

RG studies of scalar-field models of long-range interactions

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In this work, we study the long-range interactions in nongravitational field theories and their behavior in the deep infrared. To model such effects, we consider a nonlocal scalar theory obtained by adding a $\phi\Box^{-1}\phi$ term to the local action. Using the functional renormalization group, we analyze its infrared fixed-point structure. Within the local potential approximation (LPA), we show that nonlocality modifies phase-transition patterns and can induce symmetry breaking. Extending the LPA beyond polynomial truncations, we examine the convexity property of the effective potential as $k \rightarrow 0$ and find that the flow becomes singular for $\lambda^2 > 0$ before reaching the deep infrared. Including the scale-dependent wave function renormalization, Z_k (LPA'), we find that the infrared-stable fixed point is the nonlocal Gaussian fixed point. We then generalize the model to $\phi\Box^{\sigma/2}\phi$ and analyze how the infrared properties depend on σ . With appropriate scaling choices, we show that the infrared behavior remains unchanged up to $\sigma = d/2$ and follows Sak's prediction up to $\sigma = 2$. Finally, we study higher-derivative cases within the LPA, focusing on $\sigma = 4$, which corresponds to isotropic Lifshitz criticality, and obtain results consistent with earlier work.

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I. INTRODUCTION

Locality is widely considered as a fundamental principle of nature. However, quantum corrections can naturally introduce nonlocal interactions in the effective description. Such nonlocalities often emerge when massless or light degrees of freedom are integrated out of an underlying local theory. Moreover, nonlocality arises naturally in quantum gravity frameworks like string theory, where it helps address divergences and provides a mechanism for UV completion. Introducing nonlocal interactions can significantly alter the fixed-point structure of a theory, reshaping both its low-energy (IR) behavior and high-energy dynamics by introducing new stable fixed points, changes in scaling laws, or suppression of divergences [1,2]. This has significant implications for cosmological models, where nonlocal corrections to gravity can affect large-scale structure of the Universe. Similarly, in condensed matter systems, low-energy behavior of a model with long-range

interactions is effectively equivalent to a nonlocal theory [1,3]. In general, nonlocal interactions allowed by the symmetries of the theory are equally considerable as local interactions [4]. These nonlocal interactions give interesting results in the asymptotic safety of quantum gravity, where it dynamically solves the IR problems of quantum gravity [5–7]. Thus, nonlocal field theories offer a richer framework to explore novel IR phenomena across different areas of physics.

Recent studies have explored the nonlocal aspects in various contexts. Nonlocal theories allow the construction of explicit models where symmetry breaking persists at arbitrarily high temperatures due to long-range interactions [8]. These models could advance the realization of high-temperature superconductors and can also improve our understanding of phase transitions in the early Universe. The nonlocal modification of gravity is used in cosmological models to explain inflation or late-time cosmic acceleration through infrared nonlocality without the need for additional degrees of freedom [9–12]. Because of their inherent long-range correlations, nonlocal interactions can also significantly affect the entanglement structure of a theory, a topic extensively discussed in literature, including in the context of de Sitter and anti-de Sitter spacetime [13,14]. Furthermore, analyzing infrared behavior or the low-energy physics of nonlocal theories—particularly the model we will employ—has yielded promising results, notably providing a resolution

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to the IR problem of the de Sitter propagator [15]. It is also established that certain classes of nonlocal gravity theories can be constructed to be free of ghosts and singularities. Remarkably, despite their nonlocal nature, these theories recover the correct infrared limit, Einstein's general relativity in the IR limit [16,17]. Moreover, by incorporating nonlocal terms, one can construct superrenormalizable models that improve the ultraviolet behavior of gravity while maintaining unitarity [18–20]. These nonlocal terms typically introduce exponential or similar suppressive factors in the propagators at high energies, effectively softening divergences and improving the UV behavior of the theory. Additionally, the implications of nonlocality have been explored in the context of the Unruh effect and the black hole information loss paradox, offering novel insights into these foundational problems [21,22]. Together, these developments highlight the potential of nonlocal theories to address both infrared and ultraviolet challenges in a unified framework.

In this work, we are trying to explore the fixed-point structure of a single scalar-field theory with nonlocality introduced through $\phi\Box^{-1}\phi$ interactions. This model serves as a toy model for nonlocal modifications of gravity, similar to the Deser-Woodard model [12] and related approaches [7,9,23], where understanding the IR dynamics is crucial for consistency and its viability as a dark energy alternative. In condensed matter physics, the model can be interpreted as an effective description of systems with strong long-range interactions [1,3], and our focus is on exploring its universal behavior in the low-energy limit and assessing how smoothly it connects to the established long-range behavior discussed in [3,24]. The model can be reformulated as a local theory involving two scalar fields. This property allows us to validate the results obtained from the nonlocal formulation by comparing them with those of the well-established local theory. Our objective in this analysis is to understand the complete phase space structure of the theory, focus on low-energy behavior, and relate the model to known models in the literature. To achieve our goal, we use the standard renormalization group method to understand physics at different energy scales. Given the presence of long-range correlations in the model, it is most effective to employ exact renormalization group methods. In particular, we utilize the functional renormalization group (FRG) approach, which is well suited for capturing non-perturbative features of the theory [25,26]. Using the FRG, we are also able to extend the analysis to a more general class of theories with $\phi\Box^{\sigma/2}\phi$ interactions. In general, these theories are not unitary; one may have to do the complete spectral analysis to study their unitary properties. However, the renormalization analysis of such models remains well motivated where one is interested in the universal scaling properties, operator dimensions, and the structure of fixed points rather than the time evolution of the model [27–29]. These models also naturally arise as an effective description

for long-range interactions and exhibit well-defined critical behavior [1,24]. For $\sigma > 0$, the critical behavior depends on the value of σ , with a crossover from long- to short-range universality occurring at a fixed σ^* [30,31]. Recent work has employed conformal perturbation theory to analyze these models and has obtained a new weakly coupled field-theoretic description in the vicinity of σ^* [32,33]. This approach yields a deeper understanding of the crossover. Subsequent studies have further extended this framework to investigate critical phenomena over a wider range of σ [34–38].

This paper is organized as follows. In Sec. II, we introduce the details of our model and the FRG method employed in our analysis. We begin with the leading approximation in FRG, known as the local potential approximation (LPA), where the nonlocal coupling is treated as an external parameter. Within this framework, we examine the impact of the nonlocal coupling on the phase transition. Subsequently, we extend the analysis by incorporating wave function renormalization (LPA'). The section concludes by demonstrating that the infrared-stable fixed point of the theory is the nonlocal Gaussian fixed point (NL GFP). In Sec. III, we rederive the same results using the equivalent local theory formulated with two scalar fields. Then, in Sec. IV, we generalize our model to include arbitrary interactions of the form $\phi\Box^{\sigma/2}\phi$, establishing a smooth connection between our framework and existing long-range models in the literature, as well as higher-derivative theories. Finally, in Sec. V, we summarize our findings and discuss potential extensions of this work. In Appendix A, we derive the flow equations for all parameters in our nonlocal theory, both explicitly within the nonlocal formulation and through the corresponding local theory. In Appendix B, we solve the flow equation for the nonlocal model using an asymptotic solution, confirming consistency with the results presented in the main text through spike analysis. Throughout this paper, we work in natural units with $\hbar = c = 1$ and in the Euclidean framework with metric convention $(+, +, +, +)$.

II. NONLOCAL MODEL

In this section, we introduce a model of scalar-field theory with nonlocal interactions. We examine the functional space structure of the model using the functional renormalization group method. The nonlocal Lagrangian density for the scalar field is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda^2}{2}\phi\frac{1}{-\Box}\phi + U(\phi). \quad (1)$$

Here, $\Box$ represents the Laplacian operator in flat space, and λ is the real-valued nonlocal coupling. In momentum space, the nonlocal terms scale as p^{-2} , dominating the infrared limit. The theory contains complex mass poles, as discussed in [13–15], which have been extensively studied in

the literature [39–41]. In this work, we are considering the model as an effective description of a more fundamental ultraviolet complete theory. One can also describe this nonlocal theory using a local theory that involves two scalar fields. We will explore this in Sec. III.

To study the functional space structure of the theory, we introduce a scale-dependent effective action, Γ_k , which depends on an infrared cutoff k . As $k \rightarrow \infty$, the effective action approaches the classical action. In the limit as $k \rightarrow 0$, it gives the full quantum action. For intermediate scales, Γ_k provides the mean-field free energy. The flow equation for the effective average action is given by the Wetterich flow equation as [25]

$$\partial_t \Gamma_k = \frac{1}{2} \text{tr} \left(\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k} \right), \quad (2)$$

where the trace is over all the spacetime and internal indices, $t = -\log(k/k_0)$, R_k is the regulator, and $\Gamma_k^{(2)}$ is the two-point vertex function. The addition of the cutoff function has not modified the vertices; its only effect is to replace the original inverse propagator by the cutoff inverse propagator so that we can obtain a well-behaved modified propagator for the theory. The simplest ansatz for Γ_k is the LPA [26,42,43], which is given as

$$\Gamma_k = \int_x \frac{1}{2} (\partial\phi)^2 + \frac{\lambda^2}{2} \phi \frac{1}{-\square} \phi + u(\rho), \quad (3)$$

where $2\rho = \phi^2$. In order to solve the Wetterich equation (2) one has to choose the form of R_k to obtain the flow equation for different parameters. In our model we choose the regulator function as

$$R_k = \left[(k^2 - q^2) + \lambda^2 \left(\frac{1}{k^2} - \frac{1}{q^2} \right) \right] \theta(k^2 - q^2). \quad (4)$$

Note that, as $q^2 \rightarrow 0$, $R_k \rightarrow -\infty$; however, the modified propagator is well behaved, as shown in Fig. 1, which is the necessary condition [43]. Using this regulator function and constant field approximation, one can obtain

$$\begin{aligned} \partial_t \lambda^2 &= 0, \\ \partial_t u_k &= -\mu_d k^d \left(k^2 - \frac{\lambda^2}{k^2} \right) \left(k^2 + \frac{\lambda^2}{k^2} + m^2 \right)^{-1}, \end{aligned} \quad (5)$$

where $\mu_d^{-1} = (4\pi)^{d/2} \Gamma(d/2 + 1)$ and $m^2 = u_k''(\phi)$. In deriving $\partial_t \lambda^2 = 0$, we only use the analyticity of the nonlocal propagator, so the statement is independent of the cutoff function (see Appendix A).

