

$\mathcal{N} = 8$ supersymmetric mechanics with spin variables from indecomposable multiplets

Evgeny Ivanov^{1,2,*} and Stepan Sidorov^{1,†}

¹*Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow Region, Russia*

²*Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Moscow Region, Russia*



(Received 23 January 2026; accepted 13 February 2026; published 3 March 2026)

We define two new indecomposable (not fully reducible) $\mathcal{N} = 8$, $d = 1$ off-shell multiplets and consider the corresponding models of $\mathcal{N} = 8$ supersymmetric mechanics with spin variables. Each multiplet is described off shell by a scalar superfield which is a nonlinear deformation of the standard scalar superfield X carrying the $d = 1$ multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$. Deformed systems involve, as invariant subsets, two different off-shell versions of the irreducible multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. For both systems we present the manifestly $\mathcal{N} = 8$ supersymmetric superfield constraints, as well as the component off- and on-shell invariant actions, which for one version exactly match those given in [arXiv:2402.00539](#). The two models differ off shell, but prove to be equivalent to each other on shell, with the spin variables sitting in the adjoint representation of the maximal R -symmetry group $SO(8)$.

DOI: [10.1103/tcwy-y4tg](#)

I. INTRODUCTION

Supersymmetric mechanics with spin variables was proposed in [1] as a system based on the coupled dynamical and semidynamical (with the first-order kinetic terms for physical bosonic $d = 1$ fields) irreducible multiplets. Recently, a new approach to such systems based on the not fully reducible (indecomposable) multiplets was proposed [2].¹

The first example of an $\mathcal{N} = 8$ supersymmetric model with spin variables was presented in [8], using an off-shell $\mathcal{N} = 4$ superfield formalism. Later on, the authors of Ref. [9] demonstrated that this model on shell is invariant under $\mathcal{N} = 8$, $d = 1$ superconformal group $OSp(8|2)$. In the original paper [8], it was conjectured that the model constructed corresponds to an indecomposable $\mathcal{N} = 8$ supermultiplet encompassing the dynamical $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ multiplet with a pair of semidynamical $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets. The basic aim of the present paper is to prove this conjecture. We define the relevant $\mathcal{N} = 8$ superfield set subjected to

the proper manifestly $\mathcal{N} = 8$ supersymmetric nonlinear constraints and show that the off-shell field content considered in [8] naturally comes out as a solution of these constraints. We also introduce another indecomposable $\mathcal{N} = 8$ multiplet with the same number of physical bosonic and fermionic fields, construct the relevant off- and on-shell actions, and find that on shell it yields the same system as in [8].

The superfield description of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ was originally given within the $SU(2) \times SU(2) \times SU(2)$ covariant formalism in [10]. The direct superfield construction of manifestly $\mathcal{N} = 8$ invariant actions has not been accomplished as yet. The invariant actions were constructed in the framework of harmonic $\mathcal{N} = 4$, $d = 1$ superspace [10–12]. The covariant description of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ with the maximally extended R -symmetry $SO(7)$ was presented in [13], at the component level. The invariant Lagrangian was constructed there by the “brute force” method.

The multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ was studied in the $SO(4) \times SO(4)$ covariant formulation in [10]. When dealing with this multiplet in the present paper, we rely on Ref. [14], where $\mathcal{N} = 8$, $d = 1$ superalgebra was deformed to the superalgebra $su(4|1)$. Here we are interested only in the undeformed case, with the standard $\mathcal{N} = 8$ Poincaré superalgebra written in the $SU(4)$ covariant form. As was shown in [14], $\mathcal{N} = 8$ multiplets with the off-shell $d = 1$ field content $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ can exist in three nonequivalent $SU(4)$ covariant forms related to each other by triality which is the specific feature of the complete R -symmetry group $SO(8)$ of $\mathcal{N} = 8$ supersymmetry. We studied only

*Contact author: eivanov@theor.jinr.ru

†Contact author: sidorovstepan88@gmail.com

¹Such $d = 1$ supermultiplets are known under various names: “nonminimal” [3,4], “tuned” [5], “long,” and “indecomposable” [6,7]. Their characteristic feature is the presence of some subset of fields forming an irreducible supermultiplet on their own.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](#). Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Funded by SCOAP³.

two of such multiplets since the third one requires a different $d = 1$ superspace for its off-shell description. Here, we also ignore the third version for the same reason. The triality property refers also to the multiplets with the content $(\mathbf{0}, \mathbf{8}, \mathbf{8})$. On the other hand, $SU(4)$ covariant formulations of the multiplets $(\mathbf{1}, \mathbf{8}, \mathbf{7})$, $(\mathbf{2}, \mathbf{8}, \mathbf{6})$, $(\mathbf{6}, \mathbf{8}, \mathbf{2})$, and $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ are unique.

The paper is organized as follows. In Sec. II we start with the irreducible multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$. First, we present its $SU(4)$ covariant description and derive invariant off-shell component Lagrangians through the proper reductions of the Lagrangians of the $\mathcal{N} = 8$ multiplet $(\mathbf{2}, \mathbf{8}, \mathbf{6})$. We also recall the superfield construction in terms of harmonic $\mathcal{N} = 4$ superfields accommodating the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$. Further, we rewrite the constraints of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ in the maximally $SO(7)$ covariant form. In Sec. III, we describe the first version of the indecomposable $\mathcal{N} = 8$ multiplet unifying the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ with a pair of $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets. We deform the constraints of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ by adding complex constrained superfields describing these two $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets. We construct the component off-shell Lagrangian and show its identity with that of the model constructed in [8]. In Sec. IV, we consider a different version of the indecomposable

$\mathcal{N} = 8$ multiplet, in which the $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ multiplet is coupled to the pair of the second multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ from the list given in [14]. In this version, the superfield constraints of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ are deformed by chiral and tensor $\mathcal{N} = 8$ superfields. We construct its off-shell Lagrangian. We also compare the on-shell Lagrangians of both indecomposable multiplets and prove their equivalence. In Sec. V, we summarize the results and discuss some problems for the future analysis.

II. MULTIPLY (1, 8, 7)

A. $SU(4)$ covariant formulation

We deal with the $\mathcal{N} = 8$, $d = 1$ superalgebra written as

$$\{Q^I, \bar{Q}_J\} = 2\delta_J^I H, \quad [H, Q^I] = 0, \quad [H, \bar{Q}_I] = 0. \quad (2.1)$$

Here, the capital indices I, J ($I = 1, 2, 3, 4$) refer to the $SU(4)$ fundamental representation. The corresponding superspace consists of a time coordinate t and complex Grassmann coordinates $\theta_I, \bar{\theta}^I = \overline{\theta_I}$. Supersymmetry acts on them as

$$\delta\theta_I = \epsilon_I, \quad \delta\bar{\theta}^I = \bar{\epsilon}^I, \quad \delta t = i(\bar{\epsilon}^I \theta_I + \epsilon_I \bar{\theta}^I), \quad \overline{(\epsilon_I)} = \bar{\epsilon}^I. \quad (2.2)$$

One can easily define left and right time coordinates:

$$t_L = t + i\bar{\theta}^I \theta_I, \quad t_R = t - i\bar{\theta}^I \theta_I. \quad (2.3)$$

The left chiral subspace transforms as

$$\delta\theta_I = \epsilon_I, \quad \delta t_L = 2i\bar{\epsilon}^I \theta_I. \quad (2.4)$$

The covariant derivatives are

$$D^I = \frac{\partial}{\partial\theta_I} - i\bar{\theta}^I \partial_t, \quad \bar{D}_I = -\frac{\partial}{\partial\bar{\theta}^I} + i\theta_I \partial_t. \quad (2.5)$$

This worldline realization follows from the realization of [14] in the flat-superspace limit.

An unconstrained scalar superfield X has **256** components in its θ -expansion. The number of components is reduced to **16** by imposing the following quadratic constraints²:

$$D^I \bar{D}_J X - \frac{\delta_J^I}{4} D^K \bar{D}_K X = 0, \quad D^I D^J X - \frac{1}{2} \epsilon^{IJKL} \bar{D}_K \bar{D}_L X = 0. \quad (2.6)$$

The superfield X is expanded as

²The antisymmetric tensor ϵ^{IJKL} satisfies the following identities:

$$\begin{aligned} \epsilon^{1234} &= 1, & \epsilon_{1234} &= 1, & \epsilon^{IJKL} \epsilon_{IJKL} &= 24, \\ \epsilon^{IJKL} \epsilon_{IJKM} &= 6\delta_M^L, & \epsilon^{IJKL} \epsilon_{IJMN} &= 4\delta_{[M}^K \delta_{N]}^L, & \epsilon^{IJKL} \epsilon_{IMNP} &= 6\delta_{[M}^J \delta_N^K \delta_{P]}^L. \end{aligned}$$

$$\begin{aligned}
 X(t, \theta, \bar{\theta}) = & x + \theta_I \psi^I - \bar{\theta}^I \bar{\psi}_I - \frac{i}{2} \left(\theta_I \theta_J + \frac{1}{2} \varepsilon_{IJKL} \bar{\theta}^K \bar{\theta}^L \right) A^{IJ} + \bar{\theta}^I \theta_I C + i \bar{\theta}^I \theta_I (\theta_J \dot{\psi}^J + \bar{\theta}^J \dot{\bar{\psi}}_J) - \frac{i}{3} \varepsilon^{IJKL} \theta_I \theta_J \theta_K \dot{\bar{\psi}}_L \\
 & + \frac{i}{3} \varepsilon_{IJKL} \bar{\theta}^I \bar{\theta}^J \bar{\theta}^K \dot{\psi}^L + 2[(\theta)^4 + (\bar{\theta})^4] \ddot{x} - \frac{1}{2} (\bar{\theta}^I \theta_I)^2 \ddot{x} - 2i[(\theta)^4 - (\bar{\theta})^4] \dot{C} + \frac{1}{2} \bar{\theta}^K \theta_K \theta_I \theta_J \dot{A}^{IJ} + \frac{1}{2} \theta_K \bar{\theta}^K \bar{\theta}^I \bar{\theta}^J \dot{A}_{IJ} \\
 & - 2(\theta)^4 \bar{\theta}^I \ddot{\bar{\psi}}_I + 2(\bar{\theta})^4 \theta_I \ddot{\psi}^I - \frac{1}{2} (\bar{\theta}^I \theta_I)^2 (\theta_J \ddot{\psi}^J - \bar{\theta}^J \ddot{\bar{\psi}}_J) + \frac{1}{6} (\bar{\theta}^I \theta_I)^3 \ddot{C} - \frac{i}{2} [(\theta)^4 + (\bar{\theta})^4] \left(\theta_I \theta_J + \frac{1}{2} \varepsilon_{IJKL} \bar{\theta}^K \bar{\theta}^L \right) \ddot{A}^{IJ} \\
 & + \frac{i}{2} [(\theta)^4 \varepsilon_{IJKL} \bar{\theta}^I \bar{\theta}^K \bar{\theta}^L (\partial_I)^3 \psi^J - (\bar{\theta})^4 \varepsilon^{IJKL} \theta_I \theta_J \theta_K (\partial_I)^3 \bar{\psi}_L] + (\theta)^4 (\bar{\theta})^4 (\partial_I)^4 x, \\
 \overline{(x)} = & x, \quad \overline{(\psi^I)} = \bar{\psi}_I, \quad A^{IJ} = -A^{JI}, \quad \overline{(A^{IJ})} = \frac{1}{2} \varepsilon_{IJKL} A^{KL} = A_{IJ}, \quad \overline{(C)} = C,
 \end{aligned} \tag{2.7}$$

where

$$(\theta)^4 = \frac{1}{24} \varepsilon^{IJKL} \theta_I \theta_J \theta_K \theta_L, \quad (\bar{\theta})^4 = \frac{1}{24} \varepsilon_{IJKL} \bar{\theta}^I \bar{\theta}^J \bar{\theta}^K \bar{\theta}^L. \tag{2.8}$$

So this superfield describes an irreducible off-shell multiplet with the content $(\mathbf{1}, \mathbf{8}, \mathbf{7})$. The constraints (2.6) respect the R -symmetry group $\text{SO}(7) \subset \text{SO}(8)$, though only its $\text{SU}(4) \sim \text{SO}(6)$ subgroup is manifest. The $\mathcal{N} = 8$ supersymmetry transformations of the component fields read:

$$\begin{aligned}
 \delta x &= \bar{\epsilon}^I \bar{\psi}_I - \epsilon_I \psi^I, \\
 \delta \psi^I &= i \epsilon_J A^{IJ} + \bar{\epsilon}^I (i \dot{x} + C), \quad \delta \bar{\psi}_I = -i \bar{\epsilon}^J A_{IJ} - \epsilon_I (i \dot{x} - C), \\
 \delta C &= -i (\bar{\epsilon}^I \dot{\bar{\psi}}_I + \epsilon_I \dot{\psi}^I), \quad \delta A^{IJ} = 4 \bar{\epsilon}^{[I} \dot{\bar{\psi}}^{J]} - 2 \varepsilon^{IJKL} \epsilon_K \dot{\bar{\psi}}_L.
 \end{aligned} \tag{2.9}$$

1. Derivation from multiplet (2, 8, 6)

Here, we show how invariant actions can be derived from actions of the multiplet $(\mathbf{2}, \mathbf{8}, \mathbf{6})$. This way of getting the actions of $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ proves to be most convenient.

