

Hawking radiation as a flux of Noether charge

Zhi-Wei Wang^{1,2,*} and Samuel L. Braunstein^{2,†}

¹*College of Physics, Jilin University, Changchun, 130012, People's Republic of China*

²*University of York, York YO10 5GH, United Kingdom*



(Received 28 June 2026; accepted 6 August 2026; published 21 August 2026)

We present a geometric framework linking classical spacetime geometry to black hole thermodynamics beyond the stationary regime. By evaluating the Komar conserved current on an arbitrary timelike worldtube, with the off-boundary extension fixed by a Gaussian normal foliation, we obtain a local surface density κ_{Noether} that requires no Killing vector, no spatial symmetry, and no slow-evolution assumption. It reduces to the Killing surface gravity for stationary non-degenerate horizons, and, applied to a uniformly accelerating worldtube, recovers the Unruh temperature. Adapting the Parikh-Wilczek tunneling method to horizons lacking radial and temporal symmetry, we show the local tunneling temperature is $T = \kappa_{\text{Noether}}/2\pi$ at leading semiclassical order. For a black hole that is genuinely evolving, the single stationary notion of surface gravity separates, at first order in the evaporation rate, into a family of inequivalent quantities; at the event horizon, two of these carry distinct roles: κ_{Noether} governs the local tracking kinematics and the proper temperature at the horizon, while the optical peeling rate κ_{peeling} governs the asymptotic Bogoliubov spectrum. Evaluated on the exact Vaidya solution, this yields a closed-form surface gravity for the event horizon and an exact gap equation valid for arbitrary evaporation profiles. The separation has a definite causal character: the tracking rate is fixed by the instantaneous geometry, whereas the peeling rate is a functional of the horizon's entire future, so local dynamics couples only to the former. Supplying the heat flux independently of the surface gravity, which the Vaidya geometry permits, shows further that the quasilocal first law on the apparent horizon is balanced not by the tracking rate but by the Kodama-Hayward value, with which it coincides only when the horizon's acceleration vanishes. Four surface gravities on three surfaces therefore answer four distinct questions. These results give a local interpretation of the dynamical Komar charge, identifying the thermal emission of an evolving horizon with a flux of gravitational Noether charge.

DOI: [10.1103/PhysRevD.114.044065](https://doi.org/10.1103/PhysRevD.114.044065)

I. INTRODUCTION

The connection between the laws of thermodynamics and the dynamics of black holes stands as one of the central clues in the pursuit of quantum gravity. In stationary spacetimes, this synthesis is well understood. Wald demonstrated that black hole entropy can be formulated as a Noether charge associated with the diffeomorphism invariance of the gravitational action [1,2]. Furthermore, the surface gravity κ of a stationary Killing horizon uniquely determines the Hawking temperature of the emitted thermal radiation [3].

In recent years, understanding this thermodynamic framework dynamically has become increasingly central to resolving deep theoretical puzzles—such as the black hole information paradox, the computation of the Page curve via entanglement islands [4,5], and the exact nature of holographic duality away from equilibrium. A fundamental prerequisite for all these pursuits is an unambiguous, covariant geometric definition of the dynamical local temperature. However, extending this thermodynamic framework beyond the stationary regime remains a formidable challenge.

In dynamical spacetimes, the absence of a unique time-translating Killing vector obfuscates the standard definitions of surface gravity and horizon temperature. Additionally, as horizons evolve dynamically, a single notion of surface gravity splits into a family of inequivalent quantities—such as generator inaffinity (see Sec. VIC), instantaneous geometric limits, our kinematic tracking Noether rate, and the null-ray peeling rate—that coincide in stationary settings but separate at first order in the evaporation rate. Through an exact global eikonal analysis