A. Nonlocal coupling as external parameter

In this subsection, we consider the nonlocal coupling as an external parameter which is not flowing. We are interested in the fixed-point solutions of the theory, so

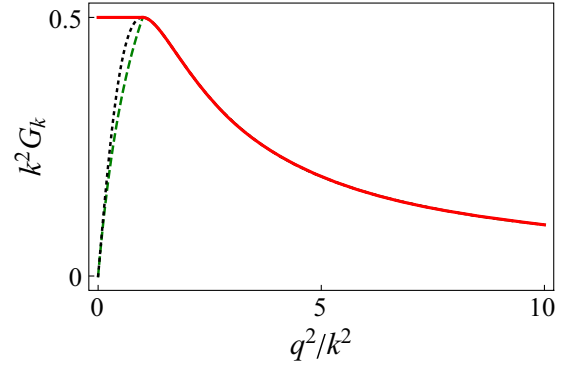


FIG. 1. Red solid line shows the modified cutoff propagator using Eq. (16), dashed green line shows the modified propagator with standard Litim cutoff, and black dotted line shows the original propagator.

we can scale the parameters in the theory as $\phi = k^{\frac{d-2}{2}} \tilde{\phi}$, $u_k = k^d \tilde{u}_k$, and $\lambda^2 = k^4 \tilde{\lambda}^2$ which leads to scaled flow equation as

$$\partial_t \tilde{u}_k = d \tilde{u}_k - \frac{d-2}{2} \tilde{\phi} \tilde{u}_k' - \mu_d (1 - \tilde{\lambda}^2) (1 + \tilde{m}^2 + \tilde{\lambda}^2)^{-1}. \quad (6)$$

Without any further approximation on the form of potential, one can solve the above equation [Eq. (6)] using the spike analysis [44] for different values of the nonlocal coupling as shown in Fig. 2. Plots in Fig. 2(a) correspond to the three-dimensional case, where the spike at nonzero value of s corresponds to the Wilson-Fisher fixed point. One can see that the spike corresponding to the Wilson-Fisher fixed point moves away as we increase the value of nonlocal coupling. The s value corresponds to each spike that gives the value of the mass parameter [$u''(\phi = 0) = s_0$] at the corresponding fixed points. The inset plot in Fig. 2(a) shows the shift in s_0 as we increase the nonlocal coupling $\tilde{\lambda}^2$. The linear shift of the non-Gaussian fixed point with increasing nonlocal coupling holds even in lower dimensions, as illustrated in Fig. 2(b). Now we do the fixed-point analysis using the polynomial truncation, where we assume that

$$\tilde{u}_k(\phi) = \sum_{n=0}^N \frac{\tilde{g}_{2n}}{(2n)!} \tilde{\phi}^{2n}. \quad (7)$$

Using the polynomial ansatz for the potential [Eq. (7)] in the flow equation for the potential [Eq. (6)] we can obtain the flow equation for the local couplings in Eq. (7). One can solve these flow equations for arbitrary truncation in Eq. (7) to obtain the fixed-point values of the local couplings as

$$\begin{aligned} \tilde{g}_0^* &= \frac{\mu_d}{d} \frac{1 - \tilde{\lambda}^2}{\Delta_*}, & \tilde{g}_4^* &= \frac{2}{\mu_d} \frac{\tilde{g}_2^* \Delta_*^2}{\tilde{\lambda}^2 - 1}, \\ \tilde{g}_6^* &= \frac{4 - d}{\mu_d} \frac{\tilde{g}_4^* \Delta_*^2}{\tilde{\lambda}^2 - 1} + \frac{6 \tilde{g}_4^{*2}}{\Delta_*}, \dots, \end{aligned} \quad (8)$$

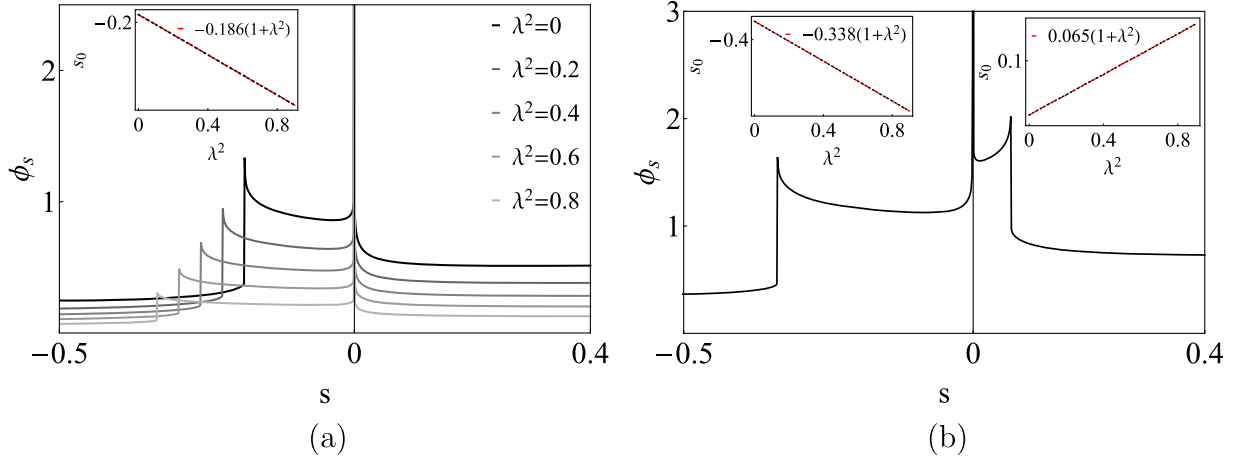


FIG. 2. (a) Spike plot for different values of $\tilde{\lambda}^2$ in $d = 3$. The black line corresponds to the local theory $\tilde{\lambda}^2 = 0$ and as we increase the $\tilde{\lambda}^2$ the spike shifts to the left. The plot corresponds to $\tilde{\lambda}^2 = 0$ to $\tilde{\lambda}^2 = 1$ at an interval of 0.2. Inset: we plot the variation s_0 that corresponds to the Wilson-Fisher fixed point as a function of nonlocal coupling ($\tilde{\lambda}^2$). We also show the best line fit for the data point, which explicitly shows the linear behavior. (b) Spike analysis and the linear shift of non-Gaussian fixed points in $d = 2.6$.

where $\Delta_* = 1 + \tilde{\lambda}^2 + \tilde{g}_2^*$. Note that these solutions are true in all orders in N and from the spike analysis we know $g_2^* = s_0$ as a function of $\tilde{\lambda}^2$, which is true for arbitrary truncation. Using the results in Eq. (8), one can obtain the fixed-point solution of the local coupling which is true for arbitrary truncations. Now we calculate the critical exponents of the fixed point, which are the physically measurable quantities. Critical exponents characterize a fixed point and provide insights into the stability of the fixed point. One can obtain the critical exponents from the eigenvalues of the stability matrix given as

$$S_{ij} = \frac{\partial \beta_i}{\partial \tilde{g}_j} \Big|_*. \quad (9)$$

For the primary analysis, we keep truncation up to quartic order in the potential [i.e., $N = 2$ in Eq. (7)], which gives the stability matrix as

$$S_{ij} = \begin{pmatrix} -d & \mu_d \frac{\lambda^2 - 1}{\Delta_*^2} & 0 \\ 0 & -2 - \frac{4\tilde{g}_2^*}{\Delta_*} & \mu_d \frac{\lambda^2 - 1}{\Delta_*^2} \\ 0 & \frac{72\tilde{g}_2^{*2}}{\mu_d(\lambda^2 - 1)} & d - 4 - \frac{24\tilde{g}_2^*}{\Delta_*} \end{pmatrix}. \quad (10)$$

Note that, for fixed dimensions, the stability matrix is only a function of $\tilde{g}_2^*$ and λ^2 . One can calculate the eigenvalues (or the critical exponents) from the above matrix as

$$\begin{aligned} \Lambda_0 &= -d, \\ \Lambda_{1,2} &= -\frac{1}{2\Delta_*} [28g_2^* + (6-d)\Delta_* \\ &\quad \pm (688g_2^{*2} - 40(d-2)^2g_2^*\Delta_* + (d-2)^2\Delta_*^2)^{1/2}]. \end{aligned} \quad (11)$$

From the spike analysis, it is evident that $g_2^* \sim (1 + \tilde{\lambda}^2)$. Using this relationship, it can be readily observed that the critical exponents remain independent of the nonlocal coupling $\tilde{\lambda}^2$ or the same as the one in the standard local theory. This holds true even for higher-order truncations and in fractional dimensions. So we are looking at the same fixed point as though the nonlocal coupling changes its values.

The nonlocal coupling affects the phase transitions (see Fig. 3). The dotted red line in Fig. 3 represents the separatrix for local theory ($\tilde{\lambda}^2 = 0$), and the point P moves to the symmetry phase for a local theory. However, once we turn on the nonlocal parameter, the flow diagram changes (dotted blue and green lines show the separatrix for different values of $\tilde{\lambda}^2$), and for $\tilde{\lambda}^2 \geq 0.2$, the point P moves to the broken phase. So, one can say that the nonlocal parameter induces symmetry breaking.

