Coarse graining holographic black holes in higher curvature gravity

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We consider the holographic description of the dynamical black hole entropy in $f(R)$ higher curvature gravity which proposed by Hollands-Wald-Zhang. On the bulk side, we show that the coarse-grained entropy (outer entropy) of a generalized marginally trapped surface corresponds precisely to the Wald entropy associated with this surface. To get this result, we first formulate the AdS/CFT correspondence in the Einstein frame and derive the correspondence between the von Neumann entropy in the Einstein frame and the $f(R)$ frame. This facilitates the derivation of the correspondence between the two outer entropies in the two frames. Furthermore, we directly derive a focusing theorem associated with generalized expansion in $f(R)$ gravity. We then formulate how to construct the stationary null hypersurface for the generalized expansion and the junction condition to glue a hypersurface in $f(R)$ gravity. Combining these results, we directly derive the expression for the outer entropy in the $f(R)$ frame and identify its holographic dual.

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I. INTRODUCTION

In the nondynamical regime, black hole entropy is conventionally governed by the area law (Bekenstein-Hawking entropy) [1],

$$S_{\mathrm{BH}} = \frac{\mathrm{Area[horizon]}}{4G}. \tag{1}$$

This prescription was elegantly extended to higher-curvature theories by Iyer and Wald [2,3], who identified the entropy as a Noether charge localized on the bifurcation surface $\mathcal{B}$. In particular, for $f(R)$ gravity,

$$S_{\mathrm{Wald}} = \frac{\int_{\mathcal{B}} f'(R)}{4G}. \tag{2}$$

Following Iyer and Wald's attempt to generalize the Noether charge formalism to dynamical black holes, an entropy formula for higher curvature theories [f(Riemann) gravity] was derived by Wall [4],

$$S_{\mathrm{Wall}} = -8\pi \int_{\mathcal{C}(v)} \left(\frac{\partial L}{\partial R_{uvuv}} - 4 \frac{\partial^2 L}{\partial R_{uiuj} \partial R_{vkvl}} K_{ij(u)} K_{kl(v)} \right). \tag{3}$$

Where $\mathcal{C}(v)$ is a cross section of the horizon, $K_{ij(a)}$ is the extrinsic curvature of the horizon in the a direction. Remarkably, the Wall entropy matches to the holographic entanglement entropy computed by Dong [5] for higher curvature gravity. Since the Dong and Wall entropies are derived in completely different contexts, it is not well understood whether their agreement is a coincidence or whether it holds more generally.

However, extending this framework to truly dynamical settings remains a nontrivial task. Away from the bifurcation surface, the evolution of the horizon introduces fundamental challenges in defining a unique, physical entropy. Two decades after the attempt of Iyer-Wald for generalizing the Noether charge formalism to dynamical black holes [2,3], significant progress in approaching the dynamical black hole entropy was made in [6,7]. By applying the improved Noether charge method, a new notion of dynamical black hole entropy in general diffeomorphism-invariant theories of gravity under first order nonstationary perturbation, denoted by S_{dyn}, was introduced,

$$S_{\mathrm{dyn}} = \frac{2\pi}{\kappa} \int_{\mathcal{C}} \tilde{\mathbf{Q}}_\xi = \frac{2\pi}{\kappa} \int_{\mathcal{C}} (\mathbf{Q}_\xi - \xi \cdot \mathbf{B}_{\mathcal{H}^+}), \tag{4}$$

where $\mathbf{Q}_\xi$ is the Noether charge of the horizon Killing field ξ^a, and $\mathbf{B}_{\mathcal{H}^+}$ is a specific $(n-1)$-form constructed on the

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future horizon $\mathcal{H}^+$. In Einstein gravity, this entropy takes the form $S_{\rm dyn} = (1 - v\partial_v)S_{\rm BH}$,[1] where v is an affine parameter along the horizon.

Significant progress about dynamical black hole entropy has also been made in various contexts, including black holes far from equilibrium [9,10], semiclassical interpretations of dynamical black holes [11–13], charged black holes [14], cosmology and dS spacetime [15–17], first-order generalized expansions[2] (or entropic expansions) and generalized focusing theorems [20–22], and dynamical black holes in the Einstein frame [21].

In this paper, we will propose a holographic interpretation of dynamical black holes entropy in $f(R)$ gravity. To do this, we will generalize the coarse grained black hole entropy proposed by Engelhardt and Wall [23,24] to $f(R)$ gravity. For these theories, we demonstrate that the dynamical entropy is $S_{\rm dyn} = (1 - v\partial_v)S_{\rm Wald}$. In particular, in $f(R)$ gravity, it has been shown in [13,20,21] that $S_{\rm dyn}$ for $f(R)$ gravity equals $S_{\rm Wald}$ of the generalized apparent horizon. Motivated by these developments, we generalize the holographic coarse-grained black hole entropy to $f(R)$ gravity; we demonstrate that the outer entropy of the generalized minimar surface μ is equal to the Wald entropy of μ,

$$S_f^{(\rm outer)}[\mu] = \frac{\int_\mu f'(R)}{4G}, \tag{5}$$

and find its boundary dual which is the simple entropy. This agrees with $S_{\rm dyn}$ in $f(R)$ gravity, and we will show that both the outer entropy and the simple entropy satisfy the second law. Therefore, it is natural to interpret the outer entropy and the simple entropy as the holographic interpretations of $S_{\rm dyn}$ in $f(R)$ gravity. This construction is nonperturbative in the classical gravity (valid, at minimum, to all orders in perturbation theory near equilibrium).

Since the matter fields will satisfy the null curvature condition (NCC) in the Einstein frame if they satisfy the null energy condition (NEC) in the $f(R)$ frame,[3] the focusing theorem will be satisfied in the Einstein frame. Therefore, we will derive the outer entropy in the Einstein frame. It follows that we can evaluate the outer entropy in the $f(R)$ frame by the von Neumann entropy correspondence between the Einstein frame and the $f(R)$ frame. We derive a nonlinear Raychaudhuri equation for the generalized expansion (6) which will guarantee the focusing theorem in $f(R)$ gravity, we will use this equation to derive the outer entropy directly in the $f(R)$ frame. This focusing theorem in $f(R)$ gravity will also help us define the boundary dual of the outer entropy.

This paper is organized as follows. In Sec. II, we derive the correspondence between the Einstein frame and the $f(R)$ frame in asymptotic anti–de Sitter (AdS) spacetime and discuss the NCC condition and junction conditions in the Einstein frame. In Sec. III, we derive the outer entropy in the Einstein frame. In Sec. IV, we will show the von Neumann entropy correspondence between the Einstein frame and the $f(R)$ frame, and using this correspondence show the result (5). In Sec. V, we derive the focusing theorem in $f(R)$ gravity and directly show (5) in the $f(R)$ frame. Finally, in Sec. VI, we show that the simple entropy is the boundary dual of the outer entropy and discuss the second law.

A. Assumptions, conventions, and definitions

This section establishes assumptions, conventions, and definitions that will be used throughout the paper.

$f(R)$ Frame: In order to distinguish from the Einstein frame and for the convenience of subsequent descriptions, we use the $f(R)$ frame to denote the frame in which the gravitational Lagrangian is $f(R)$. We use the superscripts E and f to emphasize which frame we are working in.

Generalized expansion: Let N_k be a null hypersurface with generating vector field k^a. For $f(R)$ gravity, we define the generalized expansion Θ_k,

$$\Theta_k = \theta_k + k^a\nabla_a \log f'(R), \tag{6}$$

where θ_k is the ordinary expansion (details in Sec. V A). Assume that $k^a = (\partial_v)^a$ is affinely parametrized and v is the affine parameter. We can rewrite it as

$$\Theta_k = \Theta_v = \partial_v \log\big(\sqrt{\gamma} f'(R)\big). \tag{7}$$

Generalized extremal surface: A surface X is generalized extremal if the generalized expansions of the two null orthogonal congruences fired from it both vanish,

$$\begin{aligned}\Theta_l &= 0,\\ \Theta_k &= 0.\end{aligned} \tag{8}$$

Generalized marginally trapped surface: A surface is generalized marginal if the generalized expansion of the two null orthogonal congruences fired from μ satisfy,

$$\begin{aligned}\Theta_l &< 0\\ \Theta_k &= 0.\end{aligned} \tag{9}$$

Generalized minimar surface: A generalized marginal surface μ will be called a generalized minimar surface if it additionally satisfies the following criteria:

[1]This formula was also provided in [8] along with an alternative formula which vanishes on any cross section of a light cone in Minkowski spacetime.

[2]We emphasize that the generalized here is the expansion for Wald entropy (6), not the expansion for the generalized entropy [18,19] used in quantum focusing theorem.

[3]We use the $f(R)$ frame to denote the frame of $f(R)$ gravity, see Sec. I A.

(1) μ is homologous to B [a entire connected component of the conformal field theory (CFT) at one time], and there exists a Cauchy slice $\Sigma_{\min}[\mu]$ of the outer wedge $O_W^f[\mu]$ on which μ is a minimal Wald entropy surface homologous to B.
(2) There exists a choice of normalization for l^a such that $\nabla_k\Theta_l = k^a\nabla_a\Theta_l \le 0$ on μ, with equality allowed only if $\Theta_l = 0$ everywhere on μ.

II. ADS/CFT IN THE EINSTEIN FRAME

In this section, we will assume the AdS/CFT correspondence in the original spacetime, and we work in the large-N, large-λ limit, in which the bulk $\mathcal{M}$ is well-approximated by classical gravity [25,26]. We aim to demonstrate that if the spacetime is asymptotically AdS in the $f(R)$ frame, it remains asymptotically AdS in the Einstein frame. Furthermore, we show that matter fields satisfying the NEC in the $f(R)$ frame imply the satisfaction of the NCC in the Einstein frame. Finally, we will give the junction condition for gluing spacetime regions across a codimension-2 surface.

A. AdS/CFT correspondence in the Einstein frame

Consider the action I_f for $f(R)$ gravity in the $f(R)$ frame, given by

$$I_f = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}(f(R)-2\Lambda) + I_{\text{matter}}. \tag{10}$$

The modified Einstein equation in $f(R)$ gravity is

$$f'(R)R_{ab} - \frac{1}{2}g_{ab}f(R) + (g_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)f'(R) + g_{ab}\Lambda = 8\pi G T_{ab}, \tag{11}$$

here the Latin indices abc are the abstract indices. Now we assume that

$$f'(R) > 0, \quad f''(R) \neq 0 \tag{12}$$

[4]since $f'(R) > 0$ ensures the positivity of the effective gravitational constant $G_{\text{eff}} = G/f'(R)$ and $f''(R) \neq 0$ will make our $f(R)$ gravity different from the Einstein gravity [21,27,28].

We introduce an auxiliary field Ψ and modify the action of $f(R)$ gravity as follows:

$$I_E = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}(f'(\Psi)R + f(\Psi) - \Psi f'(\Psi) - 2\Lambda) + I_{\text{matter}}. \tag{13}$$

The on shell condition for the auxiliary field Ψ is

$$\Psi = R, \tag{14}$$

reducing the modified action I_E to the original action I_f.

We now proceed to the Einstein frame by introducing the Weyl transformation,

$$g^E_{ab} = (f'(\Psi))^{\frac{2}{D-2}}g_{ab}. \tag{15}$$

One can easily verify that $\Omega^2 = (f'(\Psi))^{\frac{2}{D-2}} > 0$ (we assume $\Omega > 0$). Furthermore, the Weyl transformation preserves the causal structure of the spacetime. We redefine the auxiliary field Ψ by

$$\psi = \frac{1}{\sqrt{16\pi G}}\sqrt{\frac{2(D-1)}{D-2}}\log f'(\Psi). \tag{16}$$

We use the metric g^E_{ab} and ψ rewrite the gravitational action [21,29,30],

$$S_f = \int d^Dx\sqrt{-g^E}\left(\frac{R^E}{16\pi G} - \frac{1}{2}g^E_{ab}\partial^a\psi\partial^b\psi - V_0(\psi) - \frac{2}{16\pi G}(f'(\Psi))^{-\frac{D}{D-2}}\Lambda\right) + S_{\text{matter}}, \tag{17}$$

where

$$V_0(\psi) = \frac{(f'(\Psi))^{-\frac{D}{D-2}}}{16\pi G}(\Psi f'(\Psi) - f(\Psi)). \tag{18}$$

Suppose in the $f(R)$ frame, the on shell asymptotic AdS curvature is R_0. From (11), we can determine the vacuum solution,

$$f'(R_0)r_0 - \frac{D}{2}f(R_0) + D\Lambda = 0. \tag{19}$$

One can show that, for the on shell action, the potential and cosmological terms in the Einstein frame combine to form a new cosmological constant Λ',

$$V_0(R_0) + \frac{2}{16\pi G}(f'(R_0))^{-\frac{D}{D-2}}\Lambda = \frac{2}{16\pi G}\left(\frac{D-2}{2D}f'(R_0)^{-\frac{2}{D-2}}R_0\right) = \frac{2}{16\pi G}\Lambda'. \tag{20}$$

To obtain the standard form of the action in the Einstein frame, we define a new potential $V(\psi)$,

[4]Almost all the results in this paper will only rely on $f'(R) > 0$ except when we try to construction stationary null hypersurface (134). In that case, it will simplify our discussion if we assume $f''(R) \neq 0$, but the results actually independent of the condition $f''(R) \neq 0$. Appendix discusses the cases of $f''(R) = 0$.