*Contact author: zhiweiwang.phy@gmail.com

†Contact author: sam.braunstein@york.ac.uk

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

(detailed in Sec. V C), we prove that this separation follows, in the adiabatic regime, from the exactness of the quantum phase, where adiabatic local boundary kinematics decouple from the asymptotic spectrum. This separation is moreover of a definite causal character: the tracking rate is fixed locally, while the peeling rate depends on the global future of the horizon, so that only the former can enter local dynamical laws. Historically, addressing this gap has required restricting the spacetime symmetries, such as relying on the Kodama vector in spherically symmetric backgrounds [6,7], or employing specialized quasilocal geometric constructions like trapping, isolated, or slowly evolving horizons [8–10]. A recognized difficulty in this program is that the surface gravity of a non-Killing horizon is not unique: several natural definitions coincide in the stationary limit but differ at first order in the rate of evolution, and their relative merits have been analyzed at length [11–13]. A related persistent source of ambiguity has been identifying precisely which horizon (the apparent or the event horizon) operates as the true source of dynamical radiation (an issue we resolve in Sec. IV E).

In this paper, we present an alternative, general perspective: identifying the thermal flux of a dynamical black hole directly with a gravitational Noether charge. To achieve this, we return to the foundational Hamiltonian formulation of classical physics, where Noether's theorem dictates that symmetries generated by a system's Hamiltonian yield fundamental conserved quantities [14,15]. Already in 1959, Komar [16] found that $J^\mu \equiv \xi^{[\mu;\nu]}$ is a locally conserved 4-current for an arbitrary vector field ξ^μ . Indeed, its conservation law $J^\mu_{;\mu} \equiv 0$, which holds for general dynamical spacetimes, follows purely from geometric considerations for any torsion-free and metric-compatible theory of gravity.

Consider a spacelike foliation $\{\Sigma_t\}$ of spacetime and let $\Sigma \subset \Sigma_t$, for any particular t , be a bounded region on the hypersurface Σ_t . Then integrating the current J^μ over Σ and applying Stokes's theorem yields

$$\frac{1}{4\pi} \int_\Sigma J^\mu d\Sigma_\mu = \frac{1}{8\pi} \int_{\partial\Sigma} \xi^{[\mu;\nu]} d\Sigma_{\mu\nu}, \quad (1)$$

where $d\Sigma_\mu$ and $d\Sigma_{\mu\nu}$ are the tensorial volume and surface elements on Σ and its boundary $\partial\Sigma$, respectively. Thus, Eq. (1) immediately implies that

$$dQ = \frac{1}{2} \xi^{[\mu;\nu]} d\Sigma_{\mu\nu} \quad (2)$$

is a local Noether charge for every vector field ξ . (Note that the conventional Komar prefactor $1/4\pi$, which relates the integrated charge to the ADM mass for stationary spacetimes, is not included in our definition of dQ ; it plays no role in the determination of surface gravity or temperature.) While Eq. (2) is a confirmed Noether charge for general

relativity [2,17], a persistent roadblock has been identifying a choice of ξ^μ for which this charge is directly physically observable [18]. On asymptotically flat stationary spacetimes, substituting ξ^μ as a timelike Killing vector yields (up to the prefactor $1/4\pi$) the Komar mass [19]. Beyond the stationary setting, however, the Komar mass has largely eluded interpretation [17].

In what follows, we provide a physical interpretation for this dynamic Noether charge. We proceed in three steps: First, we evaluate the Noether charge surface density on an arbitrary, dynamically evolving timelike worldtube, defining a purely geometric local surface gravity, κ_{Noether} (Theorem 1). Second, we provide a geometric correspondence principle, demonstrating that κ_{Noether} reduces to the standard surface gravity for nondegenerate Killing horizons (Proposition 1). Finally, we transition to the semiclassical regime. By adapting the Parikh-Wilczek quantum tunneling formalism [20] to dynamically evolving horizons, we prove that the local tunneling temperature of a dynamical black hole is identically proportional to our Noether surface gravity (Theorem 2). As a direct consequence, applying this framework to Rindler spacetime exactly recovers the Unruh effect (Corollary 1). Together, these results reveal that the quasilocal Hawking radiation in general dynamical spacetimes can be understood as a flux of Noether charge, even as the asymptotic spectrum measured at spatial infinity relies upon a distinct global peeling surface gravity.

