}{2} - 1$ variables $\lambda_1, \ldots, \lambda_{\frac{N}{2}-1}$, we can set

$$x_2 = x_3 = \cdots = x_{\frac{N}{2}} = 1,$$

which reduces the hypersurface equation to

$$x_1^2 = 1,$$

which leads to $x_1 = \pm 1$. However, using the $\mathbb{Z}_2$ symmetry encoded in ϵ_a, the minus solution can be mapped to the trivial one. A similar argument applies for odd N, with minor modifications.

A. Operator product expansion of deformed algebras

It is instructive to examine the operator product expansion (OPE) between the energy-momentum tensor and the currents. The space of all possible deformed algebras is vast; however, for algebras that depend solely on δ^{ij}, it is straightforward to determine their general structure. As is standard in the literature, we consider the OPE relations on the complex plane rather than on the cylinder, using the conformal map

$$z = e^{\tau + i\sigma}. \quad (140)$$

Under this transformation, the $\mathbb{Z}_N$ action on the cylinder, $\sigma \to \sigma + \frac{2\pi}{N}$, becomes

$$z \to e^{\frac{2\pi i}{N}} z. \quad (141)$$

The $\mathbb{Z}_N$-equivariant functions $X_k(\tau, \sigma)$ and their derivatives with respect to σ exhibit the same equivariance property on the cylinder. However, on the complex plane, the holomorphic $\mathbb{Z}_N$-equivariant function

$$X_k(e^{\frac{2\pi i}{N}} z) = e^{-\frac{2\pi i k}{N}} X_k(z) \quad (142)$$

induces a different equivariance behavior on its holomorphic derivative. More precisely,

$$J^k(e^{\frac{2\pi i}{N}} z) = e^{-\frac{2\pi i}{N}} e^{-\frac{2\pi i k}{N}} J^k(z), \qquad J^k = \partial_z X_k(z), \quad (143)$$

which can be generalized to a holomorphic primary field ϕ of conformal weight h as follows:

$$\phi^k(e^{\frac{2\pi i}{N}} z) = e^{-\frac{2\pi i h}{N}} e^{-\frac{2\pi i k}{N}} \phi^k(z). \quad (144)$$

In quantum field theory, conserved charges act as the generators of the symmetry algebra and determine how operators such as ϕ transform:

$$\delta \phi^k = [Q, \phi^k]. \quad (145)$$

Specifically, in conformal field theories, the holomorphic part of the charge is expressed as

$$Q = \frac{1}{2\pi i} \sum_{k_1 + k_2 + 1 = N} \oint_C dz T^{k_1}(z) \epsilon^{k_2}(z), \quad (146)$$

where we have used the $\mathbb{Z}_N$-equivariance decomposition of the energy-momentum tensor and functions into equivariant parts. The sum is understood in a way that is compatible with the $\mathbb{Z}_N$ invariance of charge Q.

It is straightforward to verify that the deformed algebra (137) gives rise to the following OPEs between the currents and the energy-momentum tensor[9]:

$$J^{k_1 \mathbf{i}}(z) J^{k_2 \mathbf{j}}(w) = \frac{\delta^{k_1 + k_2 = 0}}{(z-w)^2} \delta^{\mathbf{ij}}, \quad (147)$$

$$T^{k_1}(z) J^{k_2 \mathbf{i}}(w) = (a\delta^{\mathbf{ij}} + b(-1)^{k_2} \epsilon^{\mathbf{ij}}) \left(\frac{J^{(k_1+k_2)\mathbf{j}}(\omega)}{(z-w)^2} + \frac{\partial_\omega J^{(k_1+k_2)\mathbf{j}}(\omega)}{z-w} \right), \quad (148)$$

[9]We denote the $U(1)^2$ index by $\boldsymbol{i}$ in order to distinguish it from the $\mathbb{Z}_N$ grading indices k.

$$T^{k_1}(z)T^{k_2}(w) = \frac{c\delta^{k_1+k_2=0}}{2(z-w)^4} + \frac{2T^{k_1+k_2}(\omega)}{(z-w)^2} + \frac{\partial_\omega T^{k_1+k_2}(\omega)}{z-w}, \tag{149}$$

where $J^{\mathbf{ki}}(z) = \sum_n z^{-n-1} j_n^{\mathbf{ki}}$ and $T^{\mathbf{k}}(z) = \sum_n z^{-n-2} S_n^{\mathbf{k}}$.

As we observe, the standard OPE between the currents and the energy-momentum tensor is deformed by the matrix $(a\delta^{\mathbf{ij}} + b(-1)^{k_2}\epsilon^{\mathbf{ij}})$ appearing in (148). Moreover, the presence of this deformation modifies the usual notion of primary fields. Consequently, one can define primary fields within the deformed $\mathbb{Z}_N$-equivariant algebra as follows.

