

Exact parametrization of a minimal seesaw model

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We propose a parametrization of neutrino masses and mixing in the minimal seesaw model (MSM). The MSM, which introduces two heavy sterile neutrinos, is the minimal extension of the Standard Model in addressing the tiny masses of active neutrinos. The parametrization includes 11 free parameters: six neutrino oscillation parameters (two mass-squared differences Δm_{21}^2 , Δm_{31}^2 , three mixing angles θ_{12} , θ_{13} , θ_{23} , and one Dirac phase δ_{CP}), one mass parameter in $0\nu 2\beta$ decay m_{ee} , and four additional parameters: two heavy neutrino masses M_1 and M_2 , one active-sterile mixing angle θ_{14} , and one CP -violating phase δ_{14} . This parametrization is derived exactly from the most general neutrino mass matrix in the MSM without any approximation. We further discuss its implications in phenomenological studies.

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I. INTRODUCTION

Neutrino masses and mixing, as proven in neutrino oscillation experiments, are definitely new physics beyond the Standard Model (SM) [1]. A straightforward way to understand these tiny masses is provided by the canonical seesaw mechanism [2–6]: the smallness of light neutrino masses is suppressed by large masses of heavy mediators beyond the SM. We take the mechanism with n copies of right-handed (RH) sterile neutrinos N_R into account. Relevant Lagrangian terms for neutrino masses are listed to be

$$-\mathcal{L}_\nu = \overline{\ell}_L Y_\nu \tilde{H} N_R + \frac{1}{2} \overline{(N_R)^c} M_R N_R + \text{H.c.}, \quad (1)$$

where ℓ_L denotes the three copies of SM lepton doublets, $\tilde{H} = i\sigma_2 H^*$ with the SM Higgs doublet H , Y_ν is a $3 \times n$ Yukawa coupling matrix, and M_R is a $n \times n$ Majorana mass matrix for n copies of RH neutrinos. After the electroweak symmetry breaking, a $(3+n) \times (3+n)$ Majorana mass matrix is obtained in the basis $((\nu_L)^c, N_R)$,

$$\mathcal{M} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, \quad (2)$$

where $M_D = Y_\nu \langle H \rangle$. This matrix is diagonalized via a unitary matrix $\mathcal{U}$ as

$$\mathcal{M} = \mathcal{U} \begin{pmatrix} D_\nu & 0 \\ 0 & D_N \end{pmatrix} \mathcal{U}^T, \quad \mathcal{U} = \begin{pmatrix} U & R \\ \tilde{S} & V \end{pmatrix}. \quad (3)$$

The first block of $\mathcal{M}$ gives the “exact seesaw formula” [7],

$$UD_\nu U^T = -RD_N R^T. \quad (4)$$

In the limit of large hierarchy between the Dirac mass matrix and Majorana mass matrix $M_D \ll M_R$, it is convenient to use the approximations $R \approx M_D M_R^{-1} V$ and $D_N \approx V^{-1} M_R (V^{-1})^T$. Then we arrive at the well-known “approximate seesaw formula,”

$$M_\nu \equiv UD_\nu U^T \approx -M_D M_R^{-1} M_D^T. \quad (5)$$

Although it is approximate, it provides a simple explanation to connect the tiny active neutrino masses with the heavy particles: While the M_D cannot be higher than the electroweak scale, the heavier Majorana masses for RH neutrinos lead to the smaller Majorana masses for light neutrinos.

Adding RH neutrinos not only constitutes the simplest possibility for the origin of neutrino masses and mixing, but also supplies sources of peculiar signatures of experiments in particle physics, such as charged lepton flavor violations (cLFVs) [8–11] and collider searches for heavy neutral

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leptons [12,13]. On the other hand, the Majorana nature of RH neutrinos and the potential CP -violating source in the Dirac Yukawa interactions provide a well-known mechanism, baryogenesis via leptogenesis [14], to explain the matter-antimatter asymmetry as observed in our Universe [1,15]. The approximate seesaw formula provides a clear vision on these phenomenological studies.

The flavor space of light and heavy neutrinos involves a lot of independent parameters and a parametrization is necessary in phenomenological studies. A widely used method is the Casas-Ibarra parametrization [16] based on the approximate seesaw formula. It is originally introduced in parametrizing the eLFV processes [16] and then turns out to be very useful in the study of leptogenesis [17]. In the minimal case, only two RH neutrinos are necessary to be introduced [18–20]. This so-called the minimal seesaw model (MSM) provides the most economical and renormalizable framework to explain two nonzero neutrino mass-squared differences and the lepton flavor mixing observed in neutrino oscillation experiments. The MSM can effectively reduce free parameters in the flavor space. Several parametrizations specifically for the MSM have been proposed and were summarized in an early review [21]. For example, the natural reconstruction parametrization [22], derived also from the approximate seesaw formula, provides a direct insight into the Dirac mass matrix in terms of the light neutrino mass matrix.

In this work, we will propose a new parametrization in the MSM framework. It is derived directly from the exact seesaw formula in Eq. (4) without any approximation. In the rest of this paper, we show the derivation of the exact parametrization of the MSM in Sec. II. The comparisons with Casas-Ibarra, natural reconstruction, and other parametrizations are given in Sec. III. The implications of this parametrization in phenomenological studies including experimental searches and leptogenesis are discussed in Sec. IV. We conclude in Sec. V.

II. THE EXACT PARAMETRIZATION OF THE MINIMAL SEESAW MODEL

The MSM introduces two RH sterile neutrinos $(N_R)_i$ ($i = 1, 2$). The mass terms in the Lagrangian are expressed in the $\nu - N_R$ basis,

$$-\mathcal{L}_{\text{MSM}} \supset \frac{1}{2} \overline{\begin{pmatrix} \nu_L & (N_R)^c \end{pmatrix}} \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} (\nu_L)^c \\ N_R \end{pmatrix} + \text{H.c.}, \quad (6)$$

where M_R is a 2×2 Majorana mass matrix of RH neutrinos, M_D is a 3×2 Dirac mass matrix after the Higgs doublet gets the vacuum expectation value $\langle H \rangle = 246/\sqrt{2}$ GeV, and Y_ν is a Dirac Yukawa coupling. The full 5×5 neutrino matrix composed of M_D and M_R is zero determinate. Thus, the lightest neutrino mass must be zero.

So far, the mass hierarchy of three light neutrinos has not been determined. There are two options: in the normal hierarchy (NH), we have $0 = m_1 < m_2 < m_3 \ll M_1 \leq M_2$ in the MSM and in the inverted hierarchy (IH), it is $0 = m_3 < m_1 < m_2 \ll M_1 \leq M_2$, where we have denoted the indices for the two heavy neutrinos following their mass ordering. The well-known seesaw formula for light neutrino masses in Eq. (5) is obtained via first-order approximation under the condition $M_D \ll M_R$. In this work, we will abandon this approximation and adopt the exact seesaw formula in Eq. (4). This exact seesaw relation has already been suggested in the study of neutrinoless double β ($0\nu 2\beta$) decay analysis [23].

With only two RH neutrinos under this consideration, the full 5×5 neutrino mass matrix $\mathcal{M}$ can be orthogonally diagonalized by a 5×5 unitary matrix $\mathcal{U}$,

$$\begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix} = \mathcal{U} \begin{pmatrix} D_\nu & 0 \\ 0 & D_N \end{pmatrix} \mathcal{U}^T, \quad (7)$$

where $D_\nu = \text{diag}\{m_1, m_2, m_3\}$ and $D_N = \text{diag}\{M_1, M_2\}$ are light and heavy neutrino masses, respectively. Here, $\mathcal{U}$ can be written as [7,24]

$$\mathcal{U} = \begin{pmatrix} I & 0 \\ 0 & U'_0 \end{pmatrix} \begin{pmatrix} A & R \\ S & B \end{pmatrix} \begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix}, \quad (8)$$

where U_0 and U'_0 are 3×3 and 2×2 unitary matrices, respectively. A , R , S , and B are 3×3 , 3×2 , 2×3 , and 2×2 matrices, respectively. Because of the unitarity property of $\mathcal{U}$, they satisfy

$$\begin{aligned} AA^\dagger + RR^\dagger &= BB^\dagger + SS^\dagger = I, \\ AS^\dagger + RB^\dagger &= A^\dagger R + S^\dagger B = 0, \\ A^\dagger A + S^\dagger S &= B^\dagger B + R^\dagger R = I. \end{aligned} \quad (9)$$

Comparing Eq. (8) with Eq. (3), we have correlations $U = AU_0$, $\tilde{S} = U'_0 S U_0$, and $V = U'_0 B$. Meanwhile, the unitary matrix $\mathcal{U}$ can be expressed by the Euler-like parametrization with ten two-dimensional unitary matrices O_{ij} ($1 \leq i < j \leq 5$),

$$\begin{aligned} \begin{pmatrix} U_0 & 0 \\ 0 & I \end{pmatrix} &= O_{23} O_{13} O_{12}, & \begin{pmatrix} I & 0 \\ 0 & U'_0 \end{pmatrix} &= O_{45}, \\ \begin{pmatrix} A & R \\ S & B \end{pmatrix} &= O_{35} O_{25} O_{15} O_{34} O_{24} O_{14}. \end{aligned} \quad (10)$$

Here, O_{ij} consists of $c_{ij} \equiv \cos \theta_{ij}$ and $\hat{s}_{ij} \equiv e^{i\delta_{ij}} \sin \theta_{ij}$. θ_{ij} is the rotation angle in the (i, j) plane. θ_{12} , θ_{13} , and θ_{23} denote the active-active mixing; θ_{14} , θ_{15} , θ_{24} , θ_{25} , θ_{34} , and θ_{35} represent the active-sterile mixing; and θ_{45} is the sterile-sterile mixing angle. δ_{ij} is the corresponding complex phase along with θ_{ij} .

The exact expression of U_0 is

$$U_0 = \begin{pmatrix} c_{12}c_{13} & \hat{s}_{12}^*c_{13} & \hat{s}_{13}^* \\ -\hat{s}_{12}c_{23} - c_{12}\hat{s}_{13}\hat{s}_{23}^* & c_{12}c_{23} - \hat{s}_{12}^*\hat{s}_{13}\hat{s}_{23}^* & c_{13}\hat{s}_{23}^* \\ \hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23} & -c_{12}\hat{s}_{23} - \hat{s}_{12}^*\hat{s}_{13}c_{23} & c_{13}c_{23} \end{pmatrix}. \quad (11)$$