To maintain a self-contained narrative, we detail these core geometric and thermodynamic demonstrations directly in the main text. We explore the path-dependence and slicing-dependent coordinate features in noncovariant WKB treatments, formulate the exact global eikonal action to analytically isolate the asymptotic peeling temperature, evaluate a purely null-adapted Noether charge to demonstrate its relationship to the gauge-dependent generator inaffinity, and show that the quasilocal dynamical first law is balanced at the apparent horizon by the Kodama-Hayward surface gravity, with which our apparent-horizon value coincides for monotonic self-similar evolution, the local tunneling temperature being carried by the event-horizon limit. The Appendix provides the canonical Gaussian normal construction that resolves the extension ambiguity of the Noether charge.

II. NOETHER CHARGE AND WORLDTUBES

Throughout this paper, we adopt the mostly plus metric signature $(-, +, +, +)$. It will be helpful for our purposes here to appreciate that conservation $J^\mu_{;\mu} = 0$ implies a more general structure beyond that within hypersurfaces. Consider a bounded 4-volume $\mathcal{V}$. To tie conservation to our Noether charge on a hypersurface (1) we shall suppose that $\mathcal{V}$ is bounded by an earlier and later hypersurfaces $\Sigma \subset \Sigma_t$ and $\Sigma' \subset \Sigma_{t'}$, respectively. In addition, we shall suppose that $\mathcal{V}$ is bounded by a section of a worldtube $\mathcal{W}$, so $\partial\mathcal{V} = \Sigma' \cup \Sigma \cup \mathcal{W}$ (see Fig. 1). Our conservation law is

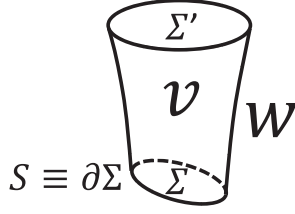


FIG. 1. Schematic of a finite region of the manifold $\mathcal{V}$ separated by a pair of spacelike hypersurfaces Σ and Σ' . Rather than taking Σ' as arbitrary, we envision a congruence of observers evolving from $\Sigma \subset \Sigma_t$ until they reach Σ' . In particular, we assume $\mathcal{W}$ is the natural evolution of these observers initially on the boundary of that initial hypersurface, corresponding to a 2-surface $\mathcal{S} \equiv \partial\Sigma$, until they reach Σ' . Cases where $\mathcal{W}$ is a null hypersurface could be treated by switching to null “observers,” or as a limiting process as we do below.

$$\text{now } 0 = \int_{\mathcal{V}} J^\mu{}_{;\mu} d\Sigma = \int_{\partial\mathcal{V}} J^\mu d\Sigma_\mu \text{ or}$$

$$\int_{\Sigma'} J^\mu d\Sigma_\mu - \int_{\Sigma} J^\mu d\Sigma_\mu = - \int_{\mathcal{W}} J^\mu d\Sigma_\mu, \quad (3)$$

with $d\Sigma$ the measure on $\mathcal{V}$ and $d\Sigma_\mu$ is outward pointing from $\mathcal{V}$ on $\mathcal{W}$ and future-directed on Σ and Σ' .

A. The boundary of a boundary vanishes

Using Eq. (1) we may further decompose Eq. (3) into a conservation law in terms of Noether charges at the mutual boundaries of each hypersurface as

$$\begin{aligned} & \frac{1}{2} \int_{\Sigma'} \xi^{[\mu;\nu]} d\Sigma_{\mu\nu}^{(\partial\Sigma')} - \frac{1}{2} \int_{\Sigma} \xi^{[\mu;\nu]} d\Sigma_{\mu\nu}^{(\partial\Sigma)} \\ &= -\frac{1}{2} \int_{\partial\mathcal{W}} \xi^{[\mu;\nu]} d\Sigma_{\mu\nu}^{(\partial\mathcal{W})}. \end{aligned} \quad (4)$$

To exclude gaps at the sharp edges in Fig. 1, we require

$$d\Sigma_{\mu\nu}^{(\partial\Sigma)} = -d\Sigma_{\mu\nu}^{(\partial\mathcal{W})}, \quad (5)$$

at each boundary, so that every term in Eq. (4) trivially cancels to yield the expression $0 = 0$, reflecting the fact that the boundary of a boundary vanishes. [The additional sign in Eq. (5) comes from the reversed order of temporal versus spatial contributions to the two bivectors $d\Sigma_{\mu\nu}$ for the spacelike and timelike hypersurfaces.]