The irreducible multiplet $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ is described by a chiral superfield [15] satisfying the constraints

$$D^I \bar{\Phi} = 0, \quad \bar{D}_J \Phi = 0, \quad D^I D^J \Phi - \frac{1}{2} \varepsilon^{IJKL} \bar{D}_K \bar{D}_L \bar{\Phi} = 0, \quad \overline{(\Phi)} = \bar{\Phi}. \tag{2.10}$$

It has the following θ -expansions over the chiral superspace (t_L, θ) :

$$\begin{aligned}
 \Phi(t_L, \theta) &= \phi + \sqrt{2} \theta_I \psi^I - \frac{i}{\sqrt{2}} \theta_I \theta_J A^{IJ} - \frac{\sqrt{2}}{3} i \varepsilon^{IJKL} \theta_I \theta_J \theta_K \dot{\bar{\psi}}_L + 4(\theta)^4 \ddot{\phi}, \\
 \overline{(\phi)} &= \bar{\phi}, \quad \overline{(\psi^I)} = \bar{\psi}_I, \quad \overline{(A^{IJ})} = A_{IJ} = \frac{1}{2} \varepsilon_{IJKL} A^{KL}.
 \end{aligned} \tag{2.11}$$

The component fields transform as

$$\begin{aligned}
 \delta \phi &= -\sqrt{2} \epsilon_I \psi^I, \quad \delta \bar{\phi} = \sqrt{2} \bar{\epsilon}^I \bar{\psi}_I, \\
 \delta \psi^I &= \sqrt{2} i \bar{\epsilon}^I \dot{\phi} + i \epsilon_J A^{IJ}, \quad \delta \bar{\psi}_I = -\sqrt{2} i \epsilon_I \dot{\bar{\phi}} - i \bar{\epsilon}^J A_{IJ}, \\
 \delta A^{IJ} &= 4 \bar{\epsilon}^{[I} \dot{\bar{\psi}}^{J]} - 2 \varepsilon^{IJKL} \epsilon_K \dot{\bar{\psi}}_L.
 \end{aligned} \tag{2.12}$$

It can be shown that these transformations reduce to the transformations (2.9) by identifying x and C as

$$x = \frac{1}{\sqrt{2}} (\phi + \bar{\phi}), \quad C = \frac{i}{\sqrt{2}} (\dot{\phi} - \dot{\bar{\phi}}). \tag{2.13}$$

This way, we reduce the multiplet $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ to $(\mathbf{1}, \mathbf{8}, \mathbf{7})$. The reversed procedure is known as the oxidation procedure [16]. We can also observe this duality between two multiplets at the superfield level. Indeed, the sum $\Phi + \bar{\Phi}$ satisfies the same constraints as the superfield X in (2.6).

The general invariant Lagrangian of the multiplet $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ can be written as

$$\mathcal{L} = -\frac{1}{4} \left[\int d^4\theta \mathcal{K}(\Phi) + \int d^4\bar{\theta} \bar{\mathcal{K}}(\bar{\Phi}) \right], \quad (2.14)$$

where $\mathcal{K}(\Phi)$ is an arbitrary holomorphic function. The component Lagrangian reads

$$\begin{aligned} \mathcal{L}_{(\mathbf{2}, \mathbf{8}, \mathbf{6})} = & g(\phi, \bar{\phi}) \left[\dot{\phi} \dot{\bar{\phi}} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{1}{4} A^{IJ} A_{IJ} \right] \\ & - \frac{i}{2} [\dot{\phi} \partial_{\bar{\phi}} g(\phi, \bar{\phi}) - \dot{\bar{\phi}} \partial_{\phi} g(\phi, \bar{\phi})] \psi^I \bar{\psi}_I \\ & - \frac{i}{2\sqrt{2}} [A_{IJ} \psi^I \psi^J \partial_{\phi} g(\phi, \bar{\phi}) + A^{IJ} \bar{\psi}_I \bar{\psi}_J \partial_{\bar{\phi}} g(\phi, \bar{\phi})] \\ & - \frac{1}{24} g(\phi, \bar{\phi}) [\varepsilon_{IJKL} \psi^I \psi^J \psi^K \psi^L \partial_{\phi} \partial_{\bar{\phi}} g(\phi, \bar{\phi}) \\ & + \varepsilon^{IJKL} \bar{\psi}_I \bar{\psi}_J \bar{\psi}_K \bar{\psi}_L \partial_{\bar{\phi}} \partial_{\phi} g(\phi, \bar{\phi})], \end{aligned} \quad (2.15)$$

where $g(\phi, \bar{\phi})$ is a special Kähler metric defined as

$$g(\phi, \bar{\phi}) = \mathcal{K}''(\phi) + \bar{\mathcal{K}}''(\bar{\phi}). \quad (2.16)$$

To gain the Lagrangian of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$, we impose the requirement that the target metric $g(\phi, \bar{\phi})$ is a function of the argument $\phi + \bar{\phi}$. There are only two solutions:

$$g_1 = 1, \quad g_2 = \frac{1}{\sqrt{2}} (\phi + \bar{\phi}). \quad (2.17)$$

The corresponding Lagrangians read

$$\mathcal{L}_{(\mathbf{2}, \mathbf{8}, \mathbf{6})}^1 = \dot{\phi} \dot{\bar{\phi}} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{A^{IJ} A_{IJ}}{8}, \quad (2.18)$$

$$\begin{aligned} \mathcal{L}_{(\mathbf{2}, \mathbf{8}, \mathbf{6})}^2 = & \frac{1}{\sqrt{2}} (\phi + \bar{\phi}) \left[\dot{\phi} \dot{\bar{\phi}} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{A^{IJ} A_{IJ}}{8} \right] \\ & - \frac{i}{2\sqrt{2}} (\dot{\phi} - \dot{\bar{\phi}}) \psi^I \bar{\psi}_I - \frac{i}{4} (A_{IJ} \psi^I \psi^J + A^{IJ} \bar{\psi}_I \bar{\psi}_J). \end{aligned} \quad (2.19)$$

Performing the redefinitions (2.13), we obtain two possible Lagrangians of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$:

$$\begin{aligned} \mathcal{L}_{(\mathbf{1}, \mathbf{8}, \mathbf{7})}^1 = & \frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{C^2}{2} + \frac{A^{IJ} A_{IJ}}{8}, \\ \mathcal{L}_{(\mathbf{1}, \mathbf{8}, \mathbf{7})}^2 = & x \left[\frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{C^2}{2} + \frac{A^{IJ} A_{IJ}}{8} \right] \\ & - \frac{C}{2} \psi^I \bar{\psi}_I - \frac{i}{4} (A_{IJ} \psi^I \psi^J + A^{IJ} \bar{\psi}_I \bar{\psi}_J). \end{aligned} \quad (2.20)$$

B. Formulation in terms of $\mathcal{N}=4$ superfields

Here, we summarize the $\mathcal{N}=4$ construction of the actions (2.20) based on the harmonic superfield description of the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ [10–12].

The $SU(2) \times SU(2) \times SU(2)$ formulation is given via the derivatives $\mathbf{D}^{i\alpha}$ and ∇^{iA} , where the indices i, α , and A ($i, \alpha, A = 1, 2$) mark doublets of three commuting $SU(2)$ groups [10]. The odd coordinates θ_I and $\bar{\theta}^I$ are split into the two sets of Grassmann coordinates $\theta^{i\alpha}$ and $\bar{\theta}^{iA}$. We define the new derivatives in terms of the original ones (2.5) as

$$\begin{aligned} \mathbf{D}^{11} &:= D^1, & \mathbf{D}^{22} &:= \bar{D}_1, & \mathbf{D}^{12} &:= D^2, & \mathbf{D}^{21} &:= -\bar{D}_2, \\ \nabla^{11} &:= \bar{D}_3, & \nabla^{22} &:= D^3, & \nabla^{12} &:= -\bar{D}_4, & \nabla^{21} &:= D^4. \end{aligned} \quad (2.21)$$

The constraints (2.6) can then be rewritten as

$$\begin{aligned} \mathbf{D}^{i(\alpha} \mathbf{D}_i^{\beta)} X &= 0, & \mathbf{D}_\alpha^{(i} \nabla_A^{j)} X &= 0, \\ \nabla^{i(A} \nabla_i^{B)} X &= 0, & \left(\mathbf{D}_\alpha^{(i} \mathbf{D}^{j)\alpha} + \nabla_A^{(i} \nabla^{j)A} \right) X &= 0. \end{aligned} \quad (2.22)$$

We define $\mathcal{N}=4$ superfields as

$$\mathcal{X} := X|_{\bar{\theta}=0}, \quad \Psi^{iA} := \nabla^{iA} X|_{\bar{\theta}=0}. \quad (2.23)$$

These $\mathcal{N}=4$ superfields satisfy

$$\mathbf{D}^{i(\alpha} \mathbf{D}_i^{\beta)} \mathcal{X} = 0, \quad \mathbf{D}_\alpha^{(i} \Psi^{j)A} = 0, \quad \mathbf{D}^{i\alpha} = \frac{\partial}{\partial \theta_{i\alpha}} + i\theta^{i\alpha} \partial_i. \quad (2.24)$$

The multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ splits into the irreducible multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ described by $\mathcal{X}$ and Ψ^{iA} , respectively. The implicit supersymmetry is realized on them through the η^{iA} transformations

$$\delta_\eta \mathcal{X} = \eta_{iA} \Psi^{iA}, \quad \delta_\eta \Psi^{iA} = \frac{1}{2} \eta_j^A \mathbf{D}_\alpha^i \mathbf{D}^{j\alpha} \mathcal{X}. \quad (2.25)$$

The Lagrangians (2.20) correspond to the following actions:

$$S_{(\mathbf{1}, \mathbf{8}, \mathbf{7})}^1 = -\frac{1}{2} \int d\zeta_{(H)} \mathcal{X}^2 - \frac{1}{2} \int d\zeta_{(A)}^{--} \Psi^{+A} \Psi_A^+, \quad (2.26)$$

$$S_{(\mathbf{1}, \mathbf{8}, \mathbf{7})}^2 = -\frac{1}{6} \int d\zeta_{(H)} \mathcal{X}^3 - \frac{1}{2} \int d\zeta_{(A)}^{--} \Psi^{+A} \Psi_A^+ \mathcal{X}_{(A)}. \quad (2.27)$$

Here, both actions were constructed within the framework of the $\mathcal{N}=4$ harmonic superspace [17] (see Appendix A). The superfield Ψ^{+A} satisfies both the Grassmann-analyticity and harmonic constraints:

$$\mathbf{D}^{+A}\Psi^{+A} = 0, \quad \mathbf{D}^{++}\Psi^{+A} = 0. \quad (2.28)$$

The superfield $\mathcal{X}$ is represented via an analytic prepotential $\mathcal{X}_{(A)}$ as [18]

$$\begin{aligned} \mathcal{X}(\zeta) &= \int du \mathcal{X}_{(A)}(\zeta_{(A)}), \\ \delta\mathcal{X}_{(A)}(\zeta_{(A)}) &= \mathbf{D}^{++}\Upsilon^{--}(\zeta_{(A)}). \end{aligned} \quad (2.29)$$

The prepotential is analytic but otherwise unconstrained (i.e. it exhibits an arbitrary harmonic dependence, $\mathbf{D}^{++}\mathcal{X}_{(A)} \neq 0$). The multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ arises as the appropriate Wess-Zumino gauge with respect to the gauge transformation in (2.29).

C. Octonionic covariant formulation

The multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$ was also described at the component level by making use of the octonionic structure constants [13]. Here, we present the relevant superfield constraints.