$$V(\psi) = V_0(\psi) + \frac{2}{16\pi G}(f'(\Psi))^{-\frac{D}{D-2}}\Lambda - \frac{2\Lambda'}{16\pi G}. \quad (21)$$

The action I_E can then be rewritten in the Einstein frame as

$$I_E = \int d^D x\sqrt{-g^E} \times \left(\frac{R_E - 2\Lambda'}{16\pi G} - \frac{1}{2}g^E_{ab}\partial^a\psi\partial^b\psi - V(\psi)\right) + I_{\text{matter}}. \quad (22)$$

This ensures that the new potential $V(\psi)$ vanishes asymptotically near the AdS boundary. Finally, to ensure the field satisfies a well-defined boundary condition, we define a shifted field ϕ,

$$\phi = \psi - \frac{1}{\sqrt{16\pi G}}\sqrt{\frac{2(D-1)}{D-2}}\log f'(R_0). \quad (23)$$

This ensures that the field ϕ vanishes asymptotically near the AdS boundary. Finally, the action in the Einstein frame should be written as

$$I_E = \int d^D x\sqrt{-g^E} \times \left(\frac{R_E - 2\Lambda'}{16\pi G} - \frac{1}{2}g^E_{ab}\partial^a\phi\partial^b\phi - V(\phi)\right) + I_{\text{matter}}. \quad (24)$$

Asymptotically, the field ϕ and the potential $V(\phi)$ satisfy the following boundary conditions:

$$\lim\phi \to 0, \quad \lim V(\phi) \to 0. \quad (25)$$

Now, from the above constructions, we can derive a correspondence between on shell asymptotic AdS solutions in the $f(R)$ frame and the Einstein frame. On the one hand, every on shell asymptotic AdS solution in the $f(R)$ frame can be translated into an asymptotic AdS solution in the Einstein frame, but the spacetime satisfies the Einstein equation (24) with an additional field ϕ satisfying the boundary condition (25). On the other hand, every on-shell solution of the action (24) with the same cosmological constant Λ' under the boundary condition (25) corresponds to an on shell solution in the $f(R)$ frame via the following steps:

(1) Reconstructing ψ by

$$\psi = \phi + \frac{1}{\sqrt{16\pi G}}\sqrt{\frac{2(D-1)}{D-2}}\log f'(R_0), \quad (26)$$

here R_0 can be obtained by (20);

(2) Using the definition of ψ (16), we can get the conformal factor,

$$f'(R) = \exp\left(\sqrt{16\pi G\frac{D-2}{2(D-1)}}\psi\right), \quad (27)$$

where we use the on shell condition $\Psi = R$;

(3) Finally, we use the conformal factor to get the geometry in the $f(R)$ frame,

$$g_{ab} = (f'(R))^{-\frac{2}{D-2}}g^E_{ab}. \quad (28)$$

Thus, we obtain a one-to-one correspondence between on shell asymptotic AdS solutions in the $f(R)$ frame and the Einstein frame.

As a consistency check, we consider the Weyl transformation of the Ricci scalar [31],

$$R^E(g) = \Omega^{-2}(R - 2(D-1)\nabla^2\log\Omega - (D-1)(D-2)(\nabla\log\Omega)^2). \quad (29)$$

Consider the on shell solution $\Psi = R$, when we approach the asymptotic boundary, Ω approaches a constant, while $\nabla^2\Omega$ and $\nabla\Omega$ vanish asymptotically. Therefore, if the original spacetime is asymptotic AdS with curvature R_0, the spacetime in the Einstein frame remains asymptotic AdS, and the asymptotic scalar curvature in the Einstein frame,

$$R^E_0 = (f'(R_0))^{-\frac{2}{D-2}}R_0. \quad (30)$$

This agrees with our previous result (24), since in the Einstein frame our theory is Einstein gravity, the asymptotic curvature in the Einstein frame,

$$R^E_0 = \frac{2D}{D-2}\Lambda' = f'(R_0)^{-\frac{2}{D-2}}R_0 < 0. \quad (31)$$

Therefore, assuming the validity of the AdS/CFT correspondence in the Einstein frame is justified.

B. NCC and junction conditions in the Einstein frame

In this subsection, we derive the NCC in the Einstein frame,

$$R^E_{ab}k^a k^b \geq 0, \quad (32)$$

assuming the matter satisfies the NEC in the $f(R)$ frame. This is necessary because the derivation of the outer entropy requires the maximin construction [32] of HRT surface and the focusing theorem in the Einstein frame. Furthermore, junction conditions in the Einstein frame are required to glue spacetime regions when we derive the outer entropy in the Einstein frame.

1. NCC condition in the Einstein frame

Consider the NCC in the Einstein frame. The action is given by

$$I_E = \int d^D x \sqrt{-g^E} \times \left(\frac{R_E - 2\Lambda'}{16\pi G} - \frac{1}{2} g^E_{ab} \partial^a \phi \partial^b \phi - V(\phi) \right) + I_{\text{matter}}. \quad (33)$$

Defining the stress tensor in the Einstein frame as T^E_{ab}, one finds that

$$T^E_{ab} = T^{\text{m}}_{ab} + \partial_a \phi \partial_b \phi - g^E_{ab} \left(\frac{1}{2} g_E^{cd} \partial_c \phi \partial_d \phi + V(\phi) \right), \quad (34)$$

where

$$T^{\text{m}}_{ab} = -\frac{2}{\sqrt{-g^E}} \frac{\delta I_{\text{matter}}}{\delta g_E^{ab}} = \frac{T_{ab}}{f'(R)}. \quad (35)$$

Consider a null vector k^a_E, we aim to show that the stress tensor in the Einstein frame satisfies the null energy condition,

$$T^E_{ab} k^a_E k^b_E \geq 0. \quad (36)$$

Since the Weyl transformation preserves the causal structure, the null vector k^a_E remains null in the $f(R)$ frame. Therefore, if the $f(R)$ frame satisfies the null energy condition, then

$$T^{\text{m}}_{ab} k^a_E k^b_E = \frac{T_{ab} k^a_E k^b_E}{f'(R)} \geq 0, \quad (37)$$

provided that $f'(R) > 0$.

It remains to verify that the residual part of the stress tensor satisfies the null energy condition. Since k^a_E is a null vector,

$$g^E_{ab} k^a_E k^b_E = 0, \quad (38)$$

we only need to ensure the kinetic term satisfies the NEC. It is easy to show that

$$k^a_E k^b_E \partial_a \phi \partial_b \phi = (k \cdot \partial \phi)^2 \geq 0. \quad (39)$$

Therefore, we find that the stress tensor in the Einstein frame satisfies the NEC,

$$T^E_{ab} k^a_E k^b_E \geq 0. \quad (40)$$

Consider the equations of motion for gravity in the Einstein frame,

$$R^E_{ab} - \frac{1}{2} g^E_{ab} R^E + \Lambda' g^E_{ab} = 8\pi G T^E_{ab}. \quad (41)$$

It follows that

$$R^E_{ab} k^a_E k^b_E = 8\pi G T^E_{ab} k^a_E k^b_E \geq 0. \quad (42)$$

Thus, we have demonstrated that if the NEC holds in the $f(R)$ frame, the NCC is satisfied in the Einstein frame.

2. Junction conditions in the Einstein frame

Deriving the outer entropy in the Einstein frame requires gluing two spacetime patches through a codimension-2 surface σ, ensuring the resulting spacetime satisfies the Einstein equation and remains free of singularities at the gluing surface σ. Consequently, establishing the proper junction conditions is essential for the gluing procedure in the Einstein frame.

We define the discontinuity of a quantity F across σ, traversing from V_{out} to V_{in}, as

$$[F] = F|_{\sigma_{\text{out}}} - F|_{\sigma_{\text{in}}}. \quad (43)$$

Codimension-2 junction conditions: Define V_{out} and V_{in} as

$$V_{\text{out}} = D[\Sigma_{\text{out}}[\sigma_{\text{out}}]] \quad V_{\text{in}} = D[\Sigma_{\text{in}}[\sigma_{\text{in}}]]. \quad (44)$$

Here, σ is a codimension-2 surface that partitions the Cauchy slice Σ into Σ_{out} and Σ_{in}, σ_{out} and σ_{in} denote the surface σ embedded in V_{out} and V_{in}, respectively. The regions V_{out} and V_{in} can be glued to one another with a finite stress-energy tensor under the following conditions [24]:

(1) The surfaces σ_{out} and σ_{in} are isometric and can thus be identified as a single surface (σ, h), where h is the induced metric of σ;
(2) There exists a choice of k^a_E and l^a_E defined on both side of σ such that the following conditions are satisfied:

$$[\theta^E_{k_E}] = 0, \quad [\theta^E_{l_E}] = 0, \quad [\chi^E_{k_E}] = -[\chi^E_{l_E}] = 0. \quad (45)$$

Here χ^E is the extrinsic twist potential. Then the null-null components of the stress tensor are finite, and the Einstein equation is distributionally well-defined.

To map these junction conditions back to the $f(R)$ frame, we must address a specific subtlety of $f(R)$ gravity; the extra scalar degree of freedom governed by $f'(R)$. In our Einstein frame construction, this degree of freedom is

entirely packaged into the scalar field ϕ via the Weyl transformation. By enforcing the standard codimension-2 junction conditions for the metric in the Einstein frame while simultaneously requiring the continuity of the scalar field ($[\phi] = 0$), we guarantee that the conformal factor $\Omega = (f'(R))^{\frac{1}{D-2}}$ is completely smooth across the junction. Consequently, the inverse Weyl transformation yields a glued geometry in the $f(R)$ frame that naturally inherits a distributionally well-defined stress-energy tensor, avoiding pathological delta-function curvature singularities at the gluing surface.

III. COARSE-GRAINED ENTROPY OF MARGINALLY TRAPPED SURFACES IN THE EINSTEIN FRAME

In the Einstein frame, consider a $(D-2)$-dimensional surface σ. The outer entropy associated with σ (homologous to B, an entire connected component of the CFT) is defined by maximizing the von Neumann entropy over all possible inner wedge data $\{\alpha\}$, while keeping the outer wedge $O_W^E[\sigma]$ fixed [24],

$$S_E^{(\text{outer})}[\sigma] = \max_{\{\alpha\}}[S_{\text{vN}}(\rho_B^{E\{\alpha\}})]. \tag{46}$$

According to the AdS/CFT correspondence in the Einstein frame, each allowed spacetime constructed by attaching an inner wedge $I_W^E[\sigma]$ corresponds to a boundary state $\rho_B^{E\{\alpha\}}$, with von Neumann entropy given by

$$S_{\text{vN}}(\rho_B^{E\{\alpha\}}) = -\text{tr}(\rho_B^{E\{\alpha\}} \log \rho_B^{E\{\alpha\}}) = \frac{\text{Area}_E[X^{\{\alpha\}}]}{4G}, \tag{47}$$

where $X^{\{\alpha\}}$ is the HRT surface homologous to the boundary component B (we will always take B as one of the conformal boundary of the AdS black hole). In the Einstein frame, the stress tensor T_{ab}^E satisfies the NEC, which implies the NCC (as the equations of motion are the Einstein equations). Together with some global assumptions, we can use the maximin construction to define the HRT surface [32]; we first find the surface $\min(B, \Sigma)$ homologous to B that minimizes the area on a given Cauchy slice Σ. We then choose Σ to maximize the area of this minimal surface over all possible Cauchy slices.

We now demonstrate that the outer entropy of a marginally trapped surface μ (satisfying the minimar condition in Einstein frame)[5] is proportional to its area in the Einstein frame. First, we show that the outer entropy of a marginally trapped surface μ is bounded from above by its area,

$$S_E^{(\text{outer})}[\mu] = \max\left[S_{\text{vN}}(\rho_B^{E\{\alpha\}}) = \frac{\text{Area}_E[X^{\{\alpha\}}]}{4G}\right] \leq \frac{\text{Area}_E[\mu]}{4G}. \tag{48}$$

Next, we demonstrate the existence of a configuration $\{\alpha\}$ and a spacetime $(\mathcal{M}^{\{\alpha\}}, g^E)$ for which the von Neumann entropy saturates this bound [23,24]; we will provide a simplified proof below.

A. The upper bound of the outer entropy

We now show that $S_E^{(\text{outer})}[\mu]$ is bounded by $\frac{\text{Area}[\mu]}{4G}$ in the Einstein frame. Assume the surface μ is marginally trapped. Let $N_{\pm k_E}(\mu)$ denote the null congruences in the Einstein frame generated from μ by the null geodesics k_E^a and $-k_E^a$, where k_E^a is future-directed. The representative of μ on Cauchy slice Σ [23,24,32],

$$\bar{\mu} = N_{\pm k_E} \cap \Sigma, \tag{49}$$

since $\bar{\mu}$ is homologous to μ and B, it is also homologous to the HRT surface $X^{\{\alpha\}}$. It follows from the NCC in the Einstein frame that the area of $\bar{\mu}$ is bounded by the area of μ [32],

$$\text{Area}_E[\bar{\mu}] \leq \text{Area}_E[\mu]. \tag{50}$$

Under the NCC, the area of μ is maximal on $N_{\pm k_E}$.