It is emphasized that this U_0 is different from the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. In the standard parametrization, the PMNS matrix V_0 is

$$V_0 = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (12)$$

where δ_{CP} is the Dirac CP -violating phase and ρ and σ are two Majorana phases. In the MSM, the phase ρ is unphysical and taken as $\rho = 0$. We can correlate U_0 in the Euler-like parametrization to V_0 in the standard parametrization, through an orthogonal transformation of the light neutrino mass basis. Hence, the relations between δ_{ij} of U_0 and $\delta_{CP}, \rho, \sigma$ of V_0 are $\delta_{CP} = \delta_{13} - \delta_{12} - \delta_{23}$, $\rho = \delta_{12} + \delta_{23} = 0$, and $\sigma = -\delta_{12}$. The exact expressions of A , R , S , and B are

$$A = \begin{pmatrix} c_{14}c_{15} & 0 & 0 \\ -c_{14}\hat{s}_{15}\hat{s}_{25}^* - \hat{s}_{14}\hat{s}_{24}^*c_{25} & c_{24}c_{25} & 0 \\ -c_{14}\hat{s}_{15}c_{25}\hat{s}_{35}^* - \hat{s}_{14}c_{24}\hat{s}_{34}^*c_{35} + \hat{s}_{14}\hat{s}_{24}^*\hat{s}_{25}\hat{s}_{35}^* & -c_{24}\hat{s}_{25}\hat{s}_{35}^* - \hat{s}_{24}\hat{s}_{34}^*c_{35} & c_{34}c_{35} \end{pmatrix}, \quad (13)$$

$$R = \begin{pmatrix} \hat{s}_{14}^*c_{15} & \hat{s}_{15}^* \\ -\hat{s}_{14}^*\hat{s}_{15}\hat{s}_{25}^* + c_{14}\hat{s}_{24}^*c_{25} & c_{15}\hat{s}_{25}^* \\ -\hat{s}_{14}^*\hat{s}_{15}c_{25}\hat{s}_{35}^* - c_{14}\hat{s}_{24}^*\hat{s}_{25}\hat{s}_{35}^* + c_{14}c_{24}\hat{s}_{34}^*c_{35} & c_{15}c_{25}\hat{s}_{35}^* \end{pmatrix}, \quad (14)$$

$$S = \begin{pmatrix} -\hat{s}_{14}c_{24}c_{34} & -\hat{s}_{24}c_{34} & -\hat{s}_{34} \\ \hat{s}_{14}c_{24}\hat{s}_{34}^*\hat{s}_{35} + \hat{s}_{14}\hat{s}_{24}^*\hat{s}_{25}c_{35} - c_{14}\hat{s}_{15}c_{25}c_{35} & \hat{s}_{24}\hat{s}_{34}^*\hat{s}_{35} - c_{24}\hat{s}_{25}c_{35} & -c_{34}\hat{s}_{35} \end{pmatrix}, \quad (15)$$

$$B = \begin{pmatrix} c_{14}c_{24}c_{34} & 0 \\ -c_{14}c_{24}\hat{s}_{34}^*\hat{s}_{35} - c_{14}\hat{s}_{24}^*\hat{s}_{25}c_{35} - \hat{s}_{14}^*\hat{s}_{15}c_{25}c_{35} & c_{15}c_{25}c_{35} \end{pmatrix}, \quad (16)$$

where A is a nonunitarity matrix. By setting N_R in the mass eigenstates, i.e., $D_N = M_R$, we can fix $\theta_{45} = 0$ and $U'_0 = I$.

For convenience, one usually adopts $U = AU_0$ in Eq. (3) to describe the flavor mixing of three light neutrinos, and the effective mass matrix of light neutrinos is $M_\nu = UD_\nu U^T$ in the neutrino flavor basis. For the following calculations, we take $\tilde{R} \equiv A^{-1}R$ as the active-sterile mixing matrix of neutrinos,

$$\tilde{R} = \begin{pmatrix} \frac{\hat{s}_{14}^*}{c_{14}} & \frac{\hat{s}_{15}^*}{c_{14}c_{15}} \\ \frac{\hat{s}_{24}^*}{c_{14}c_{24}} & \frac{\hat{s}_{14}\hat{s}_{15}^*\hat{s}_{24}^*}{c_{14}c_{15}c_{24}} + \frac{\hat{s}_{25}^*}{c_{15}c_{24}c_{25}} \\ \frac{\hat{s}_{34}^*}{c_{14}c_{24}c_{34}} & \frac{\hat{s}_{14}\hat{s}_{15}^*\hat{s}_{34}^*}{c_{14}c_{15}c_{24}c_{34}} + \frac{\hat{s}_{24}\hat{s}_{25}^*\hat{s}_{34}^*}{c_{15}c_{24}c_{25}c_{34}} + \frac{\hat{s}_{35}^*}{c_{15}c_{25}c_{34}c_{35}} \end{pmatrix}, \quad (17)$$

where c_{ik} and $\hat{s}_{ik}$ are active-sterile mixing angles defined in Eq. (10). Actually, six active-sterile mixing angles and six phase angles are not independent, so we will solve Eq. (7) to find the correlation between θ_{ik} and δ_{ik} in the following. From the upper left 3×3 zero matrix in Eq. (7), there is $UD_\nu U^T = -RD_N R^T$. We define a new effective mass matrix of light neutrinos as

$$m_{ij} \equiv (U_0 D_\nu U_0^T)_{ij} = -(\tilde{R} D_N \tilde{R}^T)_{ij}, \quad (18)$$

where $m_{ij} = (A^{-1}M_\nu(A^{-1})^T)_{ij}$. From (18), there are five independent equations,

$$\begin{aligned}(\tilde{R}D_N\tilde{R}^T)_{11} &= -m_{11}, \\(\tilde{R}D_N\tilde{R}^T)_{22} &= -m_{22}, \\(\tilde{R}D_N\tilde{R}^T)_{33} &= -m_{33}, \\w_1\tilde{R}_{11} + w_2\tilde{R}_{21} + w_3\tilde{R}_{31} &= 0, \\w_1\tilde{R}_{12} + w_2\tilde{R}_{22} + w_3\tilde{R}_{32} &= 0,\end{aligned}\quad (19)$$

where $w_i = (U_0^*)_{i1}$ for the NH and $w_i = (U_0^*)_{i3}$ for the IH with $i = 1, 2, 3$. Further, for the NH case, we solve the above five equations and obtain the following results¹:

$$\begin{aligned}\tilde{R}_{12} &= \eta_0 \sqrt{-\frac{m_{11}}{M_2} - \frac{M_1}{M_2} \tilde{R}_{11}^2}, \\ \tilde{R}_{21} &= \frac{-(m_{11}w_1^2 + m_{22}w_2^2 - m_{33}w_3^2)\tilde{R}_{11} + 2\eta_1\sqrt{m_2m_3}W_1\tilde{R}_{12}}{2m_{11}w_1w_2}, \\ \tilde{R}_{31} &= \frac{-(m_{11}w_1^2 - m_{22}w_2^2 + m_{33}w_3^2)\tilde{R}_{11} - 2\eta_1\sqrt{m_2m_3}W_1\tilde{R}_{12}}{2m_{11}w_1w_3}, \\ \tilde{R}_{22} &= \frac{-(m_{11}w_1^2 + m_{22}w_2^2 - m_{33}w_3^2)\tilde{R}_{12} - 2\eta_1\sqrt{m_2m_3}W_2\tilde{R}_{11}}{2m_{11}w_1w_2}, \\ \tilde{R}_{32} &= \frac{-(m_{11}w_1^2 - m_{22}w_2^2 + m_{33}w_3^2)\tilde{R}_{12} + 2\eta_1\sqrt{m_2m_3}W_2\tilde{R}_{11}}{2m_{11}w_1w_3},\end{aligned}\quad (20)$$

where $\eta_0, \eta_1 = \pm 1$ are arbitrary signs and W_i is an abbreviation as

$$W_i = \frac{\sqrt{M_1M_2}}{M_i} [(U_0)_{23}(U_0)_{32} - (U_0)_{22}(U_0)_{33}](U_0^*)_{21}(U_0^*)_{31}. \quad (21)$$

We can see from the above formulas that complex $\tilde{R}_{11}$ is free and others of $\tilde{R}$ are decided by $\tilde{R}_{11}, M_1, M_2$, and low energy parameters. From Eq. (17), $\tilde{R}_{11}$ is determined by the mixing angle θ_{14} and the corresponding phase δ_{14} . Thus, once θ_{14} and δ_{14} are given, we can obtain all the rest of the mixing angles and phases explicitly via Eqs. (17) and (20). Here, θ_{ik} and δ_{ik} are performed,

$$\sin \theta_{ik} = \frac{|p_{ik}|}{\sqrt{1 + |p_{ik}|^2}}, \quad \delta_{ik} = -\arg p_{ik}, \quad (22)$$

where $i = 1, 2, 3, k = 4, 5$, and

¹This involves the root operation for a complex number z , and we adopt $\sqrt{z} = \sqrt{|z|} \exp\{i \arg(z)/2\}$ and $\arg(z) \in (-\pi, \pi]$.

$$p_{24} = c_{14}\tilde{R}_{21}, \quad (23)$$

$$p_{34} = c_{14}c_{24}\tilde{R}_{31}, \quad (24)$$

$$p_{15} = c_{14}\tilde{R}_{12}, \quad (25)$$

$$p_{25} = c_{15}c_{24}(\tilde{R}_{22} - c_{14}^2\tilde{R}_{11}^*\tilde{R}_{12}\tilde{R}_{21}), \quad (26)$$

$$p_{35} = c_{15}c_{25}c_{34}[\tilde{R}_{32} - c_{14}^2c_{24}^2(\tilde{R}_{11}^*\tilde{R}_{12} + \tilde{R}_{21}^*\tilde{R}_{22})\tilde{R}_{31}]. \quad (27)$$

In other words, given values of θ_{14} and δ_{14} (as well as RH neutrinos masses M_1 and M_2 and all low energy parameters used to determine $\tilde{R}_{ij}$), we solve out θ_{24} and δ_{24} via Eq. (23) and θ_{15} and δ_{15} via Eq. (25). Then, as θ_{24} and θ_{15} are known, we can obtain θ_{34} and δ_{34} via Eq. (24) and θ_{25} and δ_{25} via Eq. (26). Finally, θ_{35} and δ_{35} are obtained via Eq. (27) after all other parameters are solved. Once all active-sterile mixing angles are derived, all submatrices A , B , R , and S are obtained. From off-diagonal terms of Eq. (7), the Dirac matrix M_D is exactly determined as

$$M_D = (AU_0)D_\nu(SU_0)^T + RD_NB^T, \quad (28)$$

where $\theta_{45} = 0$ has been considered. Therefore, we parametrize the neutrino Dirac matrix M_D in the MSM by a new method. Since we have not taken any approximation in the above derivation, we call this parametrization the exact parametrization of minimal seesaw model (EMSM).

We summarize all free parameters in this parametrization,

$$m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma, M_1, M_2, \theta_{14}, \delta_{14}, \quad (29)$$

for the NH (IH) case of light neutrino masses. Since the lightest neutrino mass is zero in the MSM, $m_2 = \sqrt{\Delta m_{21}^2}$, $m_3 = \sqrt{\Delta m_{31}^2}$, and $m_{ee} = |m_2s_{12}^2c_{13}^2e^{2\sigma} + m_3s_{13}^2e^{-i2\delta}|$ ($m_2 = \sqrt{\Delta m_{21}^2 - \Delta m_{31}^2}$, $m_1 = \sqrt{-\Delta m_{31}^2}$, and $m_{ee} = |m_1c_{12}^2c_{13}^2 + m_2s_{12}^2c_{13}^2e^{2\sigma}|$) for the NH (IH) in the MSM, where m_{ee} is the effective mass parameter in $0\nu 2\beta$ decay experiments. We might choose another set of parameters,

$$\Delta m_{21}^2, \Delta m_{31}^2, \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, m_{ee}, M_1, M_2, \theta_{14}, \delta_{14}. \quad (30)$$

Here, the first six parameters are measured in neutrino oscillation experiments, the seventh is supposed to be measured in $0\nu 2\beta$ experiments, and the last four are parameters of sterile neutrinos.

III. COMPARISON BETWEEN EMSM AND OTHER PARAMETRIZATIONS

In this section, we will compare our parametrization (EMSM) with other parametrizations. One well-known parametrization is the Casas-Ibarra (CI) parametrization, which is widely used in the study of neutrino

phenomenology and cosmology. We compare the EMSM with the CI parametrization and, importantly, check the consistency between the two parametrizations. We then turn to the comparison between the EMSM and the natural reconstruction parametrization. We show the similarity between them and point out the advantages of our parametrization. Comparisons with the rest of the parametrizations are summarized in the end of this section. Generally, most of these parametrizations adopt the approximate seesaw formula $M_\nu \simeq -M_D M_R^{-1} M_D^T$ under the condition $\sqrt{M_\nu}/M_R \sim M_D/M_R \sim O(R) \ll 1$. However, the EMSM uses the exact seesaw formula $M_\nu = -R D_N R^T$, so it is an exact parametrization, where the parameters in the EMSM have clear physical meanings.