B. Beyond the polynomial truncation

It is interesting to study the evolution of the flow without resorting to polynomial truncations. Because of the nonlinearity of the functional flow equation already in the LPA, we shall keep λ constant and explore the impact of the λ^2 term in the renormalized flow in the limit $k \rightarrow 0$. The question we would like to address is whether the presence of a nonlocal term affects the convexity properties of the potential as $k \rightarrow 0$. To investigate this problem, it is convenient to recast Eq. (5) as a flow equation for the threshold function

$$w = \left(1 + \frac{\lambda^2}{k^4} + \frac{u_k''(\phi)}{k^2} \right)^{-1}, \quad (12)$$

so that, after some algebra, one obtains the following quasilinear parabolic equation in terms of the variable $\phi = x$:

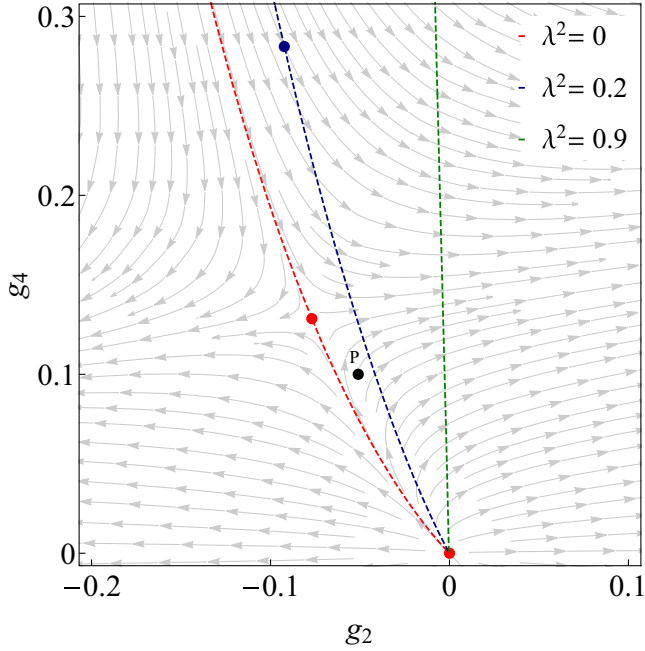


FIG. 3. RG flow diagram with parameters g_2 and g_4 for $N = 2$ truncation and $d = 3$. Separatrix of different phases for different values of λ^2 are shown in different colors. The red dashed line shows the separatrix for $\lambda^2 = 0$, the region flowing toward large positive g_2 corresponds to the symmetric phase, and the other phase corresponds to the broken symmetry phase. Green and blue dashed lines show the separatrix for nonzero values of λ^2 . The red dot corresponds to the Gaussian and Wilson-Fisher fixed point (WF FP) with $\lambda^2 = 0$ and the blue dot corresponds to the WF FP with $\lambda^2 = 0.2$. Point “P” represents a symmetric phase for $\lambda^2 = 0$ and broken phase for $\lambda^2 \geq 0.1$.

$$\dot{w} = -2w(1 - w + \lambda^2 e^{4t}w) + e^{-t(d-2)}(1 - \lambda^2 e^{4t})w''w^2, \quad (13)$$

where a dot denotes a derivative with respect to the (RG) time t , and $e^{-t} := k$, and we have set $\mu_d = 1$. In this formalism, the physical limit $k \rightarrow 0$ is therefore obtained as the large-time behavior of Eq. (13) [45–47]. Equation (13) can be interpreted as a modified heat equation with a field-dependent thermal diffusivity coefficient

$$e^{-t(d-2)}w^2(1 - e^{4t}\lambda^2), \quad (14)$$

whose sign is, however, positive definite only for $\lambda^2 \leq 0$. When this condition is violated, Eq. (13) does not define a well-posed initial value problem and the solution is not guaranteed to exist. As a consequence, the flow can become singular at a finite and positive value of t , signaling that it is not always possible to continue the evolution toward the infrared for the nonlocal theory defined by Eq. (1) for $\lambda^2 > 0$, at least in this truncation. This situation must be contrasted with the local theory, $\lambda = 0$, where the limit $t \rightarrow \infty$ is well defined below the critical line, both in $d = 3$ and $d = 4$. On the other hand, if we try to analytically continue our equation to $\lambda^2 < 0$ values, the numerical integration

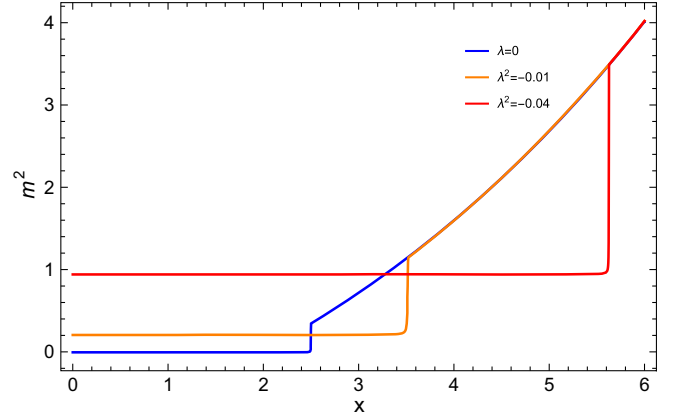


FIG. 4. The generalized “mass” $m^2 = u''(x, t)$ obtained from the integration of the flow in the limit $t \rightarrow \infty$ for $r = -0.3$ and $g = 0.01$. The solid blue line corresponds to $\lambda^2 = 0$, the orange line to $\lambda^2 = -0.01$, and the red line to $\lambda^2 = -0.04$.

does not lead to a singular flow in the infrared, and in this case the system reaches a trivial, high-temperature fixed point for $r > 0$ and an ordered phase for initial conditions below the critical line. We have therefore numerically integrated Eq. (13) using the method of lines, as implemented in *Mathematica*, as already discussed in Ref. [48], and explicitly checked this scenario considering a bare theory with a potential $rx^2/2 + gx^4$ potential. The results are shown in Fig. 4 for $d = 3$, where the recovery of the convexity properties of the effective blocked potential in the limit $t \rightarrow \infty$ is clearly visible, for $r = -0.3$ and $g = 0.01$. In contrast, it turns out that it is not possible to extend the integration to $k = 0$ ($t \rightarrow \infty$) for $\lambda^2 > 0$ as a singularity always appears at some $t_{\text{sing}} < \infty$. It would be interesting to study under which set of initial data it is possible to produce nonsingular flow in the large t limit. The question is certainly important, but because of the mathematical and numerical difficulties in treating this case, we postpone this analysis to a future investigation and focus in this analysis only on the critical properties of the theory.

C. Including the flow of nonlocal coupling

Once we consider the nonlocal coupling is also flowing, in the LPA, along with the flow equation for the potential [Eq. (6)] we have $\partial_t \tilde{\lambda}^2 = -4\tilde{\lambda}^2$, which at the fixed point gives $\tilde{\lambda}^2 = 0$, the standard local theory. So in order to capture the effects of flowing nonlocal coupling, we move to the next-order approximation by incorporating scale-dependent wave function renormalization (Z_1) as

$$\Gamma_k = \int_x \frac{Z_1}{2} (\partial\phi)^2 + \frac{\lambda^2}{2} \phi \frac{1}{-\square} \phi + u_k(\rho). \quad (15)$$

Including the wave function renormalization we rewrite our cutoff regulator function as

$$R_k = \left[Z_1(k^2 - q^2) + \lambda^2 \left(\frac{1}{k^2} - \frac{1}{q^2} \right) \right] \theta(k^2 - q^2). \quad (16)$$

Using our ansatz [Eq. (15)] and the regulator function [Eq. (16)] in the Wetterich equation (2), one can obtain the renormalization group flow equation for the parameters as (see Appendix A)

$$\begin{aligned} \partial_t \lambda^2 &= 0, \\ \partial_t Z_1 &= \mu_d k^{d+2} (u_k''')^2 \left(Z_1 - \frac{\lambda^2}{k^4} \right)^2 \left(Z_1 k^2 + \frac{\lambda^2}{k^2} + u_k'' \right)^{-4}, \\ \partial_t u_k &= -\mu_d k^{d+2} \left(Z_1 - \frac{\lambda^2}{k^4} + \frac{\partial_t Z_1}{d+2} \right) \left(Z_1 k^2 + \frac{\lambda^2}{k^2} + u_k'' \right)^{-1}, \end{aligned} \quad (17)$$

where $\mu_d^{-1} = (4\pi)^d \Gamma(d/2 + 1)$. In Eq. (5), we must specify the field configuration used to calculate $\partial_t Z_1$, which we take at the minimum of the potential ($\rho = \kappa$). We need the dimensionless flow equation for the parameters to analyze the fixed-point structure of the theory. Here we have two natural choices to choose the dimension of the parameters. First is similar to the standard local theory, where we choose Z_1 as dimensionless and absorbed into the field where the scaling is given as

$$\phi = \sqrt{\frac{\mu_d}{Z_1}} k^{\frac{d+2}{2}} \tilde{\phi}, \quad u = \mu_d k^d \tilde{u}_k, \quad z = \tilde{z}, \quad \lambda^2 = k^4 \tilde{\lambda}^2. \quad (18)$$

This leads to the definition of $\tilde{J}_{nl} = \tilde{\lambda}^2 / \tilde{Z}_1$ along with which we can write the scaled flow equation as

$$\begin{aligned} \partial_t \tilde{J}_{nl} &= (4 - \eta) \tilde{J}_{nl} \quad \text{with} \\ \eta &\equiv -\frac{\partial_t Z_1}{Z_1} = (\tilde{u}_k''')^2 (1 - \tilde{J}_{nl})^2 (1 + \tilde{J}_{nl} + \tilde{u}_k'')^{-4}, \\ \partial_t \tilde{u}_k &= d\tilde{u}_k - \frac{1}{2}(d-2)\tilde{\phi}\tilde{u}_k' \\ &\quad - \left(1 - \tilde{J}_{nl} - \frac{\eta}{d+2} \right) (1 + \tilde{J}_{nl} + \tilde{u}_k'')^{-1}. \end{aligned} \quad (19)$$

Looking at the flow equation for $\tilde{J}_{nl}$, one can see the possible solutions are $\tilde{J}_{nl} = 0$ and $\eta = 4$. The $\tilde{J}_{nl} = 0$ corresponds to the standard local theory with known fixed points in the corresponding dimensions [44]. The $\eta = 4$, cannot be studied properly using this scaling, for which we need the next scaling (see Sec. IV for more details).