To guarantee consistency of measures at the edges, the pairs of vectors forming the bivectors on either side of Eq. (5) must span the same 2-dimensional subspace. For the spacelike hypersurface Σ the natural choices are the future-directed unit normal to the hypersurface $\hat{T}^\mu$, and the outside pointing normal to its boundary, $\hat{N}^\mu$ within Σ_t (see Fig. 2). For $\mathcal{W}$ we choose the unit normal to the worldtube $\hat{R}^\mu$ and the future-directed tangent to the worldtube, and unit normal to its “boundary” where it intersects the

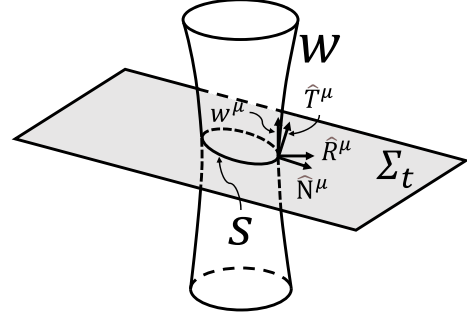


FIG. 2. A timelike worldtube $\mathcal{W}$ intersects a spacelike hypersurface Σ_t on the 2-surface $\mathcal{S} = \mathcal{W} \cap \Sigma_t$. At any point on $\mathcal{S}$ there are two pairs of orthonormal vectors: $\{\hat{T}^\mu, \hat{N}^\mu\}$ and $\{w^\mu, \hat{R}^\mu\}$ spanning the same 2-dimensional subspace. Here, $\hat{T}^\mu$ is the future-directed timelike unit normal to Σ_t , and $\hat{N}^\mu$ lies tangent to this hypersurface, but is normal to the boundary $\mathcal{S}$. Similarly, $\hat{R}^\mu$ is the outward pointing unit normal to the worldtube $\mathcal{W}$, and w^μ is the future-directed unit normal to $\mathcal{S}$ and lies tangent to the worldtube $\mathcal{W}$.

hypersurface, i.e., to $\mathcal{S} = \partial\Sigma$. We label this latter vector w^μ . By construction, the orthonormal pairs $\{\hat{T}^\mu, \hat{N}^\mu\}$ and $\{w^\mu, \hat{R}^\mu\}$ span the same 2-dimensional subspace and Eq. (5) is easily satisfied.

B. Noether charge construction

We will want to describe two vector fields on our worldtube: one field corresponding to the 4-velocity, v^μ , of an *arbitrary* congruence of observers with trajectories, $x^\mu(\tau)$, hugging the worldtube $\mathcal{W}$, and a second one corresponding to the 4-velocity, w^μ , for a class of worldtube-hugging observers whose trajectories, $y^\mu(\tau')$, flow normal to $\mathcal{S}$ as they pass it. Here, τ and τ' denote the proper time for individuals from either the arbitrarily chosen congruence or the “normal” congruence.

Consider choosing $\xi^\mu \propto v^\mu$ for an arbitrary worldtube-hugging congruence of observers. To tie the meaning of the Noether charge to quantities measured at spatial infinity we shall choose a normalization so $\xi^0 = 1$, in other words we take

$$\xi^\mu \equiv \Psi v^\mu = \frac{dx^\mu(\tau)}{dt}, \quad (6)$$

where $\Psi = d\tau/dt$ is the red-shift factor for an individual observer within this congruence.

We emphasize that tethering the parameter t to spatial infinity here is merely an interpretive choice to recover standard asymptotic thermodynamic values, rather than a mathematical necessity. The coordinate t acts as a synchronization gauge providing a common rate-of-change for the congruences across the local foliation Σ_t . Consequently, this quasilocal construction holds even in spacetimes completely lacking a spatial infinity (such as closed

cosmological backgrounds), where t simply represents an arbitrary reference clock. If we were to change our reference clock to $\tilde{t} = h(t)$, the time-flow vector ξ^μ , the redshift factor Φ , and the Noether surface gravity κ_{Noether} would all scale identically by $1/h'(t)$. As a result, the physically measured local proper temperature, $T_{\text{local}} = T_{\text{tunneling}}/\Phi$, remains a quasilocal geometric invariant.