Consider a spacetime $(\mathcal{M}^{\{\alpha\}}, g^E)$ with a fixed outer wedge $O_W^E[\mu]$. By the maximin construction [32], there exists a Cauchy slice Σ such that $X^{\{\alpha\}}$ is the minimal surface on Σ, therefore

$$S_{\text{vN}}(\rho_B^{E\{\alpha\}}) = \frac{\text{Area}_E[X^{\{\alpha\}}]}{4G} \leq \frac{\text{Area}_E[\bar{\mu}]}{4G} \leq \frac{\text{Area}_E[\mu]}{4G}. \tag{51}$$

Since the outer entropy $S^{(\text{outer})}[\mu]$ is the maximal von Neumann entropy, this implies

$$S_E^{(\text{outer})}[\mu] \leq \frac{\text{Area}_E[\mu]}{4G}. \tag{52}$$

B. Construct the geometry that saturates the upper bound

We now construct a spacetime, or equivalently an inner wedge $I_W^E[\mu]$ for which the von Neumann entropy $S_{\text{vN}}(\rho_B^E)$ saturates the upper bound $\text{Area}[\mu]/4G$ in the Einstein frame. In this subsection, we will construct an extremal surface X homologous to B with the same area as the marginally trapped surface μ in the Einstein frame by gluing stationary null hypersurface N_{-k}. We then

[5]Minimar condition: 1) μ is homologous to a connected component B of CFT at one time. And there exists a Cauchy slice Σ in $O_W^E[\mu]$ such that μ has minimal area on Σ; 2) There exists a choice of normalization for l_E^a such that $\nabla_{k_E}^E \theta_{l_E}^E \leq 0$ on μ, with equality allowed only if $\theta_{l_E}^E = 0$ everywhere on μ.

demonstrate the existence of an extremal surface X on N_{-k} and use CPT reflection across X to construct the full spacetime[6] in the Einstein frame. In this spacetime, X is the HRT surface, and this implies that the outer entropy of the marginally trapped surface in the Einstein frame is the Bekenstein-Hawking entropy of the marginally trapped surface [23,24].

We choose a gauge in null coordinates u_E, v_E and spatial coordinates x^i. Fixing μ at $u_E = 0$ and $v_E = 0$, we set $l_E^a = (\partial/\partial u_E)^a$ and $k_E^a = (\partial/\partial v_E)^a$. The gauge conditions are

$$\begin{aligned} g_{u_E v_E} &= -1, \\ g_{u_E u_E} &= g_{u_E i} = 0, \\ g_{v_E v_E}|_{u_E=0} &= g_{v_E v_E, u_E}|_{u_E=0} = g_{v_E i}|_{u_E=0} = 0. \end{aligned} \tag{53}$$

These gauge conditions imply that we are in the Gaussian null coordinates (GNC) [4,6,7,37–39],

$$ds_E^2 = -2du_E dv_E - u_E^2 \alpha dv_E^2 - 2u_E \omega_i dv_E dx^i + \gamma_{ij} dx^i dx^j. \tag{54}$$

In this gauge, the constraint equations become [24,40–44]

$$\theta_{u_E, v_E} = -\frac{1}{2}\mathcal{R}^E + \nabla^E \cdot \chi^E - \theta_{v_E}\theta_{u_E} + 8\pi G T_{u_E v_E} + \chi_E^2, \tag{55}$$

$$\theta_{v_E, v_E} = -\frac{1}{D-2}\theta_{v_E}^2 - \varsigma_{v_E}^2 - 8\pi G T_{v_E v_E}, \tag{56}$$

$$\chi^E_{i, v_E} = -\theta_{v_E}\chi_i^E + \left(\frac{D-3}{D-2}\right)\nabla_i \theta_{v_E} - (\nabla \cdot \varsigma_{v_E})_i + 8\pi G T_{i v_E}, \tag{57}$$

where we will use θ_{u_E} and θ_{v_E} to emphasize our choice of gauge.

Constructing a stationary null hypersurface requires specifying initial data on N_{-k_E} (the data on N_{-l_E} being fixed in $O_W^E[\mu]$). As the procedure differs only slightly from that in [24], we only provide a very simplified discussion. Specifically, we require that

$$\varsigma_{v_E}[N_{-k_E}] = 0, \tag{58}$$

$$T^E_{v_E v_E}[N_{-k_E}] = 0, \tag{59}$$

$$T^E_{i v_E}[N_{-k_E}] = 0, \tag{60}$$

$$T^E_{u_E v_E}[N_{-k_E}] = \text{const.} \tag{61}$$

Substituting (58) and (59) into (56), and noting that $\theta_{v_E}|_{v_E=0} = 0$ implies $\theta_{v_E} = 0$. Consequently, N_{-k_E} is stationary, and $\mathcal{R}^E$ is also constant along N_{-k_E}. Substituting (60) into (57) reveals that χ_i^E is constant on N_{-k_E}. Together with (61) and (55), it shows that $\theta_{u_E, v_E}|_{u_E=0} = \text{constant}$.

We now justify these requirements on the stress tensor. From (34), the stress tensor in the Einstein frame splits into two parts. First, consider the contribution from ϕ,

$$T^\phi_{ab} = \partial_a\phi\partial_b\phi - g^E_{ab}\left(\frac{1}{2}g_E^{cd}\partial_c\phi\partial_d\phi + V(\phi)\right). \tag{62}$$

In the characteristic initial value problem, $\partial_{v_E}\phi$ is a free data on the null hypersurface N_{-k_E} and can thus be set to zero. And in our gauge $g_{v_E v_E}|_{u_E=0} = g_{v_E i}|_{u_E=0} = 0$, therefore

$$T^\phi_{v_E v_E} = T^\phi_{i v_E} = 0. \tag{63}$$

For the matter contribution T^m_{ab} (here we consider complex scalar field for simplicity), as shown in (35),

$$\begin{aligned} T^m_{v_E v_E} &= \frac{T_{v_E v_E}}{f'(R)}, \\ T^m_{i v_E} &= \frac{T_{i v_E}}{f'(R)}. \end{aligned} \tag{64}$$

Since null geodesics are conformally invariant under the Weyl transformation (15), the null geodesic $k_E^a = (\partial_{v_E})^a$ remains null geodesic in the $f(R)$ frame but with a different affine parameter v, since $\partial_{v_E}\phi \propto \partial_{v_E}\log f'(R) = 0$, allowing us to set the parameter v [31],

$$\frac{dv_E}{dv} = 1, \tag{65}$$

or simply $v_E = v$ (further details are provided in Sec. V B). Consequently, it is physically reasonable to set the corresponding stress tensor components $T_{i v_E}$ and $T_{v_E v_E}$ in the $f(R)$ frame to zero.[7] Combining this with (63) and (64) yields

$$T^E_{v_E v_E} = T^E_{i v_E} = 0, \tag{66}$$

which is a very reasonable setting. The constancy of $T^E_{u_E v_E}$ can be derived by the Bianchi identity and the Gauss Law on the null hypersurface. We expect that a similar

[6]The geometry surfaces we construct form a null Cauchy slice $\Sigma = N'_{-l} \cup N_{-k} \cup N_{-l}$. The null Cauchy slice satisfies all the constraint equations and the junction condition. The whole spacetime $(\mathcal{M}, g^E)$ is defined by the Cauchy evolution of this slice (the characteristic initial data problem) [33–36].

[7]Since for usual matter field, for example complex scalar Φ, we can set $\partial_{v_E}\Phi = \partial_v\Phi = 0$, this will lead $T^m_{v_E v_E} = T^m_{i v_E} = 0$. And this is also true for other usual matter field such as Maxwell field [24].

prescription exists for other reasonable matter fields to satisfy (59), (60), and (61). Assuming so, it is always possible to construct a stationary null hypersurface N_{-k_E} satisfying the constraint equations. It is important to emphasize that fixing these specific stress tensor components by hand on N_{-k_E} does not restrict the generality of the physical black hole exterior being considered. The coarse-grained outer entropy is defined by maximizing over all possible inner wedge completions $I_W^E[\sigma]$ for a strictly fixed outer wedge $O_W^E[\sigma]$. Thus, our procedure serves purely as a constructive existence proof; we only need to demonstrate that at least one valid inner geometry satisfies the constraint equations and saturates the entropy upper bound. Physically, setting these null-null and null-transverse components to zero corresponds to dynamically "turning off" any additional incoming matter flux across the constructed stationary null hypersurface N_{-k_E}. In the characteristic initial value problem, this represents a well-posed and mathematically standard choice for free data that successfully captures the required stationary horizon configuration inside, without limiting the physical dynamics of the fixed exterior spacetime.

To ensure continuity across the gluing surface, we must verify the junction conditions. The junction condition for θ_{k_E} is already satisfied, as N_{-k_E} is a stationary null hypersurface where θ_{k_E} vanishes, matching its value on μ. Regarding the remaining junction conditions, our gauge fixes the transverse metric g_{ij}^E, the twist χ_i, and $\theta_{u_E,v_E}[N_{-k_E}]$ only up to functions of x^i. Even after fixing $\theta_{u_E,v_E}[N_{-k_E}]$, we retain the freedom to specify θ_{u_E}, as well as the fields ϕ and Φ. We are therefore free to choose all the quantities to be continuous across μ,

$$g_{ij}^E[N_{-k_E}] = g_{ij}^E[\mu], \tag{67}$$

$$\chi_i[N_{-k_E}] = \chi_i[\mu], \tag{68}$$

$$\theta_{u_E,v_E}[N_{-k_E}] = \theta_{u_E,v_E}[\mu], \tag{69}$$

$$\theta_{u_E}[N_{-k_E}]|_{v_E=0} = \theta_{u_E}[\mu], \tag{70}$$

$$\phi[N_{-k_E}] = \phi[\mu], \tag{71}$$

$$\Phi[N_{-k_E}] = \Phi[\mu], \tag{72}$$

and the last two conditions (71) and (72) guarantee that $T_{u_E v_E}^E[N_{-k_E}] = T_{u_E v_E}^E[\mu]$. We conclude that this choice satisfies the junction conditions and ensures field continuity in the Einstein frame.

On N_{-k_E}, θ_{u_E,v_E}^E is constant on slices of constant v. From the minimar condition on μ, it follows that $\theta_{u_E,v_E}^E|_{v_E} < 0$. In addition, we know that $\theta_{u_E}^E[\mu] < 0$, this will guarantee that there will be a surface X on N_{-k} which $\theta_{u_E}^E = \theta_{v_E}^E = 0$. This surface X need not be a slice of constant v_E; generally, X is defined by $v_E = h(x^i)$. Our conditions ensure the existence of $h(x^i)$. Then we apply the CPT reflection in the Einstein frame across X that takes $v_E \to -v_E$, $u_E \to -u_E$, $x^i \to x^i$. It is easy to show that all quantities that are odd under the CPT reflection vanish on X in our construction. Therefore, the junction conditions are automatically satisfied across X. Given the initial data constructed on the Cauchy slice $\Sigma = N_{-l}[\mu] \cup N_{-k}[\mu]\tilde{N}_{-l}[\tilde{\mu}]$, the entire spacetime $(\mathcal{M}^{\{\alpha\}}, g^E)$ is well-defined.

Finally, assuming the NEC holds for matter in the $f(R)$ frame, the NCC is satisfied in the Einstein frame. This will give us the focusing theorem in the Einstein frame,

$$\theta_{v_E,v_E} = -\frac{1}{D-2}\theta_{v_E}^2 - \varsigma_{v_E}^2 - 8\pi G T_{v_E v_E} \le 0. \tag{73}$$

This implies that the constructed extremal surface X is the HRT surface in $(\mathcal{M}^{\{\alpha\}}, g^E)$; specifically, any other extremal surface X' must have an area greater than that of X [23,24]. Then the von Neumann entropy $S_{\mathrm{vN}}(\rho_B^E)$ of $(\mathcal{M}^{\{\alpha\}}, g^E)$,

$$S_{\mathrm{vN}}(\rho_B^E) = \frac{\mathrm{Area}_E[\mu]}{4G}. \tag{74}$$

Thus, in the Einstein frame, we have

$$S_E^{(\mathrm{outer})}[\mu] = \frac{\mathrm{Area}_E[\mu]}{4G}. \tag{75}$$

IV. RELATION BETWEEN THE OUTER ENTROPY IN THE EINSTEIN FRAME AND THE $f(R)$ FRAME

In this section, we establish a correspondence between the von Neumann entropies in the Einstein and $f(R)$ frames. More precisely, we demonstrate that at a given boundary time t,

$$S_{\mathrm{vN}}(\rho_A^E(t)) = S_{\mathrm{vN}}(\rho_A^f(t)), \tag{76}$$

where A is a general subregion of the boundary Cauchy slice Σ_t, and ρ_A^E and ρ_A^f are the reduced density matrices of the subregion A in the Einstein and $f(R)$ frames, respectively. Furthermore, their corresponding bulk duals are related by the transformations described in Sec. II. We will show (76) by constructing the correspondence between the gravity dual of a Schwinger-Keldysh contour [45] in both frames. We first provide a brief review of the Schwinger-Keldysh contour on the boundary and its gravity dual, based on Xi Dong, Aitor Lewkowycz, and Mukund Rangamani's construction [45].

Next, we construct the correspondence between the gravity dual of the Schwinger-Keldysh contour, and show the correspondence of von Neumann entropy between the Einstein frame and the $f(R)$ frame. Moreover, we consider the time reflection symmetric case and derive the RT

formula of $f(R)$ gravity. And we find that our results agree with the results of Xi Dong [5] as a correctness check.