1. Definition

A field $\phi^{\mathbf{i}}(z)$ with the $\mathbb{Z}_N$ decomposition $\{\phi^{k\mathbf{i}}(z)\}$ is called *primary* with conformal dimensions h if the operator product expansions of the decompositions of energy-momentum tensor $\{T^k(z)\}$ with $\phi^{k\mathbf{i}}(z)$ take the form

$$T^{k_1}(z)\phi^{k_2\mathbf{i}}(w) = (a\delta^{\mathbf{ij}} + b(-1)^{k_2}\epsilon^{\mathbf{ij}})\left(\frac{h\phi^{(k_1+k_2)\mathbf{j}}(w)}{(z-w)^2} + \frac{\partial_w\phi^{(k_1+k_2)\mathbf{j}}(w)}{z-w}\right). \tag{150}$$

It is interesting how the parameters (a, b) affect the representations of the above Kac-Moody-Virasoro algebra.

VI. CONCLUDING REMARKS

We may be interested in replacing the epsilon tensor ϵ^{ij} with a generic antisymmetric background tensor B^{ij} and extending the construction to arbitrary dimensions. Aside from issues related to Lorentz invariance, it is straightforward to show that the general case does not differ significantly from the results obtained above. For a real nonsingular even-dimensional $2n \times 2n$ antisymmetric matrix B^{ij}, there is a real orthogonal $2n \times 2n$ matrix U such that [57–59]

$$U^T B U = \mathrm{diag}\left(\begin{bmatrix} 0 & m_1 \\ -m_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m_2 \\ -m_2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & m_n \\ -m_n & 0 \end{bmatrix}\right). \tag{151}$$

So the $2n \times 2n$ even-dimensional case reduces to a bunch of 2×2 matrices which can be studied in a similar way to the previous sections. The antisymmetric matrices are singular in odd dimensions, and the following theorem is helpful for odd-dimensional antisymmetric and singular even-dimensional matrices.

Theorem. If B is a real singular antisymmetric $d \times d$ matrix of rank $2n$, where d is even or odd and $d > 2n$, then there exist a real orthogonal $d \times d$ matrix U such that

$$U^T B U = \mathrm{diag}\left(\begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}, \ldots, \times \begin{pmatrix} 0 & m_n \\ -m_n & 0 \end{pmatrix}, \mathcal{O}_{d-2n}\right), \tag{152}$$

where $\mathcal{O}_{d-2n}$ is a $(d-2n) \times (d-2n)$ block of zeros. Therefore, the general case is similar to the previous sections, and singular matrices have some zeros which can be treated in a similar manner as the conventional case.

VII. CONCLUSION

In this paper, we have introduced and analyzed a new family of Virasoro-Kac-Moody algebras constructed from Z_N gradings of the mode indices of the Virasoro generators. We started with a concrete construction in the Z_2 case, using the antisymmetric tensor ϵ^{ij}. This extra invariant tensor exists in $D = 2$ to build sets of Virasoro generators that differ for even and odd Fourier modes n. Afterwards, this construction was extended naturally to the general case of Z_N gradings.

Partitioning the Virasoro generators into N distinct classes based on the value of their mode index n modulo N, and requiring that these new operators form a closed Virasoro algebra, imposes a set of algebraic constraints on their defining coefficients. Solving these constraints revealed the space of all possible constructions. We then solved these constraints for the special cases $N = 2$, $N = 3$, and $N = 4$ to reveal the structure of the corresponding solutions. For the $N = 2$ case, by separating the Virasoro operators into even and odd classes, we showed that the construction for the odd-indexed operators is not unique; in fact, there is an infinite family of valid constructions. The space of these solutions is described by the simple algebraic relation $x^2 + y^2 = 1$, which, in $\mathbb{C}^2$, is topologically equivalent to a cylinder. It is shown that $N = 3$ has a similar structure such as $N = 2$. For the $N = 4$ case, the space of all possible Virasoro generators corresponds to a four-dimensional manifold, $T^2 \times S^2$. We have argued that the general situation for $\mathbb{Z}_{2N}$ and $\mathbb{Z}_{2N+1}$ is similar and shares the essential features, as both have the same dimensionality.

Shifting our perspective from a purely algebraic view toward a more physical one, we introduced Lagrangians with underlying symmetries that correspond to the aforementioned Virasoro-Kac-Moody algebras. Using the concept of $\mathbb{Z}_N$ equivariance, we clarified the meaning of Z_N mode separation on a general manifold. The fields in these theories were classified based on how they transform under the action of a finite symmetry group $\mathbb{Z}_N$ on the underlying world-sheet manifold. This classification was thereby tied to a geometric symmetry of the space itself, making it an intrinsic, coordinate-independent property. Following this logic, we explicitly constructed the Z_2 case by quantizing

the theory described by such an action and derived the new Virasoro generators. The final expressions were precisely the same as those found through the purely algebraic methods. This demonstrates that the $\mathbb{Z}_N$-equivariant action principle is indeed the correct physical foundation for these new algebras.

