

Structure constants from Q systems and separation of variables

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(Received 2 April 2026; accepted 27 July 2026; published 8 October 2026)

We introduce a novel method to compute structure constants from Q functions in the scalar sector of planar $\mathcal{N} = 4$ super Yang-Mills (SYM) and related theories. The method derives from operatorial as well as functional separation of variables, and the structure constants are expressed as determinants of matrices whose entries are integrals over products of Q functions. In this framework, each operator is twisted by an external angle, mirroring the cusped Maldacena-Wilson loop. The structure constants of local single-trace operators in $\mathcal{N} = 4$ SYM are recovered in the untwisting limit, where we obtain a one-to-one correspondence between our key building blocks and those of the hexagon formalism. Retaining appropriate twists, our structure constants also perfectly match those of the orbifold points of $\mathcal{N} = 4$ SYM. Our results thus far are valid at leading order in the weak-coupling expansion, but their formulation in terms of Q functions provides a natural starting point for including loop corrections. Many of the methods we develop in this work apply more generally to the computation of correlation functions in integrable models.

DOI: [10.1103/tmj-wxq7](https://doi.org/10.1103/tmj-wxq7)

Introduction. Solving the dynamics of an interacting four-dimensional quantum field theory analytically remains an outstanding challenge in theoretical physics. In the most symmetric setting of $\mathcal{N} = 4$ super Yang-Mills theory (SYM), integrability has come a long way toward this goal. Most notably, the quantum spectral curve (QSC) [1,2] efficiently captures the spectra of local [3] and nonlocal operators [4] at finite coupling. On the contrary, extending the integrability approach to higher-point correlators remains a central open problem. The state of the art for three-point functions is an expansion in string world sheet excitations via hexagon form factors [5], which is very powerful for large-charge operators, but requires resummations that cannot be performed in practice for small charges [6].

An alternative formalism, where correlators are integrals over the Q functions naturally associated to the external operators by the QSC, would be most desirable. A natural approach is the separation of variables (SoV) framework [7], where all wave functions factorize into single-particle Q

functions, and correlators simplify drastically [8,9]. The SoV approach to structure constants thus far has been restricted to bosonic rank-one sectors [10–12]. A finite-coupling formalism will necessarily have to include the full $\mathfrak{psu}(2, 2|4)$ state space, which requires two major steps: a generalization to higher rank, and the inclusion of fermions.

In this work, we accomplish one of these major steps by combining operatorial [13–15] and functional [16–18] SoV to derive a formalism that allows us to express the leading-order structure constant of any operator in the full $\mathfrak{su}(4) = \mathfrak{so}(6)$ scalar sector with any two scalar Bogomol'nyi–Prasad–Sommerfield (BPS) operators in terms of determinants of Q functions.

A key aspect of our analysis is twists, which break the $\mathfrak{su}(4)$ symmetry and thereby lift all degeneracies. Hence our result applies uniformly to all highest-weight as well as descendant operators. Moreover, the twists allow us to access deformations of $\mathcal{N} = 4$ SYM, such as orbifold points [19].

Setup. We consider the three-point function of one excited operator $\mathcal{O}_1$ with twists z_j , $j = 1, 2, 3, 4$, and two protected operators $\mathcal{O}_2 = \text{Tr}(\mathcal{W}_{\omega, \kappa}^{L_2})$ and $\mathcal{O}_3 = \text{Tr}(\mathcal{W}_{\omega}^{L_3})$ depicted in Fig. 1.¹ $\mathcal{W}_{\omega}$ and $\mathcal{W}_{\omega, \kappa}$ are linear combinations of the six elementary scalars $Z, X, Y, \bar{Y}, \bar{X}, \bar{Z}$. In the integrable spin

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¹For general twists z_i , the cyclicity of the $\text{SU}(N_c)$ color trace is broken, and the resulting spin-chain states have no clear interpretation as field-theory operators.

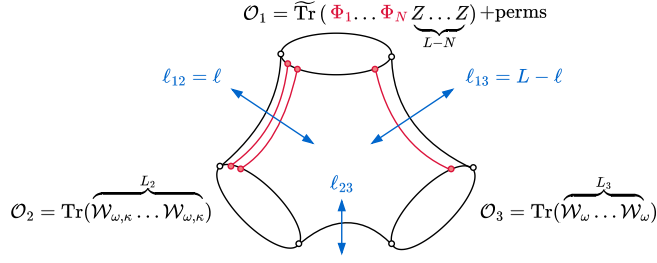


FIG. 1. Twisted three-point function of one excited operator of size L_1 with N excitations and two protected operators of sizes L_2 and L_3 . The red lines represent the propagators of the excitations $\Phi_i \in \{X, Y, \bar{X}, \bar{Y}, \bar{Z}\}$. The bridge lengths $\ell_{ij} = (L_i + L_j - L_k)/2$ denote the numbers of propagators between operators $\mathcal{O}_i$ and $\mathcal{O}_j$. In particular, we use $\ell \equiv \ell_{12}$ and $L - \ell$ to denote the bridge lengths of the excited operator.

