

Asymptotic gauge symmetry and UV extension of the nonperturbative coupling in holographic QCD

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We extend our recent analytic study of the strong coupling α_{eff} in the nonperturbative and near-perturbative regimes [*Phys. Rev. Lett.* **133**, 181901 (2024)] by imposing rigorous renormalization-group constraints from asymptotically free gauge theories at $Q^2 \rightarrow \infty$. The asymptotic boundary conditions modify the scaling properties of α_{eff} at large values of the momentum transfer Q^2 and lead to a scale-dependent confinement strength $\kappa(Q^2)$. This requires that both $\kappa(Q^2)$ and $\alpha_{\text{eff}}(Q^2, \kappa(Q^2))$ remain holomorphic in the complex Q^2 plane, except at the physical cuts associated with the heavy-quark thresholds and the singularity flow trajectory studied in our previous Letter. For color SU(3), a precise connection is found between the scaling exponent of $\kappa(Q^2)$ in the ultraviolet, the value of the infrared fixed point of the strong coupling, and the number of flavors in agreement with observations. The non-perturbative analytic model gives an accurate description of the strong coupling across all scales, up to the highest available data.

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I. INTRODUCTION

For strongly coupled quantum field theories, the AdS/CFT correspondence [1–3], the duality between gravity in

anti-de Sitter space (AdS) and conformal field theory (CFT), provides a classical gravity approximation, which is often mathematically tractable, thus an ideal tool to explore the nonperturbative content of quantum chromodynamics (QCD). In particular, the strong QCD coupling α_s at large distances, which is required to describe the basic constituents of hadrons and nuclei, is not amenable to the usual perturbative iteration at short distances, and holographic and other nonperturbative methods have been proposed to unravel its complex nature in the low-energy infrared (IR) domain of QCD [4].

The AdS/CFT duality [1] originates in string theory and is the most successful concrete realization of the holographic principle [5–8]. However, QCD as a fundamental physical theory based on color SU(3) is not supersymmetric, neither conformal nor large N_c . In practice, one is

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thus led to construct gravity duals of QCD which incorporate confinement and basic QCD properties in physical spacetime, a bottom-up approach known as AdS/QCD, or holographic QCD [9]. In particular, the holographic light-front approach to QCD (HLFQCD) is based on Dirac's Hamiltonian front form of relativistic dynamics [10]. Light-front (LF) quantization is frame independent, a property that enables a precise mapping of the equations of motion in a higher-dimensional gravity theory to semiclassical Hamiltonian equations in Minkowski spacetime at the boundary of AdS space [11]. This approach leads to an effective computational framework to describe the bound states of the fundamental constituents in the boundary quantum field theory [12].

Within the light-front holographic framework, a new procedure was introduced in Ref. [13] to map the effective coupling $g_s(z)$ in AdS space to the physical coupling $g_s(Q^2)$ in physical spacetime. It accounts for the measurements of α_s in the IR domain [4] without introducing any arbitrary parameters. The variable z , the fifth dimension of the AdS space, is also known as the holographic variable and has an exact mapping to the invariant transverse distance between the hadron constituents in the light front [13].

The analytic properties of quantum field theory require the continuity of physical observables to describe the transition between the IR and ultraviolet (UV) domains. We have therefore chosen in [14], as well as in [13], the definition of α_s as an effective charge, which is an observable [15,16]. This allowed us to extend the nonperturbative description of the strong coupling in HLFQCD into the transition region near the perturbative domain using analytic continuation methods [14]. Effective charges extend the running coupling definition to include long-range dynamics in addition to short-range processes, namely iterative quantum loops [4,17]. This scheme restores the scheme-dependent running coupling in the renormalization-group equation (RGE) to the status of an observable, as required by the analytical construction of our approach.

In this article, we describe the extension of the effective strong coupling α_{eff} introduced in [14] to arbitrarily large Q^2 , a domain in which the symmetry group structure of asymptotically free gauge theories determines the evolution of the system. In previous holographic studies of heavy-quark spectroscopy [18–20], a scale dependence of the confinement strength κ was introduced, following the results of the heavy-quark effective theory [21,22]. In the present framework, the dependence on the effective scale corresponds to the presence of heavy-quark flavors and their influence at large Q^2 . Following this assumption, we examine the flow of $\kappa(Q^2)$ as $Q^2 \rightarrow \infty$, consistent with the asymptotic boundary conditions and the analytic structure found in [14]. The effective charge at different scales has also been computed with the Dyson-Schwinger and lattice methods in [23,24].

The contents of this article are as follows. Following Ref. [13], we briefly review the holographic approach to hadronic physics in Sec. II and describe the derivation of the strong coupling in the deep IR domain by mapping the effective coupling in the AdS action to the light front. We examine the specific connection between the coupling $g_s(z)$, which encodes the IR dynamics, with the confinement potential in AdS. Following this, we briefly review the g_1 spin sum-rule scheme and the IR-UV nonanalytic point-matching procedure. We also discuss the salient points of our recent paper [14], which describes the transition regime between the IR and UV domains based on analytic continuation methods. These two sections set the stage for the extension of the model to the UV domain. Guided by the scale dependence of the confinement strength κ from heavy-quark spectroscopy, a scaling procedure is introduced in Sec. III by studying the flow of $\kappa(Q^2)$. It allows us in Sec. IV to impose rigorous renormalization-group constraints at asymptotic infinity consistent with the analytic structure of the model. It leads to precise relations between the scaling exponent with the number of open-quark flavors in the deep UV and the value of the IR fixed point, independently of any dimensionful parameter. We discuss in Sec. V the dynamics of thresholds from heavy quarks and derive UV sum rules that bound the spread of possible solutions compatible with deep UV constraints. We compare in Sec. VI our model prediction for the effective strong coupling with the available data across all scales, thereby including the region where finite-order perturbation theory becomes more and more reliable as the virtuality is increased toward the multi-TeV domain. Finally, we compare in Sec. VII our model result for the strong coupling evaluated at the Z -boson mass M_Z with the world average value of α_s in the $\overline{\text{MS}}$ scheme, $\alpha_{\overline{\text{MS}}}(M_Z)$. Conclusions are given in Sec. VIII. Some technical aspects are described in the Appendixes.

II. STRONG COUPLING IN THE IR DOMAIN AND HOLOGRAPHIC LIGHT-FRONT QCD

In holographic QCD a hadron bound state is described by a classical field propagating in AdS space. Anti-de Sitter AdS_{d+1} is the maximally symmetric $d + 1$ space with negative constant curvature and a d -dimensional flat space boundary, Minkowski spacetime with coordinates x^μ . The line element is

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N \\ &= \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \end{aligned} \quad (1)$$

where R is the AdS radius and z is the holographic variable. The asymptotic boundary of AdS space is defined at $z = 0$.

An arbitrary integer-spin J hadron is described by a bilinear action in AdS_{d+1} for a tensor field $\Phi_J = \Phi_{N_1 \dots N_J}$ in the presence of a dilaton profile $\varphi(z)$ responsible for the confinement dynamics [11,25]

$$S = \int d^d x dz \sqrt{g} e^{\varphi(z)} (D_M \Phi_J D^M \Phi_J - \mu^2 \Phi_J^2), \quad (2)$$

where a contraction over all indices in the tensor fields is to be understood. The determinant of the metric tensor g_{MN} is g , μ is the AdS mass, and D_M is the covariant derivative that includes the affine connection. An important element for our purposes is the definition of the effective coupling $g_s(z)$ in the semiclassical gravity approximation [13,26–29]

$$e^{\varphi(z)} = \frac{1}{g_s^2(z)}, \quad (3)$$

since it accounts for both the hadronic structure and the strong coupling in the IR domain.

In order to make contact with a physical hadron with four-dimensional momentum P and invariant mass $P_\mu P^\mu = M^2$, we write the AdS wave function as the product of a z -dependent wave function $\Phi(z)$ and a plane wave, $\Phi(x, z) \sim \Phi(z) e^{iP \cdot x}$, propagating in physical space-time. The variation of the AdS action leads to the wave equation

$$\left[-\frac{z^{d-1-2J}}{e^{\varphi(z)}} \partial_z \left(\frac{e^{\varphi(z)}}{z^{d-1-2J}} \partial_z \right) + \frac{(\mu R)^2}{z^2} \right] \Phi(z) = M^2 \Phi(z), \quad (4)$$

after a redefinition of the AdS mass μ , along with kinematical constraints that eliminate lower-spin degrees of freedom from the spectrum [25].

We can express the AdS wave equation (4) in its Schrödinger form after the full separation of kinematics and dynamics [11,25]. Using the substitution $\Phi_J(z) = z^{(d-1)/2-J} e^{-\varphi(z)/2} \phi(z)$ we find

$$\left(-\frac{d^2}{dz^2} - \frac{1-4\nu^2}{4z^2} + U(z) \right) \phi(z) = M^2 \phi(z), \quad (5)$$

where $\nu^2 = (\mu R)^2 + (d/2 - J)^2$. The effective interaction potential $U(z)$ can be written in terms of the gravitational coupling g_s [30]:

$$U(z) = g_s(z) z^{-\gamma-1} \partial_z (z^{\gamma+1} \partial_z g_s^{-1}(z)), \quad (6)$$

with $\gamma = 2J - d$. This expression gives an explicit and simple connection between the gravitational coupling in AdS, $g_s(z)$, and the effective potential $U(z)$, which has a wide range of applications for the hadron spectrum. We notice that the potential U is not modified by rescaling the coupling $g_s(z)$ by a constant λ : $g_s(z) \rightarrow \lambda g_s(z)$.