By modifying the Virasoro generators, the relation between the Kac-Moody currents and the Virasoro generators changed fundamentally. This led to the creation of entirely new, deformed versions of the Virasoro-Kac-Moody algebra. For the Z_2 case, we explicitly wrote down a new Lie algebra that depends on two parameters a and b, which satisfy an algebraic relation. We also showed that such novel algebras cannot be transformed into the usual ones by any redefinition of the generators.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: ALGEBRAIC PRELIMINARIES: THE $\mathbb{Z}_N$ COMMUTATORS

In this section, we derive the basic algebraic equations (19)–(22), which form the foundation for all computations in the first part of the paper.

Integer numbers can be partitioned into N classes when considered modulo N. Accordingly, the Virasoro generators can be grouped into these N classes, which we denote by S_n^k, where the index $k = 0, 1, \ldots, N-1$ specifies the class. We choose the trivial class $k = 0$ to correspond to the usual Virasoro operators:

$$S_n^0 = L_n = \frac{1}{2}\sum_q \alpha^i_{n-q}\alpha^i_q, \qquad n = N\mathbb{Z}. \tag{A1}$$

We now try to fix the most generic form of other S_n^k for $k = 1, \ldots, N-1$. Let us consider the action of these modes on the following family of generators:

$$V_m^k = \frac{1}{2}\sum_q f^k(q)\epsilon^{ij}\alpha^i_{m-q}\alpha^j_q. \tag{A2}$$

For simplicity, we ignore normal ordering subtleties at this stage, which can be determined by the Jacobi identity. Therefore, we obtain

$$\begin{aligned}[L_n, V_m^k] &= \frac{1}{2}\sum_q f^k(q)\epsilon^{ij}[L_n, \alpha^i_{m-q}\alpha^j_q] \\ &= \frac{1}{2}\sum_q f(q)\epsilon^{ij}([L_n, \alpha^i_{m-q}]\alpha^j_q + \alpha^i_{m-q}[L_n, \alpha^j_q]) \\ &= \frac{1}{2}\sum_q f^k(q)\epsilon^{ij}(-(m-q)\alpha^i_{n+m-q}\alpha^j_q - q\alpha^i_{m-q}\alpha^j_{n+q}) \\ &= \frac{1}{2}\sum_q \epsilon^{ij}(-(m-q)f^k(q) \\ &\quad - f^k(q-n)(q-n))\alpha^i_{n+m-q}\alpha^j_q, \end{aligned} \tag{A3}$$

where the equivalence class of m modulo N is denoted by $[m] = k$. For the closure of the algebra, these generators must satisfy the consistency condition

$$-(m-q)f^k(q) - f^k(q-n)(q-n) = (n-m)f^k(q). \tag{A4}$$

As n is restricted to values in $N\mathbb{Z}$, it is sufficient that the functions $f^k(q)$ be invariant modulo N to satisfy the above closure condition. Therefore, the most general solution can be expressed in the form

$$f^k(q) = \sum_{w=0}^{N-1} c_w^k e^{\frac{2\pi w i}{N}q}. \tag{A5}$$

Since f^k is a mod N invariant function, the index w runs over the range $w = 0, \ldots, N-1$. An additional constraint arises from the transformation $m - q = u$ applied to Eq. (A2), which leads to the following relation:

$$f^k(q) = -f^k(m-q). \tag{A6}$$

As m in Eq. (A2) takes values of the form $m \in N\mathbb{Z} + k$, with $k = 1, \ldots, N-1$, and since $f^k(q)$ is mod N invariant, the constraint simplifies to

$$f^k(q) = -f^k(k-q), \tag{A7}$$

which leads to the following relation:

$$\sum_{w=0}^{N-1} c_w^k e^{\frac{2\pi w i}{N}q} = -\sum_{w=0}^{N-1} c_w^k e^{\frac{2\pi w i}{N}(k-q))}. \tag{A8}$$

This condition is satisfied provided the coefficients obey the following relation:

$$c^k_{N-w} = -c^k_w e^{\frac{2\pi i w k}{N}}, \tag{A9}$$

where we assume that $c_N^k = c_0^k$. This provides the most general form of the coefficients ensuring the closedness condition $[L_n, S_m^k] = (m-n)S^k_{m+n}$. Let us focus on another class of operators as

$$U_n^k = \frac{1}{2}\sum_q h^k(q)\alpha_{m-q}^i\alpha_q^i. \tag{A10}$$

By the same reasoning as before, the function $h^k(q)$ can be expanded in the form

$$h^k(q) = \sum_{w=0}^{N-1} d_w^k e^{\frac{2\pi wi}{N}q}, \tag{A11}$$

where the coefficients d_w^k satisfy the constraint

$$d_{N-w}^k = d_w^k e^{\frac{2\pi iwk}{N}}, \tag{A12}$$

and the identification $h_N^k = h_0^k$ is assumed, ensuring periodicity modulo N. Therefore, the most general form of an operator consistent with the action of L_n is given by