A. Casas-Ibarra parametrization

The CI parametrization adopts the approximate seesaw formula $M_\nu \simeq -M_D D_N^{-1} M_D^T$ [16]. Applying this parametrization in the MSM [25], entries of the Dirac matrix M_D are approximately expressed as

$$\begin{aligned} (M_D^{\text{CI}})_{\alpha 1} &\simeq +i\sqrt{M_1}[(V_0)_{\alpha 2}\sqrt{m_2}\cos\omega + \xi \\ &\quad \times (V_0)_{\alpha 3}\sqrt{m_3}\sin\omega], \\ (M_D^{\text{CI}})_{\alpha 2} &\simeq -i\sqrt{M_2}[(V_0)_{\alpha 2}\sqrt{m_2}\sin\omega - \xi \\ &\quad \times (V_0)_{\alpha 3}\sqrt{m_3}\cos\omega], \end{aligned} \quad (31)$$

where $\alpha = 1, 2, 3$ represents the charged lepton flavor index $\{e, \mu, \tau\}$, $\xi = \pm 1$ is an arbitrary sign, ω is a complex parameter, V_0 is the standard PMNS matrix as Eq. (12), and more details are shown in Appendix A.

In the EMSM, the parametrization of M_D is shown in Eq. (28), which uses the exact seesaw formula. When considering $D_\nu \ll D_N$, there is an approximation $M_D^{\text{EMSM}} \simeq R D_N B^T$. Assuming that all active-sterile mixing angles are small enough, the submatrices of $\mathcal{U}$ approximate as $A_{\alpha i} \simeq \delta_{\alpha i} + O(\theta_{\alpha i}^2)$ and $B_{ij} \simeq \delta_{ij} + O(\theta_{ij}^2)$ via Eqs. (13) and (16). Therefore, M_D^{EMSM} in the leading-order approximation becomes

$$(M_D^{\text{EMSM}})_{\alpha i} \simeq (R D_N)_{\alpha i} \simeq M_i \theta_{\alpha i} e^{i\delta_{\alpha i}},$$

where the basis of M_D^{EMSM} is chosen as the charged lepton flavor basis and the heavy neutrino mass basis is the same as M_D^{CI} in Eq. (31).

Sequently, we will correlate the EMSM with the CI parametrization, which is equivalent to find the formula of θ_{14} , δ_{14} , and ω . Because of $(M_D^{\text{EMSM}})_{11} \simeq (M_D^{\text{CI}})_{11}$, we can directly obtain

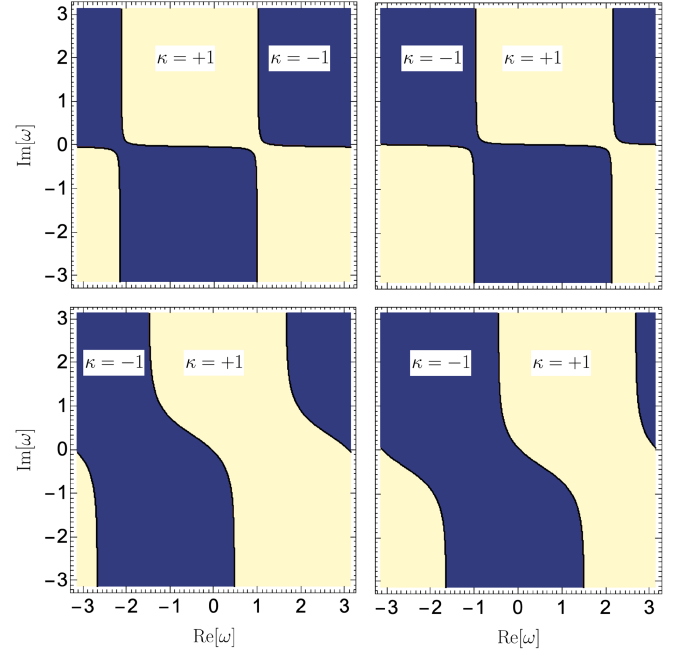


FIG. 1. The κ value about complex ω when $\xi = +1$ (left), $\xi = -1$ (right), and the Majorana phase $\sigma = 0$ (upper), $\sigma = \pi/4$ (lower). The dark blue and bright yellow regions correspond to $\kappa = +1$ and -1 , respectively.

$$\begin{cases} \theta_{14} \simeq |(V_0)_{12}\sqrt{m_2}\cos\omega + \xi(V_0)_{13}\sqrt{m_3}\sin\omega|/\sqrt{M_1} \\ \delta_{14} \simeq -\arg[i((V_0)_{12}\sqrt{m_2}\cos\omega + \xi(V_0)_{13}\sqrt{m_3}\sin\omega)]. \end{cases} \quad (32)$$

Moreover, we must check the consistency of the rest of the terms between the EMSM and the CI parametrization. From $(M_D^{\text{EMSM}})_{12} \simeq (M_D^{\text{CI}})_{12}$ and $(M_D^{\text{EMSM}})_{21} \simeq (M_D^{\text{CI}})_{21}$, there are two conditions $\eta_0 = \kappa$ and $\eta_0 \cdot \eta_1 = \kappa \cdot \xi$, where κ is defined as

$$\kappa \equiv \text{sign}\left\{\sqrt{(M_D^{\text{CI}})_{12}^2/(M_D^{\text{CI}})_{12}}\right\} = \pm 1. \quad (33)$$

Here, κ is independent of heavy neutrinos masses. Finally, it is simple to check other terms $\{22, 31, 32\}$ satisfying $M_D^{\text{EMSM}} \simeq M_D^{\text{CI}}$. We display the κ value in Fig. 1 when fixing the Majorana phase $\sigma = 0$ and $\pi/4$. Therefore, once ξ is given and then κ is obtained, the EMSM becomes consistent with the CI parametrization via Eq. (32) and conditions

$$\kappa = \eta_0 \quad \text{and} \quad \xi = \eta_1. \quad (34)$$

B. Natural reconstruction

The natural reconstruction (NR) parametrization was proposed, based on the approximate seesaw formula and the MSM, to quantitatively reconstruct CP -violating phases

TABLE I. A summary of the 11 independent parameters in the EMSM and other parametrizations of the MSM. $m_3(m_1)$ denotes the NH (or IH) scenario. We inherit conventions from [21] for names of some of these parametrizations. Note that the set of independent parameters selected in each parametrization is not unique, and that listed above is just one option.

Parametrization	11 independent parameters in parametrizations of MSM
EMSM	$m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma, M_1, M_2, \theta_{14}, \delta_{14}$
Casas-Ibarra	$m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma, M_1, M_2, \omega , \arg \omega$
Natural reconstruction	$m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma, M_1, M_2, a_1 , \arg a_1$
Blennow–Fernandez-Martinez	$m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma, M_1, M_2, \theta_y, \theta_z(\theta_x),$
Modified Casas-Ibarra-Ross	$m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma, P_{13} , P_{23} , P_{12} , \arg P_{12}$
Biunitary	$m_2, m_3(m_1), M_1, M_2, \theta_1, \theta_2, \theta_3, \delta_L, \gamma_L, \theta_R, \gamma_R$
Vector representation	$m_2, m_3(m_1), M_1, M_2, u_{e1}, u_{\mu1}, u_{e2} , u_{\mu2} ,$ $\arg u_{e2}, \arg u_{\mu2}, \arg u_{\tau2}$

for leptogenesis using only the neutrino low energy experiments [22]. The Dirac matrix M_D^{NR} can be obtained by the free parameter $(M_D^{\text{NR}})_{11}$,

$$\begin{aligned}
 (M_D^{\text{NR}})_{12} &\simeq \zeta_1 \sqrt{-(M_\nu)_{11} M_2 - \frac{M_2}{M_1} (M_D^{\text{NR}})_{11}^2}, \\
 (M_D^{\text{NR}})_{\alpha 1} &\simeq \frac{1}{(M_\nu)_{11}} \left\{ (M_D^{\text{NR}})_{11} (M_\nu)_{1\alpha} + \zeta_\alpha (M_D^{\text{NR}})_{12} \sqrt{\frac{M_1}{M_2} \sqrt{(M_\nu)_{11} (M_\nu)_{\alpha\alpha} - (M_\nu)_{1\alpha}^2}} \right\}, \\
 (M_D^{\text{NR}})_{\alpha 2} &\simeq \frac{1}{(M_\nu)_{11}} \left\{ (M_D^{\text{NR}})_{12} (M_\nu)_{1\alpha} - \zeta_\alpha (M_D^{\text{NR}})_{11} \sqrt{\frac{M_2}{M_1} \sqrt{(M_\nu)_{11} (M_\nu)_{\alpha\alpha} - (M_\nu)_{1\alpha}^2}} \right\},
 \end{aligned} \tag{35}$$

where $\alpha = 2, 3$ represents the lepton flavor index $\{\mu, \tau\}$, $\zeta_1 = \pm 1$ is an arbitrary sign, $\det(M_\nu) = 0$ constrains on $\zeta_\alpha = \pm 1$, and more details are shown in Appendix A.

The EMSM becomes consistent with the NR parametrization in the limit $M_D \ll M_R$, i.e., $M_D^{\text{NR}} \simeq M_D^{\text{EMSM}} \simeq RD_N$. Here, for all corresponding elements of both parametrizations are the same, there are two conditions $\zeta_1 = \eta_0$ and $\zeta_2 = -\zeta_3 = \eta_1$. On the other hand, the EMSM is different from the NR parametrization. The former is an exact parametrization, while the latter is an approximate method. After getting the submatrix $\tilde{R}$ from Eq. (20), our parametrization can be used to calculate $\nu - N$ mixing angles θ_{ik} and corresponding phases δ_{ik} via Eqs. (23)–(27) exactly.

C. Other parametrizations

The Dirac mass matrix M_D of neutrinos is the key in the study of neutrino phenomenology, such as leptogenesis and charged lepton flavor violations, so parametrizing M_D is the foundation of some studies and many different parametrizations have been proposed. These parametrizations, together with CI and NR, in the MSM framework are briefly reviewed in Appendix A. In the MSM, there are 11 physically independent parameters, but only seven parameters of them are observables in low energy experiments.

We summarize 11 independent parameters in different parametrizations in Table I. We stress that the choices of independent parameters in some of these parametrizations are not unique, so the parameters listed in Table I are just an option.

In Table I, the first five parametrizations contain the seven low energy parameters $m_2, m_3(m_1), \theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma$, where $m_2 \equiv \sqrt{\Delta m_{21}^2}$ and $m_3 \equiv \sqrt{\Delta m_{31}^2}$ in the NH scenario ($m_2 \equiv \sqrt{-\Delta m_{32}^2}$ and $m_1 \equiv \sqrt{-\Delta m_{32}^2 - \Delta m_{21}^2}$ in the IH case). The five parameters $\Delta m_{21}^2, \Delta m_{31}^2, \Delta m_{32}^2, \theta_{12}, \theta_{13}, \theta_{23}$ have been well measured in neutrino oscillation experiments [26], but the Dirac phase δ_{CP} has not been precisely measured and the Majorana phase σ is unknown so far, and the rest of the four parameters from the heavy sector may be constrained in the future high energy experiments. The last two parametrizations directly contain only two low energy parameters $m_2, m_3(m_1)$, and the rest $\theta_{12}, \theta_{13}, \theta_{23}, \delta_{CP}, \sigma$ are not directly presented but constitute new parameters with four high energy parameters.

All parametrizations, except EMSM, Blennow–Fernandez-Martinez, and biunitary parametrizations have used the approximate seesaw formula $M_\nu \approx -M_D M_R^{-1} M_D^T$, so their applications might be limited if the high-precision analysis is required in the future study. The Blennow–Fernandez-Martinez and biunitary parametrizations are

TABLE II. The current upper bounds and future sensitivities for branching ratios of cLFV processes in the μ - e sector.