We define $\mathcal{N} = 8$, $d = 1$ superspace in terms of real Grassmann coordinates $\theta_{\mathbf{a}}$ and θ , where the index $\mathbf{a}$ ($\mathbf{a} = 1, 2, 3, 4, 5, 6, 7$) is a vector index associated with the group $\text{SO}(7)$. The infinitesimal $\mathcal{N} = 8$ transformations are realized in this superspace as

$$\delta\theta_{\mathbf{a}} = \epsilon_{\mathbf{a}}, \quad \delta\theta = \epsilon, \quad \delta t = i(\epsilon_{\mathbf{a}}\theta_{\mathbf{a}} + \epsilon\theta). \quad (2.30)$$

The relevant covariant derivatives are defined as

$$D_{\mathbf{a}} = \frac{\partial}{\partial\theta_{\mathbf{a}}} - i\theta_{\mathbf{a}}\partial_t, \quad D = \frac{\partial}{\partial\theta} - i\theta\partial_t \quad (2.31)$$

and are related to the complex derivatives (2.5) as

$$\begin{aligned} D^1 &= \frac{1}{\sqrt{2}}(D_1 + iD_2), & D^2 &= \frac{1}{\sqrt{2}}(D_3 - iD_4), \\ D^3 &= \frac{1}{\sqrt{2}}(D_5 - iD_6), & D^4 &= \frac{1}{\sqrt{2}}(D_7 + iD_8). \end{aligned} \quad (2.32)$$

The constraints (2.6) are then rewritten in the form

$$(c_{\mathbf{abc}}D_{\mathbf{c}}D + D_{[\mathbf{a}}D_{\mathbf{b}]})X = 0, \quad (2.33)$$

where $c_{\mathbf{abc}}$ are the octonionic structure constants (see Appendix B). The octonionic covariance implies that the constraints (2.33) are invariant under the following $\text{SO}(7)$ -transformations:

$$\begin{aligned} \delta D_{\mathbf{a}} &= \lambda_{\mathbf{ab}}D_{\mathbf{b}} + \frac{1}{2}c_{\mathbf{abcd}}\lambda_{\mathbf{bc}}D_{\mathbf{d}} - \frac{1}{2}c_{\mathbf{abc}}\lambda_{\mathbf{bc}}D, \\ \delta D &= \frac{1}{2}c_{\mathbf{abc}}\lambda_{\mathbf{ab}}D_{\mathbf{c}}, \quad \lambda_{\mathbf{ab}} = -\lambda_{\mathbf{ba}}. \end{aligned} \quad (2.34)$$

Indeed, these transformations correspond to the non-standard embedding of the algebra $so(7)$ in $so(8)$ (see Sec. 2.1.3 of [19]). In fact, the total set of $\mathcal{N} = 8$ covariant derivatives form the eight-dimensional spin representation of the group $\text{Spin}(7)$ [20] and one could think that they form a vector of $\text{Spin}(8)$, implying the latter to be the full covariance group. Nevertheless, the constraints (2.33) are covariant only with respect to the $\text{SO}(7)$ internal symmetry.

The superfield X is expanded as

$$\begin{aligned} X &= x + i\theta_{\mathbf{a}}\psi_{\mathbf{a}} + i\theta\psi + i\theta\theta_{\mathbf{a}}A_{\mathbf{a}} + \frac{i}{2}\theta_{\mathbf{a}}\theta_{\mathbf{b}}c_{\mathbf{abc}}A_{\mathbf{c}} \\ &\quad - \frac{1}{2}\theta\theta_{\mathbf{a}}\theta_{\mathbf{b}}c_{\mathbf{abc}}\psi_{\mathbf{c}} + \frac{1}{6}\theta_{\mathbf{a}}\theta_{\mathbf{b}}\theta_{\mathbf{c}}c_{\mathbf{abc}}\psi + \frac{1}{6}\theta_{\mathbf{a}}\theta_{\mathbf{b}}\theta_{\mathbf{c}}c_{\mathbf{abcd}}\psi_{\mathbf{d}} \\ &\quad + \frac{1}{6}\theta\theta_{\mathbf{a}}\theta_{\mathbf{b}}\theta_{\mathbf{c}}c_{\mathbf{abc}}\ddot{x} - \frac{1}{24}\theta_{\mathbf{a}}\theta_{\mathbf{b}}\theta_{\mathbf{c}}\theta_{\mathbf{d}}c_{\mathbf{abcd}}\ddot{x} \\ &\quad + \frac{1}{6}\theta_{\mathbf{a}}\theta_{\mathbf{b}}\theta_{\mathbf{c}}\theta_{\mathbf{d}}c_{\mathbf{abc}}\dot{A}_{\mathbf{d}} + \frac{1}{6}\theta\theta_{\mathbf{a}}\theta_{\mathbf{b}}\theta_{\mathbf{c}}c_{\mathbf{abcd}}\dot{A}_{\mathbf{d}} \\ &\quad + (\text{terms with higher } \theta \text{ powers}). \end{aligned} \quad (2.35)$$

Taking into account (2.32), it is straightforward to establish the direct relation between the fermionic and auxiliary components in the $\text{SU}(4) \sim \text{SO}(6)$ and $\text{SO}(7)$ formulations:

$$\psi_{\mathbf{a}}, \psi \longleftrightarrow \psi^I, \bar{\psi}_I, \quad A_{\mathbf{a}} \longleftrightarrow C, A^{IJ}. \quad (2.36)$$

All components in (2.35) are real and transform as

$$\begin{aligned} \delta x &= -i(\epsilon_{\mathbf{a}}\psi_{\mathbf{a}} + \epsilon\psi), \\ \delta\psi_{\mathbf{a}} &= \epsilon_{\mathbf{a}}\dot{x} + \epsilon A_{\mathbf{a}} - c_{\mathbf{abc}}\epsilon_{\mathbf{b}}A_{\mathbf{c}}, & \delta\psi &= \epsilon\dot{x} - \epsilon_{\mathbf{a}}A_{\mathbf{a}}, \\ \delta A_{\mathbf{a}} &= i(\epsilon_{\mathbf{a}}\dot{\psi} - \epsilon\dot{\psi}_{\mathbf{a}}) - ic_{\mathbf{abc}}\epsilon_{\mathbf{b}}\dot{\psi}_{\mathbf{c}}. \end{aligned} \quad (2.37)$$

The Lagrangians (2.20) are rewritten as

$$\mathcal{L}_{(\mathbf{1}, \mathbf{8}, \mathbf{7})}^1 = \frac{\dot{x}^2}{2} + \frac{i}{2}(\psi_{\mathbf{a}}\dot{\psi}_{\mathbf{a}} + \psi\dot{\psi}) + \frac{A_{\mathbf{a}}A_{\mathbf{a}}}{2}, \quad (2.38)$$

$$\begin{aligned} \mathcal{L}_{(\mathbf{1}, \mathbf{8}, \mathbf{7})}^2 &= x \left[\frac{\dot{x}^2}{2} + \frac{i}{2}(\psi_{\mathbf{a}}\dot{\psi}_{\mathbf{a}} + \psi\dot{\psi}) + \frac{A_{\mathbf{a}}A_{\mathbf{a}}}{2} \right] \\ &\quad - \frac{i}{2}A_{\mathbf{a}} \left(\psi\psi_{\mathbf{a}} + \frac{c_{\mathbf{abc}}}{2}\psi_{\mathbf{b}}\psi_{\mathbf{c}} \right). \end{aligned} \quad (2.39)$$

They were found at the component level in [13]. Perhaps, the harmonic superspaces associated with the generalized flag manifolds of the groups $\text{O}(7)$ or $\text{Spin}(7)$ [21] could be used to properly rewrite the constraints on the scalar superfield X and to construct the Lagrangians (2.38) and (2.39) in terms of the appropriate analytic superfields. We will not dwell on these issues here.

1. Remark

As was emphasized, the constraints (2.33) are covariant with respect to the symmetry $\text{SO}(7)$. The free Lagrangian (2.38) is exceptional since on shell it possesses the extended symmetry $\text{SO}(8)$. Eliminating the auxiliary field A_a , we obtain the relevant on-shell Lagrangian as

$$\mathcal{L}_{(1,8,7)}^1|_{\text{on-shell}} = \frac{\dot{x}^2}{2} + \frac{i}{2}(\psi_a \dot{\psi}_a + \psi \dot{\psi}). \quad (2.40)$$

It is invariant under the hidden $\text{SO}(8)/\text{SO}(7)$ transformations of the fermionic fields:

$$\delta\psi = \rho_a \psi_a, \quad \delta\psi_a = -\rho_a \psi. \quad (2.41)$$

The field x is a trivial singlet with respect to $\text{SO}(8)$. The Lagrangian (2.40) is invariant under $\text{SO}(8)$, to which the $\text{SO}(7)$ symmetry of the off-shell model is upgraded on shell.

To show exclusivity of the Lagrangian above, let us now consider the free Lagrangian (2.18) of the multiplet $(2, \mathbf{8}, 6)$.

The on-shell Lagrangian reads

$$\mathcal{L}_{(2,8,6)}^1|_{\text{on-shell}} = \dot{\phi} \dot{\bar{\phi}} + \frac{i}{2}(\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I). \quad (2.42)$$

One can show that the fermionic part is invariant under the group $\text{SO}(8)$, while the bosonic part is invariant under its subgroup $\text{SO}(2) \sim \text{U}(1)$. Thus, the on-shell Lagrangian (2.42) possesses the symmetry $\text{SO}(2) \times \text{SO}(6)$ that just corresponds to the full covariance group of the constraints (2.10).

III. INDECOMPOSABLE MULTIPLT: VERSION I

In this and next sections, we present two modified versions of the constraints (2.6). It will be convenient to start with the version studied in [8].

We consider the version³ of $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ described by a complex superfield Z^{aI} ($a = 1, 2$). This superfield satisfies the constraints

$$\begin{aligned} D^{(I} Z^{aJ)} &= 0, & \bar{D}_{(I} \bar{Z}_{J)}^a &= 0, & \overline{(Z^{aI})} &= \bar{Z}_I^a, \\ D^I Z^{aJ} &= \frac{1}{2} \epsilon^{IJKL} \bar{D}_K \bar{Z}_L^a, & D^J \bar{Z}_I^a &= \frac{\delta_I^J}{4} D^K \bar{Z}_K^a, & \bar{D}_J Z^{aI} &= \frac{\delta_J^I}{4} \bar{D}_K Z^{aK}. \end{aligned} \quad (3.1)$$

The index a ($a = 1, 2$) is an external vector index.⁴ The field content $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ is doubled due to this vector index, i.e. the above superfields describe two irreducible multiplets $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. This doubling is necessary for getting the correct Wess-Zumino type term for bosonic components. The θ -expansion of Z^{aI} exhibits component fields as

$$\begin{aligned} Z^{aI}(t, \theta, \bar{\theta}) &= z^{aI} - 2\theta_J \chi^{aIJ} - \sqrt{2} \bar{\theta}^I \chi^a + (\text{terms with higher } \theta \text{ powers}), \\ \overline{(z^{aI})} &= \bar{z}_I^a, & \overline{(\chi^a)} &= \bar{\chi}^a, & \chi^{aIJ} &= -\chi^{aJI} \overline{(\chi^{aIJ})} = \frac{1}{2} \epsilon_{IJKL} \chi^{aKL} = \chi_{IJ}^a. \end{aligned} \quad (3.2)$$

The components transform as

$$\begin{aligned} \delta z^{aI} &= 2\epsilon_J \chi^{aIJ} + \sqrt{2} \bar{\epsilon}^I \chi^a, & \delta \bar{z}_I^a &= -2\bar{\epsilon}^J \chi_{IJ}^a - \sqrt{2} \epsilon_I \bar{\chi}^a, \\ \delta \chi^a &= -\sqrt{2} i \epsilon_I \dot{z}^{aI}, & \delta \bar{\chi}^a &= \sqrt{2} i \bar{\epsilon}^I \dot{\bar{z}}_I^a, & \delta \chi^{aIJ} &= 2i \bar{\epsilon}^{[I} \dot{z}^{aJ]} - i \epsilon^{IJKL} \epsilon_K \dot{\bar{z}}_L^a. \end{aligned} \quad (3.3)$$

Next, we consider a deformed superfield $X_{(\kappa)}$ that contains in its definition Z^{aI} as an invariant subset.

The deformed superfield $X_{(\kappa)}$ is defined by the constraints

$$\begin{aligned} D^I \bar{D}_J X_{(\kappa)} - \frac{\delta_J^I}{4} D^K \bar{D}_K X_{(\kappa)} &= i\kappa \epsilon^{ab} \left(Z^{aI} \bar{Z}_J^b - \frac{\delta_J^I}{4} Z^{aK} \bar{Z}_K^b \right), \\ D^I D^J X_{(\kappa)} - \frac{1}{2} \epsilon^{IJKL} \bar{D}_K \bar{D}_L X_{(\kappa)} &= i\kappa \epsilon^{ab} \left(Z^{a[I} Z^{bJ]} - \frac{1}{2} \epsilon^{IJKL} \bar{Z}_K^a \bar{Z}_L^b \right), \end{aligned} \quad (3.4)$$

³This version of $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ was denoted in [14] as version 2.