$$S_n^k = U_n^k + V_n^k = \frac{1}{2}\sum_q\sum_w (d_w^k e^{\frac{2\pi wi}{N}q}\alpha_{n-q}^i\alpha_q^i + c_w^k e^{\frac{2\pi wi}{N}q}\epsilon^{ij}\alpha_{n-q}^i\alpha_q^j), \tag{A13}$$

together with the constraints (A9) and (A12) on the coefficients c_w^k and d_w^k. Our goal is to determine the constants c_w^k and d_w^k in such a way that the resulting operators S_n^k satisfy the desired algebra

$$[S_n^{k_1}, S_m^{k_2}] = (n-m)S_{n+m}^{k_1+k_2}. \tag{A14}$$

Using the form (A2), the commutator can be written as

$$[V_n^k, \alpha_m^i] = -\frac{1}{2}m(f^k(n+m) - f^k(-m))\epsilon^{ij}\alpha_{n+m}^j, \tag{A15}$$

and the commutator of $[V_n^{k_1}, V_m^{k_2}]$ then becomes

$$\begin{aligned}[V_n^{k_1}, V_m^{k_2}] &= \frac{1}{2}\sum_q \epsilon^{ij} f^{k_2}(q)([V_n^{k_1}, \alpha_{m-q}^i]\alpha_q^j + \alpha_{m-q}^i[V_n^{k_1}, \alpha_q^j]) \\ &= \frac{1}{2}\epsilon^{ij} f^{k_2}(q)\Big(-\frac{1}{2}(m-q)(f^{k_1}(n+m-q) - f^{k_1}(q-m))\epsilon^{ik}\alpha_{n+m-q}^k\alpha_q^j \\ &\quad -\frac{1}{2}q\alpha_{m-q}^i(f^{k_1}(n+q) - f^{k_1}(-q))\epsilon^{jk}\alpha_{n+q}^k\Big).\end{aligned} \tag{A16}$$

By redefining $n+q \to q$ in the second term, we finally have

$$[V_n^{k_1}, V_m^{k_2}] = \Big(-\frac{1}{4}(m-q)f^{k_2}(q)(f^{k_1}(n+m-q) - f^{k_1}(q-m)) + \frac{1}{4}f^{k_2}(q-n)(q-n)(f^{k_1}(q) - f^{k_1}(n-q))\Big)\alpha_{n+m-q}^i\alpha_q^i. \tag{A17}$$

Moreover, the commutator of $[U_n^{k_1}, U_m^{k_2}]$ is given by

$$[U_n^{k_1}, U_m^{k_2}] = -\frac{1}{4}((m-q)h^{k_2}(q)(h^{k_1}(q-m) + h^{k_1}(n+m-q)) + (q-n)h^k(q-n)(h^{k_1}(q) + h^{k_1}(n-q)))\alpha_{n+m-q}^i\alpha_q^i, \tag{A18}$$

where $[n] = k_1$ and $[m] = k_2$. The $\alpha_{n+m-q}^i\alpha_q^i$ part of commutator (A14) comes from

$$[U_n^{k_1}, U_m^{k_2}] + [V_n^{k_1}, V_m^{k_2}]. \tag{A19}$$

Using the commutators (A17) and (A18), we should have the following identity:

$$\begin{aligned}&-\frac{1}{4}((m-q)h^{k_2}(q)(h^{k_1}(q-m) + h^{k_1}(n+m-q)) + (q-n)h^{k_2}(q-n)(h^{k_1}(q) + h^{k_1}(n-q))) \\ &-\frac{1}{4}((m-q)f^{k_2}(q)(f^{k_1}(n+m-q) - f^{k_1}(q-m)) - f^{k_2}(q-n)(q-n)(f^{k_1}(q) - f^{k_1}(n-q))) \\ &= \frac{1}{2}h^{k_1+k_2}(q)(n-m).\end{aligned} \tag{A20}$$

By substituting the expansions (A9) and (A12) into the commutator, after a tedious but straightforward calculation, one arrives at the following result:

$$\frac{1}{2}\sum_{q}\sum_{w_1,w_2}[(n-q)(d^{k_1}_{w_1}d^{k_2}_{w_2}-c^{k_1}_{w_1}c^{k_2}_{w_2})\zeta^{-w_2}_{k_1}+(q-m)(d^{k_1}_{w_1}d^{k_2}_{w_2}-c^{k_1}_{w_1}c^{k_2}_{w_2})\zeta^{-w_1}_{k_2}]e^{\frac{2\pi i(w_1+w_2)}{N}q}\alpha^i_{n+m-q}\alpha^i_q$$
$$+\frac{1}{2}\sum_{q}\sum_{w_1,w_2}[(n-q)(d^{k_1}_{w_1}c^{k_2}_{w_2}+c^{k_1}_{w_1}d^{k_2}_{w_2})\zeta^{-w_2}_{k_1}+(q-m)(d^{k_1}_{w_1}c^{k_2}_{w_2}+c^{k_1}_{w_1}d^{k_2}_{w_2})\zeta^{-w_1}_{k_2}]e^{\frac{2\pi i(w_1+w_2)}{N}q}\epsilon^{ij}\alpha^i_{n+m-q}\alpha^j_q. \tag{A21}$$