Processes	Current bounds	Future sensitivities
$\text{Br}(\mu \rightarrow e\gamma)$	3.1×10^{-13} [35]	6.0×10^{-14} [36]
$\text{Br}(\mu \rightarrow eee)$	1.0×10^{-12} [37]	1.0×10^{-16} [38]
$R_{\mu \rightarrow e}$ (Ti)	4.3×10^{-12} [39]	1.0×10^{-18} [40,41]

exact and analytical, but their physical meanings are not clear enough. However, the EMSM contains the advantages of other parametrizations, which is exact, analytical, physical, and clear as shown in Sec. II.

IV. PHENOMENOLOGICAL IMPLICATIONS

We further employ the EMSM parametrization in a few phenomenological implications. We will apply this parametrization in the studies of direct and indirect experimental searches for RH neutrinos including charged lepton flavor violations, nonunitarity effects of the lepton mixing, and other experiments in Sec. IV A. The parameter space in these studies will be presented. The implication in leptogenesis will be studied in Sec. IV B.

A. Experimental constraints

We first study the current constraints and future sensitivities on the parameter space of EMSM through cLFV experiments and other direct or indirect experimental searches for RH neutrinos. Subsequently, we will analyze the constraints on nonunitarity effects induced by MSM from the updated data of the neutrino oscillations and electroweak precision test (EWPT).

In the cLFV studies, the $\mu - e$ sector provides strong constraints in the current and foreseeable future experiments [27,28]. We mainly focus on the $\mu - e$ sector in the EMSM through cLFV experiments. In the MSM framework, cLFVs are triggered by two RH neutrinos via the active-sterile mixing. Hence, we will study these rare processes about the $\mu - e$ sector in the EMSM, mainly including radiative decay, $\mu \rightarrow e\gamma$, three-body decay, $\mu \rightarrow eee$, μ to e conversion in nucleus N, $\mu + N \rightarrow e + N$. The first rare process is the charged lepton radiative decay, and its branching ratio is defined as $\text{Br}(\mu \rightarrow e\gamma) \equiv \Gamma(\mu \rightarrow e\gamma)/\Gamma_\mu$, where Γ_μ is the total muon decay width. The branching ratio induced by two heavy neutrinos is presented as [2,10,11,29,30]

$$\text{Br}(\mu \rightarrow e\gamma) = \frac{3\alpha_{\text{em}}}{2\pi} \left| \sum_{i=1}^3 U_{\mu i}^* U_{ei} \times G_\gamma \left(\frac{m_i^2}{M_W^2} \right) + \sum_{j=1}^2 R_{\mu j}^* R_{ej} \times G_\gamma \left(\frac{M_j^2}{M_W^2} \right) \right|^2, \quad (36)$$

where $G_\gamma(x)$ is a one-loop function about photons as Eq. (B1) shown in Appendix B. The branching ratio of the lepton three-body decay $\mu \rightarrow eee$ is given by [31]

$$\begin{aligned} \text{Br}(\mu \rightarrow eee) = & \frac{\alpha_w^4}{24576\pi^3} \frac{m_\mu^4}{M_W^4} \frac{m_\mu}{\Gamma_\mu} \times \left\{ 2|(F_{\frac{1}{2}\text{Box}+Z}^{\mu e e e} - F_{Z-\gamma}^{\mu e})|^2 + |F_{Z-\gamma}^{\mu e}|^2 + 32s_w^4 r^{\mu e} |G_\gamma^{\mu e}|^2 \right. \\ & \left. + 16s_w^2 \text{Re} \left[G_\gamma^{\mu e*} \left(F_{\frac{1}{2}\text{Box}+Z}^{\mu e e e} - \frac{3}{2} F_{Z-\gamma}^{\mu e} \right) \right] \right\}, \end{aligned} \quad (37)$$

where μ and e are muon and electron flavor indices, respectively, and $r^{\mu e} = \ln(m_\mu^2/m_e^2) - 11/4$. Here, F and G are some functions about form factor depending on $\mathcal{U}_{\alpha i}$, $\mathcal{U}_{\beta i}$, $R_{\alpha j}$, $R_{\beta j}$, m_α , m_β , M_1 , M_2 as Eqs. (B2) and (B3) in Appendix B. Finally, denoting the atomic number of nucleus N by Z , the μ - e conversion rate in the light nucleus ($\alpha_{\text{em}}Z \ll 1$) is given in [32],

$$R_{\mu \rightarrow e} = \frac{G_F^2 \alpha_w^2 \alpha_{\text{em}}^3 m_\mu^5 Z_{\text{eff}}^4}{8\pi^4 \Gamma_{\text{capt}} Z} F_p^2 \left| \sum_{j=1}^2 R_{ej} R_{\mu j}^* [Z(2F_u + F_d) + (A - Z)(F_u + 2F_d)] \right|^2, \quad (38)$$

where F_u and F_d are functions depending on M_j^2/M_W^2 as Eq. (B4) in Appendix B. We take nucleus $^{48}_{22}\text{Ti}$ to determine the form factor $|F_p| = 0.54$, effective atomic number $Z_{\text{eff}} = 17.6$, and the capture rate $\Gamma_{\text{capt}} = 2.59 \times 10^6 \text{ s}^{-1}$ [33,34]. The current and further upper limits for the branching ratios of three cLFV processes via relevant experiments are listed in Table II. The μ - e conversion might be highly suppressed due to the cancellation inside the square brackets in Eq. (38), which is induced by the opposite signs of F_u and F_d . The mixing angle θ_{14} is hardly constrained around the cancellation point. In general, this point depends on the mass of heavy neutrinos M_1 , M_2 and the active-sterile mixing (i.e., θ_{14}). After simplifying F_u and F_d as Eq. (B5) in Appendix B for $M_j^2/M_W^2 \gg 1$, we obtain the functional relation of M_1 , M_2 , and θ_{14} at the resonant point,

$$\left\{ R_{e1} R_{\mu 1}^* \ln \left(\frac{M_1^2}{M_W^2} \right) + R_{e2} R_{\mu 2}^* \ln \left(\frac{M_2^2}{M_W^2} \right) \right\} \Big|_{\text{res}} = (R_{e1} R_{\mu 1}^* + R_{e2} R_{\mu 2}^*) f(A, Z), \quad (39)$$

where $f(A, Z)$ is a function determined by atomic nucleus,

$$f(A, Z) := \frac{\frac{9}{8}(A - Z) + (\frac{9}{8} + \frac{31}{12}s_W^2)Z}{\frac{3}{8}(A - Z) - (\frac{3}{8} - \frac{4}{3}s_W^2)Z}, \quad (40)$$

and $f(A, Z) \simeq 8.1$ for ^{48}Ti . In the case of degenerate heavy neutrinos, $M_1 \simeq M_2$, the resonant point is independent of the mixing matrix R and then the resonant mass is

$$M_{\text{res}}^2 = M_W^2 \cdot \text{Exp}(f(A, Z)). \quad (41)$$

Therefore, the resonant point for titanium satisfies $M|_{\text{res}} \simeq 3.375 \times 10^3$ GeV in the degenerate case of heavy neutrinos.

The other experimental searches for RH neutrinos include direct and indirect collider searches, meson decays, and beam-dump experiments, β decays and nuclear processes, active-sterile neutrino oscillations, EWPTs, and astrophysical observations. The data of these experiments are summarized in [42]. These experimental searches can directly constrain the active-sterile mixing matrix U . Assuming that the two RH neutrino masses are quasidegenerate $M_1 \simeq M_2$, the current constraints and future sensitivities on U_{eN} and M_1 from these experiments is shown in Fig. 2(a). Based on EMSM, current constraints and future sensitivities on the parameter space of M_1 and the exact active-sterile mixing angle θ_{14} from cLFVs and other experimental searches are exhibited in Fig. 2(b). Here, we assume the CP phases $\sigma = \delta_{14} = 0$, which reduces degrees of freedom of parameters. We give Eq. (32) to calculate the approximate active-sterile mixing angle θ_{14}^{CI} in CI parametrization. Actually, active-sterile mixing angles in former different parametrizations are the same, because these parametrizations are all derived from Eq. (5), while our parametrization is based on the exact formula of Eq. (4). Meanwhile, in Fig. 2(b), the pole on the red curve refers to the cancellation of the μ - e conversion in titanium in Eq. (38). From Fig. 2(b), current experiments suggest the active-sterile mixing angle $\theta_{14} \lesssim \mathcal{O}(10^{-3})$ and future experiments can test $\theta_{14} \sim \mathcal{O}(10^{-4})$ when the RH neutrino mass is larger than 1 GeV.

To highlight parameter regions where other commonly used parametrizations fail, we use $\delta\theta_{14}/\theta_{14} = |\theta_{14}^{\text{CI}}/\theta_{14}^{\text{EMSM}} - 1|$ to quantify the relative error of the approximate seesaw formula compared with the exact one, and the result is shown in Fig. 2(c). This error can reach as large as $\mathcal{O}(0.01 - 0.1)$ in some experimental constraints, including the established measurements: NEOS, PROSPECT, SK + IC + DC, ^3H , ^{187}Re , $^{144}\text{Ce} - ^{144}\text{Pr}$, Borexino, IHEP-JINR, BEBC, Belle, NA3, LNV decays, Higgs, L3, ATLAS, CMS, $\mu \rightarrow e(\text{Ti})$,

$\mu \rightarrow eee$, $\mu \rightarrow e\gamma$, and future proposals: HUNTER, DUNE Indirect, ILC, HL-LHC, LHeC, FCC-hh, $\mu \rightarrow eee$, $\mu \rightarrow e\gamma$. Therefore, we suggest to use EMSM in these experimental constraints/sensitivities. We further show in Table III the comparison of constraints on the upper bound of θ_{14} obtained from our parametrization and other parametrizations for the sterile neutrino masses at 1 eV.

We further check the constraints on active-sterile mixing angles in EMSM from constraints of nonunitarity effects from neutrino oscillations and EWPT. The 3×3 mixing matrix is not exactly a unitary matrix in the presence of RH neutrinos. In the MSM, the heavy neutrinos N_1, N_2 participate in the flavor mixing through nonzero active-sterile mixing angles, which induces the nonunitarity effects [7,24,43,44]. As discussed in Sec. II, the nonunitarity of the effective flavor mixing matrix $U = AU_0$ in Eq. (8) is due to the deviation of A from unit matrix I in Eq. (13). Therefore, we will use $I - A$ to evaluate the nonunitary effects of the lepton mixing. The nonunitarity behavior of neutrinos can be constrained by global fits from the neutrino oscillation experiments [45] and EWPT experiments [28] as shown in Table IV, where $\alpha_{\alpha\beta} \equiv A_{\alpha\beta}$ in Eq. (13)² and the data within 95% confidence level (C.L.) from the neutrino oscillation are obtained through Ref. [45]. Here, it is clear that the constraints from the EWPT are stricter than the neutrino oscillation. The constraints on $\alpha_{\alpha\beta}$ are straightforwardly converted into the constraints on active-sterile mixing angles θ_{ik} and the results are shown in Table V, where we consider two scenarios for RH neutrino masses, the degeneracy ($M_1 \simeq M_2$) and the hierarchy ($M_1 \ll M_2$). Table V shows that the active-sterile mixing angles are restricted mostly to lower than 10° and 1° through the neutrino oscillation and the EWPT, respectively, though the complex phases are still unconstrained.

B. Leptogenesis

The observations about the cosmic microwave background (CMB) [15] and the big-bang nucleosynthesis [1] reveal that baryons (or matter) are more than antibaryons (or antimatter), i.e., the matter-antimatter asymmetry. Baryogenesis via leptogenesis [14] is one of the most important mechanisms to explain the matter-antimatter asymmetry. In this section, we will study the parameter space of the EMSM parametrization that can realize successful leptogenesis at low energy scale $10^3 - 10^8$ GeV.