In order to capture the effect of nonlocal coupling, we consider the second scaling where we treat λ^2 as dimensionless and absorb it into the field redefinition [3,24], which gives the dimensionless parameters as

$$\phi = \left(\frac{\mu_d}{\tilde{\lambda}^2} \right)^{\frac{1}{2}} k^{\frac{d+2}{2}} \tilde{\phi}, \quad u_k = \mu_d k^d \tilde{u}_k, \quad Z_1 = \frac{\tilde{Z}_1}{k^4}, \quad \lambda^2 = \tilde{\lambda}^2. \quad (20)$$

This also leads to the definition of nonlocal parameter $\tilde{J}_l = \tilde{Z}_1 / \tilde{\lambda}^2$. Along with $\partial_t \tilde{\lambda}^2 = 0$, one can write the flow equation for the scaled parameters as

$$\begin{aligned} \partial_t \tilde{J}_l &= -4\tilde{J}_l + (\tilde{u}_k''')^2 (\tilde{J}_l - 1)^2 (1 + \tilde{J}_l + \tilde{u}_k'')^{-4}, \\ \partial_t \tilde{u}_k &= d\tilde{u}_k - \left(\frac{d+2}{2} \right) \tilde{\phi} \tilde{u}_k' \\ &\quad - \left(\tilde{J}_l - 1 + \frac{\partial_t \tilde{J}_l - 4\tilde{J}_l}{d+2} \right) (1 + \tilde{J}_l + \tilde{u}_k'')^{-1}. \end{aligned} \quad (21)$$

In the limit $\tilde{J}_l \rightarrow 0$, the fixed-point equation can be mapped to that of the local theory with an effective negative spacetime dimension, as expected from the effective dimension approach [3,24]. The limit $\tilde{J}_l \rightarrow 0$ corresponds to a regime where only nonlocal interactions remain, as one would expect in the far-infrared limit. One can solve the above equations [Eq. (21)] simultaneously as follows: first solve $\partial_t \tilde{u}_k = 0$ with $\tilde{J}_l = 0$, then use the solution of $\tilde{u}_k$ in the flow equation for $\tilde{J}_l$. Repeat this until we obtain a convergent solution for $\tilde{J}_l$. We can proceed in two ways, one is the standard spike analysis (or shooting from the origin) [44,49], where we parametrize the initial conditions using $\tilde{u}_k^{(1)}(\tilde{\phi} = 0) = 0$ and $\tilde{u}_k^{(2)}(\phi = 0) = s$ and we tune “s” such that we obtain a global solutions to the differential equation. In the second method, we consider the asymptotic form of $\tilde{u}_k$ for large values of $\tilde{\phi}$, which is a functional of arbitrary parameter “A” and we tune A to obtain the global solutions that satisfy $\tilde{u}_k'(\phi = 0) = 0$ [49] (see Appendix B).

Here we do the spike analysis starting from $J = 0$. In spike analysis we need the values of the field ϕ where the solution diverges for a choice of s. As the solution goes slowly to infinity [see Eq. (B1)], the singularity occurs for very large values of ϕ , which make it hard to obtain the spike. One can circumvent this issue by rescaling the field as $\chi = \log(1 + \tilde{\phi})$. With this rescaling, one rewrite the flow equation for the scaled potential as

$$du - \frac{1}{2}(d+2)(1 - e^{-\chi})u' + [1 + e^{-2\chi}(u'' - u')]^{-1} = 0, \quad (22)$$

where now u' is the derivative with respect to χ . With Eq. (22), we can do the standard spike analysis and the result is shown in Fig. 5, from which we can see that there only exist the Gaussian fixed point corresponding to $s = 0$. As the Gaussian fixed point corresponds to constant potential, the flow equation for $\tilde{J}_l$ gives $\tilde{J}_l = 0$. We have repeated the analysis for different values of d including the fractional dimensions where the behavior remains the same. We have further confirmed the results using the polynomial truncation of the potential with the following ansatz:

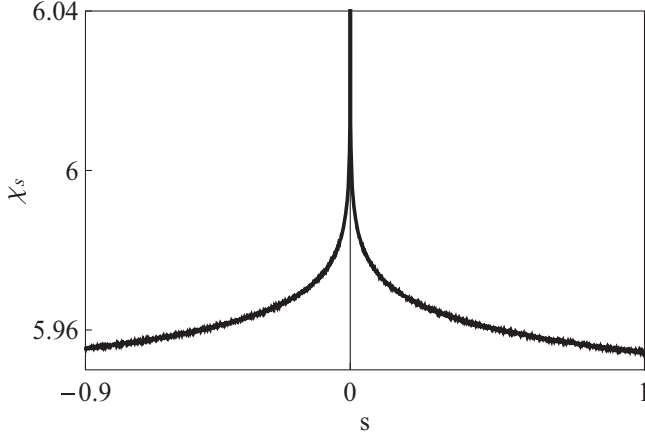


FIG. 5. Spike analysis plot for Eq. (22). Here $s = u''(\phi = 0)$ and χ_s represents the value of χ where the solution diverges. For the plot we have further scaled the field value as $\tilde{\chi} = 10^{-2}\chi$. The plot remains the same for $d = 2.2, 2.4, 2.6, 2.8, 3, 4$.

$$\tilde{u}_k = \sum_{n=2}^N \frac{\tilde{g}_n}{n!} (\tilde{\rho} - \tilde{\kappa})^n. \quad (23)$$

Note that we have changed the variable from $\tilde{\phi}$ to $\tilde{\rho} = 1/2\tilde{\phi}^2$. As a first order in approximation, one can take the lowest truncation, i.e., $N = 2$ in Eq. (23), then the flow equations are

$$\begin{aligned} \partial_t \tilde{\lambda}^2 &= 0, \\ \partial_t \tilde{J}_l &= -4\tilde{J}_l + 18\tilde{\kappa}\tilde{g}_2^2(\tilde{J}_l - 1)^2(1 + \tilde{J}_l + 2\tilde{\kappa}\tilde{g}_2)^{-4}, \\ \partial_t \tilde{g}_2 &= -(d+4)\tilde{g}_2 \\ &\quad - 18\tilde{g}_2^2 \left(\tilde{J}_l - 1 + \frac{\partial_t \tilde{J}_l - 4\tilde{J}_l}{d+2} \right) (1 + \tilde{J}_l + 2\tilde{\kappa}\tilde{g}_2)^{-3}, \\ \partial_t \tilde{\kappa} &= (d+2)\tilde{\kappa} - 3 \left(\tilde{J}_l - 1 + \frac{\partial_t \tilde{J}_l - 4\tilde{J}_l}{d+2} \right) \\ &\quad \times (1 + \tilde{J}_l + 2\tilde{\kappa}\tilde{g}_2)^{-2}, \end{aligned} \quad (24)$$

where $\tilde{\kappa} = \tilde{\rho}_0$ and we obtain the flow equation for $\tilde{\kappa}$ by scaling Eq. (A14). We find the fixed points of the theory by solving Eq. (24) for different dimensions ($2 < d \leq 4$), including fractional ones. There is only a Gaussian fixed point, corresponding to $\tilde{J}_l^* = 0$, $\tilde{g}_2^* = 0$, and $\tilde{\kappa}^* = -3/(d+2)$. Even after increasing the truncation in Eq. (23) up to $N = 13$, only the Gaussian fixed point remains. There exists a spurious fixed point which eventually converges to a Gaussian fixed point or to the infinite solution.

We can analyze the stability of this nonlocal Gaussian fixed point by calculating the eigenvalues of the stability matrix. However, in the stability matrix, at higher truncation ($N > 2$), the flow equation for κ causes divergences for the Gaussian fixed point. So we take $\kappa = 0$ with n starting

from zero in Eq. (23). In this ansatz, the nonlocal Gaussian fixed point corresponds to $\tilde{J}_l = 0$, $\tilde{g}_0 = -1/d$, and all other $\tilde{g}_n = 0$, which agrees with the spike analysis. With this truncation, we obtain that all the eigenvalues are positive except for one along the $\tilde{g}_0$ directions with eigenvalue $\theta = -d$. We have also calculated the eigenvalues of the standard local fixed point, which always has more negative eigenvalues (see Sec. IV, Table I). Also, as we are looking a Gaussian fixed point, we can check the stability of the fixed point in a more general way without assuming any form for the potential [50,51]. One can linearize the potential around the Gaussian fixed point as $\tilde{u}_k = \tilde{u}_k^* + e^{\theta t} v$, where $\tilde{u}_k^*$ is the constant solution at the Gaussian fixed point and v is the small change, which is a function of ϕ alone. Then one can find the linearized flow equation for v from Eq. (21), which gives the general solution as

$$v(\phi) = e^{-\phi^2} \left[C_1 H_{-\nu}(\phi) + C_2 {}_1F_1 \left(\frac{\nu}{2}, \frac{1}{2}, \phi^2 \right) \right], \quad (25)$$

where $\nu = (2 + 3d + 2\theta)/(d+2)$. If we further restrict the potential to be polynomial, the truncation fixes $C_1 = 0$ and requires ν to be a positive odd integer; consequently, $\theta > 0$, in agreement with our earlier result. So one can conclude that the nonlocal Gaussian fixed point is the IR-stable fixed point with this order of approximation.