A. Mapping to light-front physics and the physical strong coupling

Starting from the QCD Lagrangian, one can express the Hamiltonian operator in terms of the dynamic quark and gluon fields [31] and compute, in principle, the mass spectrum. However, its actual analytic computation is a task that has proven insurmountable in the context of physical QCD. Important insights follow from the profound holographic connection between hadronic modes in the AdS space and the semiclassical approximation to QCD using LF quantization [10].

In the chiral limit of zero quark masses, $m_q \rightarrow 0$, the dynamical problem in the semiclassical approximation to QCD reduces to a single variable, and the LF Hamiltonian equation becomes the wave equation [11,12]

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U(\zeta) \right) \phi(\zeta) = M^2 \phi(\zeta), \quad (7)$$

where ζ is the invariant physical transverse separation between partons in the light front and L is the orbital angular momentum in the transverse LF plane. Equation (7) is a frame-independent relativistic equation with a structure similar to that of Eq. (5). It leads to an exact mapping of AdS to the LF coordinates, provided that z is identified with ζ , $z = \zeta$, and the AdS parameter ν is identified with L [11,32]. The effective confinement potential $U(\zeta)$ in the Hamiltonian equation (7) acts on the valence state and comprises all interactions, including those of the higher Fock states. It is obtained from (6) by mapping the gravitational coupling to LF physics at the Minkowski boundary of AdS space, $g_s(z) \rightarrow g_s(\zeta)$; thus,

$$U(\zeta) = g_s(\zeta) \frac{1}{\zeta^{2J-3}} \frac{d}{d\zeta} \left(\zeta^{2J-3} \frac{d}{d\zeta} g_s^{-1}(\zeta) \right), \quad (8)$$

for $d = 4$, with J the total hadron spin.

The physical strong coupling α_s , measured at the momentum transfer squared $Q^2 = -q^2 > 0$, is the Fourier transform of the LF coupling $g_s(\zeta)$ written in terms of the invariant separation ζ :

$$\alpha_s(\zeta) = \frac{g_s^2(\zeta)}{4\pi} = \frac{1}{4\pi} e^{-\varphi(\zeta)}, \quad (9)$$

integrated in the transverse LF plane. It is thus given by the Bessel transform [13]

$$\alpha_s(Q^2) \propto \int_0^\infty \zeta d\zeta J_0(\zeta Q) \alpha_s(\zeta), \quad (10)$$

where a proportionality constant, irrelevant to the form of the potential U , is left unspecified.

B. Superconformal symmetry and the AdS coupling

An important additional step follows from the mechanism introduced by de Alfaro *et al.* [33] to break the conformal symmetry in the Hamiltonian while keeping the action conformal invariant. When extended to LF holographic QCD [34], this procedure leads to the appearance of a mass scale in the LF Hamiltonian. When extended to superconformal symmetry [35,36] this mechanism uniquely determines the confining interaction U [37,38] and consequently the dilaton profile that breaks the conformality in the AdS action (2). It also leads to hadronic supersymmetry among mesons, baryons, and tetraquarks [38,39].

For mesons, the confinement potential is [38]

$$U(\zeta) = \kappa_0^4 \zeta^2 + 2\kappa_0^2(J-1), \quad (11)$$

and the coupling $g_s(\zeta)$ follows from integrating (8) for the expression (11). We find [19,30]

$$g_s(z) = \frac{1}{Ae^{\kappa_0^2 z^2/2} + B\Gamma(2-J, \kappa_0^2 z^2)e^{-\kappa_0^2 z^2/2}}, \quad (12)$$

where $\Gamma(a, b)$ is the incomplete Gamma function. In the semiclassical approximation (ignoring the backreaction of the metric), the dilaton profile should depend only on the modification of AdS space, therefore independent of J . This condition implies that $B = 0$, and since the coupling $g_s(\zeta)$ is defined up to a constant, we set $A = 1$. Thus the solution

$$g_s(z) = e^{-\kappa_0^2 z^2/2}, \quad (13)$$

determined by the superconformal algebraic symmetry, where the value of the mass scale $\kappa_0 \simeq 0.5$ GeV is fixed by hadron spectroscopy [12]. From (9) $\alpha_s(\zeta) \sim e^{-\kappa^2 \zeta^2}$ and therefore

$$\alpha_s(Q^2) = \alpha_s(0)e^{-Q^2/4\kappa_0^2}, \quad (14)$$

the result from Ref. [13]. It leads to an IR fixed point $\alpha_s(0)$ whose value depends on the scheme chosen to extract $\alpha_s(Q^2)$. It is clear from (8) that the confinement potential is scheme independent, as it should.

C. The Bjorken g_1 spin sum-rule scheme

Here, as in [13,14], we define the QCD running coupling as the effective charge [15,16] from the Bjorken inelastic spin sum rule [40,41] denoted α_{g_1} :

$$\frac{\alpha_{g_1}(Q^2)}{\pi} = 1 - \frac{6}{g_A} \int_0^{1-\epsilon} dx [g_1^p(x, Q^2) - g_1^n(x, Q^2)], \quad (15)$$

which is a physical observable. It leads to a well-defined value for the IR fixed point $\alpha_{g_1}(0) = \pi$ and is well measured in the IR and UV domains [42–64].

Transforming from the g_1 scheme to any renormalization scheme (e.g., $\overline{\text{MS}}$ or MOM) can be performed in the UV domain using the Bjorken sum rule [4] or commensurate scale relations [65–67]. In particular, the effective charge α_{g_1} can be obtained from the $\overline{\text{MS}}$ results,

$$\begin{aligned} \frac{\alpha_{g_1}(Q^2)}{\pi} &= \frac{\alpha_{\overline{\text{MS}}}(Q^2)}{\pi} + a_1 \frac{\alpha_{\overline{\text{MS}}}^2(Q^2)}{\pi^2} + a_2 \frac{\alpha_{\overline{\text{MS}}}^3(Q^2)}{\pi^3} \\ &+ \dots + \frac{6}{g_A} \sum_{n=2}^{\infty} \frac{\mu_{2n}}{Q^{2n}}, \end{aligned} \quad (16)$$

where the coefficients a_i are known to fourth order in perturbation theory [68–70] and the μ_{2n} are the coefficients of the operator product expansion that appear in the expression for the Bjorken integral (15). We neglect the power correction term in (16) since it is negligible for our Q^2 range of interest: The first μ_{2n} terms have been experimentally determined to be an order of magnitude smaller than the leading twist term [61–63]. These terms are further suppressed by Q^{-2n} powers and, in addition, partly cancel each other due to their alternating signs.

Many choices for the definition of α_s are in fact available with consistent UV behavior but significantly distinct in the IR [4]. In contrast to the universally accepted quantum electrodynamics (QED) coupling definition as an effective charge [71], a consensus has yet to be reached regarding which definition should be adopted for α_s . Ultimately, such definition must be decided based on its usefulness [17]. The new expression for the potential in terms of $g_s(z)$, Eq. (6), demonstrates the pertinence of adopting the effective charge definition, Eq. (15), for the canonical definition of the QCD coupling, consistent with QED. This definition provides a well-behaved coupling in the IR, whose interpretation is clear and follows the standard understanding of a coupling, and is computable with nonperturbative approaches. Yet, its usefulness has been limited in practice limited to a few predictions because its connection to the QCD potential had not been explicitly established. Although formally derived in the IR domain, Eq. (6) provides compelling motivation for promoting the effective charge as the canonical definition for the QCD coupling. In other words, with Eq. (6), knowledge of the coupling behavior, by means of experimental data [72] or explicit calculations [13,24], provides the potential $U(z)$ from which, in turn, many QCD observables can be computed.

D. IR-UV nonanalytic point-matching procedure

The work in [13] was subsequently extended by matching, at a single point Q_0 , the holographic Gaussian

expression (14) with the perturbative QCD running coupling $\alpha_s(Q^2)$ and its β function, $\beta(\alpha_s)$. This procedure gives a good description of the IR domain and, by construction, of the UV data at all scales, since the actual matching can be performed at higher loops in the perturbative domain [73,74]. This simple nonanalytic procedure allowed us to determine the matching point $Q_0 = 1.06 \pm 0.15$ GeV [75], which was used as the initial scale of perturbative evolution of QCD in subsequent HLFQCD predictions of parton distributions [76–78] and the scale-dependent Pomeron analysis in [79]. However, the point-matching procedure does not provide further insight into the transition domain or confinement mechanisms. Recently, the holographic QCD point-matching procedure [13,73,74] has been used in the study of the holographic running coupling via the Ricci flow [80]. The infinite-order scale-setting procedure of the principle of maximum conformality was adopted [81] to eliminate renormalization-scheme and scale-dependent ambiguities in the iterative UV domain [65,67]. In [82] a holomorphic coupling (with $n_f = 3$) was constructed that follows perturbative QCD behavior at large Q^2 and for which the holographic limit at $Q^2 \rightarrow 0$ was enforced up to the second derivative.