The Boltzmann equations of heavy neutrinos N_{N_i} ($i = 1, 2$) and $B - L$ asymmetry N_{B-L} with flavor effects [46] taken into account are given by

$$\frac{dN_{N_i}}{dz} = -D_i(N_{N_i} - N_{N_i}^{\text{eq}}), \quad (42)$$

²Actually, in some literature, the definition of α is unitary transformation for A , so $\alpha^\dagger \alpha = A^\dagger A$ or $\alpha \alpha^\dagger = AA^\dagger$ is stricter.

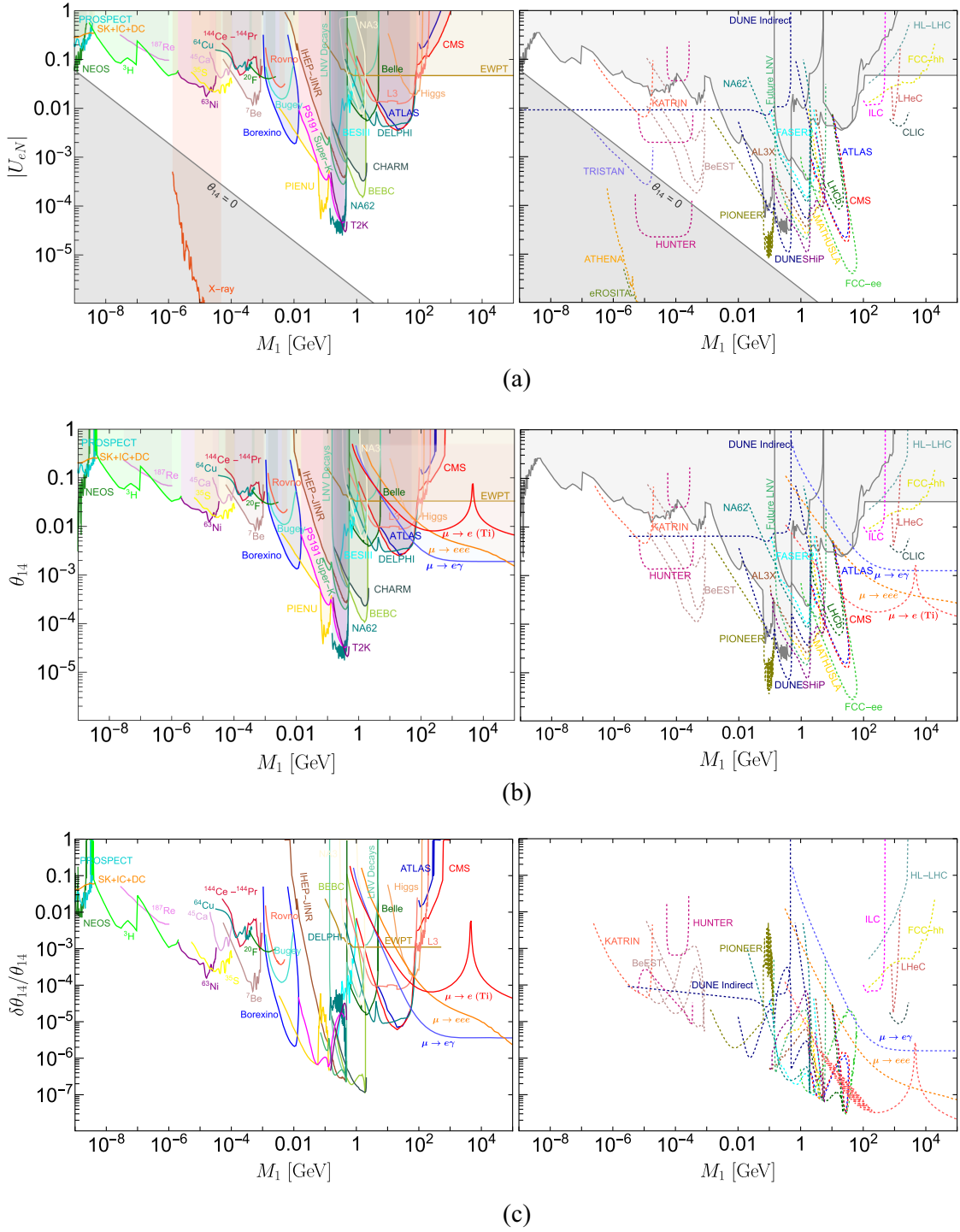


FIG. 2. Implication of EMSM in experimental searches and its advantage compared to parametrizations from approximate seesaw formula. (a) Current constraints (left) and future sensitivities (right, current constraints shown in light gray) on the heavy neutrino mass M_1 and the element of the active-sterile mixing matrix $|U_{eN}|$. $M_1 \simeq M_2$ is assumed. The dark gray region below the line $\theta_{14} = 0$ refers to the failure of the minimal seesaw model. (b) Current constraints (left) and future sensitivities (right) on the heavy neutrino mass M_1 and the active-sterile mixing angle θ_{14} . (c) Relative difference between the exact and approximate active-sterile mixing angle about the heavy neutrino mass M_1 , where $\delta\theta_{14}/\theta_{14} = |\theta_{14}^{\text{CI}}/\theta_{14}^{\text{EMSM}} - 1|$. (b), (c) $M_1 \simeq M_2$ and $\sigma = \delta_{14} = 0$ are assumed.

$$\frac{d(N_{B-L})_{\alpha\alpha}}{dz} = \sum_i \left[(-\varepsilon_{ia} D_i (N_{N_i} - N_{N_i}^{\text{eq}})) - K_{ia} W_i \sum_{\beta=e,\mu,\tau} (C_{\alpha\beta}^l + C_{\beta}^H) (N_{B-L})_{\beta\beta} \right], \quad (43)$$

TABLE III. Constraints on the upper bound of the sterile-active mixing angle θ_{14} at $M_1 = 1$ eV in EMSM and other parametrizations. All former parametrizations, due to the use of approximate seesaw formula, induce an error of $\mathcal{O}(0.01)$ – $\mathcal{O}(0.1)$.

θ_{14}	NEOS	PROSPECT	SK+IC+DC
EMSM	0.0209	0.124	0.203
Former parametrizations	0.0204	0.122	0.195
$\delta\theta_{14}/\theta_{14}$	2.47%	1.93%	4.23%

 TABLE IV. Upper bounds on the nonunitarity effects of neutrinos by global fits from neutrino oscillation experiments [45] and EWPT experiments [28] at 95% C.L., where $\alpha \equiv A$ in Eq. (13).

Nonunitarity para.	ν oscillation	EWPT
$1 - \alpha_{ee}$	3.9×10^{-2}	9.4×10^{-6}
$1 - \alpha_{\mu\mu}$	6.2×10^{-3}	1.3×10^{-4}
$1 - \alpha_{\tau\tau}$	1.4×10^{-1}	2.1×10^{-4}
$ \alpha_{\mu e} $	1.8×10^{-2}	2.4×10^{-5}
$ \alpha_{\tau e} $	4.2×10^{-2}	4.4×10^{-5}
$ \alpha_{\tau\mu} $	1.1×10^{-2}	2.6×10^{-4}

where $z \equiv M_1/T$, α is a flavor index of charged leptons, D_i and W_i are N_i 's decay and washout terms, respectively, shown in [47], $K_{i\alpha}$ is the projection probabilities between the mass and flavor states [48], and $C_{\alpha\beta}^L$ and C_{β}^H are spectator effects factors shown in [48,49]. The number densities N_{N_i} and N_{B-L} are normalized to the comoving volume containing one photon. Here, $\varepsilon_{i\alpha}$ represents the CP asymmetry defined as

$$\varepsilon_{i\alpha} \equiv \frac{\Gamma(N_i \rightarrow \ell_\alpha + H) - \Gamma(N_i \rightarrow \bar{\ell}_\alpha + \bar{H})}{\sum_\alpha [\Gamma(N_i \rightarrow \ell_\alpha + H) + \Gamma(N_i \rightarrow \bar{\ell}_\alpha + \bar{H})]}, \quad (44)$$

which is generated by the interference between the tree and one-loop effects. We denote $\varepsilon_\alpha = \sum_i \varepsilon_{i\alpha}$. In the highly degenerate mass regions where the RH neutrino mass difference is comparable with their decay widths, the CP asymmetry is resonantly increased via enhanced heavy neutrino self-energy effects in resonant leptogenesis,

$$\varepsilon_{i\alpha} = \sum_{j \neq i} \frac{\text{Im}[(Y_\nu^*)_{ai}(Y_\nu)_{\alpha j}(Y_\nu^\dagger Y_\nu)_{ij}] + \frac{M_i}{M_j} \text{Im}[(Y_\nu^*)_{ai}(Y_\nu)_{\alpha j}(Y_\nu^\dagger Y_\nu)_{ij}^*]}{(Y_\nu^\dagger Y_\nu)_{ii}(Y_\nu^\dagger Y_\nu)_{jj}} \cdot \frac{\Delta M_{ij}^2 M_i \Gamma_j}{(\Delta M_{ij}^2)^2 + M_i^2 \Gamma_j^2}, \quad (45)$$

where $\Delta M_{ij}^2 = M_i^2 - M_j^2$, $\Gamma_i = \frac{(Y_\nu^\dagger Y_\nu)_{ii}}{8\pi} M_i$ is the N_i decay width at tree level, and the second term is a resonant-enhancement factor. The matter-antimatter asymmetry is characterized by the ratio of the baryon-antibaryon number density difference to the photon number density,

$$\eta_B \equiv \frac{n_B - n_{\bar{B}}}{n_\gamma} = \frac{c_S}{f} N_{B-L}, \quad (46)$$

where c_S is the fraction of $B - L$ asymmetry converted to B asymmetry by sphaleron process shown in [50,51], f is a dilution factor, and $f = n_{\text{rec}}^\gamma a_{\text{rec}}^3 / (n_*^\gamma a_*^3) = g_*/g_{\text{rec}}$. In the MSM and standard thermal-universe history, $c_S = \frac{28}{79}$ and $f \simeq 27.65$. The matter-antimatter asymmetry η_B was measured most precisely in the CMB observation satisfying $\eta_B = (6.12 \pm 0.04) \times 10^{-10}$ in 68% C.L. [15].

 TABLE V. Upper bounds on the active-sterile mixing angles converted from Table IV. Two scenarios for RH neutrino masses, the degeneracy ($M_1 \simeq M_2$) and the hierarchy ($M_1 \ll M_2$), are considered.

Mixing angles	ν oscillation		EWPT	
	Degeneracy	Hierarchy	Degeneracy	Hierarchy
θ_{14}	$4.4 \times 10^{-2} (2.5^\circ)$	$9.0 \times 10^{-2} (5.2^\circ)$	$3.0 \times 10^{-3} (0.17^\circ)$	$4.3 \times 10^{-3} (0.25^\circ)$
θ_{15}		...		...
θ_{24}	$7.9 \times 10^{-2} (4.5^\circ)$	$1.1 \times 10^{-1} (6.4^\circ)$	$1.1 \times 10^{-2} (0.64^\circ)$	$1.6 \times 10^{-2} (0.92^\circ)$
θ_{25}		...		...
θ_{34}	$1.0 \times 10^{-1} (5.9^\circ)$	$1.8 \times 10^{-1} (10.4^\circ)$	$7.2 \times 10^{-3} (0.41^\circ)$	$8.9 \times 10^{-3} (0.51^\circ)$
θ_{35}		...		...