⁴The same type of indices was exploited for the semi-dynamical multiplets in [8]. The accompanying $\text{SO}(2)$ antisymmetric tensor satisfies

$$\epsilon^{ab} = -\epsilon^{ba}, \quad \epsilon^{12} = -\epsilon^{21} = 1, \quad \overline{(\epsilon^{ab})} = \epsilon^{ab}, \quad \epsilon^{ab} \epsilon^{cd} = \epsilon^{ac} \epsilon^{bd} - \epsilon^{ad} \epsilon^{bc}.$$

where κ is a deformation parameter. It amounts to a reducible but indecomposable representation of supersymmetry, and has an invariant subset formed by Z^{aI} . The latter appear in the expansion of $X_{(\kappa)}$,

$$\begin{aligned} X_{(\kappa)}(t, \theta, \bar{\theta}) &= X(t, \theta, \bar{\theta}) - \frac{i}{4} \kappa \epsilon^{ab} \left(\theta_I \theta_J - \frac{1}{2} \epsilon_{IJKL} \bar{\theta}^K \bar{\theta}^L \right) \left(z^{aI} z^{bJ} - \frac{1}{2} \epsilon^{IJKL} \bar{z}_K^a \bar{z}_L^b \right) \\ &\quad - i \kappa \epsilon^{ab} \bar{\theta}^J \theta_I \left(z^{aI} \bar{z}_J^b - \frac{\delta_I^J}{4} z^{aK} \bar{z}_K^b \right) + \kappa (\text{terms with higher } \theta \text{ powers}), \end{aligned} \quad (3.5)$$

and imply the following deformed transformations for the **(1, 8, 7)** field set:

$$\begin{aligned} \delta x &= \bar{\epsilon}^I \bar{\psi}_I - \epsilon_I \psi^I, \\ \delta \psi^I &= i \epsilon_J A^{IJ} + \bar{\epsilon}^I (i \dot{x} + C) - i \kappa \epsilon^{ab} \bar{\epsilon}^J \left(z^{aI} \bar{z}_J^b - \frac{\delta_I^J}{4} z^{aK} \bar{z}_K^b \right) + \frac{i}{2} \kappa \epsilon^{ab} \epsilon_J \left(z^{aI} z^{bJ} - \frac{1}{2} \epsilon^{IJKL} \bar{z}_K^a \bar{z}_L^b \right), \\ \delta \bar{\psi}_I &= -i \bar{\epsilon}^J A_{IJ} - \epsilon_I (i \dot{x} - C) - i \kappa \epsilon^{ab} \epsilon_J \left(z^{aI} \bar{z}_I^b - \frac{\delta_I^I}{4} z^{aK} \bar{z}_K^b \right) + \frac{i}{2} \kappa \epsilon^{ab} \bar{\epsilon}^J \left(\frac{1}{2} \epsilon_{IJKL} z^{aK} z^{bL} - \bar{z}_I^a \bar{z}_J^b \right), \\ \delta C &= -i (\bar{\epsilon}^I \dot{\psi}_I + \epsilon_I \dot{\psi}^I) + \frac{3\sqrt{2}}{4} i \kappa \epsilon^{ab} (\bar{\epsilon}^I \chi^a \bar{z}_I^b + \epsilon_I \bar{\chi}^a z^{bI}) + \frac{i}{2} \kappa \epsilon^{ab} (\epsilon_I \chi^{aIJ} \bar{z}_J^b + \bar{\epsilon}^I \chi_{IJ}^a z^{bI}), \\ \delta A^{IJ} &= 4 \bar{\epsilon}^{[I} \dot{\psi}^{J]} - 2 \epsilon^{IJKL} \epsilon_K \dot{\psi}_L - 2 \sqrt{2} \kappa \epsilon^{ab} \bar{\epsilon}^{[I} \bar{\chi}^a z^{bJ]} + \sqrt{2} \kappa \epsilon^{IJKL} \epsilon^{ab} \epsilon_K \chi^a \bar{z}_L^b \\ &\quad - \sqrt{2} \kappa \epsilon^{ab} \bar{\epsilon}^{[I} \chi^a z^{bJ]} + \frac{\sqrt{2}}{2} \kappa \epsilon^{IJKL} \epsilon^{ab} \epsilon_K \bar{\chi}^a \bar{z}_L^b - \kappa \epsilon^{IJKL} \epsilon^{ab} \bar{\epsilon}^M \chi_{KM}^a \bar{z}_L^b + 2 \kappa \epsilon^{ab} \bar{\epsilon}^K \chi^{aIJ} \bar{z}_K^b \\ &\quad + \kappa \epsilon^{ab} \epsilon_K \chi^{aIK} z^{bJ} - \kappa \epsilon^{ab} \epsilon_K \chi^{aJK} z^{bI} - 2 \kappa \epsilon^{ab} \epsilon_K \chi^{aIJ} z^{bK}. \end{aligned} \quad (3.6)$$

These transformations should be complemented by (3.3). The limit $\kappa = 0$ leads to X satisfying (2.6), and the multiplet becomes fully reducible.

Now, our purpose is to construct invariant Lagrangians corresponding to this multiplet. We deform the component Lagrangians (2.20) and require them to be invariant under the transformations (3.3) and (3.6). This “brute force” construction uniquely implies the deformations of both Lagrangians. It turns out that the Lagrangian $\mathcal{L}_{(1,8,7)}^2$ has no invariant deformation. At the same time, the invariant deformation of $\mathcal{L}_{(1,8,7)}^1$ exists and is written as

$$\begin{aligned} \mathcal{L}_{\text{ind. I}} &= \frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{C^2}{2} + \frac{A^{IJ} A_{IJ}}{8} + \frac{\kappa}{8} \epsilon^{ab} (z^{aI} z^{bJ} A_{IJ} + \bar{z}_I^a \bar{z}_J^b A^{IJ}) - \frac{i}{4} \kappa \epsilon^{ab} C z^{aI} \bar{z}_I^b \\ &\quad + i \kappa \epsilon^{ab} \left[\left(\frac{1}{\sqrt{2}} \bar{\chi}^a z^{bI} + \chi^{aIJ} \bar{z}_J^b \right) \bar{\psi}_I - \psi^I \left(\frac{1}{\sqrt{2}} \chi^a \bar{z}_I^b + \chi_{IJ}^a z^{bJ} \right) \right] + i \kappa \epsilon^{ab} x \left(\chi^a \bar{\chi}^b + \frac{1}{2} \chi^{aIJ} \chi_{IJ}^b \right) \\ &\quad + \frac{\kappa}{2} \epsilon^{ab} x (z^{aI} \bar{z}_I^b - z^{aI} \dot{\bar{z}}_I^b) + \frac{3\kappa^2}{32} \epsilon^{ab} \epsilon^{cd} (2 z^{aI} \bar{z}_I^b z^{cJ} \bar{z}_J^d - z^{aI} \bar{z}_I^b z^{cJ} \dot{\bar{z}}_J^d). \end{aligned} \quad (3.7)$$

It precisely coincides with the Lagrangian constructed in terms of $\mathcal{N} = 4$ superfields in Ref. [8] [Eq. (4.29) there]. One observes that the bosonic fields z^{aI} have only the first-order time derivatives, i.e. the pair of multiplets **(8, 8, 0)** are semidynamical.

A. Passing on shell

We eliminate the auxiliary fields by their equations of motion:

$$\begin{aligned} C &= \frac{i}{4} \kappa \epsilon^{ab} z^{aI} \bar{z}_I^b, \quad A^{IJ} = -\frac{\kappa}{2} \epsilon^{ab} \left(z^{aI} z^{bJ} + \frac{1}{2} \epsilon^{IJKL} \bar{z}_K^a \bar{z}_L^b \right), \\ \chi_{IJ}^a &= -\frac{1}{x} \left(\bar{\psi}_{[I} \bar{z}_{J]}^a + \frac{1}{2} \epsilon_{IJKL} \psi^K z^{aL} \right), \quad \chi^a = -\frac{1}{\sqrt{2}x} \bar{\psi}_I z^{aI}, \quad \bar{\chi}^a = -\frac{1}{\sqrt{2}x} \psi^I \bar{z}_I^a. \end{aligned} \quad (3.8)$$

Redefining the semidynamical fields as

$$z^{aI} = \frac{1}{\sqrt{x}} y^{aI}, \quad \bar{z}_I^a = \frac{1}{\sqrt{x}} \bar{y}_I^a, \quad (3.9)$$

we obtain the following on-shell Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{on-shell}} = & \frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \bar{\psi}^I \dot{\psi}_I) + \frac{\kappa}{2} \varepsilon^{ab} (\dot{y}^{aI} \bar{y}_I^b - y^{aI} \dot{\bar{y}}_I^b) \\ & + \frac{i}{4x^2} \kappa \varepsilon^{ab} (\varepsilon_{IJKL} y^{aI} y^{bJ} \psi^K \bar{\psi}^L + \varepsilon^{IJKL} \bar{y}_I^a \bar{y}_J^b \bar{\psi}_K \psi_L) + \frac{i}{2x^2} \kappa \varepsilon^{ab} (2y^{aI} \bar{y}_J^b \psi^J \bar{\psi}_I - y^{aI} \bar{y}_I^b \psi^J \bar{\psi}_J) \\ & + \frac{\kappa^2}{8x^2} \varepsilon^{ab} \varepsilon^{cd} (y^{aI} \bar{y}_J^b y^{cJ} \bar{y}_I^d - y^{aI} y^{bJ} \bar{y}_I^c \bar{y}_J^d). \end{aligned} \quad (3.10)$$

The relevant on-shell transformations are

$$\begin{aligned} \delta x &= \bar{\epsilon}^I \bar{\psi}_I - \epsilon_I \psi^I, \\ \delta \psi^I &= i\bar{\epsilon}^I \dot{x} - i\kappa \varepsilon^{ab} \left[\bar{\epsilon}^J \left(y^{aI} \bar{y}_J^b - \frac{\delta_J^I}{2} y^{aK} \bar{y}_K^b \right) + \frac{1}{2} \varepsilon^{IJKL} \epsilon_J \bar{y}_K^a \bar{y}_L^b \right] \frac{1}{x}, \\ \delta \bar{\psi}_I &= -i\epsilon_I \dot{x} - i\kappa \varepsilon^{ab} \left[\epsilon_J \left(y^{aJ} \bar{y}_I^b - \frac{\delta_I^J}{2} y^{aK} \bar{y}_K^b \right) - \frac{1}{2} \varepsilon_{IJKL} \bar{\epsilon}^J y^{aK} y^{bL} \right] \frac{1}{x}, \\ \delta y^{aI} &= \left[\frac{1}{2} (\bar{\epsilon}^K \bar{\psi}_K + \epsilon_K \psi^K) y^{aI} - \epsilon_J \psi^J y^{aI} - \bar{\epsilon}^I \bar{\psi}_J y^{aJ} - \varepsilon^{IJKL} \epsilon_J \bar{\psi}_K \bar{y}_L^a \right] \frac{1}{x}, \\ \delta \bar{y}_I^a &= -\left[\frac{1}{2} (\bar{\epsilon}^K \bar{\psi}_K + \epsilon_K \psi^K) \bar{y}_I^a - \bar{\epsilon}^J \bar{\psi}_I \bar{y}_J^a - \epsilon_I \psi^J \bar{y}_J^a - \varepsilon_{IJKL} \bar{\epsilon}^J \psi^K y^{aL} \right] \frac{1}{x}. \end{aligned} \quad (3.11)$$