A further simplification can be achieved by performing the change of summation variable $q \to n+m-q$ and applying the constraints (A9) and (A12) on the coefficients. By including the $[U^{k_1}_n, V^{k_2}_m]+[V^{k_1}, U^{k_2}_m]$ part, this leads to the following consistency conditions for the closure of the algebra:

$$\sum_{q,w_1,w_2}(d^{k_1}_{w_1}c^{k_2}_{w_2}+c^{k_1}_{w_1}d^{k_2}_{w_2})(\zeta^{-w_2}_{k_1}-\zeta^{-w_1}_{k_2})e^{\frac{2\pi i(w_1+w_2)}{N}q}\epsilon^{ij}\alpha^i_{n+m-q}\alpha^j_q=0, \tag{A22}$$

$$\sum_{q,w_1,w_2}(d^{k_1}_{w_1}d^{k_2}_{w_2}-c^{k_1}_{w_1}c^{k_2}_{w_2})(\zeta^{-w_2}_{k_1}-\zeta^{-w_1}_{k_2})e^{\frac{2\pi i(w_1+w_2)}{N}q}\alpha^i_{n+m-q}\alpha^i_q=0. \tag{A23}$$

It is crucial for the derivation of the above relation that the sum of the mode indices satisfies $[n+m]=k_1+k_2$, ensuring the consistency of the mod N structure in the algebra.

Using these simplifications, the net result of the commutator (A21) can be written as

$$=\frac{1}{2}(n-m)\sum_{q,w_1,w_2}(d^{k_1}_{w_1}d^{k_2}_{w_2}-c^{k_1}_{w_1}c^{k_2}_{w_2})\left(\frac{\zeta^{-w_2}_{k_1}+\zeta^{-w_1}_{k_2}}{2}\right)e^{\frac{2\pi i(w_1+w_2)}{N}q}\alpha^i_{n+m-q}\alpha^i_q$$
$$+\frac{1}{2}(n-m)\sum_{q,w_1,w_2}(d^{k_1}_{w_1}c^{k_2}_{w_2}+c^{k_1}_{w_1}d^{k_2}_{w_2})\left(\frac{\zeta^{-w_2}_{k_1}+\zeta^{-w_1}_{k_2}}{2}\right)e^{\frac{2\pi i(w_1+w_2)}{N}q}\epsilon^{ij}\alpha^i_{n+m-q}\alpha^j_q$$
$$+\frac{1}{2}\sum_{q,w_1,w_2}(d^{k_1}_{w_1}d^{k_2}_{w_2}-c^{k_1}_{w_1}c^{k_2}_{w_2})(\zeta^{-w_2}_{k_1}-\zeta^{-w_1}_{k_2})e^{\frac{2\pi i(w_1+w_2)}{N}q}q\alpha^i_{n+m-q}\alpha^i_q$$
$$+\frac{1}{2}\sum_{q,w_1,w_2}(d^{k_1}_{w_1}c^{k_2}_{w_2}+c^{k_1}_{w_1}d^{k_2}_{w_2})(\zeta^{-w_2}_{k_1}-\zeta^{-w_1}_{k_2})e^{\frac{2\pi i(w_1+w_2)}{N}q}q\epsilon^{ij}\alpha^i_{n+m-q}\alpha^j_q. \tag{A24}$$

To be compatible with the commutator (A14), it is necessary that the $q\alpha^i_{n+m-q}\alpha^i_q$ and $q\epsilon^{ij}\alpha^i_{n+m-q}\alpha^j_q$ terms vanish, and the first two terms satisfy the following identities:

$$\sum_{\substack{w_1,w_2\\ w_1+w_2=s \bmod N}}(d^{k_1}_{w_1}d^{k_2}_{w_2}-c^{k_1}_{w_1}c^{k_2}_{w_2})\left(\frac{\zeta^{-w_2}_{k_1}+\zeta^{-w_1}_{k_2}}{2}\right)=d^{k_1+k_2}_s,$$
$$\sum_{\substack{w_1,w_2\\ w_1+w_2=s \bmod N}}(d^{k_1}_{w_1}c^{k_2}_{w_2}+c^{k_1}_{w_1}d^{k_2}_{w_2})\left(\frac{\zeta^{-w_2}_{k_1}+\zeta^{-w_1}_{k_2}}{2}\right)=c^{k_1+k_2}_s, \tag{A25}$$

where $s=0,1,\ldots,N-1$ mod N, which are exactly Eqs. (19)–(22).