Finally, we show that the outer entropy of a codimension-2 surface σ is identical in both frames, if the outer wedge $O_W^E[\sigma]$ and $O_W^f[\sigma]$ are related by the transformation between the Einstein frame and the $f(R)$ frame. Then we show that the outer entropy of the generalized marginally trapped surface is just the Wald entropy associated with it.

A. The Schwinger-Keldysh contour and its gravity dual

In this section we will give a short review of Schwinger-Keldysh contour and its gravity dual. Since the Cauchy slice Σ_t generally lacks time-reflection symmetry, we cannot directly apply the path integral from the far past to far future. Instead, we must employ the Keldysh contour [45,46], where the expression for $\rho(t)$ is given by

$$\rho(t) = |C_t\rangle\langle C_t| = \int [D\Phi] e^{iS_\uparrow[\Phi] - iS_\downarrow[\Phi]}, \tag{77}$$

where $|C_t\rangle$ is the boundary state at time t; the up-arrow indicates forward time evolution, and the down-arrow indicates backward time evolution. This can be well appreciated in the picture, see Fig. 1(a).

After we obtain the path integral representation of $\rho(t)$, we should use this to compute entanglement entropy of subregion A with ∂A as the entangling surface. To calculate the reduced density matrix $\rho_A(t)$, we introduce a cut along A and use $t^\pm$ to denote the evolution of the two legs. And we fix some boundary conditions along A: $\Phi(t = t^+)|_A = \Phi^+$ for the forward part and $\Phi(t = t^-)|_A = \Phi^-$ for the backward part. We obtain the matrix element $(\rho_A)_{+-}$ [45], the diagram representation as shown in Fig. 1(b).

There is a useful Rindler coordinate chart in the vicinity of ∂A. Now we consider a simple example, the boundary geometry is flat spacetime $B = \mathbb{R}^{1,D-1}$. And we choose the Cartesian coordinates (t, x, y^i). In these coordinates, Σ_t is the $t = 0$ Cauchy slice and A is located at $x > 0$. Note that $D^-[A]$ corresponds to the past half of the right Rindler wedge in flat spacetime. Therefore, now we consider the Rindler chart,

$$ds^2 = -dt^2 + dx^2 + dy^i dy^i = -r^2 d\tau^2 + dr^2 + dy^i dy^i. \tag{78}$$

The advantage of choosing this coordinate chart is that we can include all spacetime regions, by allowing τ to be complex with a discrete imaginary part. To be precise, let $\tau = \tau_A + \frac{m\pi}{2} i$ where τ_A is real and $m = 0, 1, 2, 3, 4$. The regions corresponding to $m = 0$ ($\tau_A < 0$, representing the forward part of $D^-[A]$) and $m = 4$ ($\tau_A > 0$, representing the backward part of $D^-[A]$) are the most crucial. To compute $\mathrm{Tr}(\rho_A)$, we glue the $m = 0$ domain to the $m = 4$ domain along A. Therefore, we identify $\tau \sim \tau + 2\pi i$ [45].

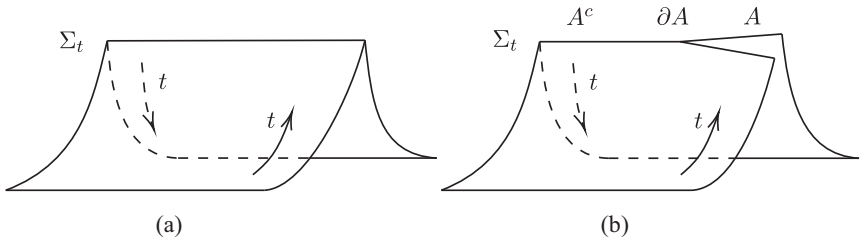
Once we obtain the reduced density matrix, we can calculate the entanglement entropy by replica trick,

$$S_{\mathrm{vN}}(\rho_A) = -\mathrm{Tr}_A(\rho_A \log \rho_A) = \lim_{q\to 1} \frac{1}{1-q} \log \mathrm{Tr}\rho_A^q = \lim_{q\to 1} S_A^{(q)}, \tag{79}$$

here $S_A^{(q)}$ is the Rényi entropy. Computing the Rényi entropy thus requires evaluating ρ_A^q. In the gluing construction, we have q different coordinates and the gluing condition is $\tau_J = \tau_{J-1} + 2\pi i$ along A. However, it is more convenient to introduce a single coordinate τ with $4q + 1$ discrete imaginary parts, that is, $\tau = \tau_A + \frac{m\pi}{2} i$ here now $m = 0, 1, 2 \cdots, 4q$. And in this coordinate, the gluing condition is $\tau \sim \tau + 2\pi i q$ [45].

There is a $\mathbb{Z}_q$ symmetry originating from the invariance under the exchange of replicas of ρ_A. Obviously, ∂A is the fixed surface under the $\mathbb{Z}_q$ symmetry. We will assume that this symmetry is unbroken. And this symmetry will ensure that every copy of the Schwinger-Keldysh contour is the same. Now we need to construct the gravity dual of the Schwinger-Keldysh contour. We assume that the bulk fields satisfy both the gluing condition ($\tau \sim \tau + 2\pi i q$) and the $\mathbb{Z}_q$ replica symmetry.

From the preceding discussion, it is evident that the gravity dual cannot be constructed in a Euclidean manifold in the absence of time-reflection symmetry. Instead, we try to construct the gravity dual of the Schwinger-Keldysh contour in Lorentzian spacetime. It was shown that in the bulk there is a Cauchy slice $\tilde{\Sigma}_t$ in the Wheeler-DeWitt patch



(a) (b)

FIG. 1. (a) is the Schwinger-Keldysh construction for $\mathrm{Tr}\rho(t)$. (b) is the Schwinger-Keldysh construction for the reduced density matrix $\rho_A(t)$.

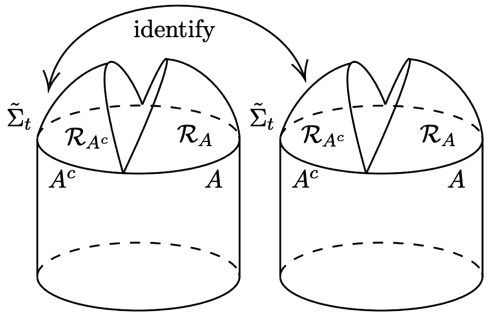


FIG. 2. This diagram is about the path integral construction of reduced density matrix $\rho_A(t)$ in the bulk. The gravity correspondence of tracing the degree of freedom of A^c is gluing the geometry across $\mathcal{R}_{A^c}$.

of Σ_t as the bulk dual of Σ_t ($\partial\tilde{\Sigma}_t = \Sigma_t$). And the bulk evolution only contains the past of the $\tilde{\Sigma}_t$, i.e., in $\tilde{J}^-[\tilde{\Sigma}_t]$. In other words the initial conditions are evolved forward from $t = -\infty$ up to $\tilde{\Sigma}_t$ and then we evolve back to construct the bulk Schwinger-Keldysh contour [45]. And there should exist a codimension 2 surface e as the bulk dual of the ∂A, and it is natural to have $\partial e = \partial A$. These can be clearly represented by Fig. 2.

Now we try to show how to construct the bulk dual of the reduced density matrix $\rho_A(t)$, we first consider the case $q = 1$. We start with the fact that we can divide the Cauchy slice $\tilde{\Sigma}_t$ into two regions $\tilde{\Sigma}_t = \mathcal{R}_{A^c} \cup \mathcal{R}_A$, with an intersection codimension-2 surface e. To understand how we extend the boundary coordinates into the bulk, we foliate the causal development of $\mathcal{R}_A$ in Rindler-like coordinates. In general, in the vicinity of any codimension 2 surface such as e, we can naturally write the metric as

$$ds^2 = -r^2 d\tau^2 + dr^2 + (\gamma_{ij} + re^{\pm\tau}K^{\pm}_{ij} + \cdots)dy^i dy^j. \tag{80}$$

For a Rindler-like observer, there will be four horizons, just like the previous construction, a horizon crossing can be understood as $\tau \to \tau + \frac{\pi}{2}i$ (with $r \to i^{-1}r$) [45]. It captures the local geometry in a neighborhood of e efficiently. When we calculate $\mathrm{Tr}_A\rho_A$, we will return to the starting point after crossing four horizons. This will be encoded in the gluing condition $\tau \sim \tau + 2\pi i$.

Now we consider the case where $q \neq 1$. Defining the corresponding bulk geometry as $\mathcal{M}_q$, it is natural to consider a geometry that has the same gluing condition ($\tau \sim \tau + 2\pi i q$) and replica $\mathbb{Z}_q$ symmetry. And the surface e_q the bulk correspondence of ∂A, should invariant under replica $\mathbb{Z}_q$ symmetry. We also assume that the partition function of boundary should be computed in the saddle point approximation of the bulk geometry $\mathcal{M}_q$ which satisfies all the symmetries and bulk EOM without any singularity,

$$\mathrm{Tr}_A\rho_A^q = I_q. \tag{81}$$

Here I_q is the corresponding on-shell action of the bulk physics, which consists of three parts,

$$I_q = I[\mathcal{M}_q] + I[B_q] + I[\tilde{\Sigma}_t]. \tag{82}$$

Due to the $\mathbb{Z}_q$ symmetry around e_q, we need only focus on a single copy of the replica geometry $\hat{\mathcal{M}}_q = \mathcal{M}_q/\mathbb{Z}_q$,

$$I_q = qI_1 = q(I[\hat{\mathcal{M}}_q] + I[\text{boundaries}]). \tag{83}$$

The one copy geometry $\hat{\mathcal{M}}_q$ has $\mathbb{Z}_q$ symmetry fixed surface e_q with conical singularity [47]. And the result of von Neumann entropy from the bulk side is

$$S_{\mathrm{vN}}(\rho_A(t)) = \lim_{q\to 1}\partial_q I_1. \tag{84}$$

To calculate this derivative ∂_q, we should regard this as a variation of the bulk solution, since the conical structure will only affect the geometry in the vicinity of e_q. And the boundary geometry remains unchanged, $\partial_q g_q|_{\partial\mathcal{M}_q}$ or $\tilde{\Sigma}_t = 0$. Therefore, the boundary terms will not contribute to the von Neumann entropy. We only need to consider the bulk term $I[\hat{\mathcal{M}}_q]$. Under the standard variation process [45],

$$\begin{aligned}\partial_q I[\hat{\mathcal{M}}_q] &= \int_{\mathcal{M}_q} (\mathrm{EOM}\cdot\partial_q g_q + d\Theta(g_q, \partial_q g_q)) \\ &= \int_{e_q(\epsilon)} \Theta(g_q, \partial_q g_q),\end{aligned} \tag{85}$$

where we have chosen to regulate the result by blowing up the singular locus to a tubular neighborhood. We will obtain the answer when $\epsilon \to 0$.

B. Correspondence of von Neumann entropy

In this section, we will use the results in the last section to prove the (76). We now aim to prove that if the geometries are related by the Weyl transformation between the Einstein and $f(R)$ frames, their corresponding Rényi entropies—or more precisely, $I[\mathcal{M}_q]$—are identical,

$$I^E[\mathcal{M}_q] = I^f[\mathcal{M}_q]. \tag{86}$$

We first construct a replica geometry (g^E_q, ϕ, matter fields, $\mathcal{M}_q$) in the Einstein frame, exhibiting gluing condition and $\mathbb{Z}_q$ replica symmetry with respect to the codimension-2 surface e_q. And surface e_q give a partition in Cauchy slice $\tilde{\Sigma}_t$ (is located in the Wheeler-DeWitt patch of Σ_t) whose boundary is Σ_t. This geometry satisfies the Einstein equation without any singularity. When $q = 1$ this

geometry will return to the geometry corresponding to boundary density matrix $\rho^E(t)$.

Now we move into the $f(R)$ frame, from the inverse transformations (26), (27), and (28), it is easy to show that:

(1) If in the Einstein frame g_q^E ϕ and other matter fields satisfy the gluing condition and replica $\mathbb{Z}_q$ symmetry, after we move into the $f(R)$ frame, the geometry (g_q, matter fields, $\mathcal{M}_q$) will also satisfy the gluing condition and replica $\mathbb{Z}_q$ symmetry;
(2) Since we do the Weyl transformation, the Wheeler-DeWitt patch of Σ_t remains unchanged. Therefore, in the $f(R)$ frame $\tilde{\Sigma}_t$ is still located in the Wheeler-DeWitt patch of Σ_t;
(3) The surface e_q remains fixed under the $\mathbb{Z}_q$ symmetry in the $f(R)$ frame. Furthermore, the entire spacetime satisfies the $f(R)$ equations of motion and is free of singularities;
(4) When $q = 1$ the geometry will return to the geometry corresponding to boundary density matrix $\rho^f(t)$, and this geometry related to the geometry corresponding to $\rho^E(t)$ by the Weyl transformation between $f(R)$ and the Einstein frame.

These will make the geometry (g_q, matter fields, $\mathcal{M}_q$) is the replica geometry of boundary density matrix $\rho^f(t)$. Conversely, the reverse implication also holds. If in the $f(R)$ frame we construct a replica geometry (g_q, matter fields, $\mathcal{M}_q$), after we move into the Einstein frame (g_q^E, ϕ, matter fields, $\mathcal{M}_q$) remains a well-defined replica geometry of boundary density matrix $\rho^E(t)$.