Defining new quantities in the adjoint SO(8) representation,

$$R_J^I = i\kappa \varepsilon^{ab} \left(y^{aI} \bar{y}_J^b - \frac{\delta_J^I}{2} y^{aK} \bar{y}_K^b \right), \quad R_{IJ} = -i\kappa \varepsilon^{ab} \bar{y}_I^a \bar{y}_J^b, \quad \bar{R}^{IJ} = i\kappa \varepsilon^{ab} y^{aI} y^{bJ}, \quad (3.12)$$

we can rewrite the on-shell transformations as

$$\begin{aligned} \delta x &= \bar{\epsilon}^I \bar{\psi}_I - \epsilon_I \psi^I, \\ \delta \psi^I &= i\bar{\epsilon}^I \dot{x} + \frac{1}{x} \left(\frac{1}{2} \varepsilon^{IJKL} \epsilon_J R_{KL} - \bar{\epsilon}^J R_J^I \right), \\ \delta \bar{\psi}_I &= -i\epsilon_I \dot{x} + \frac{1}{x} \left(\frac{1}{2} \varepsilon_{IJKL} \bar{\epsilon}^J \bar{R}^{KL} - \epsilon_J R_I^J \right), \\ \delta R_J^I &= \frac{1}{x} (\epsilon_J \psi^K R_K^I - \epsilon_K \psi^I R_J^K + \bar{\epsilon}^K \bar{\psi}_J R_K^I - \bar{\epsilon}^I \bar{\psi}_K R_J^K) \\ &\quad + \frac{1}{x} \epsilon_K \bar{\psi}_L \left(\varepsilon^{IKLM} R_{MJ} + \frac{\delta_J^I}{2} \varepsilon^{KLMN} R_{MN} \right) + \frac{1}{x} \bar{\epsilon}^K \psi^L \left(\varepsilon_{JKLM} \bar{R}^{IM} - \frac{\delta_J^I}{2} \varepsilon_{KLMN} \bar{R}^{MN} \right), \\ \delta R_{IJ} &= -\frac{2}{x} \left[\frac{1}{2} (\bar{\epsilon}^K \bar{\psi}_K + \epsilon_K \psi^K) R_{IJ} + \bar{\epsilon}^K \bar{\psi}_{[I} R_{J]K} + \epsilon_{[I} \psi^K R_{J]K} - \varepsilon_{KLM[I} \bar{\epsilon}^K \psi^L R_{J]}^M \right], \\ \delta \bar{R}^{IJ} &= \frac{2}{x} \left[\frac{1}{2} (\bar{\epsilon}^K \bar{\psi}_K + \epsilon_K \psi^K) \bar{R}^{IJ} + \epsilon_K \psi^{[I} \bar{R}^{J]K} + \bar{\epsilon}^{[I} \bar{\psi}_K \bar{R}^{J]K} - \varepsilon^{KLM[I} \epsilon_K \bar{\psi}_L R_{M]}^J \right]. \end{aligned} \quad (3.13)$$

Imposing the standard Hamiltonian constraints as

$$p = \dot{x}, \quad p_{y^{aI}} - \frac{\kappa}{2} \varepsilon^{ab} \bar{y}_I^b \approx 0, \quad p_{\bar{y}_I^a} - \frac{\kappa}{2} \varepsilon^{ab} y^{bI} \approx 0, \quad p_{\psi^I} - \frac{i}{2} \bar{\psi}_I \approx 0, \quad p_{\bar{\psi}_I} - \frac{i}{2} \psi^I \approx 0, \quad (3.14)$$

we introduce the Poisson and Dirac brackets:

$$\{x, p\}_{\text{PB}} = 1, \quad \{y^{aI}, \bar{y}_J^b\}_{\text{DB}} = \frac{1}{\kappa} \varepsilon^{ab} \delta_J^I, \quad \{\psi^I, \bar{\psi}_J\}_{\text{DB}} = -i\delta_J^I. \quad (3.15)$$

Performing the canonical quantization, we find that the generators (3.12) form the algebra $so(8)$:

$$\begin{aligned} [R_J^I, R_L^K] &= \delta_J^K R_L^I - \delta_L^K R_J^I, & [\bar{R}^{IJ}, R_{KL}] &= \delta_K^I R_L^J + \delta_L^J R_K^I - \delta_L^K R_J^I - \delta_K^J R_L^I, \\ [R_J^I, \bar{R}^{KL}] &= \delta_J^K \bar{R}^{IL} - \delta_J^L \bar{R}^{IK}, & [R_J^I, R_{KL}] &= \delta_L^K R_{JK} - \delta_K^J R_{JL}. \end{aligned} \quad (3.16)$$

The quantum Hamiltonian reads

$$H = \frac{p^2}{2} - \frac{1}{x^2} \left[R_J^I \psi^J \bar{\psi}_I + \frac{1}{4} \left(\varepsilon_{IJKL} \psi^I \psi^J \bar{R}^{KL} - \varepsilon^{IJKL} \bar{\psi}_I \bar{\psi}_J R_{KL} \right) \right] + \frac{1}{16x^2} \left(2R_J^I R_I^J + R_{IJ} \bar{R}^{IJ} + \bar{R}^{IJ} R_{IJ} \right). \quad (3.17)$$

As was shown in [9], this $SO(8)$ symmetry is enlarged to the superconformal symmetry $OSp(8|2)$. It is worth pointing out that the presence of the superconformal symmetry $OSp(8|2)$, as well as of the R -symmetry $SO(8)$, is a specific feature of the on-shell model (see also discussion in Sec. II C 1).

B. $\mathcal{N} = 4$ superfield construction

Here, we reformulate the modified constraints (3.4) in terms of $\mathcal{N} = 4$ superfields and reproduce the $\mathcal{N} = 4$ construction of the Lagrangian (3.7).

By analogy with the covariant derivatives (2.21), we redefine the complex superfield Z^{aI} as

$$\begin{aligned} \mathbf{B}^{a11} &:= Z^{a1}, & \mathbf{B}^{a22} &:= \bar{Z}_1^a, & \mathbf{B}^{a12} &:= Z^{a2}, & \mathbf{B}^{a21} &:= -\bar{Z}_2^a, & \overline{(\mathbf{B}^{aia})} &= \mathbf{B}_{ia}^a, \\ \mathbf{Z}^{a11} &:= Z^{a3}, & \mathbf{Z}^{a22} &:= \bar{Z}_3^a, & \mathbf{Z}^{a12} &:= Z^{a4}, & \mathbf{Z}^{a21} &:= -\bar{Z}_4^a, & \overline{(\mathbf{Z}^{aia})} &= \mathbf{Z}_{ia}^a. \end{aligned} \quad (3.18)$$

The new superfields satisfy the constraints

$$\begin{aligned} \mathbf{D}_i^{(\alpha} \mathbf{B}_j^{a\beta)} &= 0, & \nabla_A^{(i} \mathbf{B}_\alpha^{aj)} &= 0, & \nabla_i^{(A} \mathbf{Z}_j^{aB)} &= 0, & \mathbf{D}_\alpha^{(i} \mathbf{Z}_A^{aj)} &= 0, \\ \mathbf{D}_\alpha^i \mathbf{B}^{aj\alpha} &= -\nabla_A^i \mathbf{Z}^{ajA}, & \mathbf{D}_i^\alpha \mathbf{Z}^{aiA} &= \nabla_i^\alpha \mathbf{B}^{aia}. \end{aligned} \quad (3.19)$$

The constraints (3.4) are then equivalently rewritten as

$$\begin{aligned} \mathbf{D}^{i(\alpha} \mathbf{D}_i^{b\beta)} X_{(\kappa)} &= i\kappa \varepsilon^{ab} \mathbf{B}^{ai(\alpha} \mathbf{B}_i^{b\beta)}, & \mathbf{D}_\alpha^{(i} \nabla_A^{j)} X_{(\kappa)} &= i\kappa \varepsilon^{ab} \mathbf{B}_\alpha^{a(i} \mathbf{Z}_A^{bj)}, & \nabla^{i(A} \nabla_i^{B)} X_{(\kappa)} &= i\kappa \varepsilon^{ab} \mathbf{Z}^{ai(A} \mathbf{Z}_i^{bB)}, \\ \left(\mathbf{D}_\alpha^{(i} \mathbf{D}_j^{a)} + \nabla_A^{(i} \nabla^{j)A} \right) X_{(\kappa)} &= i\kappa \varepsilon^{ab} \left(\mathbf{B}_\alpha^{a(i} \mathbf{B}^{bj)a} + \mathbf{Z}_A^{a(i} \mathbf{Z}^{bj)A} \right). \end{aligned} \quad (3.20)$$

Defining $\mathcal{N} = 4$ superfields as

$$\mathcal{X}_{(\kappa)} := X_{(\kappa)}|_{\hat{\theta}=0}, \quad \Psi_{(\kappa)}^{iA} := \nabla^{iA} X_{(\kappa)}|_{\hat{\theta}=0}, \quad \mathcal{B}^{aia} := \mathbf{B}^{aia}|_{\hat{\theta}=0}, \quad \mathcal{Z}^{aiA} := \mathbf{Z}^{aiA}|_{\hat{\theta}=0}, \quad (3.21)$$

we obtain the following $\mathcal{N} = 4$ constraints:

$$\mathbf{D}^{i(\alpha} \mathbf{D}_i^{b\beta)} \mathcal{X}_{(\kappa)} = i\kappa \varepsilon^{ab} \mathcal{B}^{ai(\alpha} \mathcal{B}_i^{b\beta)}, \quad \mathbf{D}_\alpha^{(i} \Psi_{(\kappa)}^{j)A} = i\kappa \varepsilon^{ab} \mathcal{B}_\alpha^{a(i} \mathcal{Z}^{bj)A}, \quad \mathbf{D}_i^{(\alpha} \mathcal{B}_j^{a\beta)} = 0, \quad \mathbf{D}_\alpha^{(i} \mathcal{Z}^{aj)A} = 0. \quad (3.22)$$

The implicit supersymmetry is realized on the $\mathcal{N} = 4$ superfields (3.21) by the η^{iA} transformations:

$$\begin{aligned} \delta_\eta \mathcal{X}_{(\kappa)} &= \eta_{iA} \Psi_{(\kappa)}^{iA}, & \delta_\eta \Psi_{(\kappa)}^{iA} &= \frac{1}{2} \eta_j^A \mathbf{D}_\alpha^i \mathbf{D}^{j\alpha} \mathcal{X}_{(\kappa)} + \frac{i}{2} \kappa \varepsilon^{ab} \left(\eta_B^i \mathcal{Z}^{ajA} \mathcal{Z}_j^{bB} - \eta_j^A \mathcal{Z}_B^{ai} \mathcal{Z}^{bjB} - \eta_j^A \mathcal{B}_\alpha^{ai} \mathcal{B}^{bj\alpha} \right), \\ \delta_\eta \mathcal{B}^{aia} &= -\frac{1}{2} \eta^{iA} \mathbf{D}^{j\alpha} \mathcal{Z}_{jA}^a, & \delta_\eta \mathcal{Z}^{aiA} &= \frac{1}{2} \eta_j^A \mathbf{D}_\alpha^i \mathcal{B}^{aj\alpha}. \end{aligned} \quad (3.23)$$

The superfields $\mathcal{X}_{(\kappa)}$ and $\Psi_{(\kappa)}^{iA}$ have a mutual invariant subset $\mathcal{B}^{ai\alpha}$. Note that the harmonization of $\Psi_{(\kappa)}^{iA}$ leads to some nonanalytic superfield $\Psi_{(\kappa)}^{+A}$, and so for the time being we were not able to construct deformations of the analytic superpotentials in (2.26) and (2.27).

The $\mathcal{N} = 8$ invariant action of the indecomposable multiplet was constructed in terms of $\mathcal{N} = 4$ harmonic superfields in [8] by introducing a fermionic superfield $\mathcal{W}^a$ that satisfies the constraints

$$\mathbf{D}^{ia}\mathcal{W}^a = \mathcal{B}^{ai\alpha}, \quad \mathbf{D}^{i(\alpha}\mathbf{D}_i^{\beta)}\mathcal{W}^a = 0, \quad \overline{(\mathcal{W}^a)} = -\mathcal{W}^a. \quad (3.24)$$

Making use of this superfield we redefine the superfields $\mathcal{X}_{(\kappa)}$ and $\Psi_{(\kappa)}^{iA}$ as

$$\mathcal{X}_{(\kappa)} = \mathcal{X} - \frac{i}{2}\kappa\epsilon^{ab}\mathcal{W}^a\mathcal{W}^b, \quad \Psi_{(\kappa)}^{iA} = \Psi^{iA} + i\kappa\epsilon^{ab}\mathcal{W}^a\mathcal{Z}^{biA}, \quad (3.25)$$

and obtain that

$$\begin{aligned} \mathbf{D}^{i(\alpha}\mathbf{D}_i^{\beta)}\mathcal{X} &= 0, & \mathbf{D}_\alpha^{(i}\Psi^{j)A} &= 0, & \delta_\eta\mathcal{X} &= \eta_{iA}\Psi^{iA}, & \delta_\eta\Psi^{iA} &= \frac{1}{2}\eta_j^A\mathbf{D}_\alpha^i\mathbf{D}^{j\alpha}\mathcal{X}, \\ \mathbf{D}^{i(\alpha}\mathbf{D}_i^{\beta)}\mathcal{W}^a &= 0, & \delta_\eta\mathcal{W}^a &= \eta_{iA}\mathcal{Z}^{aiA}, & \mathbf{D}_\alpha^{(i}\mathcal{Z}^{aj)A} &= 0, & \delta_\eta\mathcal{Z}^{aiA} &= \frac{1}{2}\eta_j^A\mathbf{D}_\alpha^i\mathbf{D}^{j\alpha}\mathcal{W}^a. \end{aligned} \quad (3.26)$$

Hence, the $\mathcal{N} = 8$ multiplet (3.22) decomposes into the sum of irreducible multiplets $(\mathcal{X}, \Psi^{iA})$ and $(\mathcal{W}^a, \mathcal{Z}^{aiA})$, while the dependence on κ disappears from constraints but reappears once again in the invariant actions.