APPENDIX B: THE $\mathbb{Z}_4$-EQUIVARIANT VIRASORO EQUATIONS

In this Appendix, we present Eqs. (19)–(22) explicitly, as they were manipulated using *Mathematica*. The equations have also been simplified by incorporating the constraint (16):

$$
\begin{aligned}
(c_2^1)^2+(d_0^1)^2 &= d_0^2,\\
(-1-i)c_1^1c_2^1+(1-i)d_0^1d_1^1 &= d_1^2,\\
(1-i)c_1^1d_0^1-(1+i)c_2^1d_1^1 &= c_1^2,\\
2i((c_1^1)^2-(d_1^1)^2) &= d_2^2,\\
(1+i)c_1^1c_2^1-(1-i)d_0^1d_1^1 &= -d_1^2,\\
(1-i)c_1^1d_0^1-(1+i)c_2^1d_1^1 &= c_1^2,\\
d_0^1d_0^2-\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^1c_1^2-d_1^1d_1^2)-\left(\frac{1}{2}+\frac{i}{2}\right)(ic_1^1c_1^2+id_1^1d_1^2) &= d_0^3,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)(c_1^2d_1^1-c_1^1d_1^2)-\left(\frac{1}{2}+\frac{i}{2}\right)(ic_1^2d_1^1-ic_1^1d_1^2) &= 0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_2^1c_1^2+\left(\frac{1}{2}-\frac{i}{2}\right)d_0^1d_1^2-id_1^1d_2^2 &= d_1^3,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_1^2d_0^1-\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1d_1^2+ic_1^1d_2^2 &= c_1^3,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)(ic_1^1c_1^2-id_1^1d_1^2)-\left(\frac{1}{2}+\frac{i}{2}\right)(-c_1^1c_1^2+d_1^1d_1^2) &= 0,\\
c_2^1d_0^2-\left(\frac{1}{2}-\frac{i}{2}\right)(ic_1^2d_1^1+ic_1^1d_1^2)-\left(\frac{1}{2}+\frac{i}{2}\right)(c_1^2d_1^1+c_1^1d_1^2) &= c_2^3,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^2-\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^2-d_1^1d_2^2 &= -id_1^3,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_1^2d_0^1+\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1d_1^2-c_1^1d_2^2 &= ic_1^3,\\
c_2^1c_2^3+d_0^1d_0^3+i(-ic_1^1c_1^3-id_1^1d_1^3)-i(ic_1^1c_1^3+id_1^1d_1^3) &= 1,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^3+\left(\frac{1}{2}-\frac{i}{2}\right)c_1^1c_2^3+\left(\frac{1}{2}+\frac{i}{2}\right)d_1^1d_0^3+\left(\frac{1}{2}-\frac{i}{2}\right)d_0^1d_1^3 &= 0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_1^3d_0^1+\left(\frac{1}{2}-\frac{i}{2}\right)c_2^3d_1^1+\left(\frac{1}{2}+\frac{i}{2}\right)c_1^1d_0^3+\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1d_1^3 &= 0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^3+\left(\frac{1}{2}-\frac{i}{2}\right)c_1^1c_2^3+\left(\frac{1}{2}+\frac{i}{2}\right)d_1^1d_0^3+\left(\frac{1}{2}-\frac{i}{2}\right)d_0^1d_1^3 &= 0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_1^3d_0^1-\left(\frac{1}{2}-\frac{i}{2}\right)c_2^3d_1^1-\left(\frac{1}{2}+\frac{i}{2}\right)c_1^1d_0^3-\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1d_1^3 &= 0,\\
2(c_1^2)^2+(d_0^2)^2+2(d_1^2)^2+(d_2^2)^2 &= 1,\\
2(c_1^2)^2-2(d_1^2)^2+2d_0^2d_2^2 &= 0,\\
d_0^2d_0^3-\left(\frac{1}{2}+\frac{i}{2}\right)(-c_1^2c_1^3-d_1^2d_1^3)-\left(\frac{1}{2}-\frac{i}{2}\right)(-ic_1^2c_1^3-id_1^2d_1^3) &= d_0^1,
\end{aligned}
\tag{B1}
$$