III. LOCAL THEORY

As we mentioned before, the nonlocal theory we considered above can be obtained from a local theory with the action

$$S = \int_x \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} (\partial\psi)^2 - \lambda\phi\psi + u(\phi). \quad (26)$$

We can obtain the nonlocal theory by path integrating over the ψ field and taking $\lambda^2 \rightarrow -\lambda^2$ (see [13,14] for discussions on tachyons and complex poles in theory). Therefore, we expect the fixed-point structures of the nonlocal theory to be rederived from the local theory, which we carry out in this section. To stay consistent with the approximations made earlier, we use the following ansatz for the effective action:

$$\Gamma_k = \int_x \frac{Z_1}{2} (\partial\phi)^2 + \frac{Z_2}{2} (\partial\psi)^2 - \lambda\phi\psi + u(\phi). \quad (27)$$

We need to select a regulator function to obtain a closed-form expression for the flow equations. In this model, we use the standard Litim cutoff [52] with

$$\mathbb{R}_k = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \quad (28)$$

where $R_i = Z_i(k^2 - q^2)\theta(k^2 - q^2)$. Note that, as we have two fields, we also need to take trace over field indexes in the

Wetterich equation (2). The associated two-point vertex function is given as

$$\mathbb{F}_k^{(2)} = \begin{pmatrix} \Gamma_{11}^{(2)} & -\lambda \\ -\lambda & \Gamma_{22}^{(2)} \end{pmatrix}, \quad \text{with} \quad \Gamma_{ii}^{(2)} = Z_i q^2 + \mu_i, \quad (29)$$

where $\mu_1 = u_k^{(2)}$ and $\mu_2 = 0$. In this model, we can rewrite Wetterich equation (2) as

$$\partial_t \mathbb{F}_k = \frac{1}{2} \text{tr}(\partial_t \mathbb{R}_k (\mathbb{F}_k^{(2)} + \mathbb{R}_k)^{-1}). \quad (30)$$

Note that the two-point vertex function does not depend on ψ , making the right-hand side of Eq. (30) a function of ϕ only. First, let us focus on the flow equation for the two-point vertex function, which will provide us with the flow equations for the nonlocal coupling and the wave function renormalization. The flow equation for the two-point vertex function is

$$\partial_t \mathbb{F}_k^{(2)} = \begin{pmatrix} p^2 \partial_t Z_1 + \partial_t \mu_1 & -\partial_t \lambda \\ -\partial_t \lambda & p^2 \partial_t Z_2 \end{pmatrix}. \quad (31)$$

To derive the flow equations for λ and Z_1 , we need to take the second derivative of Eq. (30) with respect to the fields. However, since the right-hand side of Eq. (30) is independent of ψ , all derivatives with respect to ψ vanish. Thus, we conclude that $\partial_t \lambda = 0$ and $\partial_t Z_2 = 0$ are consistent with the results from the previous section. To find $\partial_t Z_1$, we focus on $\partial_t \Gamma_{11}^{(2)}$, which is the second derivative of Eq. (30) with respect to ϕ . Proceeding in the same way as for the nonlocal theory, we can show that $\partial_t Z_1$ matches the expression for the nonlocal theory in Eq. (5) (see Appendix A). Next, we calculate the flow equation for the other parameters from Eq. (30), and we can proceed in two ways: by performing the momentum integral and then taking the trace of the matrix or by taking the trace first and then performing the momentum integral. Both methods yield the same result as

$$\begin{aligned} \partial_t u_k &= -\mu_d k^{d+2} \left(2Z_1 k^2 + u_k^{(2)} + \frac{k^2 \partial_t Z_1}{d+2} \right) \\ &\times (k^2 (Z_1 k^2 + u_k^{(2)}) - \lambda^2)^{-1}, \end{aligned} \quad (32)$$

where we took $Z_2 = 1$. To compare the result with the nonlocal one, we take $\lambda^2 \rightarrow -\lambda^2$, which gives

$$\begin{aligned} \partial_t u_k &= -\mu_d k^d - \mu_d k^d \left(Z_1 k^2 - \frac{\lambda^2}{k^2} + \frac{k^2 \partial_t Z_1}{d+2} \right) \\ &\times \left(Z_1 k^2 + u_k^{(2)} + \frac{\lambda^2}{k^2} \right)^{-1}. \end{aligned} \quad (33)$$

This aligns with the flow equation for the nonlocal u_k [see Eq. (5)] up to a constant. This constant is related to the normalization of the Jacobian when performing the path integral over ψ to derive the nonlocal action.

IV. INTERACTIONS OF THE FORM $(-\square)^{\sigma/2}$

In the previous section, we observed that the study of the renormalization group structure of a nonlocal theory is consistent with the equivalent local theory. We now extend this analysis to a broader class of nonlocality by considering a theory with the action

$$S = \int_x \frac{Z_1}{2} (\partial \phi)^2 + \frac{\lambda^2}{2} \phi (-\square)^{\frac{\sigma}{2}} \phi + U(\rho). \quad (34)$$

The fractional power of $\square$ can be understood in momentum space as p^σ . For positive values of σ , this model corresponds to the study of weak long-range interactions, as discussed in [3,24]. One can repeat the analysis in Sec. II B for $Z_1 = 1$ and see that the flow equation is not singular for $\sigma > 0$. Here, we extend the critical property analysis to negative values of σ , focusing specifically on the consistency with $\sigma = -2$, which was examined in Sec. II. One can proceed in the same way as in the previous section using the similar ansatz for effective average action and the regulator by replacing $-\square$ with $(-\square)^{\sigma/2}$, where now the regulator takes the form

$$R_k = [Z_1 (k^2 - q^2) + \lambda^2 (k^\sigma - q^\sigma)] \theta(k^2 - q^2). \quad (35)$$

Similar to the previous section, we have two natural choices for the dimensionless couplings. As we are more interested in the negative values σ , we first use the one with constant nonlocal coupling. As the flow equation is analytic in p , we can obtain $\partial_t \tilde{\lambda}^2 = 0$ and along with the scaling equation (20) one can write the flow equations as

$$\begin{aligned} \partial_t \tilde{J}_l &= (\sigma - 2) \tilde{J}_l + \frac{1}{4} (\sigma + 2 \tilde{J}_l)^2 (\tilde{u}_k''')^2 (1 + \tilde{J}_l + \tilde{u}_k'')^{-4}, \\ \partial_t \tilde{u}_k &= d \tilde{u}_k - \frac{1}{2} (d - \sigma) \tilde{\phi} \tilde{u}_k' \\ &\quad - \left(\tilde{J}_l + \frac{\sigma}{2} + \frac{\partial_t \tilde{J}_l + (\sigma - 2) \tilde{J}_l}{d + 2} \right) (1 + \tilde{J}_l + \tilde{u}_k'')^{-1}. \end{aligned} \quad (36)$$

Similar to the above section, now one can solve Eq. (36) for specific values of σ and d using the spike analysis as shown in Fig. 7. First, solve the flow equation for $\tilde{u}_k$ using spike analysis with $\tilde{J}_l = 0$ and use the solution in the flow equation for $\tilde{J}_l$, and iteratively solve the system until a convergent value for $\tilde{J}_l$ is obtained. In Fig. 6, we have plotted the convergent value for $\tilde{J}_l$, and $s_0 = \tilde{u}_k''(\tilde{\phi} = 0)$ as a function of σ . We can see that $\tilde{J}_l$ and s_0 diverges as we approach $\sigma^* < 2$, where the $\tilde{\lambda}^2$ vanishes and theory is determined by the standard local theory. Also, as we move

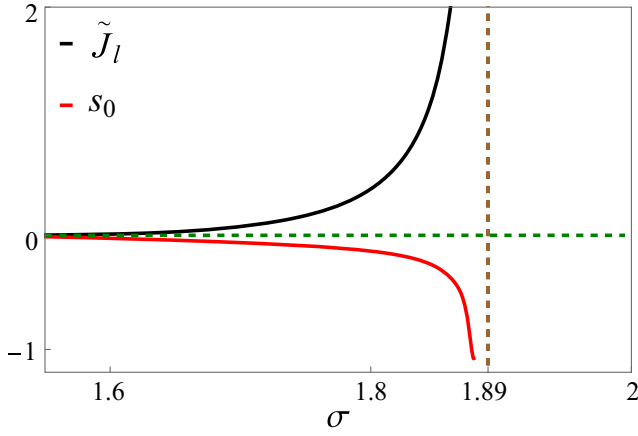


FIG. 6. Plot shows $\tilde{J}_l$ and s_0 as a function of σ for $d = 3$. The green dashed line corresponds to the nonlocal Gaussian fixed point and the value of $\sigma^* = 1.89$ is marked, after which the theory is determined by the standard local theory.

toward $\sigma = d/2$, the fixed point merges with the nonlocal Gaussian fixed point. This can also be seen the spike plot in Fig. 7. Similar conclusions are obtained in [3,24] using polynomial truncation, from which we know that $\sigma^* = 2 - \eta$ as obtained in [31] using perturbative analysis. We also verified the above statement using spike analysis, where we calculated $\eta = 0.109$ for the Wilson-Fisher fixed point, which makes $\sigma^* \sim 1.89$ as one can see in Fig. 6. In the standard local theory, the asymptotic behavior for the large values of $\tilde{\phi}$ is dominated by the scaling part of the equation [first terms in the flow equation for $\tilde{u}_k$ in Eq. (36)]. For an arbitrary values of σ and d , one can solve the scaling part to get $\tilde{u}_k \sim \tilde{\phi}^{(2d)/(d-\sigma)}$ which dominates in the $\tilde{\phi} \rightarrow \infty$ limits for $0 < \sigma < d$. Or, for large values of $\tilde{\phi}$, one can

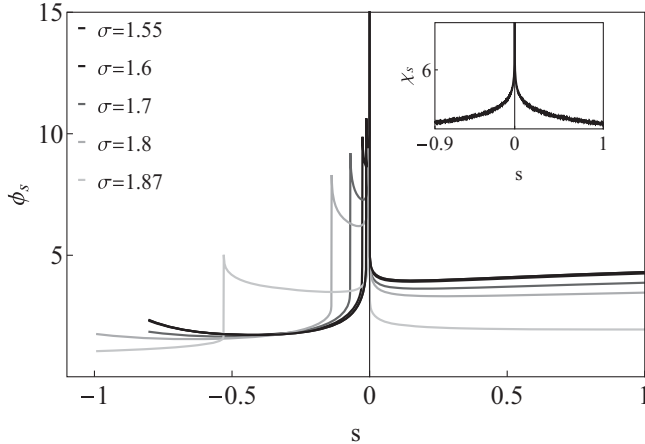


FIG. 7. The spike analysis of Eq. (36) for different values of σ for $d = 3$. The single non-Gaussian fixed point corresponds to a nonlocal fixed point that merges with the NL GFP at $\sigma = 1.5$. Inset: the spike plot for $\sigma = -2$; note the change in y axis as we scale the field. We qualitatively get the same plot for all $-2 < \sigma < 1.5$.

neglect the contribution from the quantum corrections, the nonlinear contribution [the last term in the flow equation for $\tilde{u}_k$ in Eq. (36)], as $\tilde{u}_k'' \gg 1$. However, all this behavior changes for $\sigma < 0$; the nonlinear part also contributes to the asymptotic behavior and the solution goes slowly to infinity. That shows, in the strong long-range regime ($\sigma < 0$) [1], the behavior in the $\tilde{\phi} \rightarrow \infty$ regime is also contributed by $\tilde{\phi} \sim 0$ fields. For this reason, one cannot directly do the spike analysis for $\sigma < 0$, as one has to scale the field appropriately in order to include $\tilde{\phi} \rightarrow \infty$ regions, as discussed around Eq. (22).