The $\mathcal{N} = 8$ invariant action of the indecomposable multiplet was constructed in [8] as a deformation of (2.26):

$$S_{\text{ind. I}} = -\frac{1}{2}\int d\zeta_{(\text{H})}\left(\mathcal{X} - \frac{i}{2}\kappa\epsilon^{ab}\mathcal{W}^a\mathcal{W}^b\right)^2 + \frac{1}{2}\int d\zeta_{(\text{A})}^--\left(\Psi^{+A}\Psi_A^+ + 2i\kappa\epsilon^{ab}\mathcal{W}_{(\text{A})}^a\mathcal{Z}^{b+A}\Psi_A^+ + i\kappa\epsilon^{ab}\mathcal{Z}^{a+A}\mathcal{Z}_A^{b+}\mathcal{X}_{(\text{A})}\right). \quad (3.27)$$

The superfields Ψ^{+A} and $\mathcal{Z}^{a+A}$ are analytic, while the superfields $\mathcal{X}$ and $\mathcal{W}^a$ can be represented by their analytic prepotentials (2.29) and

$$\mathcal{W}^a(\zeta) = \int du\mathcal{W}_{(\text{A})}^a(\zeta_{(\text{A})}), \quad \delta\mathcal{W}_{(\text{A})}^a = \mathbf{D}^{++}\Omega^{a--}(\zeta_{(\text{A})}). \quad (3.28)$$

The relevant component Lagrangian (3.7) was obtained by performing the oxidation procedure [16] that restores the indecomposable multiplet at the component level. The action (2.27) has no similar deformation, since we cannot simultaneously preserve invariance under the implicit symmetry (3.26) and gauge transformations of the prepotentials (2.29) and (3.28).

1. Remark

We can introduce a chiral fermionic superfields Π^a that satisfies

$$\begin{aligned} Z^{aI} &= \frac{1}{\sqrt{2}}D^I\Pi^a, & \bar{Z}_I^a &= -\frac{1}{\sqrt{2}}\bar{D}_I\bar{\Pi}^a, & \overline{(\Pi^a)} &= \bar{\Pi}^a, \\ D^I\bar{\Pi}^a &= 0, & \bar{D}_J\Pi^a &= 0, & D^ID^J\Pi^a + \frac{1}{2}\epsilon^{IJKL}\bar{D}_K\bar{D}_L\bar{\Pi}^a &= 0. \end{aligned} \quad (3.29)$$

We can then redefine the scalar superfield $X_{(\kappa)}$ as

$$X_{(\kappa)} = X - \frac{i}{4}\kappa\epsilon^{ab}(\Pi^a - \bar{\Pi}^a)(\Pi^b - \bar{\Pi}^b), \quad (3.30)$$

where X satisfies (2.6). So, the superfield $X_{(\kappa)}$ satisfying the constraints (3.4) decomposes into the sum of X and Π^a . We can write the $\mathcal{N} = 8$ counterpart of (3.24) by defining a new superfield W^a as

$$W^a := \frac{1}{\sqrt{2}}(\Pi^a - \bar{\Pi}^a), \quad \mathcal{W}^a = W^a|_{\hat{\theta}=0}. \quad (3.31)$$

Recalling Sec. II A 1, we can oxidize the multiplet $(\mathbf{1}, \mathbf{8}, 7)$ to $(\mathbf{2}, \mathbf{8}, 6)$ as

$$X = \frac{1}{\sqrt{2}}(\Phi + \bar{\Phi}). \quad (3.32)$$

Unfortunately, we cannot split $X_{(\kappa)}$ into a sum of chiral and antichiral superfields, which makes it impossible to construct the appropriate deformation of the superfield Lagrangian (2.14).

IV. INDECOMPOSABLE MULTIPLT: VERSION II

Now we consider the second version of an indecomposable multiplet associated with another formulation of the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ [14]. This multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ is described by a chiral superfield Z^a and an antisymmetric tensor superfield V^{aIJ} . They satisfy the constraints

$$\begin{aligned} D^I \bar{Z}^a &= 0, & \bar{D}_J Z^a &= 0, & D^I D^J Z^a - \frac{1}{2} \epsilon^{IJKL} \bar{D}_K \bar{D}_L \bar{Z}^a &= 0, & \overline{(Z^a)} &= \bar{Z}^a, \\ D^{(I} V^{aJ)K} &= 0, & \bar{D}_{(I} V_{J)K}^a &= 0, & V^{aIJ} &= -V^{aJI}, & \overline{(V^{aIJ})} &= V_{IJ}^a = \frac{1}{2} \epsilon_{IJKL} V^{aKL}, \\ \sqrt{2} D^J V_{IJ}^a &= 3 \bar{D}_I \bar{Z}^a, & \sqrt{2} \bar{D}_J V^{aIJ} &= 3 D^I Z^a. \end{aligned} \quad (4.1)$$

The index a plays the same role as in Sec. III. The θ -expansions of both superfields exhibit component fields as

$$\begin{aligned} Z^a(t_L, \theta) &= z^a + \sqrt{2} \theta_I \chi^{aI} + \sqrt{2} i \theta_I \theta_J \dot{v}^{aIJ} + (\text{terms with higher } \theta \text{ powers}), \\ V^{aIJ}(t, \theta, \bar{\theta}) &= v^{aIJ} - \epsilon^{IJKL} \theta_K \bar{\chi}_L^a + 2 \bar{\theta}^{[I} \chi^{aJ]} - \frac{\sqrt{2} i}{2} \epsilon^{IJKL} \theta_K \theta_L \dot{z}^a - \sqrt{2} i \bar{\theta}^I \bar{\theta}^J \dot{z}^a \\ &\quad + 2 i \bar{\theta}^{[I} \theta_K \dot{v}^{aJ]K} - i \epsilon^{IJKL} \bar{\theta}^M \theta_K \dot{v}_{LM}^a + (\text{terms with higher } \theta \text{ powers}), \\ v^{aIJ} &= -v^{aJI}, & \overline{(v^{aIJ})} &= v_{IJ}^a = \frac{1}{2} \epsilon_{IJKL} v^{aKL}, & \overline{(z^a)} &= \bar{z}^a, & \overline{(\chi^{aI})} &= \bar{\chi}_I^a. \end{aligned} \quad (4.2)$$

These components transform according to the rules

$$\begin{aligned} \delta v^{aIJ} &= \epsilon^{IJKL} \epsilon_K \bar{\chi}_L^a - 2 \bar{\epsilon}^{[I} \chi^{aJ]}, & \delta z^a &= -\sqrt{2} \epsilon_I \chi^{aI}, & \delta \bar{z}^a &= \sqrt{2} \bar{\epsilon}^I \bar{\chi}_I^a, \\ \delta \chi^{aI} &= \sqrt{2} i \bar{\epsilon}^I \dot{z}^a - 2 i \epsilon_J \dot{v}^{aIJ}, & \delta \bar{\chi}_I^a &= -\sqrt{2} i \epsilon_I \dot{z}^a + 2 i \bar{\epsilon}^J \dot{v}_{IJ}^a. \end{aligned} \quad (4.3)$$

By analogy with (3.4), we define the deformed constraints,

$$\begin{aligned} D^I \bar{D}_J X_{(\kappa)} - \frac{\delta_J^I}{4} D^K \bar{D}_K X_{(\kappa)} &= i \kappa \epsilon^{ab} V_{JK}^a V^{bIK}, \\ D^I D^J X_{(\kappa)} - \frac{1}{2} \epsilon^{IJKL} \bar{D}_K \bar{D}_L X_{(\kappa)} &= \sqrt{2} i \kappa \epsilon^{ab} (Z^a - \bar{Z}^a) V^{bIJ}. \end{aligned} \quad (4.4)$$

The superfields Z^a and V^{bIJ} form invariant subsets of the indecomposable set $(X_{(\kappa)}, Z, V)$. The solution of (4.4) can be expanded as

$$\begin{aligned} X_{(\kappa)}(t, \theta, \bar{\theta}) &= X(t, \theta, \bar{\theta}) - i \kappa \epsilon^{ab} \bar{\theta}^J \theta_I v_{JK}^a v^{bIK} - \frac{\sqrt{2}}{4} i \kappa \epsilon^{ab} \left(\theta_I \theta_J - \frac{1}{2} \epsilon_{IJKL} \bar{\theta}^K \bar{\theta}^L \right) (z^a - \bar{z}^a) v^{bIJ} \\ &\quad + \kappa (\text{terms with higher } \theta \text{ powers}). \end{aligned} \quad (4.5)$$

The full set of $\mathcal{N} = 8$ supersymmetry transformation properties of the component fields is given by (4.3) and by the following transformations:

$$\begin{aligned}
 \delta x &= \bar{\epsilon}^I \bar{\psi}_I - \epsilon_I \psi^I, \\
 \delta \psi^I &= i\epsilon_J A^{IJ} + \bar{\epsilon}^I (i\dot{x} + C) - i\kappa \epsilon^{ab} \bar{\epsilon}^J v_{JK}^a v^{bIK} + \frac{\sqrt{2}}{2} i\kappa \epsilon^{ab} \epsilon_J (z^a - \bar{z}^a) v^{bIJ}, \\
 \delta \bar{\psi}_I &= -i\bar{\epsilon}^J A_{IJ} - \epsilon_I (i\dot{x} - C) - i\kappa \epsilon^{ab} \epsilon_J v_{IK}^a v^{bJK} + \frac{\sqrt{2}}{2} i\kappa \epsilon^{ab} \bar{\epsilon}^J (z^a - \bar{z}^a) v_{IJ}^b, \\
 \delta C &= -i(\bar{\epsilon}^I \dot{\psi}_I + \epsilon_I \dot{\bar{\psi}}^I) + i\kappa \epsilon^{ab} (\bar{\epsilon}^I \chi^{aJ} v_{IJ}^b + \epsilon_I \bar{\chi}^a_J v^{bIJ}), \\
 \delta A^{IJ} &= 4\bar{\epsilon}^{[I} \dot{\psi}^{J]} - 2\epsilon^{IJKL} \epsilon_K \dot{\bar{\psi}}_L + 2\kappa \epsilon^{IJKL} \epsilon^{ab} \bar{\epsilon}^M \bar{\chi}_K^a v_{LM}^b - 4\kappa \epsilon_K \chi^{a[I} v^{bJ]K} - \kappa \epsilon^{ab} (\epsilon_K \chi^{aK} - \bar{\epsilon}^K \bar{\chi}_K^a) v^{bIJ} \\
 &\quad - \sqrt{2} \kappa \epsilon^{ab} \bar{\epsilon}^{[I} \chi^{aJ]} (z^b - \bar{z}^b) - \frac{\sqrt{2}}{2} \kappa \epsilon^{ab} \epsilon^{IJKL} \epsilon_K \bar{\chi}_L^a (z^b - \bar{z}^b). \tag{4.6}
 \end{aligned}$$

Applying the same method as for the previous indecomposable multiplet, we can construct the component Lagrangian:

$$\begin{aligned}
 \mathcal{L}_{\text{indec. II}} &= \frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{C^2}{2} + \frac{A^{IJ} A_{IJ}}{8} + \frac{\sqrt{2}}{8} \kappa \epsilon^{ab} A^{IJ} (z^a + \bar{z}^a) v_{IJ}^b + \frac{i}{2} \kappa \epsilon^{ab} C z^a \bar{z}^b \\
 &\quad + i\kappa \epsilon^{ab} \left[\frac{\sqrt{2}}{2} (\psi^I \bar{\chi}_I^a z^b + \bar{\psi}_I \chi^{aI} \bar{z}^b) - \psi^I \chi^{aJ} v_{IJ}^b - \bar{\psi}_I \bar{\chi}_J^a v^{bIJ} - x \chi^{aI} \bar{\chi}_I^b \right] + \frac{\kappa}{2} \epsilon^{ab} x (v_{IJ}^a v^{bIJ} + z^a \bar{z}^b - \dot{z}^a \bar{z}^b) \\
 &\quad + \frac{\kappa^2}{16} \epsilon^{ab} \epsilon^{cd} (z^a - \bar{z}^a) v^{bIJ} (z^c - \bar{z}^c) v_{IJ}^d + \frac{\kappa^2}{8} \epsilon^{ab} \epsilon^{cd} v_{JK}^a v^{bIK} v_{IL}^c v^{dJL}. \tag{4.7}
 \end{aligned}$$

This off-shell Lagrangian is not equivalent to (3.7). Below we demonstrate the equivalence of these Lagrangians after elimination of auxiliary fields, i.e. on shell.