chain picture, the operators $\mathcal{O}_i$ are identified with eigenstates of twisted conserved charges,

$$\mathcal{O}_1: |\Psi\rangle, \quad \mathcal{O}_2: \langle \mathcal{W}_{\omega, \kappa}^{L_2} |, \quad \mathcal{O}_3: \langle \mathcal{W}_{\omega}^{L_3} |, \quad (1)$$

where the protected operators $\mathcal{O}_2, \mathcal{O}_3$ are $SU(4)$ rotations of the vacuum $\langle Z |$ parametrized as

$$\langle \mathcal{W}_A | = \langle Z | e^{\nu_2 p_{13}^A E_{13}} e^{\nu_1 p_{23}^A E_{23}} e^{\nu_1 p_{14}^A E_{14}} e^{\nu_2 p_{24}^A E_{24}}. \quad (2)$$

The ν_j are simple ratios of z_j —the twists of the excited state—while p_{jk}^A parametrize the polarizations of the operators, both written explicitly in the Supplemental Material [20]. The associated field combination is

$$\mathcal{W}_A = \bar{Z} - \nu_1 p_{23}^A \bar{X} + \nu_2 p_{24}^A \bar{Y} + \nu_2 p_{13}^A Y + \nu_1 p_{14}^A X + (\nu_1^2 p_{14}^A p_{23}^A - \nu_2^2 p_{13}^A p_{24}^A) Z. \quad (3)$$

The polarizations are not completely independent due to the reduction $p_{jk}^{\omega, \kappa=0} = p_{jk}^{\omega}$. The reference polarizations $p_{jk} \equiv p_{jk}^{\omega=0}$ can be freely chosen up to the constraint $p_{13} p_{24} = p_{14} p_{23}$. We emphasize that while the definitions of these fields may seem *ad hoc*, they arise naturally in the SoV framework; see the section below.

In planar $\mathcal{N} = 4$ SYM with $z_j = 1$, the $\mathfrak{su}(4)$ R-symmetry allows one to rotate to a canonical frame where the BPS operators are [5,21]

$$\mathcal{O}_2 = \text{Tr}(Z + \bar{Z} + X - \bar{X})^{L_2}, \quad \mathcal{O}_3 = \text{Tr}(\bar{Z})^{L_3}, \quad (4)$$

where $\mathcal{O}_3$ is called the “reservoir state.” In the presence of twists $z_j \neq 1$, the $\mathfrak{su}(4)$ symmetry is broken; hence the structure constants depend nontrivially on the parameters ω and κ , while the freedom in choosing p_{ij} reflects the residual $\mathfrak{u}(1)^3$ symmetry. The untwisted planar $\mathcal{N} = 4$ super Yang-Mills setup is recovered in the $z_j \rightarrow 1$ limit.

Setting in addition $\omega = \kappa = 1$ and suitable polarizations p_{jk} , the BPS operators $\mathcal{O}_2, \mathcal{O}_3$ reduce to (4), up to a global rotation that leaves a highest-weight $\mathcal{O}_1$ invariant, see the Supplemental Material [20]. We refer to this as the *untwisting limit*:

$$z_i \rightarrow 1, \quad \kappa \rightarrow 1, \quad \omega \rightarrow 1. \quad (5)$$

The central object is the normalized structure constant $C_\ell \equiv C^{\omega\omega}/C^{\omega\omega\omega}$ where $C^{\omega\omega\omega}$ is the structure constant of three protected operators of the same lengths. At leading order, C_ℓ is independent of the bottom bridge ℓ_{23} , and it is equal to the spin chain overlap,

$$C_\ell = \frac{\langle \mathcal{W}_{\omega, \kappa}^\ell \otimes \mathcal{W}_{\omega}^{L-\ell} | \Psi \rangle}{\|\Psi\|}. \quad (6)$$

The state $|\Psi\rangle$ is characterized by $4 + 4$ Q functions Q_j and Q^j , defined as solutions to the fourth-order *Baxter equation* $\mathcal{B} \cdot Q = 0$ and the dual equation $\mathcal{B}^\dagger \cdot Q^j = 0$. The Baxter operator $\mathcal{B}$ acts as follows:

$$\mathcal{B} \cdot Q = u^L (u - i)^L Q^{[4]} + u^L (u + i)^L Q^{[-4]} + \tau_0^+ Q - (u - i)^L \tau_+^{[2]} Q^{[2]} - (u + i)^L \tau_- Q^{[-2]}, \quad (7)$$

where $f^{[a]} \equiv f(u + \frac{ai}{2})$ and $f^\pm \equiv f(u \pm \frac{i}{2})$ denote half-integer shifts. The functions $\tau_0(u)$ and $\tau_\pm(u)$ are the eigenvalues of the transfer matrices $t_0(u)$ and $t_\pm(u)$, which encode the complete set of $4L$ conserved quantities; see the Supplemental Material [20]. The dual Baxter operator $\mathcal{B}^\dagger$ is obtained directly from $\mathcal{B}$ by swapping the eigenvalues $\tau_\pm \rightarrow \tau_\mp$.