E. IR-UV transition domain

In [14] an analytic expression for the effective coupling was proposed which describes essential features of the short- and long-distance behavior of QCD:

$$\alpha_{\text{eff}}(Q^2) = \alpha_{\text{eff}}(0) \exp \left[- \int_0^{Q^2} \frac{du}{4\kappa_0^2 + u \ln(\frac{u}{\Lambda^2})} \right]. \quad (17)$$

It incorporates the IR fixed point $\alpha_{\text{eff}}(0)$ and the non-perturbative confinement scale κ_0^2 defined in the holographic model. The small- and large- Q^2 behavior of $\alpha_{\text{eff}}(Q^2)$ is

$$\alpha_{\text{eff}}(Q^2) = \begin{cases} e^{-Q^2/4\kappa^2}, & Q^2 \leq 4\kappa^2, \\ K / \ln(Q^2/\Lambda^2), & Q^2 \gg 4\kappa^2, \end{cases} \quad (18)$$

where $K = 0.285785$ is a dimensionless constant computed in Appendix A. We thus recover the holographic exponential dependence in the IR [13] and the logarithmic behavior characteristic of asymptotic freedom in the UV [83,84].

The effective coupling (17) leads to a continuous transition from IR to UV, where the QCD iteration methods are applicable. It is defined for $\kappa_0^2 > \Lambda^2/4e$ and leads to the maximal analyticity relation

$$\kappa_0^2 = \frac{\pi}{8} \Lambda^2, \quad (19)$$

as deduced from the analytic structure and the flow of singularities of (17) in the complex Q^2 plane [14]. The

strong coupling (17) provides an accurate description of the available data in the IR and the near-perturbative transition domains in terms of a unique scale.

III. SCALE-DEPENDENT HADRONIC STRENGTH

The boundary condition at infinity modifies the scaling properties of the effective coupling α_{eff} in the ultraviolet domain and leads to a Q^2 flow for the confinement strength, κ , constrained by the asymptotic symmetry and the analytic properties described in [14].

A. Scaling properties of the effective coupling

To properly incorporate the scale dependence of κ into the effective coupling α_{eff} , it is necessary to account for its evolution with Q^2 by introducing a dimensionless holomorphic function $f(Q^2)$ in the complex plane, except at the unitarity branch cuts. A consistent procedure amounts to rescaling the variable u in the integral (17) according to

$$u \rightarrow \frac{u}{f(u)}, \quad (20)$$

which is equivalent to performing the rescaling $\kappa_0^2 \rightarrow \kappa_0^2 f(Q^2)$, thereby embedding the scale dependence directly into the effective coupling.

From (17) and (20) we obtain

$$\begin{aligned} \alpha_{\text{eff}}(Q^2, f(Q^2)) \\ = \alpha_{\text{eff}}(0) \exp \left(- \int_0^{Q^2} \frac{d(u/f(u))}{4\kappa_0^2 + (u/f(u)) \ln(u/f(u)\Lambda^2)} \right), \end{aligned} \quad (21)$$

with $\alpha_{\text{eff}}(0, f(0)) \equiv \alpha_{\text{eff}}(0)$, the IR fixed point value. We can also express Eq. (21) in terms of a scale-dependent κ defined by

$$\kappa^2(Q^2) = \kappa_0^2 f(Q^2), \quad \text{where } f(Q^2) \rightarrow 1, \quad \text{for } Q^2 \rightarrow 0, \quad (22)$$

and its UV behavior as $Q^2 \rightarrow \infty$ is determined by the structure of the gauge group at the asymptotic boundary. From (26) and (21) we find

$$\begin{aligned} \alpha_{\text{eff}}(Q^2, \kappa(Q^2)) \\ = \alpha_{\text{eff}}(0) \exp \left(- \int_0^{Q^2} \frac{\kappa^2(u) \partial_u (u/\kappa^2(u)) du}{4\kappa^2(u) + u \ln(\rho u/4\kappa^2(u))} \right). \end{aligned} \quad (23)$$

The functional form of $\kappa(Q^2)$ must be chosen to smoothly interpolate between the IR and UV regimes, such that the effective coupling $\alpha_{\text{eff}}(Q^2)$ preserves the analytic structure found in [14]. This requires that both $\kappa(Q^2)$ and $\alpha_{\text{eff}}(Q^2)$

remain holomorphic in the complex Q^2 plane, except at the physical cuts associated with the heavy-quark thresholds for $Q^2 < 0$ and the flow of singularities studied in [14] for $Q^2 > 0$.

By further changing the integration variable to

$$v = \frac{u}{4\kappa^2(u)}, \quad (24)$$

we can express Eq. (23) as

$$\alpha_{\text{eff}}(Q^2, \kappa(Q^2)) = \alpha_{\text{eff}}(0) \exp \left(- \int_0^{Q^2/4\kappa^2(Q^2)} \frac{dv}{1 + v \ln(\rho v)} \right), \quad (25)$$

where

$$\rho = \frac{4\kappa_0^2}{\Lambda^2}, \quad \frac{1}{e} < \rho \leq \frac{\pi}{2}. \quad (26)$$

Thus, the introduction of a scale-dependent κ as given by (22), maintains, by construction, the flow of singularities described in [14] and, therefore, the maximal analyticity condition (19), since the value of ρ (26) is invariant under (20).

The large- Q^2 behavior of the effective coupling follows from the results in Appendix A by modifying the upper limit of integration from (25) without further changes. We thus obtain at large Q^2

$$\alpha_{\text{eff}}(Q^2, \kappa(Q^2)) = \alpha_{\text{eff}}(0) \frac{K_w(\rho)}{\ln(\rho \frac{Q^2}{4\kappa^2(Q^2)})} \left(1 + \mathcal{O} \left(\frac{4\kappa^2(Q^2)}{Q^2} \right) \right), \quad (27)$$

where $K_w(\rho)$ is given by (A4).

B. Scaling properties of the effective β function

From the scale-dependent expression of the effective coupling (21) we compute its β function:

$$\begin{aligned} \beta(\alpha_{\text{eff}}) &= Q^2 \frac{\alpha_{\text{eff}}(Q^2, \kappa(Q^2))}{dQ^2} \\ &= -Q^2 \frac{\kappa^2(Q^2) \partial_{Q^2} (Q^2 / \kappa^2(Q^2))}{4\kappa^2(Q^2) + Q^2 \ln(\rho Q^2 / 4\kappa^2(Q^2))} \\ &\quad \times \alpha_{\text{eff}}(Q^2, \kappa(Q^2)). \end{aligned} \quad (28)$$

It reduces for $\kappa^2(Q^2) = \kappa_0^2$ to the expression of the β function found in [14]. At large Q^2 we obtain from (27) and (28)

$$\beta(\alpha_{\text{eff}}) = -\beta_0^{\text{eff}}(Q^2) \alpha_{\text{eff}}^2(Q^2, \kappa(Q^2)), \quad (29)$$

where

$$\beta_0^{\text{eff}}(Q^2) = \frac{1}{\alpha_{\text{eff}}(0) K_w(\rho)} f(Q^2) \frac{d}{dQ^2} (Q^2 / f(Q^2)). \quad (30)$$

Equation (29) has a structure similar to the leading RGE result (B1) but with a Q^2 -dependent factor $\beta_0^{\text{eff}}(Q^2)$.

IV. ASYMPTOTIC GAUGE SYMMETRY, NUMBER OF FLAVORS AND UV SCALING BEHAVIOR

The actual comparison of the effective (27) and the perturbative QCD coupling (B4) requires a power-law scaling dependence of the dimensionless function $f(Q^2)$ to reach a finite value for their ratio in the asymptotic limit. Thus, for $Q^2 \gg \kappa_0^2$ the scaling function $f(Q^2)$ should behave as

$$f(Q^2) \rightarrow (\sigma Q^2)^r, \quad r \geq 0, \quad (31)$$

where r is the scaling exponent of f and $\sigma > 0$ is a dimensionful parameter, except for the trivial case $r = 0$. For a given gauge group, the power-law scaling (31) leads to a relationship between the number of flavors n_f , the scaling exponent r and $\alpha_{\text{eff}}(0)$, the value of the IR fixed point.