We may similarly consider the same derivative for our normal observers

$$\frac{dy^\mu(\tau')}{dt} = \Phi w^\mu, \quad (7)$$

where we use Φ to label the redshift factors for our normal observers.

Before stating our primary theorem, we must address a fundamental geometric prerequisite. Evaluating a local surface charge density involves the Komar integrand $\xi_{[\mu;\nu]}$, whose evaluation at a point on $\mathcal{S}$ requires a derivative of ξ_μ pointing infinitesimally off the 3-dimensional worldtube $\mathcal{W}$ and into the 4-dimensional bulk spacetime. Because our congruence generator ξ^μ is initially defined strictly on the worldtube, its transverse derivative into the bulk is mathematically undefined. To compute a unique and physically meaningful Noether charge, one must explicitly postulate a geometric rule for extending the vector field off the boundary. As detailed in the Appendix, we resolve this uniquely by adopting a canonical Gaussian normal coordinate extension into the bulk, which imposes the exact geometric boundary condition $[\hat{R}, \Phi w]^\mu = 0$.

Theorem 1. Consider a timelike worldtube $\mathcal{W}$ and a spacelike hypersurface Σ_t , which intersects at the 2-surface $\mathcal{S} = \mathcal{W} \cap \Sigma_t$. Let $x^\mu(\tau)$ be the trajectory of an arbitrary observer lying on $\mathcal{W}$, that intersects a spacelike hypersurface Σ_t at some particular point. Further, let $y^\mu(\tau')$ be the trajectory of an observer that passes this same point normally to the 2-surface $\mathcal{S}$ while also remaining constrained to lie on $\mathcal{W}$. Choosing $\xi^\mu = dx^\mu(\tau)/dt$ for a congruence of such observers, and assuming this vector field is extended infinitesimally into the bulk spacetime via a Gaussian normal foliation (such that $\mathcal{L}_{\hat{R}}(\Phi w^\mu) = 0$, as established in the Appendix), and writing dA for the proper area element of the 2-surface $\mathcal{S}$ over which the Noether charge dQ of Eq. (2) is evaluated, we find that

$$\begin{aligned} \kappa_{\text{Noether}} &\equiv \frac{dQ}{dA} = \frac{1}{2} \frac{\xi^{[\mu;\nu]} d\Sigma_{\mu\nu}}{dA} \\ &= \Phi a_\mu^w \hat{R}^\mu = \Phi_{;\mu} \hat{R}^\mu = -\Phi K_{\mu\nu}^\mathcal{W} w^\mu w^\nu, \end{aligned} \quad (8)$$

where Φ and a_μ^w are the redshift factor and 4-acceleration of a normal worldtube-hugging observer and $\hat{R}^\mu$ is the unit normal to the worldtube at the point under consideration

(see Fig. 2), and finally $K_{\mu\nu}^\mathcal{W}$ is the extrinsic curvature of the worldtube $\mathcal{W}$.

Although rather complicated in formulation, this theorem tells us that the Noether charge surface density of an arbitrary worldtube-hugging vector field is given by the worldtube-normal component of the 4-acceleration (as measured by an observer at spatial infinity) of worldtube-hugging observers passing normally through an element of surface area lying at the intersection of our timelike worldtube $\mathcal{W}$ and the spacelike hypersurface Σ_t . We shall see that this Noether charge surface density is an intrinsic local property of the worldtube itself as seen (and “foliated”) by an observer at spatial infinity.

Proof. The theorem explicitly assumes the Gaussian normal extension into the bulk spacetime (detailed in the Appendix), yielding $\mathcal{L}_{\hat{R}}(\Phi w^\mu) = [\hat{R}, \Phi w]^\mu = 0$. By definition, the arbitrary worldtube-hugging congruence can be decomposed as $\xi^\mu = \Phi w^\mu + \zeta^\mu$, where $\zeta^\mu \equiv \Psi \beta s^\mu$ is the purely tangential drift along the 2-surface $\mathcal{S}$, with s^μ a unit vector tangent to $\mathcal{S}$, β the drift speed and Ψ the redshift factor of the congruence. As explicitly verified in the tangential gauge invariance section below, this geometric condition identically extinguishes the charge contribution from the tangential drift. Consequently, we only need to evaluate the Noether charge density for the normal flow component, $\xi^\mu = \Phi w^\mu$.