The Q functions are twisted polynomials of the form

$$Q_j(u) = z_j^{-iu} q_j(u), \quad Q^j(u) = z_j^{iu} q^j(u), \quad (8)$$

where $z_j = e^{i\phi_j}$ are the twist parameters of the excited state, and q_j and q^j are polynomials whose degrees λ_j and $\lambda^j = L - \lambda_j$ encode its $\mathfrak{su}(4)$ quantum numbers; see the Supplemental Material [20]. In terms of the Dynkin labels $[p, q, r]$,

$$p = \lambda_2 - \lambda_1, \quad q = \lambda_3 - \lambda_2, \quad r = \lambda_4 - \lambda_3, \quad (9)$$

with the constraint $2L = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.

For simplicity, we restrict to twists satisfying $z_1 z_4 = z_2 z_3 = 1$, treating the general case in the Appendixes. For generic real angles ϕ_1 and ϕ_2 , the spectrum of conserved charges is nondegenerate [22,23]: each Hamiltonian eigenstate is uniquely associated with the four Q functions Q_j . The situation simplifies further for our representation of

$\mathfrak{su}(4)$: Knowing just two Q functions, e.g. Q_1 and Q_2 , suffices to reconstruct all others [15].

The variables p_{jk} parametrize the $\mathfrak{u}(1)^3 \subset \mathfrak{su}(4)$ Cartan symmetry; thus C_ℓ depends on them rather trivially:

$$C_\ell(p_{jk}) = |p_{13}|^{K-\bar{K}} |p_{14}|^{\bar{K}} |p_{23}|^{M-K} \times C_\ell^0, \quad (10)$$

where $C_\ell^0 \equiv C_\ell(p_{jk} = 1)$, and $K, M, \bar{K}$ are Cartan quantum numbers that are related to the Dynkin labels (9) via

$$[p, q, r] = [M - 2K, L + K - 2M + \bar{K}, M - 2\bar{K}]. \quad (11)$$

Unless otherwise stated, we assume $p_{ij} = 1$ in what follows.

Result. We now present our main result: The twisted structure constant (6) admits the following SoV formulation, factorized into simple universal building blocks:

$$C_\ell^0 = \mathcal{N}_\ell \times \frac{\mathbb{W}_\omega \mathbb{A}_{\ell, \kappa}}{\sqrt{\mathbb{B}}}, \quad (12)$$

where $\mathcal{N}_\ell$ is a normalization defined below, and the remaining factors are SoV inner products built from the Q functions of the excited state:

$$\mathbb{B} = \det_{(\alpha, j, k), (\beta, a, b)} \langle Q_j^{[a]} u^{\beta-1} Q^{[b]} \rangle_{L, \alpha}, \quad (13)$$

$$\mathbb{W}_\omega = \det_{\alpha, \beta} \langle \omega^{-iu} u^{\beta-1} Q_{12} \rangle_{L, \alpha}, \quad (14)$$

$$\mathbb{A}_{\ell, \kappa} = \det_{(\gamma, j), (\delta, a)} \langle \kappa^{-iu} u^{\delta-1} Q_j^{[a]} \rangle_{\ell, \gamma}. \quad (15)$$

The notation is as follows:

- (1) Q_j and Q^k are normalized Q functions:

$$Q_j(u) = \frac{Q_j(u)}{\sqrt{Q_{12}(-i/2)}}, \quad Q^k(u) = \frac{Q^k(u)}{\sqrt{Q^{34}(+i/2)}}. \quad (16)$$

- (2) $\langle \cdot \rangle$ are contour integrals:

$$\langle f \rangle_{\ell, \alpha} = \oint \frac{du}{(-2\pi)^\alpha} \frac{e^{2\pi u(\alpha-1)}}{(u+i/2)^\ell (u-i/2)^\ell} f(u), \quad (17)$$

which encircle the poles at $\pm i/2$ counterclockwise.