From the discussions in Sec. II, it is evident that there exists a correspondence between the bulk actions,

$$I^E[\mathcal{M}_q] = I^f[\mathcal{M}_q], \tag{87}$$

if the geometries correspondent to $\rho^E(t)$ and $\rho^f(t)$ are related by the Weyl transformation between the $f(R)$ frame and the Einstein frame. Since $\mathbb{Z}_q$ is satisfied in both frames we can write

$$qI^E[\hat{\mathcal{M}}_q] = qI^f[\hat{\mathcal{M}}_q]. \tag{88}$$

Following this, we perform an analytic continuation in q. Finally, the direct consequence of (87) and (88) is

$$S_{\mathrm{vN}}(\rho_A^E(t)) = S_{\mathrm{vN}}(\rho_A^f(t)). \tag{89}$$

Now we apply our construction to a special case, when the spacetime satisfies the time reflection symmetry $t \to -t$ and consider the $t = 0$ surface. Now we move into the Einstein frame, and from the above construction the time reflection symmetry remains unchanged. Since the time reflection symmetry, the extrinsic curvature of t direction of e_q, $K^{E,t}$ will automatically be zero. We also require that the spacetime $\mathcal{M}_q$ contains no singularities. We will find that the extrinsic curvature of x direction of e_q, $K^{E,x}$ will be zero [45]. Here, x and t are bulk coordinates correspondent to boundary Cartesian coordinates. The constraint is equivalent to the minimal surface condition [48] in the Einstein frame and the minimal Wald entropy in the $f(R)$ frame. And the von Neumann entropy is

$$S_{\mathrm{vN}}(\rho_A^E) = S_{\mathrm{vN}}(\rho_A^f) = \min_{e_q \in \tilde{\Sigma}_0} \frac{\mathrm{Area}_E[e_q]}{4G} = \min_{e_q \in \tilde{\Sigma}_0} \frac{\int_{e_q} f'(R)}{4G}, \tag{90}$$

this agrees with the result in [5]. This is a self consistency check of our theory.

C. Relation between outer entropy in two frames

Now we consider the outer entropy of a codimension 2 surface σ in the $f(R)$ frame. The outer entropy associated with a surface σ homologous to B is defined by maximizing the von Neumann entropy over all possible inner wedge data $\{\alpha\}$, while keeping the outer wedge $O_W^f[\sigma]$ fixed in the $f(R)$ frame,

$$S_f^{(\mathrm{outer})}[\sigma] = \max_{\{\alpha\}} \left[S_{\mathrm{vN}}(\rho_B^{f\{\alpha\}}) \right]. \tag{91}$$

Here fixing the outer wedge $O_W^f[\sigma]$ means that we fix the geometry g and matter fields in $O_W^f[\sigma]$. Transforming to the Einstein frame, we note that the Weyl transformation leaves the outer wedge invariant,

$$O_W^f[\sigma] = O_W^E[\sigma]. \tag{92}$$

From the discussions in Sec. II, when we fix g and matter fields in $O_W^f[\sigma]$, g^E, ϕ and matter fields in the outer wedge $O_W^E[\sigma]$ in the Einstein frame will be fixed at the same time. Therefore, when we change the inner wedge data and fix the outer wedge in the $f(R)$ frame, in the Einstein frame the outer wedge $O_W^E[\sigma]$ is automatically fixed.

From Sec. II, for every asymptotically AdS solution in the $f(R)$ frame with a fixed outer wedge $O_W^f[\sigma]$ (satisfying the EOM and free of singularities), there exists a corresponding asymptotically AdS solution in the Einstein frame with a fixed outer wedge $O_W^E[\sigma]$ and identical on shell action (satisfying the Einstein-frame EOM and free of singularities). As demonstrated in the previous subsection, their von Neumann entropies are identical,

$$S_{\mathrm{vN}}(\rho_B^{f\{\alpha\}}) = S_{\mathrm{vN}}(\rho_B^{E\{\alpha\}}) = \frac{\mathrm{Area}_E[X]}{4G} = \frac{\int_X f'(R)}{4G}, \tag{93}$$

where X is the HRT surface ($\theta_{k_E}^E = \theta_{l_E}^E = 0$) in the Einstein frame and the generalized extremal surface ($\Theta_k = \Theta_l = 0$) in the $f(R)$ frame. Therefore, if we are maximizing von

Neumann entropy in the $f(R)$ frame with fixed outer wedge $O_W^f[\sigma]$, then we maximize von Neumann entropy in the Einstein frame with fixed outer wedge $O_W^E[\sigma]$ at the same time. Finally, we find that the outer entropy of surface σ are the same in $f(R)$ and the Einstein frame,

$$S_f^{(\text{outer})}[\sigma] = S_E^{(\text{outer})}[\sigma], \tag{94}$$

if the outer wedge $O_W^f[\sigma]$ and $O_W^E[\sigma]$ are related by the transformation between the $f(R)$ frame and the Einstein frame.

In Sec. III, we have already shown that in the Einstein frame, the outer entropy of the minimar surface μ is just the Bekenstein-Hawking entropy of the minimar surface,

$$S_E^{(\text{outer})}[\mu] = \frac{\text{Area}_E[\mu]}{4G}. \tag{95}$$

And from (94), we can write

$$S_E^{(\text{outer})}[\mu] = S_f^{(\text{outer})}[\mu] = \frac{\text{Area}_E[\mu]}{4G} = \frac{\int_\mu f'(R)}{4G}. \tag{96}$$

Recall that the minimar surface μ is defined by the vanishing of the outer null expansion, $\theta_{v_E} = 0$, in the Einstein frame. Since null geodesics are conformally invariant, $k_E^a = (\partial_{v_E})^a$ remains a null geodesic in the $f(R)$ frame, but a different affine parameter v. The relationship between these parameters is given by [31]

$$\frac{dv_E}{dv} = c\Omega^2. \tag{97}$$

Therefore, we can show that the generalized expansion of μ in the $f(R)$ frame vanishes [21]

$$\begin{aligned}\theta_{v_E} &= \frac{\partial}{\partial v^E}\log\sqrt{\gamma^E} = \frac{1}{c(f'(R))^{\frac{2}{D-2}}}\frac{\partial}{\partial v}\log(\sqrt{\gamma}f'(R)) \\ &= \frac{1}{c(f'(R))^{\frac{2}{D-2}}}\Theta_v = 0,\end{aligned} \tag{98}$$

here c is a positive constant, and this shows that $\Theta_v = 0$. And the minimar condition of μ in the Einstein frame implies that μ satisfies the generalized minimar condition[8] (see Sec. I A) in the $f(R)$ frame. Therefore, μ is just the generalized minimar surface in the $f(R)$ frame, and we finally show that the outer entropy of the generalized marginally trapped surface with generalized minimar condition in $f(R)$ gravity is just the Wald entropy of it

$$S_f^{(\text{outer})}[\mu] = \frac{\int_\mu f'(R)}{4G}. \tag{99}$$

V. DERIVING THE RESULTS IN THE $f(R)$ FRAME

In this section, we will derive the outer entropy directly in the $f(R)$ frame. To do this, we should first derive a focusing theorem for the generalized expansion Θ_k in $f(R)$ gravity. We will construct the stationary null hypersurface N_{-k} for the generalized expansion. Finally, we will construct the spacetime $(\mathcal{M}', g')$ that the von Neumann entropy that saturates the upper bound of the outer entropy.

A. Nonlinear Raychaudhuri equation for the generalized expansion

Consider a null hypersurface N_k generated by an affinely parametrized null geodesic vector field k^a, equipped with a "rigging" vector field l^a satisfying $k_a l^a = -1$ (note that l^a is not unique). The null and transverse extrinsic curvatures of N_k are defined as

$$\begin{aligned}K_{ab}^{(k)} &= \gamma_a^c\gamma_b^d\nabla_c k_d, \\ K_{ab}^{(l)} &= \gamma_a^c\gamma_b^d\nabla_c l_d,\end{aligned} \tag{100}$$

where $\gamma_{ab} = g_{ab} + 2l_{(a}k_{b)}$. And we define the expansion θ_k as

$$\theta_k = \gamma^{ab}K_{ab}^{(k)}. \tag{101}$$

Assuming k^a is an affinely parametrized null geodesic, the Raychaudhuri equation for the expansion θ_k is given by

$$k^a\nabla_a\theta_k = -\frac{1}{D-2}\theta_k^2 - \varsigma^{ab}\varsigma_{ab} - R_{ab}k^ak^b. \tag{102}$$

This equation is purely geometrical.

For the $f(R)$ gravity, we define the generalized expansion Θ_k,

$$\Theta_k = \gamma^{ab}K_{ab}^{(k)} + k^a\nabla_a\log f'(R). \tag{103}$$

Assuming $k^a = (\partial_v)^a$ is affinely parametrized with affine parameter v, we can rewrite (103) as

$$\Theta_k = \Theta_v = \partial_v\log(\sqrt{\gamma}f'(R)). \tag{104}$$

Using the equations of motion for $f(R)$ gravity (11) and assuming $f'(R) > 0$, we find

$$R_{ab}k^ak^b = \frac{8\pi G T_{ab}k^ak^b}{f'(R)} + \frac{1}{f'(R)}k^ak^b\nabla_a\nabla_b f'(R). \tag{105}$$

[8] It can be shown that $\nabla_{l_E}\theta_{k_E}^E|_\mu \propto \frac{f'(R)\partial_u\Theta_v - \Theta_v\partial_u f'(R)}{f'(R)^2}|_\mu = \frac{\partial_u\Theta_v}{f'(R)}|_\mu \le 0$. Since $f'(R) > 0$, this implies $\nabla_l\Theta_k = \partial_u\Theta_v \le 0$.

We notice that

$$\frac{1}{f'(R)}k^a k^b \nabla_a \nabla_b f'(R) = k^a \nabla_a (k^b \nabla_b \log f'(R)) + (k^a \nabla_a \log f'(R))^2. \quad (106)$$

Substituting the $f(R)$ equations of motion into the Raychaudhuri equation yields

$$k^a \nabla_a \Theta_k = -\frac{\theta_k^2}{D-2} - (k^a \nabla_a \log f'(R))^2 - \varsigma^{ab}\varsigma_{ab} - 8\pi G \frac{T_{ab}k^a k^b}{f'(R)}. \quad (107)$$

Since we assume $f'(R) > 0$, if the matter fields satisfy the NEC ($T_{ab}k^a k^b \geq 0$), we obtain a focusing theorem for the generalized expansion Θ_k,

$$k^a \nabla_a \Theta_k = \partial_v \Theta_v \leq 0. \quad (108)$$

We define (107) as the nonlinear Raychaudhuri equation. It is crucial to distinguish this result from the standard geometric focusing in General Relativity (GR). In classical GR, the Raychaudhuri equation guarantees the strict decrease of the standard geometric expansion θ_k under the NEC. However, in $f(R)$ gravity, θ_k does not necessarily focus due to effective stress-energy contributions from higher-curvature terms. Yet, the thermodynamically relevant quantity in $f(R)$ gravity is not the bare geometric area, but the $f'(R)$-weighted area, which corresponds to the Wald entropy.

The inequality $\partial_v \Theta_v \leq 0$ demonstrates that the generalized expansion Θ_k strictly monotonically decreases as long as the physical matter satisfies the NEC and $f'(R) > 0$. Therefore, (108) acts as a full focusing theorem for $f(R)$ gravity: it provides the exact monotonic behavior required to prove that the Wald entropy strictly shrinks along null hypersurfaces. Importantly, this is a fully nonperturbative classical geometric result within $f(R)$ gravity, providing the necessary mathematical foundation to legitimately extend the maximin construction and outer entropy bounds from purely area-based frameworks (GR) to higher-curvature theories.

We now use the nonlinear Raychaudhuri equation to calculate the upper bound of the outer entropy of the generalized marginally trapped surface. As discussed in Sec. III, the NCC is satisfied in the Einstein frame. Therefore, under the AdS/CFT and some global assumptions, the von Neumann entropy $S_{\mathrm{vN}}(\rho_B^{E\{\alpha\}})$ satisfies the maximin construction in the Einstein frame. Then we move into the $f(R)$ frame (two geometries are related by the Weyl transformation), this construction is equivalent to first finding the minimal Wald entropy surface $\min_f(B,\Sigma)$ homologous to B on a given Cauchy slice Σ. We then choose Σ to maximize the Wald entropy of $\min_f(B,\Sigma)$ over all possible Cauchy slices, the von Neumann entropy is the Wald entropy of $\min_f(B,\Sigma)$.