We now evaluate the charge surface density for Φw^μ . Noting that the outward-pointing surface binormal for the 2-surface $\mathcal{S}$ is explicitly given by $d\Sigma_{\mu\nu} = (\hat{R}_\mu w_\nu - \hat{R}_\nu w_\mu) dA$, the contraction with the Komar integrand $dQ = \frac{1}{2} \xi^{[\mu;\nu]} d\Sigma_{\mu\nu}$ yields

$$\begin{aligned} \frac{dQ}{dA} &= \frac{1}{2} [(\Phi w_\mu)_{;\nu} w^\nu \hat{R}^\mu - (\Phi w_\nu)_{;\mu} w^\mu \hat{R}^\nu] \\ &= \frac{1}{2} [(\Phi_{;\nu} w_\mu + \Phi w_{\mu;\nu}) w^\nu \hat{R}^\mu \\ &\quad - (\Phi_{;\mu} w_\nu + \Phi w_{\nu;\mu}) w^\mu \hat{R}^\nu]. \end{aligned} \quad (9)$$

Since w^μ and $\hat{R}^\mu$ are orthogonal ($w_\mu \hat{R}^\mu = 0$), the first term vanishes and the third term simplifies. Furthermore, since w^μ is a unit timelike vector ($w_\nu w^\nu = -1$), its covariant derivative is orthogonal to itself ($w_{\nu;\mu} w^\nu = 0$). The expression therefore reduces to

$$\frac{dQ}{dA} = \frac{1}{2} [\Phi a_\mu^w \hat{R}^\mu - \Phi_{;\mu} (-1) \hat{R}^\mu] = \frac{1}{2} [\Phi a_\mu^w \hat{R}^\mu + \Phi_{;\mu} \hat{R}^\mu], \quad (10)$$

where $a_\mu^w = w_{\mu;\nu} w^\nu$ is the 4-acceleration of the normal worldtube-hugging observer.

Next, we directly apply the geometric assumption $[\hat{R}, \Phi w]^\mu = 0$. This is not automatic and is what the Gaussian normal construction of the Appendix supplies: for an extension not built that way the two forms in Eq. (10)

are inequivalent, which is precisely the extension ambiguity the construction removes. On Vaidya, for instance, the family of constant-area-radius tubes fails the condition whenever $\dot{M} \neq 0$. Lowering the index and contracting with w_μ yields

$$0 = w_\mu (\hat{R}^\nu \nabla_\nu (\Phi w^\mu) - \Phi w^\nu \nabla_\nu \hat{R}^\mu). \quad (11)$$

Distributing the derivative in the first term gives $w_\mu \hat{R}^\nu (w^\mu \nabla_\nu \Phi + \Phi \nabla_\nu w^\mu)$. Since $w_\mu w^\mu = -1$, we have $w_\mu \nabla_\nu w^\mu = 0$, so this evaluates exactly to $-\hat{R}^\nu \Phi_{,\nu}$. For the second term, differentiating the orthogonality condition $w_\mu \hat{R}^\mu = 0$ gives $w_\mu \nabla_\nu \hat{R}^\mu = -\hat{R}^\mu \nabla_\nu w_\mu$. Substituting these back yields the exact geometric identity

$$\begin{aligned} 0 &= -\Phi_{,\nu} \hat{R}^\nu - \Phi w^\nu (-\hat{R}^\mu \nabla_\nu w_\mu) \\ &= -\Phi_{,\nu} \hat{R}^\nu + \Phi a_\mu^w \hat{R}^\mu \\ \Rightarrow \Phi_{,\mu} \hat{R}^\mu &= \Phi a_\mu^w \hat{R}^\mu. \end{aligned} \quad (12)$$