- (3) Indices run over

$$j \in \{1, 2\}, \quad k \in \{3, 4\}, \quad a, b \in \{\pm 1\}, \\ \alpha, \beta \in \{1, \dots, L\}, \quad \gamma, \delta \in \{1, \dots, \ell\}.$$

- (4) $\mathcal{N}_\ell$ is the state-independent normalization, only depending on the twists and lengths of the operators,

$$\mathcal{N}_\ell = \omega^{L/2} \kappa^\ell \frac{(1-z_1^2)^{\frac{\ell}{2}} (1-z_2^2)^{\frac{\ell}{2}} (1-z_1 z_2)^L}{(1-\kappa z_1)^\ell (1-\kappa z_2)^\ell (1-\omega z_1 z_2)^L}, \quad (18)$$

which is valid up to a pesky (irrelevant) phase (computed in the Supplemental Material [20]) depending on the normalization of the three operators.

An important feature of our result (12) is *duality invariance* [24], i.e. invariant under the replacement of Q_1, Q_2 with any other pair of solutions Q_j of the Baxter equation. Physically, this means the structure constant is independent of whichever vacuum (e.g. $\text{Tr}(Z^L)$ or $\text{Tr}(\bar{Z}^L)$) the excited state is built upon—an important consistency requirement. We prove duality invariance in the Supplemental Material [20].

Left-right quantum symmetry. Structure constants in $\mathcal{N} = 4$ SYM possess an infinite tower of conservation laws beyond the usual R-charge conservation, called *left-right (LR) symmetry*, which will be important in the derivation of (12). This quantum symmetry was established in [25] for the untwisted setup (5), and it enforces $C_\ell = 0$ unless the excited state $|\Psi\rangle$ is such that $\tau_+ = \tau_-$.

We now demonstrate that in our twisted SoV formulation (12), this property is still present for generic z_j and κ once we set $\omega = 1$. Following the approach of [26–28], consider a state A and its image under the map $A \rightarrow \bar{A}$, where $\bar{A}$ is obtained from A by taking $Q_j^A = Q_{\bar{A}}^{5-j}$ and $\tau_\pm^A = \tau_\mp^{\bar{A}}$. Then the Baxter operators satisfy $\mathcal{B}_A^\dagger = \mathcal{B}_{\bar{A}}$, and so

$$((Q_2(\mathcal{B} - \mathcal{B}^\dagger)Q_1))_\alpha = 0, \quad (19)$$

where $((\cdot))$ is the scalar product under which $\mathcal{B}$ and $\mathcal{B}^\dagger$ are adjoint (see Sec. V), and we dropped the state index as now all objects refer to state A . By shifting contours appropriately together with the QQ relation $Q_{12} = Q_1^+ Q_2^- - Q_1^- Q_2^+$ we obtain the linear system

$$\sum_{\beta=1}^L \langle u^{\beta-1} Q_{12} \rangle_{L, \alpha} (I_{+, \beta} - I_{-, \beta}) = 0, \quad \alpha = 1, \dots, L, \quad (20)$$

where the coefficients $I_{\pm, \beta}$ are defined by

$$\tau_\pm \left(u + \frac{i}{2} \right) = \chi_\pm u^L + \sum_{\beta=1}^L I_{\pm, \beta} u^{\beta-1}, \quad (21)$$

with $\chi_+ = \sum_j z_j = \sum_j z_j^{-1} = \chi_-$ for $z_1 z_4 = z_2 z_3 = 1$. For a state which is not LR symmetric, at least one difference $I_{+, \beta} - I_{-, \beta}$ is nonzero; hence the linear system has a

nontrivial null vector and its determinant vanishes.

$$\det_{1 \leq \alpha, \beta \leq L} \langle u^{\beta-1} Q_{12}^A \rangle_{L, \alpha} \propto \delta_{A\bar{A}}. \quad (22)$$

Hence the structure constant (12) vanishes unless the excited state $|\Psi\rangle$ is LR symmetric.

Derivation. In the following, we outline the derivation of our main result (12). All details of the derivation can be found in the Supplemental Material [20].

SoV bases: The structure constant (6) is computed from spin chain overlaps of transfer matrix eigenstates. To convert these to SoV expressions, we resolve these overlaps in terms of left and right SoV basis states $\langle \mathbf{x} |$ and $| \mathbf{y} \rangle$, which are generated by acting with transfer matrices $t_{\pm}(u)$, $t_0(u)$ on sufficiently generic left and right SoV vacuum states $\langle 0 |$ and $| 0 \rangle$, respectively [15]. In this basis, the wave functions $\langle \mathbf{x} | \Psi \rangle$ and $\langle \Psi | \mathbf{y} \rangle$ of left and right transfer matrix eigenstates $\langle \Psi |$ and $|\Psi\rangle$ completely factorize [15]. An overlap $\langle \Psi_A | \Psi_B \rangle$ can then be computed by inserting $1 = \sum_{\mathbf{x}, \mathbf{y}} \mathcal{M}_{\mathbf{x}, \mathbf{y}} | \mathbf{y} \rangle \langle \mathbf{x} |$, where $\mathcal{M}_{\mathbf{x}, \mathbf{y}}$ is obtained by inverting the matrix of $\langle \mathbf{x} | \mathbf{y} \rangle$ overlaps. We circumvent that by employing the functional separation of variables method (FSOV) [16, 18].