Now consider the generalized marginally trapped surface μ, let $N_{\pm k}(\mu)$ be the null congruence generated from μ by null geodesics k^a and $-k^a$ in the $f(R)$ frame. Defining $\bar{\mu}$ as the representative of μ on Cauchy slice Σ, following the definition in Sec. III A. Assuming $\bar{\mu}$ lies on N_k, since $\Theta_v|_\mu = 0$ and the nonlinear Raychaudhuri equation dictates $\partial_v \Theta_v \leq 0$, it follows that

$$\Theta_v|_{N_k} \leq 0. \quad (109)$$

In this case the Wald entropy of $\bar{\mu}$ is smaller than the Wald entropy of μ, since

$$\partial_v \int_{\mathcal{T}} f'(R) dA = \int_{\mathcal{T}} \Theta_v f'(R) dA \leq 0, \quad (110)$$

here $\mathcal{T}$ is the cross section of N_k. If the representative $\bar{\mu}$ is on the N_{-k}, since $\partial_v \Theta_v \neq 0$, it is easy to show that

$$\Theta_v|_{N_{-k}} \geq 0. \quad (111)$$

Therefore, in this case the Wald entropy of $\bar{\mu}$ is smaller than the Wald entropy μ, now we have already shown that

$$\int_{\bar{\mu}} f'(R) \leq \int_{\mu} f'(R). \quad (112)$$

Now consider a spacetime $(\mathcal{M}^{\{\alpha\}}, g)$ with a fixed outer wedge $O_W^f[\mu]$. By the maximin construction [32], there exist a Cauchy slice Σ such that $X^{\{\alpha\}}$ (the surface that we calculate the von Neumann entropy in the $f(R)$ frame) has the minimal Wald entropy surface of Σ, therefore

$$S_{\mathrm{vN}}(\rho_B^f) = \frac{\int_{X^{\{\alpha\}}} f'(R)}{4G} \leq \frac{\int_{\bar{\mu}} f'(R)}{4G} \leq \frac{\int_{\mu} f'(R)}{4G}. \quad (113)$$

Since the outer entropy $S_f^{(\mathrm{outer})}$ is defined as the maximal von Neumann entropy, this implies

$$S_f^{(\mathrm{outer})}[\mu] \leq \frac{\int_{\mu} f'(R)}{4G}. \quad (114)$$

B. Construction of stationary null hypersurface for the generalized expansion

Similar to Sec. III, constructing a spacetime with a fixed $O_W^f[\mu]$ that maximizes $S_{\mathrm{vN}}(\rho_B^f)$ requires gluing a stationary null hypersurface N_{-k} for the generalized expansion, subject to specific initial data on N_{-k}. This will show that there exists a generalized extremal surface X with zero

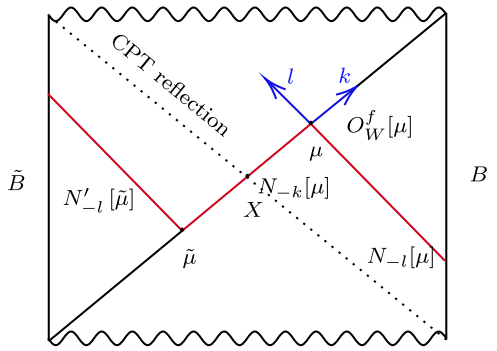


FIG. 3. This is the full maximizing spacetime in the $f(R)$ frame. The characteristic Cauchy slice that we construct to obtain this geometry consists of $\Sigma = N_{-l}[\mu] \cup N_{-k}[\mu] \cup N'_{-l}[\tilde{\mu}]$.

generalized expansions of two null directions,[9] the Wald entropy of this surface is equal to the Wald entropy of μ. The spacetime is then completed via a *CPT* reflection across X, analogous to the procedure in Sec. III (see Fig. 3).

We now first construct the stationary null hypersurface N_{-k} for generalized expansion. Adopting the same gauge as in Sec. III, we use null coordinates u, v and spatial coordinates x^i, fixing μ at $u = v = 0$ and placing N_{-k} at $u = 0$. $l^a = (\partial_u)^a$ and $k^a = (\partial_v)^a$ are the generating null vectors of N_{-l} and N_{-k}. We use the GNC gauge, that is,

$$\begin{aligned} g_{uv} &= -1, \\ g_{uu} &= g_{ui} = 0, \\ g_{vv}|_{u=0} &= g_{vv,u}|_{u=0} = g_{vi}|_{u=0} = 0. \end{aligned} \tag{115}$$

We can write the line element under these gauge conditions (GNC) [4,6,7,37–39],

$$ds^2 = -2dudv - u^2\alpha dv^2 - 2u\omega_i dv dx^i + \gamma_{ij}dx^i dx^j. \tag{116}$$

In this gauge, the twist and the null extrinsic curvatures are

$$2\chi_i = g_{iv,u} = \omega_i + u\partial_u\omega_i, \tag{117}$$

$$2K^{(l)}_{ij} = g_{ij,u}, \tag{118}$$

$$2K^{(k)}_{ij} = g_{ij,v}|_{u=0}. \tag{119}$$

In this gauge, the constraint equations on the stationary null hypersurface N_{-k} for twist and generalized expansions reduce to [24,40–44]

$$\Theta_{u,v} = -\frac{1}{2}\mathcal{R} + \nabla\cdot\chi - \theta_u\theta_v + \partial_v\partial_u \log f'(R) + R_{uv} + \chi^2, \tag{120}$$

$$\Theta_{v,v} = -\frac{\theta_k^2}{D-2} - (\partial_v \log f'(R))^2 - \varsigma_v^2 - 8\pi G\frac{T_{vv}}{f'(R)}, \tag{121}$$

$$\chi_{i,v} = -\theta_v + \left(\frac{D-3}{D-2}\right)\nabla_i\theta_v - (\nabla\cdot\varsigma_v)_i + R_{iv}. \tag{122}$$

We now specify the initial data on N_{-k}. Based on our analysis in the Einstein frame, $\partial_v \log f'(R)$ represents free data that can be set to zero. Since in the Einstein frame, $\partial_{v_E}\phi$ is a free data on the null hypersurface N_{-k_E},

$$\partial_{v_E}\phi = \frac{1}{\sqrt{16\pi G}}\sqrt{\frac{2(D-1)}{D-2}}\partial_{v_E}\log f'(R) \propto \partial_v \log f'(R). \tag{123}$$

Therefore, in the $f(R)$ frame $\partial_v \log f'(R)$ should be free data that we set it to zero. Since we assume $f'(R) > 0$, this is equivalent to $\partial_v f'(R) = 0$. We will require

$$\partial_v f'(R)[N_{-k}] = 0, \tag{124}$$

$$\varsigma_v[N_{-k}] = 0, \tag{125}$$

$$T_{vv}[N_{-k}] = 0, \tag{126}$$

$$T_{iv}[N_{-k}] = 0, \tag{127}$$

$$T_{uv}[N_{-k}] = \text{const}, \tag{128}$$

$$T_{ij}[N_{-k}] = \text{const}. \tag{129}$$

Since $\partial_v \log f'(R) = 0$, it is easy to show that $\Theta_v[N_{-k}] = \theta_v[N_{-k}]$. Substituting the first three equalities into (121) reveals that $\Theta_v = \theta_v = 0$ on N_{-k}, implying that R is also constant along N_{-k}. Now we consider the quantity R_{iv}, from the EOM of $f(R)$ gravity, we can show that

$$R_{iv} = \frac{1}{2}g_{iv}\frac{f(R)}{f'(R)} + \frac{1}{f'(R)}(\nabla_i\nabla_v - g_{iv}\nabla^2)f'(R) + 8\pi G\frac{T_{iv}}{f'(R)}. \tag{130}$$

Since $g_{iv}|_{N_{-k}} = T_{iv}[N_{-k}] = 0$, we only need to evaluate $\frac{1}{f'(R)}\nabla_i\nabla_v f'(R)$. One can show that [7]

$$\nabla_i\nabla_v f'(R)|_{u=0} = \partial_i\partial_v f'(R)|_{u=0} - K^j_i\partial_j f'(R)|_{u=0} = 0, \tag{131}$$

[9]In the Einstein frame, the surface X where we evaluate the von Neumann entropy is the extremal surface. The extremal condition of X in the Einstein frame will lead X to a generalized extremal surface in the $f(R)$ frame.

and we already use that on N_{-k}, $\theta_v = \varsigma_v = 0$, this will imply $K_i^j[N_{-k}] = 0$. Therefore, we have shown that on the stationary null hypersurface N_{-k},

$$R_{iv}[N_{-k}] = 0. \tag{132}$$

Combined with (125), this shows that $\chi_{i,v} = 0$, which restricts the twist to be constant along N_{-k}. From (117), we find that

$$\chi_{i,v}[N_{-k}] = \frac{1}{2}\partial_v \omega_i[N_{-k}] = 0. \tag{133}$$

Therefore, on the stationary null hypersurface N_{-k}, ω_i is constant, that is, $\partial_v \omega_i = 0$.

Next, consider $\Theta_{u,v}$. Since $\theta_v = \varsigma_v = 0$ on the stationary null hypersurface N_{-k}, it follows that $\partial_v \gamma_{ij} = 0$. Therefore, $\mathcal{R}$ will be a constant on N_{-k}. Since we have already shown that χ_i is constant and $\theta_v = 0$ on N_{-k}, the combination $\partial_v \partial_u \log f'(R) + R_{uv}$ is the only term that could potentially be nonconstant on N_{-k}. Now we use the EOM of $f(R)$ gravity (11) to show that this term is also a constant on N_{-k}. We first consider R_{uv},

$$R_{uv} = \frac{1}{2} g_{uv} \frac{f(R)}{f'(R)} + \frac{1}{f'(R)}(\nabla_u \nabla_v - g_{uv}\nabla^2) f'(R) + 8\pi G \frac{T_{uv}}{f'(R)}. \tag{134}$$

Since $f'(R)$ and g_{uv} are constant on N_{-k}, we must determine whether $f(R)$ is also constant. We know that $\partial_v f(R) = \partial_v R \cdot f'(R)$, since $\partial_v f'(R) = \partial_v R \cdot f''(R) = 0$ and for simplicity we assume $f''(R) \neq 0$ [21] [in Appendix we will discuss the cases of $f''(R) = 0$, where we show that $\partial_v R = 0$ even without $f''(R) \neq 0$ condition], this will show that $\partial_v R = 0$. Therefore, $f(R)$ is constant on N_{-k}. And we know that T_{uv} is a constant on N_{-k}, then $8\pi G(T_{uv}/f'(R))$ is also a constant. Finally, we consider the remaining terms,

$$\frac{1}{f'(R)}\partial_v \partial_u f'(R) + \frac{1}{f'(R)}(\nabla_u \nabla_v - g_{uv}\nabla^2) f'(R). \tag{135}$$

It is easy to show that [7]

$$\nabla_u \nabla_v f'(R)|_{u=0} = \partial_u \partial_v f'(R) + \frac{1}{2}\omega^i \partial_i f'(R). \tag{136}$$

And we can show that

$$\begin{aligned} \nabla^2 f'(R)|_{u=0} &= -2\partial_u \partial_v f'(R) + \Delta_\gamma f'(R) \\ &\quad - \partial_v \log\sqrt{\gamma}\partial_u f'(R) \\ -\partial_u \log\sqrt{\gamma}\partial_v f'(R) - \omega^i \partial_i f'(R) &= -2\partial_u \partial_v f'(R) + \Delta_\gamma f'(R) \\ &\quad - \omega^i \partial_i f'(R), \end{aligned} \tag{137}$$

where Δ_γ is the laplacian of x^i. Then the remaining terms are reduced to

$$\begin{aligned} &\frac{1}{f'(R)}\partial_v \partial_u f'(R) + \frac{1}{f'(R)}(\nabla_u \nabla_v - g_{uv}\nabla^2) f'(R) \\ &= \frac{1}{f'(R)}\left(\Delta_\gamma f'(R) - \frac{1}{2}\omega^i \partial_i f'(R)\right). \end{aligned} \tag{138}$$

Since $\partial_v \gamma_{ij} = 0$ and χ_i is constant on N_{-k}, it can be shown that $\Delta_\gamma f'(R)$ and $\omega^i \partial_i f'(R)$ are constant on N_{-k}. Finally, we show that R_{iv} is a constant on N_{-k}, this will lead to $\Theta_{u,v}$ is a constant on N_{-k}.