A. Asymptotic boundary conditions for the effective coupling

We incorporate the asymptotic gauge symmetry at $Q^2 \rightarrow \infty$ by imposing the boundary condition

$$\lim_{Q^2 \rightarrow \infty} \frac{\alpha_{\text{eff}}(Q^2, \kappa(Q^2))}{\alpha_s(Q^2)} = 1, \quad (32)$$

where $\kappa^2(Q^2) = \kappa_0^2 f(Q^2)$. From (27), (31) and (B4) we find

$$\lim_{w \rightarrow \infty} K(\rho) = \frac{12\pi(1-r)}{\alpha_{\text{eff}}(0)(11N_c - 2n_f)}, \quad (33)$$

where the value of the scaling exponent r in (31) can be determined based on certain assumptions. Specifically, for a given value of the parameter ρ , the scaling exponent r depends on the value of the IR fixed point $\alpha_{\text{eff}}(0)$, the Casimir group invariant $C_A = N_c$, and the number of flavors n_f , independently of any dimensionful parameter. By imposing maximal analyticity, $\rho = \pi/2$, we find for $K(\rho = \pi/2)$ in (A5), and $n_f = 6$ in color SU(3), the result $1 - r = 0.500124$ for $\alpha_{\text{eff}}(0) = \pi$. This agrees with the exponent $r = 1/2$, from heavy-quark bound state systems in holographic QCD [18–20], the only possible solution of r for $N_c = 3$ and $n_f = 6$ [see Eq. (C2) in Appendix C].

Alternatively, we can start with the exponent $r = 1/2$ following [18–22], leaving the number of flavors n_f and the value of the IR fixed point unspecified. From (33),

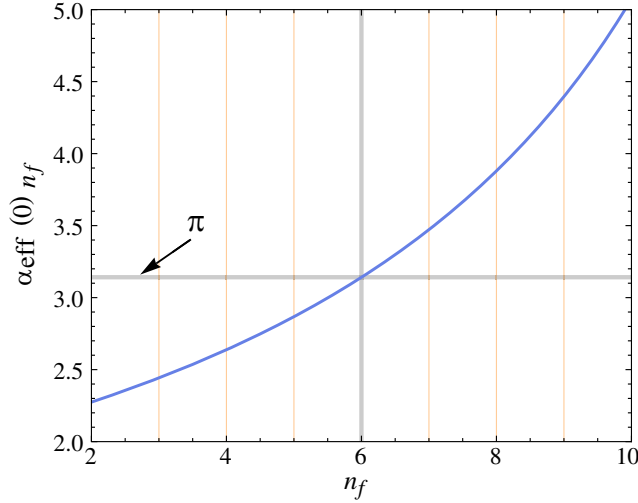


FIG. 1. The light blue curve represents the value of the infrared fixed point value $\alpha_{\text{eff}}(0)_{n_f}$ from (34) as a function of the number of flavors for $N_C = 3$ and the maximal analyticity condition $\rho = \pi/2$. The value $\alpha_{\text{eff}}(0) = \pi$ imposed by the Bjorken sum rule is depicted by the horizontal gray line: It crosses the vertical line precisely at $n_f = 6$. This result is consistent with the power scaling $r = 1/2$ from heavy-quark spectroscopy in holographic QCD.

$$\alpha_{\text{eff}}(0)_{n_f} = \frac{2\pi}{(11 - \frac{2}{3}n_f)K_{w \rightarrow \infty}(\pi/2)}, \quad (34)$$

for $N_C = 3$ and maximal analyticity $\rho = \pi/2$. We show in Fig. 1 that the IR fixed point $\alpha_{\text{eff}}(0) = \pi$, imposed by the Bjorken sum rule [40,41], requires six flavors in the deep UV, if we demand $r = 1/2$. Thus, fixing $n_f = 6$ and $\alpha_{\text{eff}}(0) = \pi$ we can evaluate $K_w(\rho)$ (A4) with very good precision. From (34)

$$K_{w \rightarrow \infty}(\pi/2) = \frac{2}{7} = 0.285714..., \quad (35)$$

compared with the value from numerical integration 0.285785.

From (33) and (35) we find for $\alpha_{\text{eff}} = \pi$ and maximal analyticity, $\rho = \pi/2$, the relation

$$\frac{11N_c - 2n_f}{42} = 1 - r \quad (36)$$

in the asymptotic limit. For asymptotically free gauge theories $11N_c - 2n_f > 0$, thus the upper bound $r < 1$. Other values of n_f , N_C and r are also compatible with the asymptotic boundary condition (32). This is discussed in Appendix C.

The power $r = 1/2$, which was previously observed in [18–20], leads for SU(3) color to the IR fixed point $\alpha_{\text{eff}}(0) = \pi$ and the physical solution $N_c = 3$, $n_f = 6$ (Fig. 1). Another interesting case occurs for $r = 0$, the

solution for constant κ , as demanded in a chiral holographic model. In this case $n_f = 1$ corresponds to $N_C = 4$ and $\beta_0 = 7/2\pi$, twice the QCD value $\beta_0 = 7/4\pi$ for $n_f = 6$ and $r = 1/2$ (40). The expression (36) establishes an interesting connection between chiral symmetry breaking and the dynamics of color and flavor.

B. Asymptotic boundary conditions for the effective β function

The incorporation of the gauge symmetry at the asymptotic boundary discussed above can be readily introduced in terms of the leading RGE results for the β function. Indeed, comparing the effective β function (29) and (30) with the RGE expansion of the QCD β function in powers of α_s (B1) lays out the condition

$$f(u) \frac{d}{du} \left(\frac{u}{f(u)} \right) = C, \quad (37)$$

with C a constant. It has the solution

$$f(u) = \sigma u^r, \quad \text{with } r = 1 - C. \quad (38)$$

Substituting the power solution (38) in (30) we obtain

$$\beta_0^{\text{eff}} = \frac{1 - r}{\alpha_{\text{eff}}(0)K_{w \rightarrow \infty}(\rho)}, \quad (39)$$

which we compare with the leading coefficient of the QCD β function β_0 (B3) with six open flavors:

$$\beta_0 = \frac{7}{4\pi}. \quad (40)$$

Thus, we recover the solution $r = 1/2$ obtained above for maximal analyticity, $\rho = \pi/2$, $K(\pi/2) = 2/7$ from (35), and $\alpha_{\text{eff}}(0) = \pi$, consistent with the results in Sec. IV A.

V. THRESHOLD DYNAMICS AND THE IR-UV ANALYTIC FLOW

We begin by considering a simple functional form for the scaling function $f(u)$ introduced in Sec. III:

$$f(Q^2) = (1 + \sigma Q^2)^r, \quad \sigma > 0, \quad (41)$$

which, respectively, incorporates the IR and UV limits (22) and (31):

$$f(Q^2) = \begin{cases} 1, & Q^2 \rightarrow 0, \\ (\sigma Q^2)^r, & Q^2 \rightarrow \infty, \end{cases} \quad (42)$$

and interpolates smoothly between both. For the possible values of the scaling exponent r in the interval $0 \leq r < 1$

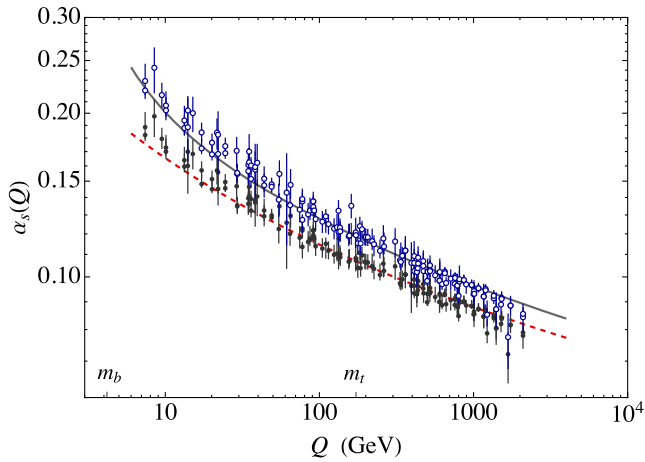


FIG. 2. World data for $\alpha_{\overline{\text{MS}}}$ as compiled by the PDG [85] (filled black circles) is compared with the $\overline{\text{MS}}$ data transformed to the g_1 scheme in the perturbative domain (16) (open blue circles). The lower dashed red curve corresponds to the NLO approximation (B4) for $n_f = 6$ and $\Lambda_{\overline{\text{MS}}}^{n_f=6} = 87$ GeV. The result for the effective coupling (21) (light gray curve) corresponds to $\kappa_0 = 0.534$ GeV [86] and $\sigma = 0.015$ GeV $^{-2}$ in (41) for $r = 1/2$ and maximal analyticity. The bottom and top quark masses are also indicated for reference.

(see Appendix C), $f(Q^2)$ in (41) has a branch cut in the timelike domain of the complex plane $q^2 = -Q^2 > 0$, from $q^2 = 1/\sigma$ to $q^2 \rightarrow \infty$. Since there are no overlapping singularities arising from this additional cut, our previous results for the flow of singularities in the spacelike domain $Q^2 > 0$ are not modified [14]. Consequently, maximal analyticity is maintained by (41), independently of the value of Q^2 .