$$
\begin{aligned}
\left(-\frac{1}{2}+\frac{i}{2}\right)(ic_1^3d_1^2-ic_1^2d_1^3)-\left(\frac{1}{2}+\frac{i}{2}\right)(-c_1^3d_1^2+c_1^2d_1^3)&=0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_1^2c_2^3+\left(\frac{1}{2}+\frac{i}{2}\right)d_1^2d_0^3+id_2^2d_1^3&=d_1^1,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_2^3d_1^2-ic_1^3d_2^2+\left(\frac{1}{2}+\frac{i}{2}\right)c_1^2d_0^3&=c_1^1,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)(-ic_1^2c_1^3+id_1^2d_1^3)-\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^2c_1^3+d_1^2d_1^3)&=0,\\
c_2^3d_0^2-\left(\frac{1}{2}+\frac{i}{2}\right)(-ic_1^3d_1^2-ic_1^2d_1^3)-\left(\frac{1}{2}-\frac{i}{2}\right)(c_1^3d_1^2+c_1^2d_1^3)&=c_2^1,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_1^2c_2^3-\left(\frac{1}{2}-\frac{i}{2}\right)d_1^2d_0^3-d_2^2d_1^3&=id_1^1,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3d_1^2-c_1^3d_2^2+\left(\frac{1}{2}-\frac{i}{2}\right)c_1^2d_0^3&=-ic_1^1,\\
(c_2^3)^2+(d_0^3)^2&=d_0^2,\\
(-1+i)c_1^3c_2^3+(1+i)d_0^3d_1^3&=d_1^2,\\
(1+i)c_1^3d_0^3-(1-i)c_2^3d_1^3&=c_1^2,\\
-i((c_1^3)^2-(d_1^3)^2)+i(-(c_1^3)^2+(d_1^3)^2)&=d_2^2,\\
(1-i)c_1^3c_2^3-(1+i)d_0^3d_1^3&=-d_1^2,\\
(1+i)c_1^3d_0^3-(1-i)c_2^3d_1^3&=c_1^2,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1c_1^2+d_1^1d_0^2-\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^2&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_1^2d_0^1+c_1^1d_0^2+\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1d_1^2&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)(ic_1^1c_1^2-id_1^1d_1^2)+\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^1c_1^2+d_1^1d_1^2)-d_0^1d_2^2&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)(ic_1^2d_1^1+ic_1^1d_1^2)+\left(\frac{1}{2}-\frac{i}{2}\right)(c_1^2d_1^1+c_1^1d_1^2)&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^2+id_1^1d_0^2+\left(\frac{1}{2}-\frac{i}{2}\right)d_0^1d_1^2&=0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_1^2d_0^1-ic_1^1d_0^2-\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1d_1^2&=0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1c_1^3-\left(\frac{1}{2}+\frac{i}{2}\right)c_1^1c_2^3+\left(\frac{1}{2}-\frac{i}{2}\right)d_1^1d_0^3-\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^3&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_1^3d_0^1-\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3d_1^1+\left(\frac{1}{2}-\frac{i}{2}\right)c_1^1d_0^3+\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1d_1^3&=0,\\
-c_2^3d_0^1+c_2^1d_0^3+i(-c_1^3d_1^1-c_1^1d_1^3)-i(c_1^3d_1^1+c_1^1d_1^3)&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)(c_1^2d_1^1-c_1^1d_1^2)+\left(\frac{1}{2}-\frac{i}{2}\right)(ic_1^2d_1^1-ic_1^1d_1^2)-c_2^1d_2^2&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)(-c_1^1c_1^2-d_1^1d_1^2)+\left(\frac{1}{2}-\frac{i}{2}\right)(ic_1^1c_1^2+id_1^1d_1^2)&=0,
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
\left(-\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^3+\left(\frac{1}{2}+\frac{i}{2}\right)c_1^1c_2^3-\left(\frac{1}{2}-\frac{i}{2}\right)d_1^1d_0^3+\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^3&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_1^3d_0^1-\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3d_1^1+\left(\frac{1}{2}-\frac{i}{2}\right)c_1^1d_0^3+\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1d_1^3&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)(-c_1^1c_1^2-d_1^1d_1^2)-\left(\frac{1}{2}-\frac{i}{2}\right)(ic_1^1c_1^2+id_1^1d_1^2)&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)(c_1^2d_1^1-c_1^1d_1^2)-\left(\frac{1}{2}-\frac{i}{2}\right)(ic_1^2d_1^1-ic_1^1d_1^2)+c_2^1d_2^2&=0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^2-d_1^1d_0^2+\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^2&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_1^2d_0^1-c_1^1d_0^2-\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1d_1^2&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)(ic_1^1c_1^2-id_1^1d_1^2)-\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^1c_1^2+d_1^1d_1^2)+d_0^1d_2^2&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)(ic_1^2d_1^1+ic_1^1d_1^2)-\left(\frac{1}{2}-\frac{i}{2}\right)(c_1^2d_1^1+c_1^1d_1^2)&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_2^1c_1^2-id_1^1d_0^2-\left(\frac{1}{2}-\frac{i}{2}\right)d_0^1d_1^2&=0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_1^2d_0^1+ic_1^1d_0^2+\left(\frac{1}{2}+\frac{i}{2}\right)c_2^1d_1^2&=0,
\end{aligned}
\tag{B3}
$$