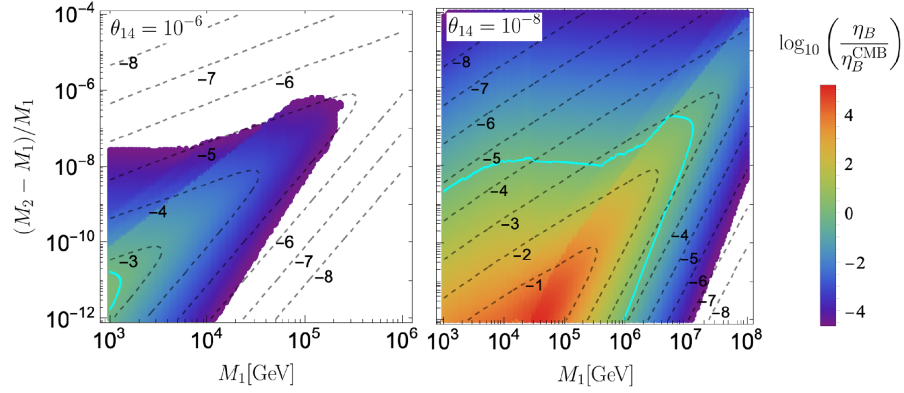


FIG. 3. The heat map of the matter-antimatter asymmetry η_B and the counterplot of the CP asymmetry $\varepsilon_{\max}$ along the heavy neutrino mass M_1 and the degeneracy parameter $(M_2 - M_1)/M_1$ when the active-sterile mixing angle $\theta_{14} = 10^{-6}$ (left) and 10^{-8} (right). Here, $\eta_B^{\text{CMB}} = 6.12 \times 10^{-10}$ is the best-fit value of the CMB observation. The dashed curves are counterlines of $\log_{10} \varepsilon_{\max}$ and corresponding values are marked along the curves. The colors represent the values of $\log_{10} \eta_B/\eta_B^{\text{CMB}}$, where the narrow cyan band satisfies the CMB observation $\eta_B \in (6.12 \pm 0.20) \times 10^{-10}$ in 5σ C.L.

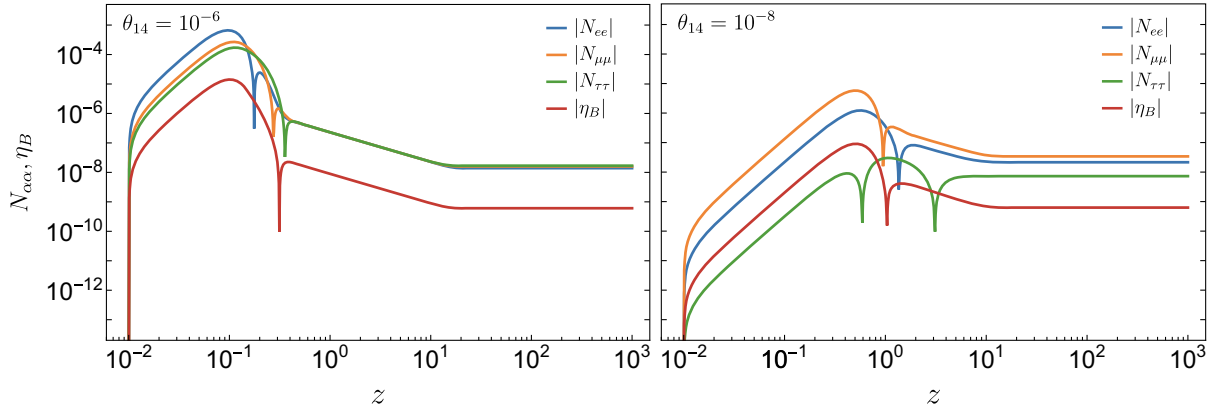


FIG. 4. The matter evolutions satisfying CMB observations for resonant leptogenesis. Here, $N_{\alpha\alpha} \equiv (N_{B-L})_{\alpha\alpha}$. Left: $\theta_{14} = 10^{-6}$, $M_1 = 10^3$ GeV, $|M_2/M_1 - 1| \simeq 1.74 \times 10^{-12}$, $\delta_{14} = 1.72\pi$, and $\sigma = 1.52\pi$. Right: $\theta_{14} = 10^{-8}$, $M_1 = 10^3$ GeV, $|M_2/M_1 - 1| \simeq 2.09 \times 10^{-9}$, $\delta_{14} = 0.72\pi$, and $\sigma = 1.64\pi$.

We apply the EMSM parametrization in leptogenesis as below. As discussed in Sec. II, $Y_\nu = \sqrt{2}M_D/v$ and the exact expression of M_D is Eq. (28). After inputting low energy parameters and taking heavy neutrino masses M_1 , M_2 and fundamental parameters θ_{14} , δ_{14} of EMSM, we can calculate the unitary matrix $\mathcal{U}$ of Eq. (8) through the method in Sec. II and then obtain Y_ν from Eq. (28). Finally, the CP asymmetry is calculated via Eq. (45), and subsequently, the baryon-antibaryon asymmetry is evaluated by Eqs. (42) and (43).

Numerically, the PYTHON package ULYSSES [52,53] can successfully calculate the evolution of particle number densities and give the final η_B . The procedure is arranged in the following. First, we should obtain the Yukawa coupling Y_ν . Hence, taking the $\nu_1 - N_1$ mixing angle $\theta_{14} = 10^{-6}, 10^{-8}$, the heavy neutrino mass $M_1 \in [10^3, 10^8]$ GeV and the degeneracy of heavy

neutrinos $(M_2 - M_1)/M_1 \in [10^{-12}, 10^{-4}]$, and inputting known low energy parameters, we scan the two phase angles $\delta_{14}, \sigma \in (-\pi, \pi]$ and then choose the values of phase angles to make $\varepsilon_{\max} \equiv \max\{-\varepsilon_e, -\varepsilon_\mu, -\varepsilon_\tau\}$ maximized. Next, we use ULYSSES to calculate the matter-antimatter asymmetry η_B after inputting physical parameters and these Yukawa couplings at different heavy neutrino mass M_1 and degeneracy $(M_2 - M_1)/M_1$. We use the mode 2RESsp (considering spectator effects) in resonant leptogenesis.³

The matter-antimatter asymmetry η_B and the CP asymmetry $\varepsilon_{\max}$ are shown in Fig. 3, and the representative figures of matter evolutions are given in Fig. 4. As we show in Fig. 3, leptogenesis at low energy scale can successfully

³Here, we have used the PYTHON GNU parallel from the website: <https://doi.org/10.5281/zenodo.15071920>.

generate the matter-antimatter asymmetry satisfying the CMB observation.

V. CONCLUSIONS

The seesaw mechanism provides a natural way in the explanation of tiny neutrino masses by introducing heavy mediators. In the MSM, only two RH neutrinos are introduced beyond the Standard Model, two active neutrinos gain small masses via the active-sterile mixing, and the lightest active neutrino keeps massless. Several parametrizations of the MSM have been provided successively in the literature. Most of them employ the approximate seesaw formula $M_\nu \simeq -M_D M_R^{-1} M_D^T$ under the assumption $M_D \ll M_R$. In this work, we propose a parametrization without considering this approximation, called EMSM.

In the MSM, we can parametrize the most general 5×5 (3 active +2 sterile) neutrino mass matrix in terms of 11 independent parameters. In the EMSM parametrization, these parameters include six neutrino oscillation parameters Δm_{21}^2 , Δm_{31}^2 , θ_{12} , θ_{13} , θ_{23} , δ_{CP} , one mass parameter m_{ee} used in $0\nu 2\beta$ decay processes, two sterile neutrino masses M_1 , M_2 , one active-sterile mixing angle θ_{14} , and one relevant phase δ_{14} for the NH or IH case. The other parameters, e.g., other active-sterile mixing angles and phases and the Dirac mass matrix, can be represented by these 11 parameters exactly without any approximation. The results are given in the last part of Sec. II. For better performing advantages of this parametrization, we compared it with other parametrizations in Sec. III. In particular, correlations with Casas-Ibarra and natural reconstruction parametrizations are presented.

We applied this parametrization in several phenomenological studies in Sec. IV. The active-sterile mixing angle θ_{14} is constrained to $\theta_{14} \lesssim \mathcal{O}(10^{-3})$ from direct and indirect experimental searches if the RH neutrino masses are larger than 1 GeV. In some experimental searches, the relative error of active-sterile mixing in parametrizations from the approximative seesaw formula can reach percent level. Regions of these large errors motivate the use of exact seesaw formula in the presentation of experimental

constraints and sensitivities. Meanwhile, we employed the EMSM to nonunitarity effects and then limited $\nu_i - N_k$ mixing angles $\theta_{ik} \lesssim \mathcal{O}(1^\circ)$. Finally, we showed examples of successful leptogenesis through the EMSM parametrization with sterile neutrino masses below 10^8 GeV.

The EMSM parametrization is exact and analytical and its parameters have clear physical meanings. This parametrization will be helpful for high-precision measurements in the near future.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [54].

APPENDIX A: SEVERAL PARAMETRIZATIONS OF THE MINIMAL SEESAW MODEL

1. Casas-Ibarra parametrization [16,25]

The well-known Casas-Ibarra parametrization [16] is

$$M_D \simeq iV_0 \sqrt{D_\nu} \Omega \sqrt{D_N}, \quad (\text{A1})$$

where the approximate seesaw formula, $M_\nu \simeq -M_D D_N^{-1} M_D^T$, has been employed and Ω is an orthogonal matrix. Applying this parametrization to the special case of only two RH neutrinos [25], called the Casas-Ibarra-Ross parametrization, Ω satisfies $\Omega \Omega^T = \text{diag}\{0, 1, 1\}$ or $\{1, 1, 0\}$ in the NH or IH, respectively, and $\Omega^T \Omega = \mathbf{I}_{2 \times 2}$, so it can be represented as

$$\Omega = \begin{pmatrix} 0 & 0 \\ \cos \omega & -\sin \omega \\ \xi \sin \omega & \xi \cos \omega \end{pmatrix} \text{ (NH), \quad or \quad } \Omega = \begin{pmatrix} \cos \omega & -\sin \omega \\ \xi \sin \omega & \xi \cos \omega \\ 0 & 0 \end{pmatrix} \text{ (IH),} \quad (\text{A2})$$

where $\xi = \pm 1$ and $\omega = |\omega| e^{i \arg \omega}$. The entries of M_D for the NH and IH cases are shown,

$$(M_D)_{a1} \simeq \begin{cases} +i\sqrt{M_1}((V_0)_{a2}\sqrt{m_2} \cos \omega + \xi(V_0)_{a3}\sqrt{m_3} \sin \omega) & \text{(NH),} \\ +i\sqrt{M_1}((V_0)_{a1}\sqrt{m_1} \cos \omega + \xi(V_0)_{a2}\sqrt{m_2} \sin \omega) & \text{(IH),} \end{cases} \quad (\text{A3})$$

$$(M_D)_{a2} \simeq \begin{cases} -i\sqrt{M_2}((V_0)_{a2}\sqrt{m_2} \sin \omega - \xi(V_0)_{a3}\sqrt{m_3} \cos \omega) & \text{(NH),} \\ -i\sqrt{M_2}((V_0)_{a1}\sqrt{m_1} \sin \omega - \xi(V_0)_{a2}\sqrt{m_2} \cos \omega) & \text{(IH),} \end{cases} \quad (\text{A4})$$

where $\alpha = e, \mu, \tau$ is a flavor index. This parametrization contains 11 real parameters in total, i.e., seven parameters in light neutrinos [two masses $m_2, m_3(m_1)$, three mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$, one Dirac phase δ_{CP} , and one Majorana phase σ], and four new independent parameters from heavy neutrinos ($M_1, M_2, |\omega|$, and $\arg \omega$), as summarized in Table I.