Overlaps and norm: The starting point [16, 28] is the observation that the operator $\mathcal{B}^\dagger$ of the dual Baxter equation $\mathcal{B}^\dagger Q^j = 0$ satisfied by the Hodge dual Q function Q^j is adjoint to $\mathcal{B}$,

$$((f\mathcal{B}^\dagger g))_\alpha = ((g\mathcal{B}f))_\alpha, \quad (23)$$

under the scalar product defined via

$$((f))_\alpha = \oint \frac{du}{(-2\pi)^\alpha} \frac{e^{2\pi u(\alpha-1)}}{Q_\theta^{[-2]}(u) Q_\theta(u) Q_\theta^{[2]}(u)} f(u), \quad (24)$$

where the contour encircles all zeros of the denominator factors, $Q_\theta(u) \equiv \prod_{\alpha=1}^L (u - \theta_\alpha)$, and we have introduced inhomogeneities θ_α , $\alpha = 1, \dots, L$ that shift the local Lax operators $\mathcal{L}^{(\alpha)}(u)$ used to construct the conserved charges (see the Supplemental Material [20]) as $\mathcal{L}^{(\alpha)}(u) \rightarrow \mathcal{L}^{(\alpha)}(u - \theta_\alpha)$; they are required for regularization, and are set to $\theta_\alpha = 0$ at the end. This allows us to write $0 = ((Q_A^j (\mathcal{B}_A - \mathcal{B}_B) Q_k^B))_\alpha$ for any two states A and B with their associated Baxter operators $\mathcal{B}_A$ and $\mathcal{B}_B$. Since the transfer matrix eigenvalues generate integrals of motion, this equation can be expanded to

$$0 = \sum_{\beta=1}^L \sum_{a, b=\pm 1} \left\langle \left\langle Q_A^{j, [a]} u^{\beta-1} Q_k^{B, [b]} \right\rangle \right\rangle_\alpha I_{a, b, \beta}^{AB}, \quad (25)$$

where the bracket $\langle \langle \cdot \rangle \rangle_\alpha$ is obtained from $((\cdot))_\alpha$ by replacing $Q_\theta^{[-2]} Q_\theta Q_\theta^{[2]}$ with $Q_\theta^- Q_\theta^+$. The $I_{a, b, \beta}^{AB}$ form a complete set of $4L$ independent differences of integrals of motion for the two states A and B . Since the vector of I^{AB} is nonzero for different states, this implies $\det \mathbb{M}_{AB} \propto \delta_{AB}$, with

$$\mathbb{M}_{AB} = \left[\langle Q_{B, j}^{[a]} u^{\beta-1} Q_A^{k, [b]} \rangle_{L, \alpha} \right]_{(\alpha, j, k), (\beta, a, b)} \quad (26)$$

with $j \in \{1, 2\}$, $k \in \{3, 4\}$ and $a, b \in \{\pm 1\}$. Here we have already taken the homogeneous limit $\theta_\alpha \rightarrow 0$, which reduces $\langle \langle \cdot \rangle \rangle_\alpha$ to the bracket $\langle \cdot \rangle$ in (17). The orthogonal pairing $\det \mathbb{M}_{AB}$ must equal $\langle \Psi_A | \Psi_B \rangle$, up to an overall factor. We fix the latter by computing the brackets $\langle \langle \cdot \rangle \rangle_\alpha$ by residues, and identifying the occurring products of factorized wave functions $\langle \Psi_A | \mathbf{x} \rangle \langle \mathbf{y} | \Psi_B \rangle$ in the determinant, finally arriving at

$$\langle \Psi_A | \Psi_B \rangle = \left(\frac{\mathcal{N}}{Q_{12}^B (-i/2)^2 Q_A^{34} (+i/2)^2} \right)^L \det \mathbb{M}_{AB}, \quad (27)$$

with $\mathcal{N} = (z_{13} z_{23} z_{14} z_{24})^{-1}$, $z_{ij} \equiv z_i - z_j$. To compute the norm, we must note that in our SoV setup, $\langle \Psi | \Psi \rangle$ equals $||\Psi^2||$ only up to an overall normalization factor. Up to a complex phase, the norm is

$$||\Psi^2|| = |p_{13}|^{-2(K-\bar{K})} |p_{14}|^{-2\bar{K}} |p_{23}|^{-2(M-K)} \mathcal{N}^L \mathbb{B}, \quad (28)$$

Structure constant: To compute the overlap $\langle \mathcal{W}_{\omega, \kappa}^\ell \otimes \mathcal{W}_\omega^{L-\ell} | \Psi \rangle$, we first construct $\langle \mathcal{W}_\omega^L |$ with $\langle \mathcal{W}_\omega |$ as in (2), where $\mathbb{E}_{jk}$ are generators of $\mathfrak{su}(4)$, and p_{jk}^ω have 4 degrees of freedom that parametrize a general global SU(4) rotation.