$$
\begin{aligned}
\left(-\frac{1}{2}+\frac{i}{2}\right)(-c_1^2c_1^3-d_1^2d_1^3)-\left(\frac{1}{2}+\frac{i}{2}\right)(-ic_1^2c_1^3-id_1^2d_1^3)&=0,\\
c_2^3d_2^2-\left(\frac{1}{2}+\frac{i}{2}\right)(ic_1^3d_1^2-ic_1^2d_1^3)-\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^3d_1^2+c_1^2d_1^3)&=0,\\
\left(-\frac{1}{2}-\frac{i}{2}\right)c_1^2c_2^3+\left(\frac{1}{2}-\frac{i}{2}\right)d_1^2d_0^3-d_0^2d_1^3&=0,\\
-c_1^3d_0^2-\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3d_1^2+\left(\frac{1}{2}-\frac{i}{2}\right)c_1^2d_0^3&=0,\\
d_2^2d_0^3-\left(\frac{1}{2}-\frac{i}{2}\right)(-ic_1^2c_1^3+id_1^2d_1^3)-\left(\frac{1}{2}+\frac{i}{2}\right)(-c_1^2c_1^3+d_1^2d_1^3)&=0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)(-ic_1^3d_1^2-ic_1^2d_1^3)-\left(\frac{1}{2}+\frac{i}{2}\right)(c_1^3d_1^2+c_1^2d_1^3)&=0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_1^2c_2^3-\left(\frac{1}{2}+\frac{i}{2}\right)d_1^2d_0^3+id_0^2d_1^3&=0,\\
-ic_1^3d_0^2+\left(\frac{1}{2}-\frac{i}{2}\right)c_2^3d_1^2+\left(\frac{1}{2}+\frac{i}{2}\right)c_1^2d_0^3&=0,\\
\left(-\frac{1}{2}+\frac{i}{2}\right)c_2^1c_1^3+\left(\frac{1}{2}+\frac{i}{2}\right)c_1^1c_2^3-\left(\frac{1}{2}-\frac{i}{2}\right)d_1^1d_0^3+\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^3&=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_1^3d_0^1+\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3d_1^1-\left(\frac{1}{2}-\frac{i}{2}\right)c_1^1d_0^3-\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1d_1^3&=0,\\
c_2^3d_0^1-c_2^1d_0^3-i(-c_1^3d_1^1-c_1^1d_1^3)+i(c_1^3d_1^1+c_1^1d_1^3)&=0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1c_1^3-\left(\frac{1}{2}+\frac{i}{2}\right)c_1^1c_2^3+\left(\frac{1}{2}-\frac{i}{2}\right)d_1^1d_0^3-\left(\frac{1}{2}+\frac{i}{2}\right)d_0^1d_1^3&=0,
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{2}+\frac{i}{2}\right)c_1^3 d_0^1+\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3 d_1^1-\left(\frac{1}{2}-\frac{i}{2}\right)c_1^1 d_0^3-\left(\frac{1}{2}-\frac{i}{2}\right)c_2^1 d_1^3=0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^2 c_1^3-d_1^2 d_1^3)+\left(\frac{1}{2}+\frac{i}{2}\right)(-ic_1^2 c_1^3-id_1^2 d_1^3)=0,\\
-c_2^3 d_2^2+\left(\frac{1}{2}+\frac{i}{2}\right)(ic_1^3 d_1^2-ic_1^2 d_1^3)+\left(\frac{1}{2}-\frac{i}{2}\right)(-c_1^3 d_1^2+c_1^2 d_1^3)=0,\\
\left(\frac{1}{2}+\frac{i}{2}\right)c_1^2 c_2^3-\left(\frac{1}{2}-\frac{i}{2}\right)d_1^2 d_0^3+d_0^2 d_1^3=0,\\
c_1^3 d_0^2+\left(\frac{1}{2}+\frac{i}{2}\right)c_2^3 d_1^2-\left(\frac{1}{2}-\frac{i}{2}\right)c_1^2 d_0^3=0,\\
-d_2^2 d_0^3+\left(\frac{1}{2}-\frac{i}{2}\right)(-ic_1^2 c_1^3+id_1^2 d_1^3)+\left(\frac{1}{2}+\frac{i}{2}\right)(-c_1^2 c_1^3+d_1^2 d_1^3)=0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)(-ic_1^3 d_1^2-ic_1^2 d_1^3)+\left(\frac{1}{2}+\frac{i}{2}\right)(c_1^3 d_1^2+c_1^2 d_1^3)=0,\\
\left(\frac{1}{2}-\frac{i}{2}\right)c_1^2 c_2^3+\left(\frac{1}{2}+\frac{i}{2}\right)d_1^2 d_0^3-id_0^2 d_1^3=0,\\
ic_1^3 d_0^2-\left(\frac{1}{2}-\frac{i}{2}\right)c_2^3 d_1^2-\left(\frac{1}{2}+\frac{i}{2}\right)c_1^2 d_0^3=0.
\end{aligned}
\tag{B4}
$$

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