To extract the value of the parameter σ in (41), we compare in Fig. 2 the available data for $\alpha_{\overline{\text{MS}}}$ in the perturbative domain [85], with the same dataset transformed into the g_1 scheme (16). The lower red dashed curve corresponds to the next-to-leading-order (NLO) scheme-independent approximation (B4) for $n_f = 6$ and $\Lambda_{\overline{\text{MS}}}^{n_f=6} = 87$ GeV. The UV data in Fig. 2 above the m_t mass threshold are used to determine the values of $\Lambda_{\overline{\text{MS}}}$ and σ (41). The model result for the effective coupling (21) (light gray curve) is also shown in this figure for the effective scaling function (41). We find the value $\sigma = 0.015$ GeV $^{-2}$ for the value $\kappa_0 = 0.534$ GeV obtained from the hadron spectrum [86], $r = 1/2$ and the maximal analyticity condition (19), namely $\rho = \pi/2$.

A. Physical thresholds and UV sum rules

The scaling function (41) can be seen as an effective approximation that encodes the dynamics from the quark thresholds. Actually, one may expect a distinct contribution from each heavy-quark flavor:

$$f(Q^2) = \sum_{f=c,b,t} C_f (1 + \sigma_f Q^2)^r$$

$$= \sum_{f=c,b,t} C_f \left(1 + \frac{1}{m_f^2} Q^2\right)^r, \quad C_f \geq 0, \quad (43)$$

with branch cuts in the timelike domain, $q^2 = -Q^2$, from $q^2 = m_f^2$ to $q^2 \rightarrow \infty$. Only the heavy quarks c , b and t contribute to the sums in (43) since the ultrarelativistic u , d and s light quarks are a component of the confinement dynamics, thus embedded in the hadron scale κ_0 . This distinction is not so clear for the charm quark with a mass $m_c \simeq 1.3$ GeV. It is located near the boundary of the IR-UV transition domain at $Q^2 \simeq 4\kappa_0^2 \simeq 1$ GeV ^{2} [14,74]. Consequently, the contribution from the charm threshold may be nonzero but relatively smaller than the combined b and t contributions, $C_c \ll C_b + C_t$.

The boundary conditions (42) in the IR and UV lead to the sum rules

$$\sum_f C_f = 1,$$

$$\sum_f C_f \sigma_f^r = \sigma^r, \quad (44)$$

where $\sigma_f = 1/m_f^2$. The system of equations (44) results in two independent solutions, which we choose as the b and t coefficients, C_b and C_t , expressed in terms of the masses of the heavy quarks, plus two independent parameters σ and C_c . We find

$$C_b = \frac{\sigma^r - C_c \sigma_c^r - (1 - C_c) \sigma_t^r}{\sigma_b^r - \sigma_t^r},$$

$$C_t = -\frac{\sigma^r - (1 - C_c) \sigma_b^r - C_c \sigma_c^r}{\sigma_b^r - \sigma_t^r}. \quad (45)$$

If we fix the value of σ from the fit to the UV data (Fig. 2), $\sigma = 0.015$ GeV $^{-2}$, we are left with the solution for C_b and C_t given in terms of the charm coefficient C_c . This is a parametric solution that determines the locus of the coefficients of C_b and C_t in terms of C_c , as shown in Fig. 3. The upper curve corresponds to C_t , $0.5 \leq C_t \leq 0.85$, and the lower curve corresponds to C_b , $0.5 \geq C_b \geq 0$. Each value of C_c in the interval $0 \geq C_c \geq 0.15$ in the upper figure corresponds to one specific value for C_b and C_t , which, in turn, are mapped in the lower figure to a specific trajectory of the scaling function $f(Q^2)$. For example, the value $C_c = 0.075$ intercepts the curves for C_b and C_t , pinpointing the values $C_b = 0.250$ and $C_t = 0.675$. This particular solution corresponds to the red curve in the lower figure. Likewise, since the red vertical line sweeps the interval from the initial to the final values of C_c , $0 \geq C_c \geq 0.15$, the locus of trajectories for $f(Q)$ (43) is built in the lower figure, from the lowest gray curve corresponding to $C_s = 0$ to the

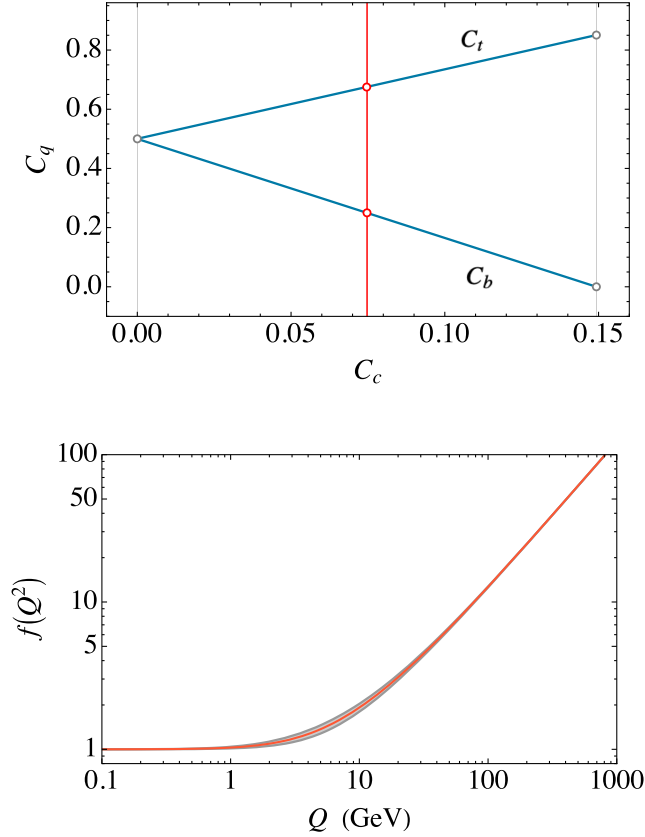


FIG. 3. The locus of possible values of the charm, bottom and top weight coefficients, C_c , C_b and C_t (upper figure) is mapped into the locus of the possible trajectories of $f(Q^2)$ determined by the sum rules for $\sigma = 0.015 \text{ GeV}^{-2}$, as explained in the text.

highest one corresponding to $C_s = 0.15$. The locus of the trajectories in Fig. 3 corresponds to the values $\sigma_c = 0.62 \text{ GeV}^{-2}$, $\sigma_b = 0.057 \text{ GeV}^{-2}$ and $\sigma_t = 3.4 \times 10^{-5} \text{ GeV}^{-2}$, the inverse of the square masses of the charm, bottom and top quark, respectively, $m_c = 1.27 \text{ GeV}$, $m_b = 4.18 \text{ GeV}$ and $m_t = 172.76 \text{ GeV}$.

VI. COMPARISON WITH THE DATA ACROSS ALL SCALES

We compare the predictions from the present model (21) for the middle trajectory shown in Fig. 3 (solid red line), with those from the IR-UV transition model (17) (blue dashed curve) and the HLFQCD prediction (14) (gray dot-dashed curve). We distinguish in Fig. 4 three distinct domains: the IR nonperturbative holographic domain [13] for $0 < Q^2 \leq 4\kappa_0^2 \simeq 1 \text{ GeV}^2$, the IR-UV transition domain studied in [14] for $4\kappa_0^2 < Q^2 \lesssim 10^2 \text{ GeV}^2$, and the UV extension described in this article for $10^2 \text{ GeV}^2 \lesssim Q^2$, up to the deep UV domain. We notice that the value $4\kappa_0^2 = 1.14 \text{ GeV}^2$ is compatible with the value $Q_0^2 = 1.12 \text{ GeV}^2$ [75] from the IR-UV point-matching procedure described in Sec. II D.

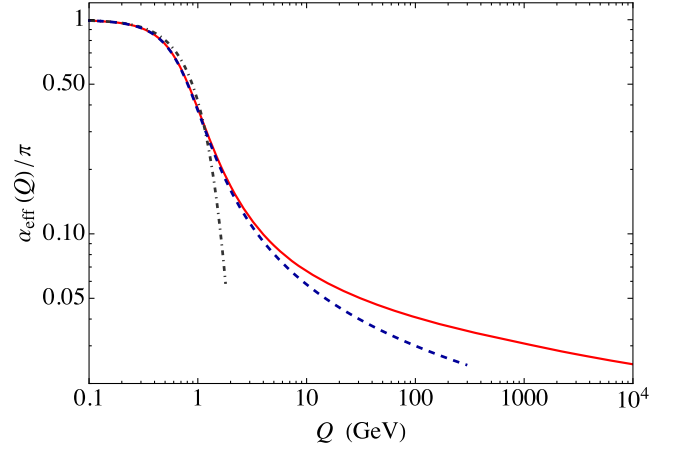


FIG. 4. The present model prediction from (21) and (43) (red solid line) is compared with the IR-UV transition model results (17) (blue dashed curve) and the holographic exponential form (14) (gray dot-dashed curve) for $\kappa_0 = 0.534 \text{ GeV}$. The values for C_q and σ_q correspond to the red curve in Fig. 3, as explained in the text.