2. Natural reconstruction [22]

In the MSM, the Dirac matrix M_D of this parametrization including six arbitrary complex entries is expressed as

$$M_D = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}. \quad (\text{A5})$$

The approximate seesaw formula, $M_\nu \simeq -M_D D_N^{-1} M_D^T$, together with $\det(M_\nu) = 0$ in the MSM, provides 6 – 1 identities. If M_1, M_2 are given and any one entry of M_D is known, all the rest of the entries can be solved analytically. Without loss of generality, we assume $a_1 = |a_1|e^{i \arg a_1}$ is known. Then, the rest of the entries are solved to be

$$\begin{aligned} b_1 &\simeq \zeta_1 \sqrt{-(M_\nu)_{11} M_2 - \frac{M_2}{M_1} a_1^2}, \\ a_\alpha &\simeq \frac{1}{(M_\nu)_{11}} \left\{ a_1 (M_\nu)_{1\alpha} + \zeta_\alpha b_1 \sqrt{\frac{M_1}{M_2}} \sqrt{(M_\nu)_{11} (M_\nu)_{\alpha\alpha} - (M_\nu)_{1\alpha}^2} \right\}, \quad \alpha = 2, 3, \\ b_\alpha &\simeq \frac{1}{(M_\nu)_{11}} \left\{ b_1 (M_\nu)_{1\alpha} - \zeta_\alpha a_1 \sqrt{\frac{M_2}{M_1}} \sqrt{(M_\nu)_{11} (M_\nu)_{\alpha\alpha} - (M_\nu)_{1\alpha}^2} \right\}, \quad \alpha = 2, 3. \end{aligned} \quad (\text{A6})$$

Here, $\alpha = 1, 2, 3$ is a flavor index representing e, μ, τ . Therefore, the four independent parameters from heavy neutrinos are $M_1, M_2, |a_1|$, and $\arg a_1$, as summarized in Table I. One can replace $|a_1|$ and $\arg a_1$ by any $|a_i|$ and $\arg a_i$ or $|b_1|$ and $\arg b_1$. This parametrization is called the natural reconstruction in the review [21].

3. Blennow–Fernandez-Martinez parametrization [55]

In MSM, the 5×5 unitary matrix $\mathcal{U}$ which diagonalizes the neutrino mass matrix is parametrized as

$$\mathcal{U} = \begin{pmatrix} c & s \\ -s^\dagger & \hat{c} \end{pmatrix} \begin{pmatrix} V_0 & 0 \\ 0 & I \end{pmatrix}, \quad (\text{A7})$$

where the nonphysical heavy-heavy mixing angle and phase have been omitted and V_0 is the standard PMNS matrix (taking $\rho = 0$). The first matrix on the right side of the above equation is a unitary rotation matrix expressed as a series expansion of the matrix exponential,

$$\begin{pmatrix} c & s \\ -s^\dagger & \hat{c} \end{pmatrix} \equiv \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-\Theta\Theta^\dagger)^n}{(2n)!} & \sum_{n=0}^{\infty} \frac{(-\Theta\Theta^\dagger)^n}{(2n+1)!} \Theta \\ -\sum_{n=0}^{\infty} \frac{(-\Theta^\dagger\Theta)^n}{(2n+1)!} \Theta^\dagger & \sum_{n=0}^{\infty} \frac{(-\Theta^\dagger\Theta)^n}{(2n)!} \end{pmatrix}, \quad (\text{A8})$$

where the 3×2 matrix $\Theta \equiv V_L \theta V_R^\dagger$ is parametrized as

$$\Theta \equiv V_L \begin{pmatrix} 0 & 0 \\ \theta_y & 0 \\ 0 & \theta_z \end{pmatrix} V_R^\dagger \text{ (NH)}, \quad \text{or} \quad \Theta \equiv V_L \begin{pmatrix} \theta_x & 0 \\ 0 & \theta_y \\ 0 & 0 \end{pmatrix} V_R^\dagger \text{ (IH)}. \quad (\text{A9})$$

Here, $(V_L)_{3 \times 3}$ and $(V_R)_{2 \times 2}$ are unitary matrices written as

$$\begin{aligned}
 V_L &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\alpha_2^L} & 0 \\ 0 & 0 & e^{-i\alpha_3^L} \end{pmatrix} \begin{pmatrix} c_1 c_3 & s_1 c_3 & s_3 e^{-i\delta^L} \\ -s_1 c_2 - c_1 s_2 s_3 e^{i\delta_L} & +c_1 c_2 - s_1 s_2 s_3 e^{i\delta^L} & s_2 c_3 \\ +s_1 s_2 - c_1 c_2 s_3 e^{i\delta^L} & -c_1 s_2 - s_1 c_2 s_3 e^{i\delta^L} & c_2 c_3 \end{pmatrix}, \\
 V_R &= \begin{pmatrix} c_R & s_R e^{-i\delta^R} \\ -s_R e^{i\delta^R} & c_R \end{pmatrix},
 \end{aligned} \tag{A10}$$

where $c_i \equiv \cos(\theta_i^L)$, $s_i \equiv \sin(\theta_i^L)$ ($i = 1, 2, 3$), $c_R \equiv \cos(\theta_R)$, and $s_R \equiv \sin(\theta_R)$. The diagonalization of the neutrino mass matrix tells us that M_D can be parametrized as

$$M_D = -V_L^* \cos(\hat{\theta}_L) (V_L^T V_0^* D_\nu V_0^\dagger V_L) \sin(\theta) V_R^\dagger + V_L^* \sin(\theta) (V_R^T D_N V_R) \cos(\hat{\theta}_R) V_R^\dagger, \tag{A11}$$

where $\hat{\theta}_L \equiv \text{diag}\{0, \theta_y, \theta_z\}$ (NH) or $\text{diag}\{\theta_x, \theta_y, 0\}$ (IH), and $\hat{\theta}_R \equiv \text{diag}\{\theta_y, \theta_z\}$ (NH) or $\text{diag}\{\theta_x, \theta_y\}$ (IH). Following the diagonalization procedure, the unitary V_R and V_L can be connected through

$$V_L^T (V_0^* D_\nu V_0^\dagger) V_L = -\tan \theta (V_R^T D_N V_R) \tan \theta^T. \tag{A12}$$

After given the introduced θ_y and $\theta_z(\theta_x)$, we can solve other parameters in V_L and V_R .⁴ If considering $D_N \gg M_D$ and assuming θ is tiny, M_D can be approximated to $(V_L^* \theta V_R^T) D_N$ and (A12) will return to the well-known approximate seesaw formula. Therefore, in this parametrization, M_D is parametrized by seven parameters in light neutrinos, two heavy masses M_1 and M_2 , and θ_y and $\theta_z(\theta_x)$, as summarized in Table I.

4. Modified Casas-Ibarra-Ross parametrization [56]

Based on M_D parametrized as Eq. (A1), a Hermitian matrix P is introduced,

$$P \equiv M_D M_D^\dagger \simeq V_0 \sqrt{D_\nu} (\Omega D_N \Omega^\dagger) \sqrt{D_\nu} V_0^\dagger. \tag{A14}$$

In the MSM, due to $m_{\text{lightest}} = 0$, the orthogonal condition $(V_0^\dagger P)_{1\alpha} = 0$ in the NH and $(V_0^\dagger P)_{3\alpha} = 0$ in the IH ($\alpha = e, \mu, \tau$) can correlate the diagonal entries $P_{\alpha\alpha}$ with P_{12} , P_{13} , P_{23} . For example, in the NH scenario, there are

$$\begin{aligned}
 P_{11} &= -\frac{P_{12}^*(V_0^*)_{21} + P_{13}^*(V_0^*)_{31}}{(V_0^*)_{11}}, \\
 P_{22} &= -\frac{P_{12}(V_0^*)_{11} + P_{23}^*(V_0^*)_{31}}{(V_0^*)_{21}}, \\
 P_{33} &= -\frac{P_{13}(V_0^*)_{11} + P_{23}(V_0^*)_{21}}{(V_0^*)_{31}},
 \end{aligned} \tag{A15}$$

and the two complex phases $\arg P_{13}$ and $\arg P_{23}$ can be solved by P_{12} , $|P_{13}|$, $|P_{23}|$,

$$\begin{aligned}
 \arg P_{13} &= \frac{-i \text{Im}[P_{12}(V_0)_{21}(V_0^*)_{11}] + \xi \sqrt{|P_{13}|^2 |(V_0)_{11}|^2 |(V_0)_{31}|^2 - [\text{Im}(P_{12}(V_0)_{21}(V_0^*)_{11})]^2}}{|P_{13}| (V_0)_{31} (V_0^*)_{11}}, \\
 \arg P_{23} &= \frac{+i \text{Im}[P_{12}(V_0)_{21}(V_0^*)_{11}] + \xi \sqrt{|P_{23}|^2 |(V_0)_{21}|^2 |(V_0)_{31}|^2 - [\text{Im}(P_{12}(V_0)_{21}(V_0^*)_{11})]^2}}{|P_{23}| (V_0)_{31} (V_0^*)_{21}},
 \end{aligned} \tag{A16}$$

⁴We can define $X \equiv V_L^* \tan(\theta) V_R^T$ and then the above Eq. (A12) is equal to $M_\nu^* \equiv V_0^* D_\nu V_0^\dagger = -X D_N X^T$. Here, if we regard $|X_{11}|$ and $\arg X_{11}$ as independent parameters, the rest of the entries of $(X)_{3 \times 2}$ can be solved algebraically. Because the matrix θ satisfies $\lambda(X^\dagger X) = \lambda((\tan \theta)^\dagger \tan \theta)$, where $\lambda(A)$ means eigenvalues of A , we can obtain the eigenvalues of $X^\dagger X$,

$$n_\pm^2 = \frac{1}{2} \sum_{i,j} |X_{ij}|^2 \pm \sqrt{\left(\frac{1}{2} \sum_{i,j} |X_{ij}|^2\right)^2 - \sum_{i \neq j} |X_{i1} X_{j2} - X_{i2} X_{j1}|^2}, \tag{A13}$$

where $n_\pm^2 = \tan \theta_y^2, \tan \theta_z^2$ for the NH or $\tan \theta_x^2, \tan \theta_y^2$ for the IH, respectively, and $i = 1, 2, 3$ and $j = 1, 2$. Combining (A13) with $X = V_L^* \tan(\theta) V_R^T$, V_R and V_L can be solved through $XX^\dagger = V_L \tan \theta (\tan \theta)^\dagger V_L^\dagger$ and $X^\dagger X = V_R (\tan \theta)^\dagger \tan \theta V_R^\dagger$.

where $\xi = \pm 1$. The analogous procedure can be taken in the IH. This parametrization includes the seven parameters in light neutrinos and the four new parameters $|P_{12}|$, $|P_{13}|$, $|P_{23}|$, and $\arg P_{12}$, as summarized in Table I.

5. Bi-unitary parametrization [57]

The Dirac matrix M_D can be parametrized as

$$M_D = V_L \begin{pmatrix} 0 & 0 \\ d_2 & 0 \\ 0 & d_3 \end{pmatrix} U_R \text{ (NH)} \quad \text{or} \quad M_D = V_L \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \\ 0 & 0 \end{pmatrix} U_R \text{ (IH)} \quad (\text{A17})$$

through the two unitary matrixes V_L and U_R expressed as

$$V_L = \begin{pmatrix} c_1 c_3 & s_1 c_3 & s_3 \\ -c_1 s_2 s_3 - s_1 c_2 e^{-i\delta_L} & -s_1 s_2 s_3 + c_1 c_2 e^{-i\delta_L} & s_2 c_3 \\ -c_1 c_2 s_3 + s_1 s_2 e^{-i\delta_L} & -s_1 c_2 s_3 - c_1 s_2 e^{-i\delta_L} & c_2 c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\gamma_L} & 0 \\ 0 & 0 & e^{+i\gamma_L} \end{pmatrix},$$

$$U_R = \begin{pmatrix} c_R & s_R \\ -s_R & c_R \end{pmatrix} \begin{pmatrix} e^{-i\gamma_R} & 0 \\ 0 & e^{+i\gamma_R} \end{pmatrix}, \quad (\text{A18})$$

where $c_i = \cos \theta_i$, $s_i = \sin \theta_i$ ($i = 1, 2, 3$), $c_R = \cos \theta_R$, and $s_R = \sin \theta_R$. This parametrization contains four rotation angles $\theta_1, \theta_2, \theta_3, \theta_R$, three CP -violating phases $\delta_L, \gamma_L, \gamma_R$, two light masses in the mass eigenvalue equation of $M_D^\dagger M_D$, and two parameters $d_1(d_3)$ and d_2 , as summarized in Table I. It is emphasized that V_L is not the PMNS matrix, which is approximately given by $V_0 \simeq V_L K_R$. Here, K_R is obtained by diagonalizing the following mass matrix, e.g., in the NH case,

$$\begin{pmatrix} 0 & 0 \\ d_2 & 0 \\ 0 & d_3 \end{pmatrix} U_R D_N^{-1} U_R^T \begin{pmatrix} 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} = K_R D_\nu K_R^T, \quad (\text{A19})$$

where the approximate seesaw formula has been applied.