A. Passing on shell

We eliminate the auxiliary fields by their equations of motion:

$$\begin{aligned}
 C &= -\frac{i}{2} \kappa \epsilon^{ab} z^a \bar{z}^b, \quad A^{IJ} = -\frac{\sqrt{2}}{2} \kappa \epsilon^{ab} (z^a + \bar{z}^a) v_{IJ}^b, \\
 \chi^{aI} &= -\frac{1}{x} \left(\bar{\psi}_J v^{aIJ} + \frac{\sqrt{2}}{2} \psi^J z^a \right), \quad \bar{\chi}_I^a = -\frac{1}{x} \left(\psi^J v_{IJ}^a + \frac{\sqrt{2}}{2} \bar{\psi}_I \bar{z}^a \right), \tag{4.8}
 \end{aligned}$$

and redefine the semidynamical fields as

$$z^a = \frac{1}{\sqrt{x}} y^a, \quad \bar{z}^a = \frac{1}{\sqrt{x}} \bar{y}^a, \quad v^{aIJ} = \frac{1}{\sqrt{x}} u^{aIJ}. \tag{4.9}$$

The on-shell Lagrangian reads

$$\begin{aligned}
 \mathcal{L}_{\text{on-shell}} &= \frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) + \frac{\kappa}{2} \epsilon^{ab} (u_{IJ}^a \dot{u}^{bIJ} + y^a \dot{\bar{y}}^b - \dot{y}^a \bar{y}^b) + \frac{\sqrt{2}i}{2x^2} \kappa (y^a u_{IJ}^b \psi^I \psi^J + \bar{y}^a u^{bIJ} \bar{\psi}_I \bar{\psi}_J) \\
 &\quad + \frac{i}{x^2} \kappa \left(u_{JK}^a u^{bIK} \psi^J \bar{\psi}_I + \frac{1}{2} y^a \bar{y}^b \psi^I \bar{\psi}_I \right) - \frac{\kappa^2}{8x^2} \epsilon^{ab} \epsilon^{cd} (2y^a u^{bIJ} \bar{y}^c u_{IJ}^d - u_{JK}^a u^{bIK} u_{IL}^c u^{dJL} - y^a \bar{y}^b y^c \bar{y}^d). \tag{4.10}
 \end{aligned}$$

The on-shell transformations are

$$\begin{aligned}
 \delta x &= \bar{\epsilon}^I \bar{\psi}_I - \epsilon_I \psi^I, \\
 \delta \psi^I &= i \bar{\epsilon}^I \dot{x} - i \kappa \epsilon^{ab} \left[\bar{\epsilon}^J \left(u_{JK}^a u^{bIK} + \frac{\delta_J^I}{2} y^a \bar{y}^b \right) + \sqrt{2} \epsilon_J \bar{y}^a u^{bIJ} \right] \frac{1}{x}, \\
 \delta \bar{\psi}_I &= -i \epsilon_I \dot{x} - i \kappa \epsilon^{ab} \left[\epsilon_J \left(u_{IK}^a u^{bJK} + \frac{\delta_J^I}{2} y^a \bar{y}^b \right) - \sqrt{2} \bar{\epsilon}^J y^a u_{IJ}^b \right] \frac{1}{x}, \\
 \delta u^{aIJ} &= \left[\frac{1}{2} (\bar{\epsilon}^K \bar{\psi}_K + \epsilon_K \psi^K) u^{aIJ} + 2 \epsilon_K \psi^{[I} u^{aJ]K} + 2 \bar{\epsilon}^{[I} \bar{\psi}_K u^{aJ]K} + \sqrt{2} \bar{\epsilon}^{[I} \psi^J y^a - \frac{\sqrt{2}}{2} \epsilon^{IJKL} \epsilon_K \bar{\psi}_L \bar{y}^a \right] \frac{1}{x}, \\
 \delta y^a &= \left[\frac{1}{2} (\bar{\epsilon}^I \bar{\psi}_I + \epsilon_I \psi^I) y^a + \sqrt{2} \epsilon_I \bar{\psi}_J u^{aIJ} \right] \frac{1}{x}, \quad \delta \bar{y}^a = - \left[\frac{1}{2} (\bar{\epsilon}^I \bar{\psi}_I + \epsilon_I \psi^I) \bar{y}^a + \sqrt{2} \bar{\epsilon}^I \psi^J u_{IJ}^a \right] \frac{1}{x}.
 \end{aligned} \tag{4.11}$$

The Poisson and Dirac brackets are defined as follows:

$$\{x, p\}_{\text{PB}} = 1, \quad \{u^{aIJ}, u_{KL}^b\}_{\text{DB}} = -\frac{1}{\kappa} \epsilon^{ab} \delta_{[K}^I \delta_{L]}^J, \quad \{y^a, \bar{y}^b\}_{\text{DB}} = -\frac{1}{\kappa} \epsilon^{ab}, \quad \{\psi^I, \bar{\psi}_J\}_{\text{DB}} = -i \delta_J^I. \tag{4.12}$$

Performing the canonical quantization, we define generators satisfying the algebra relations (3.16) as

$$R_J^I = i \kappa \epsilon^{ab} \left(u_{JK}^a u^{bIK} + \frac{\delta_J^I}{2} y^a \bar{y}^b \right), \quad R_{IJ} = -\sqrt{2} i \kappa \epsilon^{ab} \bar{y}^a u_{IJ}^b, \quad \bar{R}^{IJ} = \sqrt{2} i \kappa \epsilon^{ab} y^a u^{bIJ}. \tag{4.13}$$

Finally, we obtain the same Hamiltonian (3.17).

One observes that the on-shell transformations (4.11) are written via the quantities (4.13) in the same form (3.13). This means that the on-shell Lagrangians (3.10) and (4.10) can be uniquely written in terms of the adjoint representation “currents” R_J^I , R_{IJ} , and $\bar{R}^{IJ}$. We guess an ansatz invariant under (3.13) (up to total time derivatives):

$$\begin{aligned}
 \mathcal{L}_{\text{on-shell}} &= \frac{\dot{x}^2}{2} + \frac{i}{2} (\psi^I \dot{\bar{\psi}}_I - \dot{\psi}^I \bar{\psi}_I) - \dot{R}_J^I \mathcal{A}_I^J(R) - \dot{\bar{R}}^{IJ} \bar{\mathcal{A}}_{IJ}(R) - \dot{R}_{IJ} \mathcal{A}^{IJ}(R) \\
 &\quad + \frac{1}{x^2} \left[R_J^I \psi^J \bar{\psi}_I + \frac{1}{4} (\epsilon_{IJKL} \psi^I \psi^J \bar{R}^{KL} - \epsilon^{IJKL} \bar{\psi}_I \bar{\psi}_J R_{KL}) \right] - \frac{1}{8x^2} (R_J^I R_I^J + R_{IJ} \bar{R}^{IJ}),
 \end{aligned} \tag{4.14}$$

where $\mathcal{A}_I^J$, $\bar{\mathcal{A}}_{IJ}$, and $\mathcal{A}^{IJ}$ are some functions in the adjoint representation. Calculating the explicit expressions for these functions is a technically complicated problem which will be tackled elsewhere.

B. $\mathcal{N} = 4$ superfields

In accordance with the notations of Sec. II B, we make the redefinitions:

$$\begin{aligned}
 \mathbf{Q}^{a11} &:= V^{a14}, & \mathbf{Q}^{a22} &:= V^{a23}, & \mathbf{Q}^{a12} &:= V^{a24}, & \mathbf{Q}^{a21} &:= V^{a13}, \\
 \mathbf{V}^{a11} &:= V^{a12}, & \mathbf{V}^{a22} &:= V^{a34}, & \mathbf{V}^{a12} &:= \frac{1}{2\sqrt{2}} (Z^a - \bar{Z}^a), & \mathbf{V}^a &:= \frac{1}{\sqrt{2}} (Z^a + \bar{Z}^a), \\
 \overline{(\mathbf{Q}^{aA\alpha})} &= \mathbf{Q}_{A\alpha}^a, & \mathbf{V}^{aij} &= \mathbf{V}^{aji}, & \overline{(\mathbf{V}^{aij})} &= \mathbf{V}_{ij}^a, & \overline{(\mathbf{V}^a)} &= \mathbf{V}^a.
 \end{aligned} \tag{4.15}$$

Then, the constraints (4.1) are rewritten as

$$\begin{aligned}
 \mathbf{D}_i^{(\alpha} \mathbf{Q}_A^{a\beta)} &= 0, & \nabla_i^{(A} \mathbf{Q}_\alpha^{aB)} &= 0, \\
 \mathbf{D}_\alpha^{(i} \mathbf{V}^{ajk)} &= 0, & \mathbf{D}^{(i} \mathbf{D}_i^{j\beta)} \mathbf{V}^a &= 0, & \mathbf{D}_{j\alpha} \mathbf{V}^{aij} &= -\frac{3}{2} \mathbf{D}_\alpha^i \mathbf{V}^a = \frac{3}{4} \nabla_A^i \mathbf{Q}_\alpha^{aA}, \\
 \nabla_A^{(i} \mathbf{V}^{ajk)} &= 0, & \nabla^{(A} \nabla_i^{B)} \mathbf{V}^a &= 0, & \nabla_{jA} \mathbf{V}^{aij} &= \frac{3}{2} \nabla_A^i \mathbf{V}^a = -\frac{3}{4} \mathbf{D}_\alpha^i \mathbf{Q}_A^{a\alpha},
 \end{aligned} \tag{4.16}$$

and the superfield $X_{(\kappa)}$ satisfies now the constraints

$$\begin{aligned} \mathbf{D}^{i(\alpha}\mathbf{D}_i^{b\beta)}X_{(\kappa)} &= -2i\kappa\epsilon^{ab}\mathbf{Q}^{aA(\alpha}\mathbf{Q}_A^{b\beta)}, & \nabla^{i(A}\nabla_i^{B)}X_{(\kappa)} &= -2i\kappa\epsilon^{ab}\mathbf{Q}^{a(A\alpha}\mathbf{Q}_\alpha^{bB)}, \\ (\mathbf{D}_\alpha^{(i}\mathbf{D}^{j)\alpha} + \nabla_A^{(i}\nabla^{j)A})X_{(\kappa)} &= -4i\kappa\epsilon^{ab}\mathbf{V}_k^{a(i}\mathbf{V}^{bj)k}, & \mathbf{D}_\alpha^{(i}\nabla_A^{j)}X_{(\kappa)} &= -2i\kappa\epsilon^{ab}\mathbf{Q}_\alpha^a\mathbf{V}^{bj}. \end{aligned} \quad (4.17)$$

We pass to $\mathcal{N} = 4$ superfields,

$$\begin{aligned} \mathcal{X}_{(\kappa)} &:= X_{(\kappa)}|_{\hat{\theta}=0}, & \Psi_{(\kappa)}^{iA} &:= \nabla^{iA}X_{(\kappa)}|_{\hat{\theta}=0}, \\ \mathcal{Q}^{aA\alpha} &:= \mathbf{Q}^{aA\alpha}|_{\hat{\theta}=0}, & \mathcal{V}^{aij} &:= \mathbf{V}^{aij}|_{\hat{\theta}=0}, & \mathcal{V}^a &:= \mathbf{V}^a|_{\hat{\theta}=0}, \end{aligned} \quad (4.18)$$

and find the $\mathcal{N} = 4$ form of the constraints

$$\begin{aligned} \mathbf{D}^{i(\alpha}\mathbf{D}_i^{b\beta)}\mathcal{X}_{(\kappa)} &= -2i\kappa\epsilon^{ab}\mathcal{Q}^{aA(\alpha}\mathcal{Q}_A^{b\beta)}, & \mathbf{D}_\alpha^{(i}\Psi_{(\kappa)}^{j)A} &= -2i\kappa\epsilon^{ab}\mathcal{Q}_\alpha^a\mathcal{V}^{bj}, \\ \mathbf{D}_i^{(\alpha}\mathcal{Q}_A^{a\beta)} &= 0, & \mathbf{D}_\alpha^{(i}\mathcal{V}^{ajk)} &= 0, & \mathbf{D}^{i(\alpha}\mathbf{D}_i^{b\beta)}\mathcal{V}^a &= 0, & \mathbf{D}_{j\alpha}\mathcal{V}^{aij} &= -\frac{3}{2}\mathbf{D}_\alpha^i\mathcal{V}^a. \end{aligned} \quad (4.19)$$

The implicit $\mathcal{N} = 4$ supersymmetry transformations look like

$$\begin{aligned} \delta_\eta\mathcal{X}_{(\kappa)} &= \eta_{iA}\Psi_{(\kappa)}^{iA}, & \delta_\eta\Psi_{(\kappa)}^{iA} &= \frac{1}{2}\eta_j^A\mathbf{D}_\alpha^i\mathbf{D}^{j\alpha}\mathcal{X}_{(\kappa)} - i\kappa\epsilon^{ab}(\eta_B^i\mathcal{Q}^{aA\alpha}\mathcal{Q}_\alpha^{bB} + 2\eta_j^A\mathcal{V}^{aik}\mathcal{V}_k^{bj}), \\ \delta_\eta\mathcal{Q}^{aA\alpha} &= -\frac{2}{3}\eta^{iA}\mathbf{D}^{j\alpha}\mathcal{V}_{ij}^a, & \delta_\eta\mathcal{V}^{aij} &= \frac{1}{2}\eta_A^{(i}\mathbf{D}^{j)A}\mathcal{Q}^{aA\alpha}, & \delta_\eta\mathcal{V}^a &= -\frac{1}{2}\eta_{iA}\mathbf{D}_\alpha^i\mathcal{Q}^{aA\alpha}. \end{aligned} \quad (4.20)$$

Recalling the action (3.27), we suspect that there should exist $\mathcal{N} = 4$ superfield construction of the component action (4.7).