We confront in Fig. 5 the model predictions with all available data, thus including the IR and transition domains. Therefore, in addition to the PDG data [85] transformed to the g_1 scheme in the perturbative domain (blue open circles), we have included the data obtained

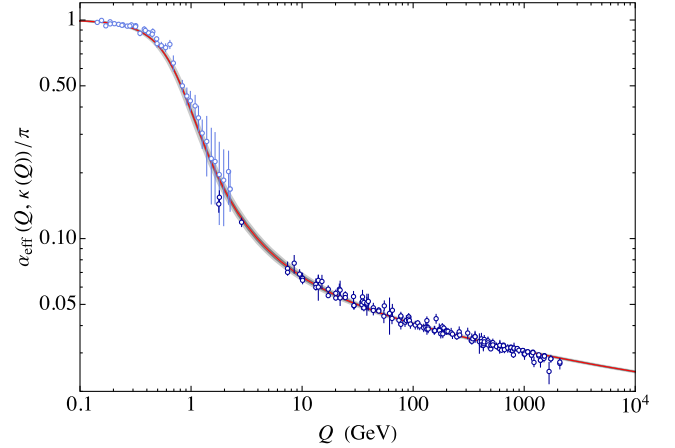


FIG. 5. The model predictions from (21) and (43) for $\kappa_0 = 0.534 \text{ GeV}$ are compared with the Bjorken spin sum-rule data [42–64] (light-blue open circles), combined with the PDG data [85] transformed to the g_1 scheme (blue open circles). The red curve corresponds to the value $C_c = 0.075$, $C_b = 0.250$ and $C_t = 0.675$. The lower and upper curves at the edge of the gray narrow band for $C_c = 0$, $C_b = 0.5$, $C_t = 0.5$ and $C_c = 0.15$, $C_b = 0$, and $C_t = 0.85$, respectively, define the boundaries of the possible solutions in Fig. 3. The values of $\sigma_c = 0.62 \text{ GeV}^{-2}$, $\sigma_b = 0.057 \text{ GeV}^{-2}$ and $\sigma_t = 3.4 \times 10^{-5} \text{ GeV}^{-2}$ are computed from the heavy-quark masses in (43). The light-gray wider band includes the uncertainty in the confinement scale $\kappa_0 = 0.534 \pm 0.025 \text{ GeV}$.

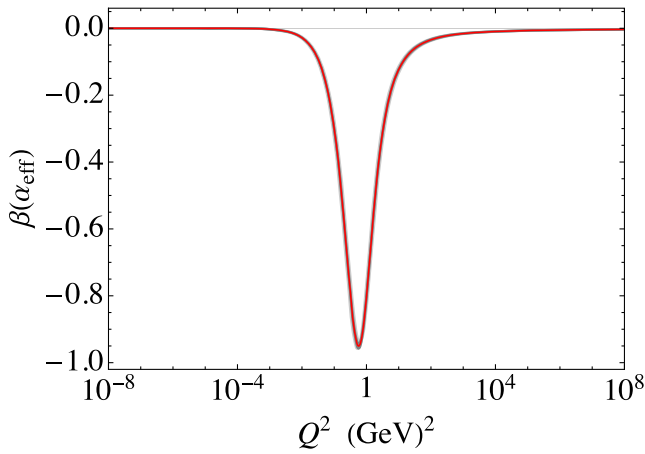


FIG. 6. The behavior of the β function (28) is computed using the same conventions as those in Fig. 5. The figure illustrates the basic conformality of QCD, except at the confinement scale $Q^2 \simeq 4\kappa_0^2 \simeq 1 \text{ GeV}^2$.

directly from the Bjorken spin sum rule [42–64] (light blue open circles). The solid red curve for $\kappa_0 = 0.534 \text{ GeV}$ corresponds to the central red curve in Fig. 3 for $f(Q)$, while the lower and upper curves at the edge of the gray narrow band correspond to the lower and upper curves for the locus of the solutions of $f(Q)$ in the same figure. The wider light-gray band in Fig. 5 also includes the uncertainty in κ_0 , $\kappa_0 = 0.534 \pm 0.025 \text{ GeV}$, obtained independently in [86]. The prediction of the model for the effective coupling (21), with the heavy-quark scaling function (43), gives a precise description of the data across all scales. The model results displayed in Fig. 5 are also provided in tabulated form in Supplemental Material [87].

The β function (28) shown in Fig. 6 captures the essential conformal properties of QCD, except at the confinement scale $Q^2 \simeq 4\kappa_0^2 \simeq 1 \text{ GeV}^2$. The conventions used in this figure are the same as those in Fig. 5.

VII. COMPUTATION OF THE STRONG COUPLING AT THE Z-BOSON MASS SCALE

In this section, we compare our results for the strong coupling, α_s , evaluated at the Z-boson mass scale, M_Z , with its standard value [85]. The quantity $\alpha_s(M_Z)$ has become the benchmark in high-energy physics. This energy scale is precisely measured at colliders and is sufficiently high for perturbative QCD to be reliable. Given the value of $\alpha_s(M_Z)$, one can compute α_s at any other energy scale within the perturbative regime using the renormalization-group equations, enabling comparisons with QCD predictions and global fits.

The locus of the scaling function $f(Q^2)$ in Fig. 3 indicates that all possible solutions converge to a unique trajectory in the deep UV, determined by the parameter $\sigma = 0.015 \text{ GeV}^{-2}$ extracted from the UV data in Sec. V. Furthermore, the equality of the coefficients C_b and C_t for

$C_c = 0$ in Fig. 3 indicates that σ can be computed from the equality of the bottom and top coefficients, independently of the data fit. Thus, by imposing the additional condition $C_b = C_t$, for $C_c = 0$, we can solve (45) exactly to determine the value of σ from the point of common loci in Fig. 3. We find

$$\sigma = 2^{-1/r}(\sigma_b^r + \sigma_t^r)^{1/r}. \quad (46)$$

For $r = 1/2$ we can write (46) as

$$\sigma = \frac{1}{4\mu^2}, \quad \frac{1}{\mu} = \frac{1}{m_b} + \frac{1}{m_t}, \quad (47)$$

with the value $\sigma = 0.01501 \text{ GeV}^{-2}$. Therefore, by imposing the condition $C_b = C_t$, $C_c = 0$, one can determine the value of σ that controls the deep UV behavior and, as a second step, reconstruct the locus but independently of the data fitting procedure in Fig. 2.

Having determined σ precisely with the procedure described in this section, we compute the strong coupling at the Z-boson mass $M_Z = 91.2 \text{ GeV}$. We find

$$\alpha_{\overline{\text{MS}}}^{\text{HLF}}(M_Z) = 0.1161 \pm 0.0015, \quad (48)$$

which corresponds to $\alpha_{\text{eff}}(M_Z, \kappa(M_Z)) = 0.1302 \pm 0.018$ transformed to the $\overline{\text{MS}}$ scheme. Our prediction is compatible with the world average value $\alpha_{\overline{\text{MS}}}(M_Z) = 0.1180 \pm 0.0009$, which includes lattice QCD results, and $\alpha_{\overline{\text{MS}}}(M_Z) = 0.1175 \pm 0.0010$ without lattice QCD [85]. The total uncertainty ± 0.0015 at the M_Z scale, assuming uncorrelated errors, is dominated by the uncertainty in the confinement scale κ_0 (see Table I). This is not the case, however, at lower scales where other errors have to be taken into account.

To illustrate the uncertainty dependence on the energy scale, we also provide a prediction for $\alpha_{\overline{\text{MS}}}(m_b)$, corresponding to the bottom quark mass scale m_b , which lies within an energy range lacking direct measurements of α_s (see Fig. 5). We find

$$\alpha_{\overline{\text{MS}}}^{\text{HLF}}(m_b) = 0.2267 \pm 0.0067, \quad (49)$$

which corresponds to $\alpha_{\text{eff}}(m_b, \kappa(m_b)) = 0.3045 \pm 0.0114$ in the g_1 scheme. The uncertainty dependence in the energy scale is also apparent from the error bands shown in Fig. 5. The sources of uncertainty in the computation of α_s using the present model are summarized in Table I and discussed below.

- (1) The value of the confinement scale $\kappa = 0.534 \pm 0.025 \text{ GeV}$ follows from the Regge trajectories of the light vector mesons relevant for the computation of the electromagnetic form factors and quark distribution functions [86]. This value is compatible

TABLE I. The identifiable sources of uncertainty in the strong coupling $\alpha_s(Q)$ at the Z-boson mass scale, M_Z , and the bottom quark mass scale, m_b , are displayed in the table. These include the uncertainty in the confinement scale κ_0 , the model uncertainty in the coefficients C_q , the uncertainty in the heavy-quark mass thresholds m_q , the errors arising from the g_1 -to- $\overline{\text{MS}}$ scheme transition, and the uncertainty in the energy scale Q , where applicable. The total uncertainty at the M_Z scale is dominated by the uncertainty in the confinement scale κ_0 .