The flow equation (36) reduces to standard local theory for $\sigma = 2$ in the $\tilde{J}_l \rightarrow 0$ limit (at $\sigma = 2$ both $\tilde{\lambda}^2$ and Z_1 act as wave function renormalization and $\tilde{J}_l \rightarrow 0$ means $Z_1 \rightarrow 0$ and $\tilde{\lambda}^2$ act as wave function renormalization) and we recover the spike plots for the local theory (also see the effective dimension method as discussed in [24]). As we decrease the σ value from $\sigma = 2$ to $\sigma = d/2$, the nonlocal non-Gaussian fixed point moves toward the nonlocal Gaussian one and merges with it at $\sigma = d/2$. After $\sigma = d/2$, there only exists the Gaussian fixed point and in order to see this for $\sigma < 0$, we need to do the spike analysis after rescaling the field value (see the plot in the inset of Fig. 7). The behavior remains the same for arbitrary dimensions $2 \leq d \leq 4$ including the fractional dimensions.

Similar to Sec. II C, one can also solve Eq. (36), using the polynomial truncation as in Eq. (23), which for $N = 2$ gives

$$\begin{aligned} \partial_t \tilde{J}_l &= (\sigma - 2)\tilde{J}_l + \frac{9}{2}\tilde{g}_2^2 \kappa(\sigma + 2\tilde{J}_l)(1 + \tilde{J}_l + 2\kappa\tilde{g}_2)^{-4}, \\ \partial_t \tilde{g}_2 &= (2\sigma - d)\tilde{g}_2 \\ &\quad - 18\tilde{g}_2^2 \left(\tilde{J}_l + \frac{(\sigma - 2)\tilde{J}_l}{d + 2} + \frac{\sigma}{2} \right) (1 + \tilde{J}_l + 2\kappa\tilde{g}_2)^{-3}, \\ \partial_t \tilde{\kappa} &= (d - \sigma)\kappa - 3 \left(\tilde{J}_l + \frac{\sigma}{2} + \frac{(\sigma - 2)\tilde{J}_l}{d + 2} \right) (1 + \tilde{J}_l + 2\kappa\tilde{g}_2)^{-2}. \end{aligned} \quad (37)$$

One can solve Eq. (37) numerically for different values of σ and d to obtain a qualitatively similar plot as in Fig. 6, as obtained in [24]. Now we consider the other scaling where we take Z_1 as dimensionless [see Eq. (18)], which is the best to study the transition at σ^* . With this scaling, one can write the scaled flow equation as

$$\begin{aligned} \partial_t \tilde{J}_{nl} &= (2 - \sigma - \eta)\tilde{J}_{nl} \quad \text{with} \\ \eta &\equiv -\frac{\partial_t Z_1}{Z_1} = \frac{1}{4}(\tilde{u}_k''')^2(2 + \sigma\tilde{J}_{nl})^2(1 + \tilde{J}_{nl} + \tilde{u}_k'')^{-4}, \\ \partial_t \tilde{u}_k &= d\tilde{u}_k - \frac{1}{2}(d - 2 + \eta)\tilde{\phi}\tilde{u}_k' - \left(1 + \frac{\sigma}{2}\tilde{J}_{nl} - \frac{\eta}{d + 2} \right) \\ &\quad \times (1 + \tilde{J}_{nl} + \tilde{u}_k'')^{-1}. \end{aligned} \quad (38)$$

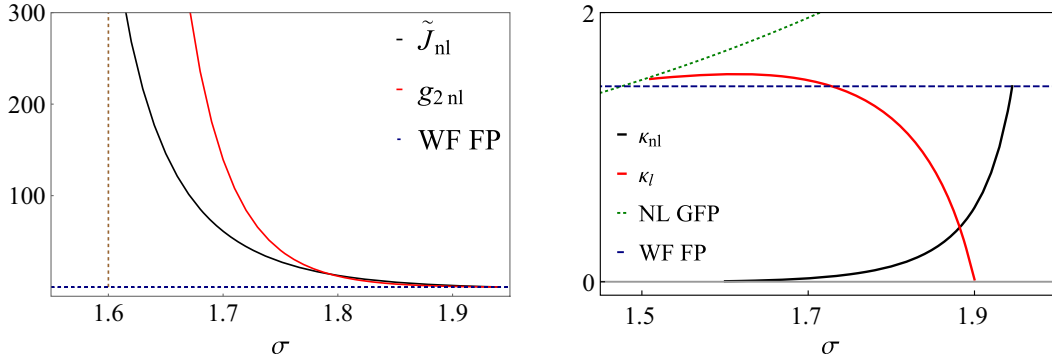


FIG. 8. Plotted values of $\tilde{J}_{nl}$, g_{2nl} , κ_{nl} , and κ_l as functions of σ for $d = 3$. The $\tilde{J}_{nl}$ and g_{2nl} diverge near $\sigma \sim d/2$ as the fixed point merges with the nonlocal Gaussian fixed point.

We can do similar analysis using truncation like Eq. (23) and values of $\tilde{J}_{nl}$, κ , and $\tilde{g}_2$ as a function of σ is given in Fig. 8. From Figs. 8 and 9, one can see that, as we move toward σ^* , the nonlocal fixed point emerges from the standard Wilson-Fisher fixed point. The plot diverges near $\sigma = 1.5$ ($d/2$) as the nonlocal fixed point merges with the nonlocal Gaussian fixed point, which we studied using the first scaling. The standard local theory corresponds to $\tilde{J}_{nl} = 0$ in the flow equation for $\tilde{J}_{nl}$ in Eq. (38) and the $\eta = 2 - \sigma$ solution corresponds to the nonlocal fixed point. In our analysis of $\sigma = -2$, we mention that we are not taking the $\eta = 4$ case, which we can understand as invalid.

Now one can study the stability of different fixed points in different regions of σ using the eigenvalues of the stability matrix. The critical exponents of different fixed points at different values of σ for $N = 2$ truncation for $d = 3$ is plotted in Fig. 9 (also see Table I). One can see that for $\sigma > \sigma^*$ the Wilson-Fisher fixed point is the IR-stable one with less negative eigenvalues. As we cross the σ^* , one of the positive eigenvalues of the WF point turns negative, which makes the new nonlocal fixed point as the stable one. As we cross $\sigma < d/2$, then the nontrivial nonlocal fixed point merges with the NL GFP and it is the stable one in the region. Note that NL GFP corresponds to theory $Z_1 \rightarrow 0$ and nonzero value of $\tilde{\lambda}^2$, however, the standard local GFP

corresponds to $\tilde{\lambda}^2 \rightarrow 0$ with nonzero Z_1 . As expected, these fixed points merge at $\sigma = 2$.

Note that, as discussed before, in order to obtain the critical exponents for both local and nonlocal Gaussian FPs with $N > 2$, we use the polynomial truncation with zero vacuum expectation value as

$$\tilde{u}_k(\tilde{\rho}) = \sum_{n=1}^N \frac{g_n}{n!} \tilde{\rho}^n. \quad (39)$$

A. Isotropic Lifshitz fixed point

In this section, we extend our analysis to $\sigma > 2$, resulting in a higher-derivative theory. We compare our findings with those in [29] on isotropic Lifshitz fixed point, which is equivalent to $\sigma = 4$ in our model. A full analysis of the transition from $\sigma = 2$ to $\sigma = 4$ may require higher-order approximations, incorporating additional higher-derivative terms. However, a leading-order approximation can be

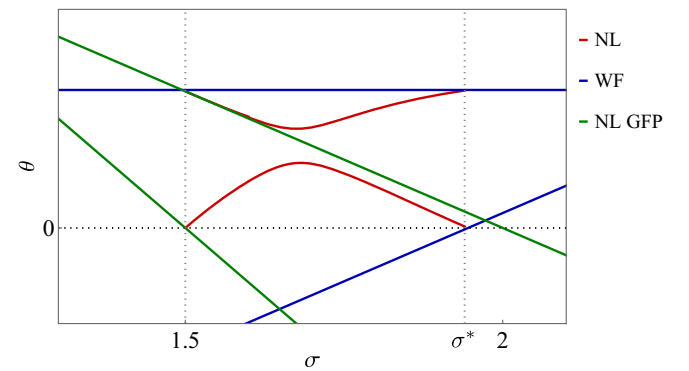


TABLE I. Shows the number of negative eigenvalues for different FPs for different ranges of σ . More negative eigenvalues means the FP is unstable in the IR region. The symbol $\dagger$ means as we increase the truncation number changes without changing the IR-stable FP (we have more number of negative eigenvalues for $\sigma > \sigma^*$ and all eigenvalues turn positive for $\sigma < d/2$).