When $z_1 z_4 = z_2 z_3 = 1$, the planar $\mathcal{N} = 4$ SYM structure constant obeys $C_\ell = 0$ unless $|\Psi\rangle$ is *left-right symmetric*, that is $(t_+ - t_-)|\Psi\rangle = 0$; see the section above. Thus it is sufficient to consider BPS states $\langle \mathcal{W}_\omega^L |$ with $\langle \mathcal{W}_\omega^L | (t_+ - t_-) = 0$. These are exactly the states with $\omega = 1$, where the parameters $p_{jk}^{\omega=1}$ retain 3 degrees of freedom. Keeping ω general, $\langle \mathcal{W}_\omega^L |$ is a completely general BPS state, $\langle \mathcal{W}_\omega^L | (t_+ - t_-) \neq 0$, but remarkably, our result (12) still remains correct. By a similar analysis as above, we find the overlap

$$\langle \mathcal{W}_\omega^L | \Psi \rangle = \frac{\omega^{L/2}}{(1 - \omega z_1 z_2)^L} \det \langle \omega^{-iu} u^{\beta-1} Q_{12} \rangle_{L, \alpha}. \quad (29)$$

Next, we deform $\langle \mathcal{W}_\omega^L |$ to $\langle \mathcal{W}_{\omega, \kappa}^\ell \otimes \mathcal{W}_\omega^{L-\ell} |$ by using a crucial feature that we call *localization*: By direct computation, we find that for any global rotation $\langle \mathcal{W}_\omega^L |$ of $\langle Z^L |$, the

following action of transfer matrices localizes to the first ℓ sites of the L -fold product:

$$\langle \mathcal{W}_\omega^L | \prod_{k=1}^{\ell} \hat{t}_{s_k}(\theta_k) = \langle \mathcal{W}_\omega^\ell | \prod_{k=1}^{\ell} \hat{t}_{s_k}^{(\ell)}(\theta_k) \otimes \langle \mathcal{W}_\omega^{L-\ell} |, \quad (30)$$

where $s_k \in \{+, 0, -\}$, $\hat{t}_{s_k}$ are properly normalized and shifted transfer matrices, and $\hat{t}_s^{(\ell)}$ are the transfer matrices of an ℓ -site chain. We want to consider linear combinations of (30) where the first ℓ sites equal $\langle \mathcal{W}_\omega^\ell | K$, with $K \in \text{GL}(4)$ a global rotation. By considering $\ell = 1$, it is easy to see that there is a one-parameter family of solutions $\langle \mathcal{W}_{\omega, \kappa}^\ell | = \langle \mathcal{W}_\omega^\ell | K_\kappa$. K_κ is a polynomial of transfer matrices $t_\pm$, t_0 that are diagonalized by $|\Psi\rangle$; hence

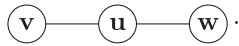
$$\langle \mathcal{W}_{\omega, \kappa}^\ell \otimes \mathcal{W}_\omega^{L-\ell} | \Psi \rangle = K_\kappa(\tau) \langle \mathcal{W}_\omega^L | \Psi \rangle. \quad (31)$$

Again considering $\ell = 1$, we recognize $K_\kappa(\tau)$ as the known SoV determinant of the antifundamental $\mathfrak{su}(3)$ spin chain [17,18], which straightforwardly generalizes to arbitrary ℓ (15),

$$K_\kappa(\tau) = \frac{\kappa^\ell}{(1 - \kappa z_1)^\ell (1 - \kappa z_2)^\ell} \det_{(\gamma, j), (\delta, a)} \langle \kappa^{-iu} u^{\delta-1} \mathcal{Q}_j^{[a]} \rangle_{\ell, \gamma}. \quad (32)$$

Combining (27), (29), and (32), we arrive at (10), (12), where the various prefactors combine to (18), up to a complex phase that depends on the normalization of $|\Psi\rangle$, and is spelled out in the Supplemental Material [20].