V. CONCLUSIONS

In this paper we have defined new indecomposable multiplets of $\mathcal{N} = 8$ mechanics by deforming the constraints (2.6) of the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$. There are two nonlinear deformations (3.4) and (4.4) related to different versions of the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$.⁵ In both cases we have constructed the invariant off-shell Lagrangians (3.7) and (4.7), and shown that the Lagrangian (3.7) exactly coincides with the one constructed in [8]. We have shown that the relevant on-shell models are equivalently described by the Lagrangians (3.13) and (4.14), where the spin variables enter through the functions in the adjoint representation of $\text{SO}(8)$.

Among the problems for the future study we mention the manifestly $\mathcal{N} = 4$ supersymmetric superfield construction of various off-shell component actions presented in this paper (perhaps, using the appropriate harmonic formalism) and the proof of possible duality between the two models at the full superfield level. It would be also interesting to consider $\mathcal{N} = 4$ analog of the $\mathcal{N} = 8$ indecomposable multiplets as a nonlinear deformation of $\mathcal{N} = 4$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ by two spin $\mathcal{N} = 4$ multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$.

⁵These deformations are the first known examples of the nonlinear $\mathcal{N} = 8$, $d = 1$ superfield constraints.

The original work [1] was formulated in terms of matrix superfields and aimed to derive $\mathcal{N} = 4$ supersymmetric spin Calogero models by gauging [18,22]. An intriguing question is whether we can generalize the constraints (3.4) and (4.4) to matrix superfields and derive from them $\mathcal{N} = 8$ supersymmetric spin Calogero models.

ACKNOWLEDGMENTS

The authors thank Sergey Fedoruk for useful discussions and comments.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: BASICS OF $\mathcal{N} = 4$, $d = 1$ HARMONIC SUPERSPACE

The $\mathcal{N} = 4$, $d = 1$ superspace $\zeta = \{t, \theta^{i\alpha}\}$ is defined by standard transformation properties:

$$\begin{aligned} \delta\theta^{i\alpha} &= \epsilon^{i\alpha}, & \delta t &= -ie^{i\alpha}\theta_{i\alpha}, \\ \overline{(\theta^{i\alpha})} &= -\theta_{i\alpha}, & \overline{(\epsilon^{i\alpha})} &= -\epsilon_{i\alpha}. \end{aligned} \quad (A1)$$

We extend it by the harmonic variables $u_i^\pm$:

$$u^{+i}u_j^- - u^{-i}u_j^+ = \delta_j^i. \quad (A2)$$

Defining the new coordinates as

$$t_{(A)} = t - i\theta_\alpha^{(i}\theta^{j)\alpha}u_i^+u_j^-, \quad \theta^{\pm\alpha} = \theta^{i\alpha}u_i^\pm, \quad (\text{A3})$$

we obtain the harmonic superspace $\zeta_{(H)} = \{t_{(A)}, \theta^{\pm\alpha}, u_i^\pm\}$. It has an analytic harmonic subspace $\zeta_{(A)} = \{t_{(A)}, \theta^{+\alpha}, u_i^\pm\}$, which is closed under the supersymmetry transformations:

$$\begin{aligned} \delta\theta^{+\alpha} &= \epsilon^{+\alpha}, & \delta t_{(A)} &= 2i\epsilon^{-\alpha}\theta_\alpha^+, \\ \delta u_i^\pm &= 0, & \epsilon^{\pm\alpha} &:= \epsilon^{i\alpha}u_i^\pm. \end{aligned} \quad (\text{A4})$$

The covariant derivatives are

$$\begin{aligned} \mathbf{D}^{+\alpha} &= \frac{\partial}{\partial\theta_\alpha^-}, & \mathbf{D}^{-\alpha} &= -\frac{\partial}{\partial\theta_\alpha^+} + 2i\theta_\alpha^{-\alpha}\frac{\partial}{\partial t_{(A)}}, \\ \mathbf{D}^{++} &= u_i^+\frac{\partial}{\partial u_i^-} + i\theta^{+\alpha}\theta_\alpha^+\frac{\partial}{\partial t_{(A)}} + \theta_\alpha^+\frac{\partial}{\partial\theta_\alpha^-}, \\ \mathbf{D}^{--} &= u_i^-\frac{\partial}{\partial u_i^+} + i\theta^{-\alpha}\theta_\alpha^-\frac{\partial}{\partial t_{(A)}} + \theta_\alpha^-\frac{\partial}{\partial\theta_\alpha^+}, \\ \mathbf{D}^0 &= u_i^+\frac{\partial}{\partial u_i^+} - u_i^-\frac{\partial}{\partial u_i^-} + \theta_\alpha^+\frac{\partial}{\partial\theta_\alpha^+} - \theta_\alpha^-\frac{\partial}{\partial\theta_\alpha^-}. \end{aligned} \quad (\text{A5})$$

The integration measures $d\zeta_{(H)}$ and $d\zeta_{(A)}$ over the full and analytic $\mathcal{N} = 4$ harmonic superspaces are defined as

$$\begin{aligned} d\zeta_{(H)} &= \frac{1}{4} du dt_{(A)} \mathbf{D}_\alpha^- \mathbf{D}^{-\alpha} \mathbf{D}_\beta^+ \mathbf{D}^{+\beta}, \\ d\zeta_{(A)} &= \frac{1}{2} du dt_{(A)} \mathbf{D}_\alpha^- \mathbf{D}^{-\alpha}. \end{aligned} \quad (\text{A6})$$

Unconstrained harmonic superfields have an infinite number of components in their harmonic expansion. In order to gain the finite number of fields one needs to impose harmonic constraint $\mathbf{D}^{++}q = 0$ together with the Grassmann-analyticity condition:

$$\mathbf{D}^{+\alpha}q(\zeta_{(A)}) = 0. \quad (\text{A7})$$

More details can be found in Ref. [17].

APPENDIX B: OCTONIONIC STRUCTURE CONSTANTS

The third-rank antisymmetric tensor c_{abc} is also referred to as the octonionic tensor (or the octonionic structure constants) [13,19,23,24]. It has the following nonzero components:

$$c_{123} = c_{147} = c_{165} = c_{246} = c_{257} = c_{354} = c_{367} = 1. \quad (\text{B1})$$

The fourth-rank tensor c^{abcd} is defined as

$$\begin{aligned} c_{abcd} &= \frac{1}{6} \epsilon_{abcdefg} c_{efg}, & \epsilon_{abcdefg} c_{defg} &= 24c_{abc}, \\ \epsilon_{1234567} &= 1, & c_{4567} &= c_{2356} = c_{2437} = c_{1357} = c_{1346} = c_{1276} = c_{1245} = 1. \end{aligned} \quad (\text{B2})$$

We list a few useful identities for these tensors:

$$\begin{aligned} c_{abc}c_{dec} &= 2\delta_{a[d}\delta_{e]b} - c_{abde}, & c_{abcd}c_{efcd} &= 8\delta_{a[e}\delta_{f]b} - 2c_{abef}, & c_{abc}c_{dbc} &= 6\delta_{ad}, \\ c_{abcd}c_{ebcd} &= 24\delta_{ae}, & c_{abcd}c_{efcd} &= 2(\delta_{a[e}\delta_{f]bc} + \delta_{b[e}\delta_{f]ca} + \delta_{c[e}\delta_{f]ab}), & c_{abcd}c_{ecd} &= -4c_{abe}. \end{aligned} \quad (\text{B3})$$

-
- [1] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Supersymmetric Calogero models by gauging, *Phys. Rev. D* **79**, 105015 (2009).
 - [2] S. Sidorov, Dual superfield approach to supersymmetric mechanics with spin variables, *J. Phys. A* **58**, 025210 (2025).
 - [3] M. Gonzales, S. Khodae, and F. Toppan, On non-minimal $\mathcal{N} = 4$ supermultiplets in 1D and their associated sigma-models, *J. Math. Phys. (N.Y.)* **52**, 013514 (2011).
 - [4] M. Gonzales, K. Iga, S. Khodae, and F. Toppan, Pure and entangled $\mathcal{N} = 4$ linear supermultiplets and their one-dimensional sigma-models, *J. Math. Phys. (N.Y.)* **53**, 103513 (2012).
 - [5] T. Hübsch and G. Katona, A Q-continuum of off-shell supermultiplets, *Adv. High Energy Phys.* **2016**, 7350892 (2016).
 - [6] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, and D. Martelli, The Casimir energy in curved space and its supersymmetric counterpart, *J. High Energy Phys.* **07** (2015) 043.
 - [7] E. Ivanov and S. Sidorov, Long multiplets in supersymmetric mechanics, *Phys. Rev. D* **93**, 065052 (2016).
 - [8] S. Fedoruk and E. Ivanov, $\mathcal{N} = 8$ invariant interaction of dynamical and semi-dynamical $\mathcal{N} = 4$ multiplets, *Phys. Rev. D* **109**, 085007 (2024).

- [9] E. Khastyan, S. Krivonos, and A. Nersessian, Note on $\mathcal{N} = 8$ supersymmetric mechanics with dynamical and semi-dynamical multiplets, *Int. J. Mod. Phys. A* **40**, 2450165 (2025).
- [10] S. Bellucci, E. Ivanov, S. Krivonos, and O. Lechtenfeld, ABC of $N = 8$, $d = 1$ supermultiplets, *Nucl. Phys.* **B699**, 226 (2004).
- [11] F. Delduc and E. Ivanov, New model of $\mathcal{N} = 8$ superconformal mechanics, *Phys. Lett. B* **654**, 200 (2007).
- [12] S. Fedoruk and E. Ivanov, Multiparticle $\mathcal{N} = 8$ mechanics with $F(4)$ superconformal symmetry, *Nucl. Phys.* **B938**, 714 (2019).
- [13] Z. Kuznetsova, M. Rojas, and F. Toppan, Classification of irreps and invariants of the N -extended supersymmetric quantum mechanics, *J. High Energy Phys.* 03 (2006) 098.
- [14] E. Ivanov, O. Lechtenfeld, and S. Sidorov, Deformed $\mathcal{N} = 8$ mechanics of $(8, 8, 0)$ multiplets, *J. High Energy Phys.* 08 (2018) 193.
- [15] E. Ivanov, O. Lechtenfeld, and S. Sidorov, Deformed $N = 8$ supersymmetric mechanics, *Symmetry* **11**, 135 (2019).
- [16] S.J. Gates, Jr. and L. Rana, Ultramultiplets: A new representation of rigid 2D, $N = 8$ supersymmetry, *Phys. Lett. B* **342**, 132 (1995).
- [17] E. Ivanov and O. Lechtenfeld, $N = 4$ supersymmetric mechanics in harmonic superspace, *J. High Energy Phys.* 09 (2003) 073.
- [18] F. Delduc and E. Ivanov, Gauging $\mathcal{N} = 4$ supersymmetric mechanics II: $(1, 4, 3)$ models from the $(4, 4, 0)$ ones, *Nucl. Phys.* **B770**, 179 (2007).
- [19] S. Krivonos and A. Nersessian, Two faces of $\mathcal{N} = 7$, 8 superconformal mechanics, *Phys. Rev. D* **111**, 125025 (2025).
- [20] V. S. Varadarajan, Spin(7)-subgroups of SO(8) and Spin(8), *Exp. Math.* **19**, 163 (2001).
- [21] P. S. Howe and M. I. Leeming, Harmonic superspaces in low dimensions, *Classical Quantum Gravity* **11**, 2843 (1994).
- [22] F. Delduc and E. Ivanov, Gauging $\mathcal{N} = 4$ supersymmetric mechanics, *Nucl. Phys.* **B753**, 211 (2006).
- [23] N. Aizawa, Z. Kuznetsova, and F. Toppan, The quasi-nonassociative exceptional $F(4)$ deformed quantum oscillator, *J. Math. Phys. (N.Y.)* **59**, 022101 (2018).
- [24] F. Toppan, The octonionically-induced $\mathcal{N} = 7$ exceptional $G(3)$ superconformal quantum mechanics, [arXiv:1912.05596](#).