6. Vector representation [58]

The M_D can be parametrized as

$$M_D = (m_{D1} \mathbf{u}_1, m_{D2} \mathbf{u}_2), \quad (\text{A20})$$

where m_{D1} and m_{D2} are two real mass parameters and $\mathbf{u}_i = (u_{ei}, u_{\mu i}, u_{\tau i})^T$ (for $i = 1, 2$) are two unit vectors satisfying $\mathbf{u}_1^\dagger \mathbf{u}_1 = \mathbf{u}_2^\dagger \mathbf{u}_2 = 1$. Without loss of generality, $\mathbf{u}_1$ and $\mathbf{u}_2$ are taken as real and complex, respectively. The seven independent parameters in M_D under the normalized condition can be set as $u_{e1}, u_{\mu 1}, |u_{e2}|, |u_{\mu 2}|$ and three phases $\arg u_{e2}, \arg u_{\mu 2}, \arg u_{\tau 2}$. Then, $u_{\tau 1}$ and $|u_{\tau 2}|$ can be solved through

$$u_{\tau 1} = \xi_1 \sqrt{1 - u_{e1}^2 - u_{\mu 1}^2},$$

$$|u_{\tau 2}| = \xi_2 \sqrt{1 - |u_{e2}|^2 - |u_{\mu 2}|^2}, \quad (\text{A21})$$

where $\xi_{1,2} = \pm 1$. After applying the approximate seesaw formula, $M_\nu \simeq -M_D D_N^{-1} M_D^T$, m_{D1} and m_{D2} can be obtained via $\lambda(M_D D_N^{-1} M_D^T M_D^* D_N^{-1} M_D^\dagger) = \lambda(M_\nu M_\nu^\dagger)$, where $\lambda(A)$ means the eigenvalues of A matrix, so there are

$$m_{D1,D2}^4 \simeq M_{1,2}^2 \left(\frac{n_-^2 + n_+^2}{2} - \text{Re}[\mathbf{u}_1^\dagger \mathbf{u}_2]^2 \frac{n_- n_+}{1 - |\mathbf{u}_1^\dagger \mathbf{u}_2|^2} \right) \pm M_{1,2}^2 \sqrt{\left(\frac{n_-^2 + n_+^2}{2} - \text{Re}[\mathbf{u}_1^\dagger \mathbf{u}_2]^2 \cdot \frac{n_- n_+}{1 - |\mathbf{u}_1^\dagger \mathbf{u}_2|^2} \right)^2 - \left(\frac{n_- n_+}{1 - |\mathbf{u}_1^\dagger \mathbf{u}_2|^2} \right)^2}. \quad (\text{A22})$$

Here, $n_\pm^2$ are eigenvalues of $M_\nu M_\nu^\dagger$, i.e., $n_- = m_2$, $n_+ = m_3$ in the NH and $n_- = m_1$, $n_+ = m_2$ in the IH. This parametrization contains four mass parameters $m_2, m_3(m_1), M_1, M_2$ and seven independent parameters in unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, as summarized in Table I.

APPENDIX B: RELATED LOOP FUNCTIONS

The related functions about the one-loop correction are listed below to analyze the cLFV processes in Sec. IV A and more details are shown in Ref. [32]. Here, $\mathcal{U}$ is the unitary matrix as Eq. (8), R is the active-sterile mixing matrix as Eq. (14), $\alpha, \beta = e, \mu, \tau$ are flavor indices, $i, j = 1, 2, 3, 4, 5$ represent all neutrinos, and $C_{ij} = \sum_{\alpha} \mathcal{U}_{\alpha i}^* \mathcal{U}_{\alpha j}$. The related loop functions are

$$\begin{aligned}
G_{\gamma}(x) &= -\frac{x(2x^2 + 5x - 1)}{4(1-x)^3} - \frac{3x^3}{2(1-x)^4} \ln x, \\
F_{\gamma}(x) &= \frac{x(7x^2 - x - 12)}{12(1-x)^3} - \frac{x^2(x^2 - 10x + 12)}{6(1-x)^4} \ln x, \\
F_Z(x) &= -\frac{5x}{2(1-x)} - \frac{5x^2}{2(1-x)^2} \ln x, \\
G_Z(x, y) &= -\frac{1}{2(x-y)} \left[\frac{x^2(1-y)}{1-x} \ln x - \frac{y^2(1-x)}{1-y} \ln y \right], \\
H_Z(x, y) &= \frac{\sqrt{xy}}{4(x-y)} \left[\frac{x^2 - 4x}{1-x} \ln x - \frac{y^2 - 4y}{1-y} \ln y \right], \\
F_{\text{Box}}(x, y) &= \frac{1}{x-y} \left\{ \left(4 + \frac{xy}{4} \right) \left[\frac{1}{1-x} + \frac{x^2}{(1-x)^2} \ln x - \frac{1}{1-y} - \frac{y^2}{(1-y)^2} \ln y \right] \right. \\
&\quad \left. - 2xy \left[\frac{1}{1-x} + \frac{x}{(1-x)^2} \ln x - \frac{1}{1-y} - \frac{y}{(1-y)^2} \ln y \right] \right\}, \\
F_{\text{XBox}}(x, y) &= \frac{-1}{x-y} \left\{ \left(1 + \frac{xy}{4} \right) \left[\frac{1}{1-x} + \frac{x^2}{(1-x)^2} \ln x - \frac{1}{1-y} - \frac{y^2}{(1-y)^2} \ln y \right] \right. \\
&\quad \left. - 2xy \left[\frac{1}{1-x} + \frac{x}{(1-x)^2} \ln x - \frac{1}{1-y} - \frac{y}{(1-y)^2} \ln y \right] \right\}, \\
G_{\text{Box}}(x, y) &= \frac{-\sqrt{xy}}{x-y} \left\{ (4 + xy) \left[\frac{1}{1-x} + \frac{x}{(1-x)^2} \ln x - \frac{1}{1-y} - \frac{y}{(1-y)^2} \ln y \right] \right. \\
&\quad \left. - 2 \left[\frac{1}{1-x} + \frac{x^2}{(1-x)^2} \ln x - \frac{1}{1-y} - \frac{y^2}{(1-y)^2} \ln y \right] \right\}. \tag{B1}
\end{aligned}$$

In terms of these functions, the formal factors are

$$\begin{aligned}
G_{\gamma}^{\alpha\beta} &= \sum_i \mathcal{U}_{\alpha i}^* \mathcal{U}_{\beta i} G_{\gamma}(x_i) = \sum_{N_i} R_{\alpha N_i}^* R_{\beta N_i} G_{\gamma}(x_{N_i}), \\
F_{\gamma}^{\alpha\beta} &= \sum_i \mathcal{U}_{\alpha i}^* \mathcal{U}_{\beta i} F_{\gamma}(x_i) = \sum_{N_i} R_{\alpha N_i}^* R_{\beta N_i} F_{\gamma}(x_{N_i}), \tag{B2} \\
F_Z^{\alpha\beta} &= \sum_{ij} \mathcal{U}_{\alpha j}^* \mathcal{U}_{\beta i} [\delta_{ij} F_Z(x_i) + C_{ij} G_Z(x_i, x_j) + C_{ij}^* H_Z(x_i, x_j)] \\
&= \sum_{N_i N_j} R_{\alpha N_j}^* R_{\beta N_i} \{ \delta_{N_i N_j} (F_Z(x_{N_i}) + 2G_Z(0, x_{N_i})) \\
&\quad + C_{N_i N_j} [G_Z(x_{N_i}, x_{N_j}) - G_Z(0, x_{N_i}) - G_Z(0, x_{N_j})] + C_{N_i N_j}^* H_Z(x_{N_i}, x_{N_j}) \}, \\
F_{\text{Box}}^{\alpha\beta\beta\beta} &= \sum_{ij} \mathcal{U}_{\alpha j}^* \mathcal{U}_{\beta i} [\mathcal{U}_{\beta j}^* \mathcal{U}_{\beta i} G_{\text{Box}}(x_i, x_j) - 2\mathcal{U}_{\beta j} \mathcal{U}_{\beta i}^* F_{\text{XBox}}(x_i, x_j)] \\
&= -2 \sum_{N_i} R_{\alpha N_i}^* R_{\beta N_i} [F_{\text{XBox}}(x_{N_i}, 0) - F_{\text{XBox}}(0, 0)] + \sum_{N_i N_j} R_{\alpha N_j}^* R_{\beta N_i} \{ R_{\beta N_j}^* R_{\beta N_i} G_{\text{Box}}(x_{N_i}, x_{N_j}) \\
&\quad - 2R_{\beta N_j} R_{\beta N_i}^* [F_{\text{XBox}}(x_{N_i}, x_{N_j}) - F_{\text{XBox}}(0, x_{N_j}) - F_{\text{XBox}}(x_{N_i}, 0) + F_{\text{XBox}}(0, 0)] \}, \\
F_{\frac{1}{2}\text{Box}+Z}^{\alpha\beta\beta\beta} &= \frac{1}{2} F_{\text{Box}}^{\alpha\beta\beta\beta} + F_Z^{\alpha\beta}, \quad \text{and} \quad F_{Z-\gamma}^{\alpha\beta} = 2s_w^2 (F_Z^{\alpha\beta} - F_{\gamma}^{\alpha\beta}), \tag{B3}
\end{aligned}$$

where $x_i = m_i^2/M_W^2$ ($m_4 = M_1, m_5 = M_2$) and $x_{N_i} = M_{N_i}^2/m_W^2$. The functions about quarks are

$$\begin{aligned}
 F_u(x) &= \frac{2}{3}s_W^2 G_\gamma(x) + \frac{2}{3}s_W^2 [F_\gamma(x) - F_Z(x) - 2G_Z(0, x)] \\
 &\quad + \frac{1}{4}[F_Z(x) + 2G_Z(0, x) + F_{\text{Box}}(x, 0) - F_{\text{Box}}(0, 0)], \\
 F_d(x) &= -\frac{1}{3}s_W^2 G_\gamma(x) - \frac{1}{3}s_W^2 [F_\gamma(x) - F_Z(x) - 2G_Z(0, x)] \\
 &\quad - \frac{1}{4}[F_Z(x) + 2G_Z(0, x) - F_{\text{XBox}}(x, 0) + F_{\text{XBox}}(0, 0)].
 \end{aligned} \tag{B4}$$

When $x \gg 1$, $F_u(x)$ and $F_d(x)$ can be simplified as

$$\begin{aligned}
 F_u(x) &\xrightarrow{x \gg 1} \frac{2}{3}s_W^2 \frac{16 \ln(x) - 31}{12} - \frac{3 + 3 \ln(x)}{8}, \\
 F_d(x) &\xrightarrow{x \gg 1} -\frac{1}{3}s_W^2 \frac{16 \ln(x) - 31}{12} - \frac{3 - 3 \ln(x)}{8}.
 \end{aligned} \tag{B5}$$

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