Comparison with hexagons. We apply the untwisting limit (5) in order to compare our result with the hexagon formalism [5,25]. Here, highest-weight states are characterized by sets of *Bethe roots* $\mathbf{v}$, $\mathbf{u}$, and $\mathbf{w}$ associated to the three nodes of the $\mathfrak{su}(4)$ Dynkin diagram:



The structure constants are given by

$$C_\ell^{\text{hex}} = \frac{\langle \mathbf{v} | \mathbf{w} \rangle}{\sqrt{\langle \mathbf{u} | \mathbf{u} \rangle}} \mathcal{A}_\ell, \quad \mathcal{A}_\ell = \sum_{\alpha \cup \bar{\alpha} = \mathbf{u}} (-\tau)^{|\alpha|} \prod_{j \in \alpha} f(u_j) e^{-ip(u_j)^\ell} \prod_{i \in \alpha, j \in \bar{\alpha}} \frac{1}{h(u_i, u_j)}, \quad (33)$$

where $\langle \mathbf{u} | \mathbf{u} \rangle$ is the *Gaudin norm*, $\langle \mathbf{v} | \mathbf{w} \rangle$ is the *wing Gaudin norm*, $\mathcal{A}_\ell$ is a sum over distributions of the roots $\mathbf{u}$ onto the two hexagon form factors and $\tau = 1$ was fixed in the

original untwisted setting [25]. Precise definitions are given in the Supplemental Material [20].

Notice that the form of our SoV result (12) looks very similar to (33), with three building blocks appearing in both expressions. Remarkably, we also get an exact match between (12) and (33) for $\omega = 1$ but general κ and z_j by setting $\tau = \kappa z_2$ in $\mathcal{A}_\ell$, and keeping both Gaudin norms unchanged. Indeed the match of the approaches works at the level of the individual building blocks! Namely, by setting

$$p_{13} = 1, \quad p_{14} = \frac{1 - \kappa z_1}{1 - z_1^2}, \quad p_{23} = \frac{1 - \kappa z_2}{1 - z_2^2}, \quad (34)$$

we find that $C_\ell(p_{jk})|_{\omega=1} = C_\ell^{\text{hex}}$, with

$$\langle \mathbf{u} | \mathbf{u} \rangle = \prod_{j=1}^3 \prod_{k=j+1}^4 z_{j,k}^{\lambda_j - \lambda_k} \times \mathbb{B}, \quad (35)$$

$$\langle \mathbf{v} | \mathbf{w} \rangle = \prod_{j=2}^3 z_{1,j}^{\lambda_1 - \lambda_j} \times \mathbb{W}_{\omega=1}, \quad (36)$$

$$\mathcal{A}_\ell = \kappa^\ell (1 - \kappa z_1)^{\lambda_1 - \ell} (1 - \kappa z_2)^{\lambda_2 - \ell} \times \mathbb{A}_{\ell, \kappa}. \quad (37)$$

Recall that the λ_i 's are related to the numbers $K, M, \bar{K}$ of Bethe roots in the sets $\mathbf{u}, \mathbf{v}, \mathbf{w}$, which matches the order $K, M, \bar{K}$ by (11) and (9).

While the match between the norms $\mathbb{B}$ and $\langle \mathbf{u} | \mathbf{u} \rangle$ is not too surprising, it yields interesting interpretations for the other two factors: $\mathbb{W}_{\omega=1}$ equals the norm $\langle \mathbf{v} | \mathbf{w} \rangle$ between auxiliary Bethe roots, whereas $\mathbb{A}_{\ell, \kappa}$ is identified as the SoV determinant representation of the product $\mathcal{A}_\ell$ of two hexagon form factors. We expect that the above match extends to $\omega \neq 1$ by including the corresponding $\mathfrak{su}(4)$ rotation in the hexagon computation.

Untwisting limits and orbifolds.

Twisted theory: The SoV formulation (12) can compute twisted structure constants at arbitrary values of ω, κ and twists z_i . For example, the state with $L = 2$ and Dynkin labels $[1, 0, 1]$, detailed in the Supplemental Material [20], has structure constant [in the “hexagon gauge” (34)]

$$|C_{\ell=1}| = \frac{(1 + \kappa)(z_2 - \omega z_1)(1 - z_1 z_2)}{\sqrt{2} z_2 (z_1 - z_2)(1 + z_2)(1 - \omega z_1 z_2)}, \quad (38)$$

and momentum $e^{iP} = Q_{12}(i/2)/Q_{12}(-i/2) = -z_1$. By specializing the twists z_k to

$$z_1 = e^{2\pi i n_1/N}, \quad z_2 = e^{2\pi i n_2/N}, \quad n_i \in \{0, 1, \dots, N-1\}, \quad (39)$$

one obtains structure constants at the $\mathbb{Z}_N$ orbifold points of $\mathcal{N} = 4$ super Yang-Mills [19,29,30]. For states in the $\mathfrak{su}(2)$

subsector, $\mathbb{Z}_N$ orbifold tree-level structure constants were computed from hexagons with many $\mathbb{Z}_2$, $\mathbb{Z}_3$ examples in [31], and all-loop Coulomb branch operator structure constants in [32]. Our formalism reproduces those results and at the same time extends them to the full $\mathfrak{su}(4)$ sector, yielding new predictions for general $\mathbb{Z}_N$ orbifold structure constants. For example, (38) with (39) reduces, with $\omega = 1$ and $\kappa = e^{2\pi i n/N}$ for simplicity, to

$$|C_{\ell=1}| = \frac{1}{\sqrt{2}} \frac{\cos(\pi n/N)}{\cos(\pi n_2/N)}. \quad (40)$$

away from the degenerate points where $\kappa z_2 = 1$ or $z_1 = z_2$.