We now need to show that our construction satisfies the junction conditions. Since the junction conditions have beautiful forms in the Einstein frame, we now first move back to the Einstein frame. Given that we have set $\partial_v f'(R) = 0$, one can easily show that $k^a|_{N_{-k}} = (\partial_v)^a|_{N_{-k}}$ remains an affinely parametrized null geodesic [31],

$$k^a \nabla_a^E k^b|_{N_{-k}} = (2k^a \nabla_a^E \log\Omega)\cdot k^a = 0, \tag{139}$$

where $\Omega = (f'(R))^{\frac{1}{D-2}}$ and we already use $k^a \nabla_a \Omega = \partial_v \Omega = 0$. Therefore, it is reasonable to define $k_E^a|_{N_{-k}} = (\partial_{v_E})^a|_{N_{-k}} = (\partial_v)^a|_{N_{-k}}$. Now let us define the "rigging" vector field l_E^a. Weyl transformation preserves the null geodesic and the relation between affine parameter before and after Weyl transformation is [31]

$$\frac{du_E}{du} = c\Omega^2, \tag{140}$$

here c is a positive constant and now we choose $c = 1$. Therefore, $l_E^a = (\partial_{u_E})^a = (1/\Omega^2)(\partial_u)^a$, and we define N_{-k} is located at $u_E = 0$. This will guarantee that on N_{-k},

$$g_{ab}^E k_E^a l_E^b = \frac{1}{\Omega^2}\Omega^2 g_{ab} k^a l^b = -1, \tag{141}$$

with $(\partial_{u_E})^a$ is affine null geodesic everywhere under the metric $g_{ab}^E = \Omega^2 g_{ab}$. Since Weyl transformation preserves the orthogonal relationship between l, k, and x. On N_{-k}, we define $(x_i^E)^a = (\partial_{x_E^i})^a = (\partial_{x^i})^a$. We now extend v_E and x_E^i by keeping the parameter v_E and x_E^i fixed along the null geodesics generated by l_E^a, and demanding that $k_E^a = (\partial_{v_E})^a$ and $(x_i^E)^a = (\partial_{x_E^i})^a$ everywhere [7],

$$[k_E, l_E]^a = [x_i^E, l_E]^a = 0, \tag{142}$$

and the line element of the Einstein frame,

$$\begin{aligned} ds_E^2 &= -2du_E dv_E - u_E^2 \alpha_E dv_E^2 - 2u_E \omega_i^E dv_E dx_E^i \\ &\quad + \gamma_{ij}^E dx_E^i dx_E^i. \end{aligned} \tag{143}$$

This implies that we reconstruct a GNC coordinate near N_{-k} in the Einstein frame. In the Einstein frame, the junction conditions are (2). Since $\theta^E_{v_E}[\mu] = \Theta_v[\mu] = \theta^E_{v_E}[N_{-k}] = \Theta_v[N_{-k}] = 0$, the first junction condition is satisfied automatically. Now for the second junction condition, $\theta^E_{u_E,v_E}[N_{-k}] = (1/\Omega^2)\Theta_{u,v}[N_{-k}]$ is obviously a constant along N_{-k}. Since $\theta^E_{u_E,v_E}[N_{-k}]$ is defined up to functions of the transverse $x^i_E|_{N_{-k}} = x^i|_{N_{-k}}$ directions. Even after fixing $\theta^E_{u_E,v_E}[N_{-k}]$, we retain the freedom to choose $\theta^E_{u_E}$ at $v = v_E = 0$ on N_{-k}. For χ^E_i, we have

$$\chi^E_i[N_{-k}] = \partial_{u_E} g^E_{v_E i}|_{N_{-k}} = \omega^E_i[N_{-k}] = \frac{1}{\Omega^2}\partial_u(u\Omega^2\omega_i)|_{N_{-k}} = \omega_i[N_{-k}], \quad (144)$$

remains defined up to functions of the transverse $x^i_E|_{N_{-k}} = x^i|_{N_{-k}}$ directions, we are always free to choose all of these quantities to be continuous across the junction at μ,

$$[\theta^E_{v_E}] = 0, \quad (145)$$

$$[\theta^E_{u_E}] = 0, \quad (146)$$

$$[\chi^E_i] = 0. \quad (147)$$

Similarly, since ϕ and the matter field Φ are defined up to transverse functions, we are always free to enforce these quantities to be continuous across the junction at μ, i.e., $[\phi] = [\Phi] = 0$. These choices satisfy the junction conditions for the metric (2) and for the matters. This implies that we will get a continuous spacetime and well defined stress tensor across μ in the Einstein frame, and therefore in the $f(R)$ frame.

Therefore, we glue a stationary null hypersurface N_{-k} for the generalized expansion Θ_k to μ. Our next step is to find the generalized extremal surface X with zero generalized expansions of two null directions.

C. Construction of the generalized extremal surface and the outer entropy in the $f(R)$ frame

We are going to find the generalized extremal surface X on N_{-k}. Since the generalized expansion $\Theta_k[N_{-k}] = 0$ on the stationary null hypersurface N_{-k}, we only need to find a surface or a cross section of N_{-k} on which $\Theta_l = 0$. Similarly to Sec. III B, on a constant v slice $\Theta_{u,v} =$ const and from the condition of the generalized minimal surface, $\Theta_{u,v} < 0$. This will ensure that there exists a surface on N_{-k} with $\Theta_l = 0$, but this surface does not need to be a constant v slice. Consider a slice β with varying v, defined by $v = h(x^i)$. By definition, $k^a = (\partial_v)^a$ is normal to β. We define the second null normal to β, denote by w^a, it can be shown that [24]

$$w^a = l^a + \sum_i x^a_i \nabla_i h + \frac{1}{2}k^a \Delta_\gamma h, \quad (148)$$

where $\nabla_i = x^a_i \nabla_a$. The generalized expansion of w^a is defined as $\Theta_w = \theta_w + w^a \nabla_a \log f'(R)$, it can be shown that [24]

$$\Theta_w[\beta] = \Theta_u[\beta] + \Delta_\gamma h(x^i) + (2\chi + \nabla \log f'(R)) \cdot \nabla h(x^i). \quad (149)$$

Since we already show that $\Theta_{u,v}$ is independent of v on constant v slice, we can show that [49,50]

$$\Theta_w[\beta] = \Theta_u[\mu] + \Theta_{u,v}[\mu]h(x^i) + (2\chi + \nabla \log f'(R)) \cdot \nabla h(x^i) + \Delta_\gamma h(x^i) = L^\mu[h] + \Theta_u[\mu]. \quad (150)$$

Since χ and γ are independent of v, the linear operator L^μ depends only on quantities evaluated at μ, this linear operator is known as the stability operator [49,50], and now we generalized it to $f(R)$ gravity.

To locate the generalized extremal surface X where $\Theta_w[X] = 0$, we must solve the second-order partial differential equation (PDE),

$$L^\mu[h] = -\Theta_u[\mu]. \quad (151)$$

From the results of a second-order PDE on a closed manifold (codimension 2 surface μ), if $\Theta_{u,v}[\mu] < 0$ and $\Theta_u[\mu] > 0$, this second-order PDE has a nontrivial solution and $h(x^i) < 0$ [24,49,50]. Finally, we conclude that there exists a generalized extremal surface X on N_{-k} with two null generalized expansions zero. Since N_{-k} is stationary for the generalized expansion Θ_k,

$$\int_X f'(R) = \int_\mu f'(R). \quad (152)$$

Similarly to Sec. III B, we complete our construction by a CPT reflection across X that takes $v \to -v$, $u \to -u$, $x^i \to x^i$. Under the CPT transformation, Φ, $f'(R)$ and χ_i are even, whereas Θ_u, Θ_v and $\partial_v f'(R)$ are odd. All quantities that are odd under CPT vanish on X by construction. Therefore, the CPT-conjugate data satisfies the requisite junction conditions. The results is a second boundary $\tilde{B}$, B, and $\tilde{B}$ are connected by a Cauchy slice with three null segments $\Sigma = N_{-l}[\mu] \cup N_{-k}[\mu] \cup N'_{-l}[\tilde{\mu}]$, we are using prime to represent CPT-conjugated submanifolds; see Fig. 3. We have now specified all data necessary to uniquely evolve characteristic initial data via the EOM of $f(R)$ gravity. The resulting spacetime $(\mathcal{M}', g)$ has a generalized minimal surface μ with a fixed outer wedge $O^f_W[\mu]$. This spacetime contains a generalized extremal

surface X on the boundary of the inner wedge $I_W^f[\mu]$, which is homologous to μ and therefore to the boundary B.

Finally, we need to show that this generalized extremal surface X has the least Wald entropy. Given the duality established in (76), X must have a minimal area in the Einstein frame if X is the extremal surface. Therefore, in the $f(R)$ frame, X should have the least Wald entropy if there exist other generalized extremal surfaces. If the constructed spacetime $(\mathcal{M}', g')$ contains no other generalized extremal surfaces, the proof is complete. If there exists other generalized extremal surfaces we denote it by X', define $\overline{X'}[\Sigma]$ as the representative of X' on Cauchy slice $\Sigma = N_{-l}[\mu] \cup N_{-k}[\mu] \cup N'_{-l}[\tilde{\mu}]$. There are in total three cases; If $\overline{X'}[\Sigma]$ lies entirely on N_{-k}, then, because N_{-k} is stationary with respect to the generalized expansion, we have

$$\int_{\overline{X'}[\Sigma]} f'(R) = \int_X f'(R). \tag{153}$$

If $\overline{X'}[\Sigma]$ lies on $N_{-l}[\mu]$ or $N'_{-l}[\tilde{\mu}]$, from the focusing theorem (107) for $f(R)$ gravity,

$$\int_{\overline{X'}[\Sigma]} f'(R) \geq \int_X f'(R). \tag{154}$$

And for the same reason, in both cases,

$$\int_{X'} f'(R) \geq \int_{\overline{X'}[\Sigma]} f'(R) \geq \int_X f'(R). \tag{155}$$

If $\overline{X'}[\Sigma]$ lies on $N_{-l}[\mu]$ and N_{-k}, define $s_1 = \overline{X'}[\Sigma] \cap N_{-l}[\mu]$, $s_2 = \overline{X'}[\Sigma] \cap N_{-k}[\mu]$, $\mu_1 = \mu \cap O_W^f[X']$ and μ_2 as the complement in μ. It is easy to show that $\mu_1 \cup s_1$ and $\mu_2 \cup s_2$ homologous to μ, then by the focusing theorem,

$$\int_{s_1} f'(R) + \int_{\mu_1} f'(R) \geq \int_{\mu} f'(R), \tag{156}$$

$$\int_{s_2} f'(R) + \int_{\mu_2} f'(R) = \int_{\mu} f'(R). \tag{157}$$

Then we can combine (156) and (157), we find

$$\int_{X'} f'(R) \geq \int_{\overline{X'}[\Sigma]=s_1\cup s_2} f'(R) \geq \int_{\mu} f'(R) = \int_X f'(R). \tag{158}$$

Therefore, the preceding discussion establishes that X minimizes the Wald entropy among all generalized extremal surfaces in $(\mathcal{M}', g)$. Consequently, for the spacetime $(\mathcal{M}', g)$,

$$S_{\mathrm{vN}}(\rho') = \frac{\int_X f'(R)}{4G}, \tag{159}$$

which saturates the upper bound of the outer entropy. This shows that

$$S_f^{(\mathrm{outer})}[\mu] = \frac{\int_\mu f'(R)}{4G}. \tag{160}$$

VI. THE BOUNDARY DUAL OF THE OUTER ENTROPY IN $f(R)$ GRAVITY

Our constructions so far have focused on the bulk side constructions. To achieve a fully holographic generalization of the coarse-grained entropy for a black hole in $f(R)$ gravity, we must define a dual quantity on the boundary, namely the simple entropy [23,24]. In this section, we demonstrate that in $f(R)$ gravity, the simple entropy serves as the exact boundary dual to the outer entropy of the generalized marginally trapped surface. Furthermore, we discuss the second law as applied to both the simple and outer entropy.

The simple entropy is defined via the coarse-graining of S_{vN}, holding fixed the expectation values of "simple" boundary operators in the presence of all possible "simple" sources. Sources are defined as "simple" if the bulk fields they generate propagate causally into the bulk spacetime;[10] correspondingly, operators are "simple" if their associated sources are simple [24]. The coarse-graining procedure consists of three steps. First, we select a boundary initial time t_i and a late-time cutoff t_f. Second, we fix all one-point functions of the local operators for times $t > t_i$, in the presence of all possible simple sources after t_i. Finally, find the state ρ that maximizes the von Neumann entropy, more precisely [23,24],

$$S^{(\mathrm{simple})}[t_i] = \max_\rho [S_{\mathrm{vN}}(\rho) | \mathrm{fixed} \langle E\mathcal{O}E^\dagger \rangle_\rho], \tag{161}$$

where E represents the presence of all possible simple sources.

A. Evaluating the simple entropy

To establish that the simple entropy is the correct boundary dual to the outer entropy, we first demonstrate that the outer entropy acts as the lower bound for the simple entropy. The first step is relating the simple entropy $S^{(\mathrm{simple})}[t_i]$ to the outer entropy of a bulk surface μ. Consider the slice t_i and shoot a future direct null

[10]It is worth noting that in $f(R)$ gravity there exists a dynamical operator $f'(R)$ in the bulk, which satisfy the Klein-Gordan equation $\Box f'(R) + V(f'(R)) = (8\pi G/(D-1))T$. Therefore, $f'(R)$ operator propagate causally in the bulk spacetime, that is, a simple operator.

hypersurface $N_l[t_i] = \partial I^+[t_i]$. For an $f(R)$ black hole, there must exist a surface μ on $N_l[t_i]$ with vanishing Θ_k. For the outer most μ, the condition $\nabla_l \Theta_k < 0$[11] should be generically satisfied. By the focusing theorem in $f(R)$ gravity (107), μ minimizes the Wald entropy on $N_l[t_i]$ relative to any surface outside it. Therefore, μ is the generalized minimar surface.

To compare the simple entropy and the outer entropy, it is natural to compare the information that is fixed under the change of state ρ. More precisely, compare the bulk region that can be reconstructed by the one-point data after t_i with the outer wedge of μ. For the fixed simple source J, following the HKLL prescription[12] [53–55] and EOM in the bulk, we can use all one-point data after t_i, namely $\langle E\mathcal{O}E^\dagger\rangle$, to reconstruct $R_J[t_i] = D[I^+[t_i] \cap I^-[\partial\mathcal{M}]]$. The information outside the $R_J[t_i]$ cannot be reconstructed from one-point data after t_i. More precisely, since we only consider the simple operators which propagate causally in bulk, the modification outside the $R_J[t_i]$ cannot affect boundary operators after t_i.