	$\Delta\alpha_{g_1}(m_b)$	$\Delta\alpha_{\overline{\text{MS}}}(m_b)$	$\Delta\alpha_{g_1}(M_Z)$	$\Delta\alpha_{\overline{\text{MS}}}(M_Z)$
κ	$\pm 9.5 \times 10^{-3}$	$\pm 5.2 \times 10^{-3}$	$\pm 1.8 \times 10^{-3}$	$\pm 1.5 \times 10^{-3}$
C_q	$\pm 6.3 \times 10^{-3}$	$\pm 3.4 \times 10^{-3}$	$\pm 1.9 \times 10^{-4}$	$\pm 1.6 \times 10^{-4}$
m_c	$\pm 6.7 \times 10^{-5}$	$\pm 3.9 \times 10^{-5}$	$\pm 3.2 \times 10^{-5}$	$\pm 2.7 \times 10^{-5}$
m_b	$\pm 2.3 \times 10^{-5}$	$\pm 1.3 \times 10^{-5}$	$\pm 1.5 \times 10^{-5}$	$\pm 1.3 \times 10^{-5}$
m_t	$\pm 5.6 \times 10^{-8}$	$\pm 3.3 \times 10^{-8}$	$\pm 4.9 \times 10^{-7}$	$\pm 4.1 \times 10^{-7}$
Q	$\pm 2.8 \times 10^{-4}$	$\pm 2.4 \times 10^{-4}$		
$g_1 \rightarrow \overline{\text{MS}}$		$\pm 2.4 \times 10^{-3}$		$\pm 2.5 \times 10^{-5}$
	$\alpha_{g_1}(m_b) = 0.3045$	$\alpha_{\overline{\text{MS}}}(m_b) = 0.2267$	$\alpha_{g_1}(M_Z) = 0.1302$	$\alpha_{\overline{\text{MS}}}(M_Z) = 0.1161$

with the value $\kappa = 0.523 \pm 0.024$ obtained from the light hadron spectrum including all radial and orbital excitations [39].

- (2) The wide variation in the coefficients $C_q = \{C_s, C_b, C_t\}$ shown in Fig. 3 (top panel) $\{0, 0.15, 0.85\}$, $\{0.075, 0.25, 0.675\}$ and $\{0, 0.5, 0.5\}$ is determined by the asymptotic sum rules (44). It corresponds to the upper, central and lower trajectories in Fig. 3 (bottom panel) for the possible trajectories. Compared with the uncertainty in κ , the error is an order of magnitude smaller at the M_Z scale but of the same order at the m_b scale.
- (3) The uncertainty in the dynamic function $f(Q)$ (43), arising from the heavy-quark masses $m_c = 1.2730 \pm 0.0046$ GeV, $m_b = 4.183 \pm 0.007$ GeV, and $m_t = 172.56 \pm 0.31$ GeV [85], is indicated in the table. The charm and bottom quark masses are the renormalized $\overline{\text{MS}}$ values, and the top quark mass is extracted from event kinematics [85]. The effect from transforming to a different renormalization scheme is an order of magnitude smaller. Therefore, the correction from the quark mass thresholds in any renormalization scheme is negligible for the determination of α_s at both the M_Z and m_b scales.
- (4) The uncertainty in α_s from the measurement of the Z-boson mass, $m_Z = 91.1180 \pm 0.0020$ GeV is negligible. The bottom quark mass scale, $m_b = 4.183 \pm 0.007$ GeV, introduces an error that is an order of magnitude smaller compared to the leading uncertainty in α_s .
- (5) The uncertainty in transforming from the g_1 scheme $\overline{\text{MS}}$ is negligible at the M_Z mass scale but contributes at leading order at the m_b mass scale.

The transition from g_1 to the $\overline{\text{MS}}$ scheme is carried out using the perturbative relation (16), evaluated at leading twist and N⁴LO. The truncation of Eq. (16) is carried out at N⁴LO by including in $\alpha_{\overline{\text{MS}}}$ an estimate of the N⁵LO

contribution.¹ The uncertainty arising from truncating at leading-twist order in Eq. (16) can be estimated by considering the effect of the twist-4 contribution measured in Ref. [63]. In our Q^2 range of interest, the impact of higher-twist effects is found to be at least an order of magnitude smaller than the N⁵LO estimate and can therefore be neglected. Furthermore, while we accounted for the n_f dependence of the coefficients a_n in Eq. (16) [89], the associated uncertainty was assessed to be at the 10^{-4} level and was therefore neglected.

VIII. CONCLUSIONS AND OUTLOOK

A unified, nonperturbative analytical description of the QCD running coupling at all scales is a key step toward understanding the confinement mechanism in strongly coupled gauge theories and bridging the gap between short-distance predictions and long-distance hadronic observables. To be physically meaningful, the running coupling must capture the probability of gluon emission in the UV, and in the IR, encode the QCD potential that underlies hadronic phenomenology. The latter connection is explicitly established in the present article by Eq. (6).

In QCD, in addition to the perturbative scale Λ_s , heavy-quark masses also influence the asymptotic behavior. Thus, it is not surprising that the massless theory described in [14] fails to reproduce the data in the UV domain. To address this limitation, we have extended the domain of applicability of [14] by incorporating the symmetry group structure of asymptotically free gauge theories in the limit $Q^2 \rightarrow \infty$, with observable consequences in the UV domain. In the context of holographic QCD, a symmetry at infinity corresponds to the symmetries of the dual quantum field theory at the AdS boundary, where QCD exhibits conformal symmetry.

¹See [88]. See also Table I in Ref. [4].

The specific scaling behavior in the deep UV, determined from asymptotic boundary conditions, requires a scale-dependent κ , the only dimensionful dynamical quantity in our model, with the same power scaling law in the UV as in previous heavy-quark spectroscopic results from holographic QCD [18–20]. This procedure requires introducing a specific scaling function, while maintaining the analytic structure of α_{eff} . For $\text{SU}(3)_C$, the deep UV boundary conditions require six flavors and an IR fixed point value $\alpha_{\text{eff}}(0) = \pi$.

The scaling properties of the present model are compatible with the introduction of branch cuts in the timelike domain of the complex plane, with no overlapping singularities in the spacelike domain [90], therefore preserving the flow of singularities and the maximal analyticity properties in the spacelike domain [14]. Comparing with the QCD data for the running coupling in the UV, we are able to determine the locus of the threshold weights for the charm, bottom, and top quarks, which converge to a unique solution in the deep UV, as shown in Fig. 3. One can also choose one specific solution, for example equal bottom and top weights and zero weight for the charm, to determine the UV scaling behavior. If we follow this alternative, we can accurately describe the strong coupling data across all scales—including the highest available data in the multi-TeV range [85]—in terms of one unified dynamical scale: the confinement scale κ_0 , which also governs the decreasing logarithmic behavior in the UV.

We also compare our prediction for the strong coupling at the Z -boson mass scale, M_Z , with the current value of $\alpha_s(M_Z)$ [85], finding very good agreement. It may appear surprising that our nonperturbative approach, originally developed to describe the all-scale behavior of α_s , achieves a precision comparable to that of standard perturbative QCD methods. This agreement likely arises from the exact gauge symmetry condition imposed at the asymptotic UV boundary, far above any physical scale. This condition largely governs the deep UV behavior, where model uncertainties are very small. The experimental precision on α_s is expected to improve significantly in the coming years [91], partly due to the upcoming Electron-Ion Collider facility [92,93], which targets a relative uncertainty of $\Delta\alpha_s/\alpha_s = \pm 0.0001$. This level of precision will provide a stringent test of our prediction in Eq. (48) and the underlying model assumptions.

In our previous work, we focused on characterizing the IR-UV transition regime by modifying the IR holographic input at larger virtualities. To this end, we incorporated the logarithmic behavior associated with asymptotic freedom, though without introducing a mechanism in the deep UV that uniquely distinguishes QCD from the various possible solutions [14]. Interestingly, the UV extension developed in the present article shares similarities with the modern bootstrap program, which—beyond enforcing unitarity and analyticity—also incorporates dynamical UV information to

single out QCD from other quantum field theories with comparable IR spectra [94].

The limit on the number of flavors, $n_f = 6$, follows from the incorporation of QCD asymptotic boundary conditions, together with the well-determined value of the coupling's IR fixed point and the constraint of maximal analyticity. The flavor limit is both surprising and remarkable, but it raises a fundamental question about the nature of the approach: Why should a nonperturbative model be sensitive to the number of flavors? More specifically, why do the model's requirements—independent of detailed dynamics and dimensionful scales—imply $n_f = 6$ for $\text{SU}(3)$? Is $N_c = 3$, $n_f = 6$ a necessary condition for the application of the holographic QCD model across all scales?

The number of quark flavors, $n_f = 6$, required by the consistency of the model agrees with cosmological observations. The bounds on the number of generations of quark and leptons, parametrized by the effective number of neutrino species, N_{eff} , is obtained from the measurements of the primordial light elements in big bang nucleosynthesis (BBN) and the pattern of fluctuations in the cosmic microwave background (CMB). The precise values are $N_{\text{eff}} = 2.88 \pm 0.27$ from combined BBN and CMB [95] and $N_{\text{eff}} = 2.99 \pm 0.17$ from CMB combined with baryon acoustic oscillations in galaxy surveys [96]. Observations in collider experiments also indicate that $n_f = 6$. Those include the decay width of the Z_0 boson [97] and the unitarity of the Cabibbo-Kobayashi-Maskawa matrix, where three generations are required [85].