Taking the untwisting limit: To recover structure constants of untwisted $\mathcal{N} = 4$ SYM, we take the limit (5). However, this limit is subtle. When $z_i \rightarrow 1$, entire lines in the matrices (13)–(15) vanish. These zeros are compensated by the twist prefactor (18), which diverges as $z_i \rightarrow 1$. The combination (12) is finite, and results in the untwisted structure constant.

The untwisted limit can be taken analytically in the $\mathfrak{su}(2)$ subsector; see the Supplemental Material [20]. The vanishing determinants are controlled by a single matrix element, reducing the determinants to single minors.

For the full $\mathfrak{su}(4)$ sector, this simplification no longer occurs: several matrix elements vanish at the same rate, and the determinant decomposes into a state-dependent sum of minors rather than a single minor. Only in the presence of twists does the SoV representation collapse to a single determinant for higher-rank sectors.

To extract untwisted structure constants, we opt for a simple strategy: We parametrize $z_k = e^{i\epsilon\phi_k}$, expand the Q functions around the $\epsilon \rightarrow 0$ limit, and plug their expansions into (13), (14) and (15). The leading term of (12) in the small twist expansion is finite, and results in the untwisted structure constant. This procedure applies equally to both primaries and descendants, as detailed in the Supplemental Material [20].

Counting untwisted structure constants: After parametrizing the twists as $z_k = e^{i\epsilon\phi_k}$ and taking $\epsilon \rightarrow 0$, the rotations acting on the BPS states still retain their dependence on ϕ_k , while the excited state becomes independent of ϕ_k . Accounting for $z_1 z_2 z_3 z_4 = 1$, we have five parameters: ϕ_1 , ϕ_2 , ϕ_3 , ω , and κ . Two generic BPS states are parametrized by eight free parameters. However, we have three Cartan conservation laws from the $\mathfrak{u}(1)^3$ residual R-symmetry. Hence, the number of parameters matches the number of degrees of freedom. By expanding in each of these parameters, we are able to extract the structure constant of *any* excited state $|\Psi\rangle$ (a conformal primary or descendant), and *any* BPS states (a rotated vacuum or descendant).

Conclusion. We took an important step toward a Q-system formulation of general structure constants in planar $\mathcal{N} = 4$ SYM. Our result (10) for the $\mathfrak{su}(4)$ scalar structure constant is in particular valid for any values of the twists z_i , ω and κ . Setting these to appropriate values, we recover the structure constants of twisted $\mathbb{Z}_N$ orbifold points. Notably, we find a one-to-one correspondence between our SoV determinants and the building blocks of the hexagon formalism. This direct match hints at a potential unification of the all-loop results of [33,34] with SoV.

Our formalism may be adapted to settings where hexagons are not fully developed, such as ABJM theory [35], where recent progress on orthogonal group Q systems can be exploited [36], or marginal deformations of planar $\mathcal{N} = 4$ SYM [37,38].

Structurally, the twists z_i , ω and κ play a similar role as the cusp angles in the Maldacena-Wilson loop [39,40]. Strikingly, our result indeed takes the same form as the finite coupling results of [9,28], where each inner product is a pairing between the excited state Q functions and twisted vacuum Q functions represented by the factors ω^{-iu} and κ^{-iu} in (14) and (15).

Our result is the ideal starting point for loop corrections, where the Q functions get promoted to their higher-loop QSC counterparts. It also enables analytic continuation in spin, and direct access to light-ray operators [41], bypassing the integer-spin restriction inherent in the hexagon picture. Finally, the functional SoV techniques developed here pave the way toward a supersymmetric SoV formalism for the full $\mathfrak{psu}(2, 2|4)$ superalgebra at finite coupling—the ultimate goal of this program.

Acknowledgments. We are grateful to S. Ekhammar, N. Gromov, C. Kristjansen, A. Liashyk, T. Skrzypek, and P. Vieira for discussions. T. B. and P. R. are grateful to M. Baroni and V. Schomerus for discussion and collaboration on related topics. A preliminary version of these results was presented at “New Frontiers of Quantum Field and Gravity” in Beijing in January 2026. P. R. is grateful to the organizers and participants, and especially B. Basso, Y. Jiang and D. Serban for many stimulating discussions related to this work. The work of C. B. was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program-60-(Grant Agreement No. 865075) EXACTC. The work of T. B. and P. R. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—460391856. T. B. and P. R. acknowledge support from DESY (Hamburg, Germany), a member of the Helmholtz Association HGF, and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy—EXC 2121 “Quantum Universe”—390833306.

Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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