σ values	No. of negative eigenvalues				IR-stable FP
	GFP	WF	NL	NL GFP	
$\sigma > \sigma^*$	2	1	...	3 †	WF
$d/2 < \sigma < \sigma^*$	3	2	1	2	NL
$\sigma < d/2$	3	2	...	1 †	NL GFP

FIG. 9. The plot shows two eigenvalues (θ) of the RG stability matrix as a function of σ in three dimensions for the local WF fixed point (blue lines), the NL GFP (green lines), and the NL fixed point (red lines). The WF is stable in $\sigma > \sigma^*$ with two positive eigenvalues, NL is stable in $1.5 < \sigma < \sigma^*$ as one of the positive eigenvalues of WF turns negative for $\sigma < \sigma^*$, and NL GFP is stable for $\sigma < 1.5$ with two positive eigenvalues. As we move from left to right, the IR-stable theory becomes more local.

performed by setting $Z = 0$ and adopting a scaling where λ^2 remains dimensionless and is absorbed through a field redefinition. This approach is equivalent to the LPA discussed in [24], and the analysis can be carried out using an effective dimensional method [24]. At this order, one can write the flow equation for the potential as

$$\partial_t \tilde{u}_k = D_{\text{eff}} \tilde{u}_k - \frac{1}{2} (D_{\text{eff}} - 2) \tilde{\phi} \tilde{u}_k - (1 + u_k'')^{-1}, \quad (40)$$

where $D_{\text{eff}} = 2d/\sigma$. The above flow equation (40) can be mapped onto a standard local theory in an effective spacetime dimension D_{eff} , which allows us to extract key results using known properties of local models. In such local theories, it is well established that a discrete set of fixed points exists for $2 < D_{\text{eff}} < 4$, depending on the precise value of D_{eff} . Translating this to the isotropic Lifshitz case, one finds a corresponding discrete set of fixed points for $4 < d < 8$, identifying $d = 4$ and $d = 8$ as the lower and upper critical dimensions, respectively, as discussed in [29]. Based on this correspondence, we conclude that, within this approximation, a unique non-Gaussian fixed point exists in the range $6 \leq d < 8$. So, the use of effective dimensional analysis indeed gives the leading-order behavior of the isotropic Lifshitz fixed point.

V. CONCLUSION

We studied the fixed-point structure of a single scalar-field theory with interactions of the form $\phi \square^{\sigma/2} \phi$, focusing primarily on $\sigma = -2$ using the functional renormalization group approach. The theory is a useful toy model for various nonlocal extensions of gravity and provides an effective description of systems with strong long-range interactions. A comprehensive analysis of its phase space structure using the exact renormalization group method is a novel contribution to literature. Our analysis shows that, in the LPA, the flow equation becomes singular for $\lambda^2 > 0$ when $\sigma < 0$, including the case $\sigma = -2$. We then examined the critical properties and fixed-point structure of the theory. Within the LPA', we determine the fixed points and analyze their stability for all $-2 \leq \sigma \leq 2$. We further perform a qualitative examination of the fixed points for $\sigma > 0$ in the LPA.

We obtained closed-form expressions for the flow equations of all parameters, including the wave function renormalization within the LPA', valid in arbitrary spacetime dimensions. Notably, we found that the nonlocal coupling remains scale invariant, $\partial_t \lambda^2 = 0$ [see Eqs. (5) and (17)]. Initially, in LPA, we treated the nonlocal coupling as an external parameter and solved the scaled flow equation using spike analysis (see Fig. 2). At this stage, we found that the fixed-point structure remains identical to that of the local theory, with the nonlocal parameters merely shifting the fixed-point locations. Additionally, we computed the critical

exponents—physically measurable quantities—using the truncation method and confirmed that they remain independent of the nonlocal parameter [see Eq. (11)]. We also look at the RG flow diagram for the parameters and show that nonlocal coupling affects the phase transition as the nonlocality induces symmetry breaking (Fig. 3). We analyze how the nonlocal term influences the convexity of the potential as $k \rightarrow 0$ within the LPA beyond polynomial truncations. The flow equation is found to become singular for $\lambda^2 > 0$ before the far-infrared limit ($k \rightarrow 0$). Obtaining a smooth flow may therefore require going beyond these approximations, which is computationally demanding. We further investigate the fixed-point structure of the theory.

Next, we considered the flow of the nonlocal parameter, which yielded trivial results at the LPA level. This motivated us to move to the next-order approximation by incorporating wave function renormalization. We adopted a natural scaling where the nonlocal parameter λ^2 remains dimensionless. Under this scaling, we demonstrated—using spike analysis—that the only fixed point present is the nonlocal Gaussian fixed point (see Fig. 5 and Appendix B, where we independently confirm this result using another method). Additionally, we validated these findings through the truncation method. In addition, we computed the critical exponents using both polynomial truncations and without assuming the polynomial form. These results confirm that the nonlocal Gaussian fixed point is the stable infrared fixed point of the model under consideration (see Fig. 5). So, in the low-energy limit, only the nonlocal term contributes.

Moreover, we confirmed our results by rederiving the flow equations for the parameters using the corresponding local theory formulated with two scalar fields. We found that the flow equations match those of the nonlocal theory up to a constant factor. This discrepancy arises from the normalization of the Jacobian in the path integral of the auxiliary field.

Additionally, we explored how infrared physics changes as a function of σ . We considered two scaling schemes: one in which the wave function renormalization remains dimensionless, as in standard local theories, and another in which the nonlocal coupling is dimensionless. We did analysis using the appropriate scaling for different values of σ . Using both scalings, we solved the scaled flow equation via spike analysis. Starting from $\sigma = -2$, we found that the fixed-point structure remains unchanged up to $\sigma = d/2$, with the NL GFP continuing as the infrared-stable fixed point. At $\sigma = d/2$, a new nonlocal fixed point emerges from the NL GFP, which becomes the IR-stable fixed point (see Figs. 7 and 9). This fixed point persists until $\sigma = \sigma^* = 2 - \eta$ (η is the anomalous dimension), where it merges with the local fixed point, specifically, in three dimensions, with the Wilson-Fisher fixed point. For $\sigma^* \leq \sigma \leq 2$, the local fixed points become the infrared-stable one. This behavior aligns with Sak's prediction and has been previously

demonstrated using the truncation method in [24]. Additionally, we numerically computed σ^* using spike analysis. We also repeated the analysis using the truncation method to verify the results.

A complete understanding of the nonlocal fixed point depends on the choice of scaling. If the wave function renormalization is treated as dimensionless, the quantity J_{nl} diverges near $\sigma = d/2$ (see Fig. 8), where the nonlocal fixed point merges with the nonlocal Gaussian fixed point. Conversely, if λ^2 is taken to be dimensionless, the divergence of Z for $\sigma \geq \sigma^*$ cannot be absorbed through a field redefinition, resulting in a divergence of J_l near $\sigma = \sigma^*$ (see Fig. 6). So, we use both scalings to understand the nature of nonlocal fixed points. Finally, using the effective dimensional method, we studied the fixed-point structure of higher-derivative theories, particularly for $\sigma = 4$. The model corresponds to the isotropic Lifshitz critical behavior, and we find that our results align with those in [29] at this level of approximation.

Along with the results we obtained, we can conclude that the FRG is a suitable tool to study the fixed-point structure of a nonlocal theory. The method gives the advantage of keeping the spacetime dimension and the type of nonlocality (value of σ) as arbitrary. We will further do a careful analysis of the effect of nonlocality on the phase transition, including higher-order approximation, especially studying the phase transition, including the wave function renormalization. One can also include the effect of temperature in the same way as in [53,54], where we can study the interplay between temperature and nonlocality in phase transition. Another interesting direction is the effect of nonlocality on the Mermin-Wagner theorem by including N scalar fields with $O(N)$ symmetry.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATION OF THE FLOW EQUATIONS

In this appendix, we derive the flow equation for different parameters in the theory. Starting from the Wetterich equation (2), one can write the flow equation for the two-point vertex function as

$$\partial_t \Gamma^{(2)} = \text{tr} \left[\partial_t R_k \left(\frac{(\Gamma^{(3)})^2}{(\Gamma^{(2)} + R_k)^3} \right) \right] - \frac{1}{2} \text{tr} \left[\partial_t R_k \frac{\Gamma^{(4)}}{(\Gamma^{(2)} + R_k)^2} \right]. \quad (\text{A1})$$

For the ansatz [Eq. (15)], we used for the Γ_k , lhs of the above equation is given as

$$\partial_t \Gamma^{(2)} = \partial_t Z_1 p^2 + \frac{\partial_t \lambda^2}{p^2} + \partial_t u_k''(\phi). \quad (\text{A2})$$

So, one can obtain $\partial_t Z_1$ and $\partial_t \lambda^2$ by taking a ∂^2 and $1/p^2$ coefficients of the rhs. We only focus on the first term in the rhs of Eq. (A1), as in the same way one can show that the second term is independent of external momenta p , which is the same for local theory. Now,

$$\begin{aligned} \text{tr} \left[\partial_t R_k \left(\frac{(\Gamma^{(3)})^2}{(\Gamma^{(2)} + R_k)^3} \right) \right] &= \text{tr} \left[\sum_{q_1 \dots q_4} G|q_1\rangle \langle q_1| \Gamma^{(3)} |q_2\rangle \langle q_2| G|q_3\rangle \langle q_3| \Gamma^{(3)} |q_4\rangle \langle q_4| G \partial_t R_k \right] \\ &= \int_{q, q_1 \dots q_2} G(q, -q_1) \Gamma^{(3)}(p, q, -q_2) G(q_2, -q_3) \Gamma^{(3)}(-p, q_3, -q_4) G(q_4, -q) \partial_t R_k(q). \end{aligned} \quad (\text{A3})$$

Now, using the momentum conservation of the propagators in each vertex and noting that the $\Gamma^{(3)} = u_k^{(3)}$, we can rewrite this as

$$\text{tr} \left[\partial_t R_k \left(\frac{(\Gamma^{(3)})^2}{(\Gamma^{(2)} + R_k)^3} \right) \right] = (u_k^{(3)})^2 \int_q G_q^2 G_{p+q} \partial_t R_k, \quad (\text{A4})$$

where

$$\begin{aligned} G(q, -q) &= G_q = \frac{1}{Z_1 q^2 + \frac{q^2}{\lambda^2} + u_k'' + R_k} \\ &= \frac{\theta(k^2 - q^2)}{f(k)} + \frac{\theta(q^2 - k^2)}{f(q)}, \end{aligned} \quad (\text{A5})$$

where $f(q) = Z_1 q^2 + u_k'' + \lambda^2/q^2$. Note that all the p dependence is coming from G_{p+q} alone. However, G_{p+q} is known to be analytic in $p+q$ [15], as one can write G_q as the sum of propagators of two local theories as

$$G_q = \frac{A}{q^2 + r_-^2} + \frac{B}{q^2 + r_+^2}. \quad (\text{A6})$$

So one can conclude that $\partial_t \lambda^2 = 0$. In order to obtain the p^2 coefficient, we Taylor expand G_{p+q} around $p = 0$ as

$$G_{p+q} = G_q + p_\mu \frac{\partial}{\partial q_\mu} G_q + \frac{1}{2} p_\mu p_\nu \frac{\partial^2}{\partial q_\mu \partial q_\nu} G_q + O(p^3). \quad (\text{A7})$$