We now apply arbitrary simple sources after t_i, since sources are simple, the change of bulk geometry should be constrained in $I^+[t_i]$ which does not affect $N_l[t_i]$. Therefore, when we need to compare R_J and $R_{J'}$, we can compare the part of $N_l[t_i]$ contained in R_J and $R_{J'}$. When the simple source J is turned on, we either introduce or remove infalling matter. This will make the event horizon $I^-[\partial\mathcal{M}]$ move outward or inward along $N_l[t_i]$. In Einstein gravity, it is well-known that there are no sources that can move the event horizon so that the marginally trapped surface μ lies outside if the matter corresponds to sources satisfying the NEC [56]. However, $f(R)$ gravity differs from Einstein gravity in that the well-defined expansion satisfying the focusing theorem is the generalized expansion (6), and the correspondence focusing theorem is

$$k^a\nabla_a\Theta_k = -\frac{\theta_k^2}{D-2} - (k^a\nabla_a \log f'(R))^2 - \varsigma^{ab}\varsigma_{ab} - 8\pi G\frac{T_{ab}k^ak^b}{f'(R)}. \tag{162}$$

Therefore, it is natural to consider that:

(i) In $f(R)$ gravity, the event horizon cannot move so far inward such that generalized marginally trapped surface μ lies outside if the matter satisfies NEC.

Proof. Suppose there exists a generalized marginally trapped surface outside the event horizon. This implies that one can fire a light ray reaching the asymptotic AdS boundary; however, as the light ray approaches the boundary, its expansion will diverge. And this is in contradiction to the focusing theorem (162). Therefore, the generalized marginally trapped surface μ must lie behind the event horizon. ∎

The above discussions show that the generalized marginally trapped surface μ cannot lie inside $R_J[t_i]$. This shows that

$$R_J[t_i] \subseteq O_W^f[\mu]. \tag{163}$$

Thus, the one-point data $\langle O\rangle_J$ allows us to reconstruct at most $O_W^f[\mu]$, that is, the set of data used to compute the simple entropy is a subset of the set of data used to compute the outer entropy. This will lead to

$$S^{(\text{simple})}[t_i] \geq S_f^{(\text{outer})}[\mu]. \tag{164}$$

Therefore, we show that the outer entropy is the lower bound of the simple entropy.

Finally, we are going to show that for the near equilibrium state, on the bulk side corresponding to near stationary black holes, the simple entropy coincides with the outer entropy of the generalized marginally trapped surface μ. For thermal equilibrium state ρ_{stat}, the corresponding geometry is the stationary black hole in $f(R)$ gravity, denote as $\mathcal{M}_{\text{stat}}$. For a near thermal equilibrium state ρ with simple source J be turned on after time t_i, on the bulk side, we can "turn off" the matter falling into the event horizon by attaching a stationary null hypersurface $N_k[\mu]$, analogous to the procedure in Sec. V B. Using the HKLL prescription, we can then reconstruct the boundary simple source J' for the new geometry $\mathcal{M}'$. In this geometry, the generalized marginally trapped surface μ is just on the event horizon,[13] then we can fix all dynamical information on $N_l[t_i]$ outside μ. Together with original source J, we can fully reconstruct all the data in the outer wedge $O_W^f[\mu]$, this shows that for the $f(R)$ gravity,

$$S^{(\text{simple})}[t_i] = S_f^{(\text{outer})}[\mu]. \tag{165}$$

B. Discussion of the second law

We first demonstrate that both the outer and simple entropy satisfy the second law. In the bulk, due to the $f(R)$ gravity focusing theorem (162), a later-time generalized marginally trapped surface μ_2 (corresponding to boundary time t_2) must lie within the outer wedge of an earlier-time surface μ_1 (corresponding to boundary time t_1), and

[11]This is the generalized minimar condition, see Sec. I A.

[12]It is also possible to include interactions, at least perturbatively in 1/N [51,52].

[13]In the Einstein frame, for big perturbation from thermal equilibrium state, even we turn off all the "matter" [include the $f'(R)$ field, in the Einstein frame is the ϕ] on event horizon, we are not guaranteed that μ lies on event horizon [24]. This is the reason why we only talk about near thermal equilibrium state.

naturally $O_W^f[\mu_2] \subset O_W^f[\mu_1]$. Therefore, there are fewer data in $O_W^f[\mu_2]$ compared to $O_W^f[\mu_1]$, and naturally

$$S_f^{(\text{outer})}[\mu_2] \geq S_f^{(\text{outer})}[\mu_1]. \tag{166}$$

On the boundary side, things become more natural. Consider the simple entropy at two different times t_1 and t_2, where $t_2 > t_1$. The boundary argument is analogous: since there are fewer constraints (regarding simple sources and one-point functions) after t_2 than after t_1, the entropy is monotonically nondecreasing. Therefore, the simple entropy satisfies the second law,

$$S^{(\text{simple})}[t_2] \geq S^{(\text{simple})}[t_1]. \tag{167}$$

For the near equilibrium black hole in $f(R)$ gravity, under the first order perturbation the dynamical entropy proposed by Hollands-Wald-Zhan [6,7],

$$S_{\text{dyn}} = (1 - v\partial_v)\frac{1}{4G}\int_{\mathcal{C}(v)} f'(R) = (1 - v\partial_v)S_{\text{Wald}}[\mathcal{C}(v)], \tag{168}$$

here v is the affine null parameter along the future horizon and $\mathcal{C}(v)$ is an arbitrary cross section. Under a first-order perturbation, it can be shown that the dynamical black hole entropy takes the form of the Wald entropy of the generalized marginally trapped surface [13,20,21],

$$S_{\text{dyn}}(v) = \frac{1}{4G}\int_{\mu(v)} f'(R), \tag{169}$$

where $\mu(v)$ is the associate generalized marginally trapped surface or the generalized apparent horizon of $\mathcal{C}(v)$. This agrees with our result (99) in the first-order sense,

$$S_{\text{dyn}}(v) = S_f^{(\text{outer})}[\mu(v)] = S^{(\text{simple})}[t], \tag{170}$$

where t is the boundary time corresponding to $\mu(v)$. And if the matter satisfies the NEC, the dynamical black entropy satisfies the second law,

$$\partial_v \delta S_{\text{dyn}} = \frac{2\pi}{\kappa}\int_{\mathcal{C}(v)} \delta T_{ab}\xi^a k^a \geq 0, \tag{171}$$

where κ is the surface gravity and $\xi^a = \kappa v k^a$ is the killing field (after turning on falling matter ξ^a will no longer be the killing field) on the future horizon. Hence, this result agrees with the (166) and (167).

From the above discussions, we can therefore conclude that the outer entropy and its boundary dual, which is the simple entropy, are the correct holographic interpretation of the dynamical black hole entropy of $f(R)$ gravity, at least in the first-order perturbation near stationary black hole in the large-N limit where the correspondent bulk physics are classical gravity. In particular, our construction is even a nonperturbative construction, at least valid order by order in perturbation to a stationary black hole in $f(R)$ gravity, which can be viewed as a generalization of the dynamical black hole entropy of $f(R)$ gravity in AdS/CFT.

VII. CONCLUSION

In this paper, we revisit and develop the holographic description of S_{dyn} in $f(R)$ gravity proposed by Hollands-Wald-Zhang [6], placing it into a more systematic AdS/CFT framework and tracking carefully how the entropy dictionary is modified by higher-curvature dynamics. On the gravity (bulk) side, we establish that the outer entropy of a generalized marginally trapped surface is simply equal to the Wald entropy evaluated on that surface. In other words, once the "outer wedge" data are held fixed, the maximal coarse-grained entropy compatible with the exterior geometry is controlled by the same higher-curvature functional that governs black-hole entropy in $f(R)$ theories.

A key ingredient of our analysis is a careful treatment of Einstein frames and their holographic interpretation. We first derive holographic description in the Einstein frame, where the theory can be written as Einstein gravity coupled to an auxiliary scalar degree of freedom, and where standard holographic techniques apply directly. Within this setup, we then establish an explicit correspondence between the boundary von Neumann entropy computed in the Einstein frame description and the corresponding entropy in the $f(R)$ frame. Concretely, we track how the reduced density matrix and its entropy transform under the field redefinitions and Weyl rescalings relating the two frames, and we identify the precise map that preserves the physical content of entanglement. Using this entropy dictionary, we translate the outer entropy construction between frames, obtaining a robust outer entropy correspondence that allows us to import the Einstein frame maximization logic and re-express it in purely geometric language in the $f(R)$ frame. This provides the conceptual bridge needed to equate the coarse-grained outer entropy with the Wald functional of the generalized marginally trapped surface.

Building on this foundation, we proceed to derive the outer entropy directly in $f(R)$ gravity. We prove a focusing theorem for an appropriately defined generalized expansion, which serves as a higher-curvature generalization of the Raychaudhuri equation under the usual null energy conditions. With this theorem in hand, we construct a stationary null hypersurface characterized by vanishing generalized expansion, and we formulate the necessary junction conditions for consistently stitching geometries across a hypersurface within $f(R)$ gravity. These tools allow us to explicitly construct a bulk spacetime whose boundary von Neumann entropy saturates the upper bound implied by the outer entropy maximization i.e., we realize

the extremal configuration achieving the coarse-grained maximum compatible with the fixed exterior. Finally, using the $f(R)$ focusing theorem, we show that the simple entropy defined on the boundary is indeed the holographic dual of the outer entropy in $f(R)$ gravity, thereby extending the outer entropy and simple entropy correspondence to this higher-curvature setting and show the second law of the two entropy which agree with the dynamical black hole entropy [6].

Looking forward, our results suggest several promising future directions. Since our construction of the coarse graining entropy is nonperturbative in classical gravity (at least to all orders in perturbation theory around equilibrium), it suggests that the nonperturbative construction of the dynamical black hole entropy is highly related to the generalized marginally trapped surface (generalized minimar surface). In a recent paper [10] it was suggested that quasilocal horizon (QLH) is the right geometric concept for far from equilibrium black hole thermodynamics in Einstein gravity. We hope that similar concepts will appear in general higher curvature gravity. Recently there are some discussions about higher curvature focusing theorem near first order dynamic perturbative killing horizon [22,57], our construction of a nonperturbative focusing theorem in $f(R)$ gravity suggests similar theorem may exists in more general higher curvature gravity, for example, in $f(\text{Riemann})$ gravity. This will help us define HRT surface in general higher curvature gravity, and explain why the Wall entropy and Dong entropy [4,5] are the same but with different physical original.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: CASES OF $f''(R)=0$

For points on the stationary null surface N_{-k} satisfying $f''(R)\neq 0$, the condition $\partial_v R=0$ is ensured. We now consider regions on N_{-k} where $f''(R)=0$. We first introduce relevant definitions. For the considered spacetime, the prespacetime regions for discrete zeros R_α (see Fig. 4) are defined as the subregions of spacetime where the curvature equals R_α, namely,

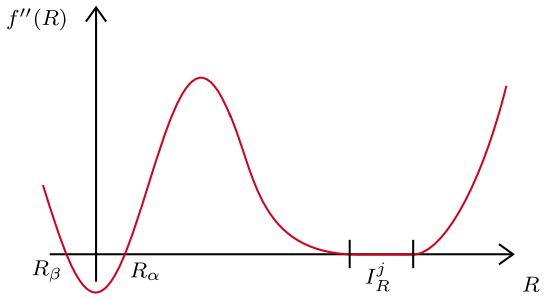


FIG. 4. This is an example of $f''(R)$. We use R_α, R_β denote the discrete zeros, α and β are used to label different zeros. And use I_R^j denote the interval zero, that is, if R is belong to the interval I_R^j, then $f''(R)=0$. Here j is used to label different interval zero.

$$\mathcal{M}_{R_\alpha}=\{p\in\mathcal{M}|R|_p=R_\alpha\}. \tag{A1}$$

Similarly, the prespacetime regions for interval zero I_R^j (see Fig. 4) are defined as the subregion of spacetime with curvature belonging to interval zero I_R^j,

$$\mathcal{M}_{I_R^j}=\{p\in\mathcal{M}|R|_p\in I_R^j\}. \tag{A2}$$

In the first case, we define $V_{R_\alpha}=N_{-k}\cap\mathcal{M}_{R_\alpha}$ and consider its k^a congruence, denoted as $N_{\pm k}[V_{R_\alpha}]$. In the complement region of the congruence, that is, $N_{\pm k}[V_{R_\alpha}]/V_{R_\alpha}$, since $f''(R)\neq 0$ we have $\partial_v R=0$. Assuming spacetime continuity, the curvature over the entire congruence must match that of V_{R_α}. Therefore, this will imply that $\partial_v R|_{N_{\pm k}[V_{R_\alpha}]}=0$ and naturally $\partial_v R|_{V_{R_\alpha}}=0$.

For the other case, we define $V_{I_R^j}=N_{-k}\cap\mathcal{M}_{I_R^j}$. If the spacetime curvature falls within the interval I_R^j, the $f(R)$ gravity reduces to Einstein gravity, that is, $f(R)=cR+\text{constant}$. Together with the equation of motion of $f(R)$ gravity, we can show that

$$\partial_v R\propto\partial_v T=\partial_v(g^{ab}T_{ab}). \tag{A3}$$

Since $g^{uu}=g^{ui}=g^{vi}=0$ on N_{-k}, we only need to consider T_{uv} and T_{ij}. Imposing the initial free data conditions (129) and (128) yields $\partial_v T=0$. Therefore, this will imply $\partial_v R|_{V_{I_R^j}}=0$. In conclusion, even if there exist subregions of N_{-k} where $f''(R)=0$, the result $\partial_v R=0$ remains correct.

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