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DATA AVAILABILITY

The data displayed in Fig. 5 are openly available Supplemental Material [87].

APPENDIX A: LARGE Q^2 BEHAVIOR OF THE EFFECTIVE COUPLING

In this appendix we compute the large- Q^2 behavior of the effective coupling (17)

$$\alpha_{\text{eff}}(Q^2) = \alpha_{\text{eff}}(0) \exp \left(- \int_0^{Q^2} \frac{du}{4\kappa_0^2 + u \ln(\frac{u}{\Lambda^2})} \right). \quad (\text{A1})$$

By changing the integration variable $v = u/4\kappa_0^2$, the integral in (A1) can be written as

$$\begin{aligned} \int_0^{Q^2/4\kappa_0^2} \frac{dv}{1 + v \ln(\rho v)} &= \int_0^w \frac{dv}{1 + v \ln(\rho v)} + \int_w^{Q^2/4\kappa_0^2} \frac{dv}{1 + v \ln(\rho v)} \\ &= \int_0^w \frac{dv}{1 + v \ln(\rho v)} + \int_w^{Q^2/4\kappa_0^2} \frac{dv}{v \ln(\rho v) (1 + \frac{1}{v \ln(\rho v)})} \\ &= \int_0^w \frac{dv}{1 + v \ln(\rho v)} + [\ln(\ln(\rho v))]_w^{Q^2/4\kappa_0^2} - \int_w^{Q^2/4\kappa_0^2} \frac{dv}{v^2 \ln^2(\rho v)} + \dots, \end{aligned} \quad (\text{A2})$$

where $\rho = 4\kappa_0^2/\Lambda^2$ and w is an intermediate integration point in the interval $0 < w < Q^2/4\kappa_0^2$. At large Q^2 ,

$$\alpha_{\text{eff}}(Q^2) = \alpha_{\text{eff}}(0) \frac{K_w(\rho)}{\ln(\rho \frac{Q^2}{4\kappa_0^2})} \left(1 + \mathcal{O}\left(\frac{4\kappa_0^2}{Q^2}\right) \right), \quad (\text{A3})$$

where $K_w(\rho)$ is computed numerically. It is given by the integral

$$K_w(\rho) = \ln(\rho w) \exp \left(- \int_0^w \frac{dv}{1 + v \ln(\rho v)} \right) \left(1 + \mathcal{O}\left(\frac{1}{w}\right) \right), \quad (\text{A4})$$

which is well defined for $\rho > 1/e \simeq 0.3679$. As shown in Table II, $K_w(\rho)$ has a fast convergence to its asymptotic value for maximal analyticity (19), $\rho = \pi/2$:

$$K_{w \rightarrow \infty}(\pi/2) = 0.285785; \quad (\text{A5})$$

thus, it basically depends on the value of ρ for w sufficiently large.

APPENDIX B: LARGE- Q^2 BEHAVIOR OF PERTURBATION THEORY

The expression of the β function in higher-order perturbation theory for an $\text{SU}(N_c)$ gauge theory is given by the RGE [83,84]

TABLE II. Numerical results for $K_w(\rho)$ show a rapid convergence for the maximal analyticity value $\rho = \pi/2$. For comparison, we present results for $\rho = \pi/4$, as an example. The second column indicates the corresponding Q scale in GeV units for $4\kappa_0^2 \simeq 1$.

w	$\sqrt{4\kappa_0^2 w}$	$K_w(\pi/4)$	$K_w(\pi/2)$
10^2	10	0.0570879	0.285702
10^4	10^2	0.0571092	0.285785
10^6	10^3	0.0571093	0.285785

$$\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s(\mu^2)}{\partial \mu^2} \quad (\text{B1})$$

$$= -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - \beta_2 \alpha_s^4 + \dots, \quad (\text{B2})$$

where the coefficients $\beta_0, \beta_1, \beta_2, \dots$ depend on the number of colors N_c and n_f , the number of active quark flavors at the renormalization scale μ . The one- and two-loop coefficients β_0 and β_1 , respectively, are independent of the renormalization scheme and are given by [85]

$$\begin{aligned} \beta_0 &= \frac{11N_c - 2n_f}{12\pi}, \\ \beta_1 &= \frac{1}{24\pi^2} \left[17N_c^2 - n_f \left(5N_c + 3 \frac{N_c^2 - 1}{2N_c} \right) \right]. \end{aligned} \quad (\text{B3})$$

From the recursive integration of (B1) one obtains at the NLO the scheme-independent expression

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda_s^2)} \left[1 - \frac{\beta_1 \ln \ln(Q^2/\Lambda_s^2)}{\beta_0^2 \ln(Q^2/\Lambda_s^2)} + \dots \right], \quad (\text{B4})$$

where the omitted higher-order scheme-dependent terms decrease as $1/\ln^2(Q^2/\Lambda_s)$ [4]. We have also identified the renormalization scale μ with the momentum transfer Q in (B4). Close to the subtraction point $\mu^2 \simeq Q^2$, Q^2 becomes the virtuality for a given process. Notice that Λ_s refers to the QCD scale in a perturbative iteration, while Λ , without a subscript, refers to the effective scale that controls the logarithmic dependence (17). It was shown in [14] that, to a good approximation, $\Lambda_s = \Lambda$ for maximal analyticity.

APPENDIX C: HIGHER-RANK GAUGE GROUPS, SCALING, AND ASYMPTOTIC SYMMETRY

The relation between the asymptotic gauge group symmetry at the asymptotic limit, the number of open flavors n_f and the scaling exponent r in (31) follows from Eq. (36):

TABLE III. Number of colors and flavors, N_c and n_f , from the solutions of the diophantine equation (C1) for the possible power scaling values of r in the interval $0 \leq r < 1$. For $N_c = 3$ and $n_f = 6$ there is only one solution $r = 1/2$.

N_c	n_f													
2				4						2	5	8		
3		6				13								
4	1		8	15			4	7	10	13	16	19		
5		17			10	24							8	11
6	12		19	26			15	18	21	24	27	30		
7		28			21	35							19	22
...														
r	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{1}{14}$	$\frac{3}{14}$

N_c	n_f													
2											1	3	6	7
3	3	9	12	15										
4					2	3	5	6	9	11	12	14	17	18
5	14	20	23	26										
6					13	14	16	17	20	22	23	25	28	29
7	25	31	34	37										
...														
r	$\frac{5}{14}$	$\frac{9}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{8}{21}$	$\frac{10}{21}$	$\frac{11}{21}$	$\frac{13}{21}$	$\frac{16}{21}$	$\frac{17}{21}$

N_c	n_f													
2	9	10												
3					1	2	4	5	7	8	10	11	14	16
4	20	21												
5			7	9	12	13	15	16	18	19	21	22	25	27
6	31	32												
7			18	20	23	24	26	27	29	30	32	33	36	38
...														
r	$\frac{19}{21}$	$\frac{20}{21}$	$\frac{1}{42}$	$\frac{5}{42}$	$\frac{11}{42}$	$\frac{13}{42}$	$\frac{17}{42}$	$\frac{19}{42}$	$\frac{23}{42}$	$\frac{25}{42}$	$\frac{29}{42}$	$\frac{31}{42}$	$\frac{37}{42}$	$\frac{41}{42}$

$$\frac{11N_c - 2n_f}{42} = 1 - r, \quad 0 \leq r < 1, \quad (\text{C1})$$

in the Bjorken sum rule scheme where $\alpha_{\text{eff}}(0) = \pi$, with the value of ρ in (A4) set by the analytic constraints of the model, $\rho = \pi/2$.

Possible values of r in the interval $0 \leq r < 1$ are found by restricting the solutions of (C1) to integer values of N_c and n_f . Specifically, this means considering the integer factorization of 42, which includes the factors 2, 3, and 7 as well as their products 6, 14, 21, and 42. This is equivalent to the solution of a linear diophantine equation, which is a linear equation in two or more unknowns with integer coefficients [98]. We find 42 possible values for r :

$$r = 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots, \frac{41}{42}, \quad (\text{C2})$$

shown in Table III with the corresponding values for N_c and n_f .

The value of r depends only on the difference $11N_c - 2n_f$ and not on N_c and n_f separately. For SU(3), for example, all combinations for $N_c = 3 + 2j, n_f = 6 + 11j, j = 0, 1, 2, \dots$, lead to the same value $r = 1/2$ with three colors and six flavors, thus generating the corresponding column in Table III:

$$(N_c, n_f) = (3, 6), (5, 17), (7, 28), \dots, (3 + 2j, 6 + 11j). \quad (\text{C3})$$

We notice that, while the values of the leading coefficient of the β function (B3) are the same for all the values in the table entry generated by (C3), $\beta_0 = 7/4\pi$, the higher coefficients can be drastically different from the ones of real QCD. For example, $\beta_1 = 13/8\pi^2$ for $(N_c, n_f) = (3, 6)$, but the sequence

$$\beta_1 = +\frac{13}{8\pi^2}, -\frac{51}{10\pi^2}, -\frac{145}{8\pi^2}, \dots \quad (\text{C4})$$

increasingly diverges for the higher-rank groups in the same column.

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