However, in our model, G_q is a function of q^2 so we can use the modified expansion in which

$$p_\mu p_\nu = \frac{p^2}{d} \delta_{\mu\nu}. \quad (\text{A8})$$

Then, by comparing the coefficients in Eq. (A1),

$$\partial_t Z_1 = (u_k^{(3)})^2 \int_q G_q^2 \partial_t R_k \left(\frac{4q^2}{2d} G'' + G' \right), \quad (\text{A9})$$

where G' is the derivative with respect to q^2 . With our regulator choice [Eq. (16)], one can show that the G' term will not contribute and we can do the remaining momentum integral using the identity

$$\int_x \theta(x-a) \delta(x-a) = \frac{1}{2}. \quad (\text{A10})$$

We can obtain

$$\partial_t Z_1 = \mu_d k^{d+2} (u_k^{(3)})^2 \left(Z_1 - \frac{\lambda^2}{k^4} \right)^2 \left(Z_1 k^2 + \frac{\lambda^2}{k^2} + u_k^{(2)} \right)^{-4}, \quad (\text{A11})$$

where $\mu_d^{-1} = (4\pi)^d \Gamma(d/2 + 1)$. Now, directly from the Wetterich equation (2), using the leading-order term in the LPA, one can write

$$\begin{aligned} \partial_t u_k &= \frac{1}{2} \int_q G_q \partial_t R_k \\ &= -\mu_d k^{d+2} \left(Z_1 k^2 + \frac{\lambda^2}{k^2} + u_k^{(2)} \right)^{-1} \left(Z_1 - \frac{\lambda^2}{k^4} + \frac{\partial_t Z_1}{d+2} \right). \end{aligned} \quad (\text{A12})$$

In our analysis, we are considering the potential with nonzero vacuum expectation value. So, we also need to consider the flow equation for the vacuum expectation value (κ). One can obtain this from the condition for

minimum of the potential, i.e.,

$$\left. \frac{\partial u_k}{\partial \rho} \right|_{\rho=\kappa} = 0. \quad (\text{A13})$$

Taking the total derivative over t one can get

$$\begin{aligned} \partial_t \kappa &= -\mu_d k^{d+2} \left(Z_1 - \frac{\lambda^2}{k^4} + \frac{\partial_t Z_1}{d+2} \right) (3u_k'' + 2\rho u_k''') \\ &\times \left[u_k'' \left(Z_1 k^1 + \frac{\lambda^2}{k^2} + u_k' + 2\rho u_k'' \right) \right]^{-1}. \end{aligned} \quad (\text{A14})$$

1. Flow equation of wave function coefficient from local theory

In this subsection, we derive the flow equation for the Z_1 from the local theory discussed in Sec. III. Consider

$$\partial_t \mathbb{F}_k^{(2)} = \begin{pmatrix} \partial_t \Gamma_{11}^{(2)} & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{A15})$$

where we took $\partial_t \lambda = 0$ and $\partial_t Z_2 = 0$. Taking the second derivative of Eq. (30) with respect to ϕ gives

$$\begin{aligned} \partial_t \Gamma_{11}^{(2)} &= p^2 \partial_t Z_1 + \partial_t \mu_1 \\ &= \frac{1}{2} \text{tr} \left[\partial_t \mathbb{R}_k \left(\frac{2(\mathbb{F}_k^{(3)})^2}{(\mathbb{F}_k^{(2)} + \mathbb{R}_k)^3} - \frac{\mathbb{F}_k^{(4)}}{(\mathbb{F}_k^{(2)} + \mathbb{R}_k)^2} \right) \right]. \end{aligned} \quad (\text{A16})$$

Similar to the nonlocal theory (see Appendix A), the second term is independent of the external momentum. So in order to obtain $\partial_t Z_1$, we only need to consider the first term. Then,

$$\partial_t \Gamma_{11}^{(2)} = \text{tr} \begin{pmatrix} G_{11}^3 (\Gamma^{(3)})^2 \partial_t R_1 & G_{11}^2 G_{12} (\Gamma^{(3)})^2 \partial_t R_2 \\ G_{11}^2 G_{12} (\Gamma^{(3)})^2 \partial_t R_1 & G_{11} G_{12} G_{21} (\Gamma^{(3)})^2 \partial_t R_2 \end{pmatrix}, \quad (\text{A17})$$

where

$$(\mathbb{F}_k^{(2)} + \mathbb{R}_k)^{-1} \equiv \mathbb{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \quad (\text{A18})$$

with

$$\begin{aligned}
G_{11} &= \frac{\Gamma_{22}^{(2)} + R_2}{(\Gamma_{11}^{(2)} + R_1)(\Gamma_{22}^{(2)} + R_2) - \lambda^2} \\
&= \frac{\theta(k^2 - q^2)}{f(k)} + \frac{\theta(q^2 - k^2)}{f(q)}, \\
G_{12} &= G_{21} = \frac{\lambda}{(\Gamma_{11}^{(2)} + R_1)(\Gamma_{22}^{(2)} + R_2) - \lambda^2} \\
&= \frac{\lambda\theta(k^2 - q^2)}{k^2 f(k)} + \frac{\lambda\theta(q^2 - k^2)}{q^2 f(q)}, \\
G_{22} &= \frac{\Gamma_{11}^{(2)} + R_1}{(\Gamma_{11}^{(2)} + R_1)(\Gamma_{22}^{(2)} + R_2) - \lambda^2}, \tag{A19}
\end{aligned}$$

where we have used the explicit form of R_i [see Eq. (28)] with $f(q) = Z_1 q^2 + u_k'' - \lambda^2/q^2$ and

$$\mathbb{F}_k^{(3)} = \begin{pmatrix} \Gamma^{(3)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{A20}$$

Note that G_{11} is equivalent to the nonlocal propagator once we take $\lambda^2 \rightarrow -\lambda^2$ [see Eq. (A5)]. After taking the trace over the matrix index and introducing the momentum basis (see above section), we can write that

$$\partial_i \Gamma_{11}^{(2)} = (u_k^{(3)})^2 \int_q G_{11}(p+q) (G_{11}^2(q) \partial_i R_1 + G_{12}^2(q) \partial_i R_2). \tag{A21}$$

Now, using the explicit form for Eq. (A19), one can see that

$$\begin{aligned}
&G_{11}^2(q) \partial_i R_1 + G_{12}^2(q) \partial_i R_2 \\
&= f^{-2}(k) \left(\partial_i Z_1 (k^2 - q^2) + 2k^2 Z_1 + \frac{2\lambda^2}{k^2} \right). \tag{A22}
\end{aligned}$$

With this we can see that Eq. (A4) is equivalent to Eq. (A21) when $\lambda^2 \rightarrow -\lambda^2$. Then, following the same steps as in the nonlocal theory, one can rederive the formula for $\partial_i Z_1$ from the local theory itself.

APPENDIX B: SHOOTING FROM INFINITY

In this appendix, we solve the flow equation of the scaled potential [Eq. (21)] using the shooting from infinity

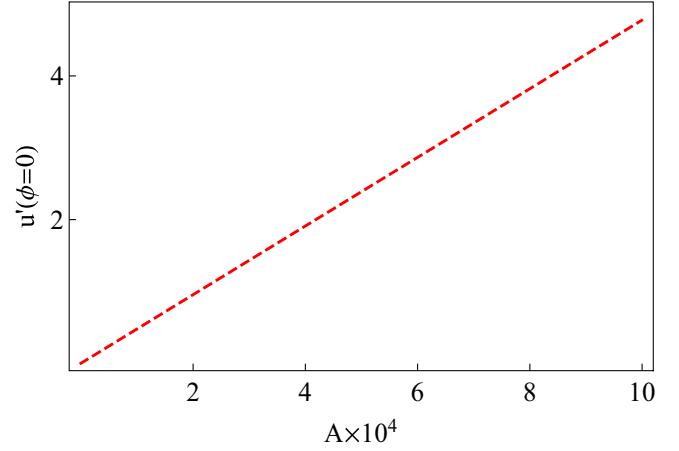


FIG. 10. Plot shows the $u'(\phi = 0)$ as function of A . From the plot, one can see that there is only one value ($A = 0$), which satisfies the required condition, i.e., $u'(\phi = 0) = 0$.

method [49] in $d = 3$ dimensions and the analysis can be extended to other dimensions also. First, we look at the asymptotic solution of the differential equation for large field values, which one can write as

$$\lim_{\phi \rightarrow \infty} \tilde{u}_k(\phi) \sim A\phi^{\frac{6}{5}} - \frac{1}{3} + \frac{6A}{125\phi^{\frac{4}{5}}} - \frac{36}{4375} \frac{A^2}{\phi^{\frac{8}{5}}} + \dots \tag{B1}$$

Compare to the standard local theory, where the leading part of the asymptotic solution goes as $\sim \phi^6$, our solution goes slowly to infinity. Now we numerically solve Eq. (21) with the initial conditions calculated from Eq. (B1) for some large values of ϕ such that the series is convergent. As our asymptotic solution [Eq. (B1)] goes slowly to infinity, we require to take large values of $\phi \sim 10^6$, which can be achieved by scaling the field value appropriately. With this one can get the solution of Eq. (21) as a function of A . Now we plot the solution for different values of A and choose the ones that satisfy $u'(\phi = 0) = 0$ (see Fig. 10). From Fig. 10, we can see that there is only one solution that satisfies $u'(\phi = 0) = 0$, which is $A = 0$, which corresponds to the constant solution or the Gaussian fixed point.

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