Substituting Eq. (12) into Eq. (10) eliminates the geometric factor of $1/2$ and yields

$$\frac{dQ}{dA} = \Phi a_\mu^w \hat{R}^\mu = \Phi_{,\mu} \hat{R}^\mu. \quad (13)$$

Crucially, by Stokes's theorem, the integrated Noether charge $Q = \oint_S (dQ/dA) dA$ for a general, non-Killing vector field is highly sensitive to the choice of transverse extension off the 2-surface, as the normal derivative directly dictates the boundary flux. Concretely, the Komar integrand $\xi_{[\mu;\nu]}$ at a point on S involves the covariant derivative of ξ_μ in the direction $\hat{R}^\mu$ normal to $\mathcal{W}$, which is mathematically undefined until an extension into the bulk is specified. Historically, this extreme extension-dependence has been a primary obstacle making the definition of an off-equilibrium Komar charge ambiguous. By adopting the Gaussian normal extension, we do not merely select a representative density for an already-fixed total charge; rather, we provide the canonical geometric constraint that physically fixes both the local surface gravity and the total Noether charge itself. As we shall see in Theorem 2, this exact same extension arises in the tunneling calculation, where the tunneling path is naturally evaluated along a geodesic normal to the horizon worldtube.

Finally, we relate this normal acceleration to the extrinsic curvature of the worldtube, $K_{\mu\nu}^{\mathcal{W}}$. The extrinsic curvature tensor for $\mathcal{W}$ is defined by $K_{\mu\nu}^{\mathcal{W}} = h_\mu^\alpha h_\nu^\beta \hat{R}_{\alpha\beta}$, where $h_{\mu\nu} = g_{\mu\nu} - \hat{R}_\mu \hat{R}_\nu$ is the induced metric on $\mathcal{W}$. Projecting this tensor along the normal worldtube-hugging velocity w^μ , we have

$$-\Phi K_{\mu\nu}^{\mathcal{W}} w^\mu w^\nu = -\Phi \hat{R}_{\mu;\nu} w^\mu w^\nu. \quad (14)$$

Because the velocity w^μ is strictly tangent to $\mathcal{W}$ and $\hat{R}^\mu$ is its normal, they are orthogonal everywhere on the worldtube ($w^\mu \hat{R}_\mu = 0$). Differentiating this orthogonality condition along the trajectory w^ν gives

$$(w^\mu \hat{R}_\mu)_{;\nu} w^\nu = 0 \Rightarrow w^\mu_{;\nu} w^\nu \hat{R}_\mu + w^\mu \hat{R}_{\mu;\nu} w^\nu = 0. \quad (15)$$

Recognizing the normal observer's 4-acceleration $a_\mu^w \equiv w_{\mu;\nu} w^\nu$, we find $\hat{R}_{\mu;\nu} w^\mu w^\nu = -a_\mu^w \hat{R}^\mu$. Substituting this relation into Eq. (14) yields

$$-\Phi K_{\mu\nu}^{\mathcal{W}} w^\mu w^\nu = \Phi a_\mu^w \hat{R}^\mu. \quad (16)$$

Combining Eqs. (12), (13), and (16), we confirm the exact chain of equalities stated in the theorem

$$\frac{dQ}{dA} = \Phi a_\mu^w \hat{R}^\mu = \Phi_{,\mu} \hat{R}^\mu = -\Phi K_{\mu\nu}^{\mathcal{W}} w^\mu w^\nu. \quad (17)$$

This completes the proof of Theorem 1. $\blacksquare$

C. Tangential gauge invariance

An immediate physical consequence of this geometric extension is that the tangential terms vanish, yielding the reduction $\kappa_{\text{Noether}}[\xi^\mu] \equiv \kappa_{\text{Noether}}[\Phi w^\mu]$. We prove this by evaluating the Noether charge purely for the tangential drift component, $\zeta^\mu \equiv \Psi \beta s^\mu$. Since ζ^μ lies entirely within the 2-surface S , it is orthogonal to both normal vectors: $\zeta_\mu \hat{R}^\mu = 0$ and $\zeta_\mu w^\mu = 0$.

Using these orthogonality conditions, we can pull the covariant derivatives off ζ^μ in the charge density formula

$$\begin{aligned} \kappa_{\text{Noether}}[\zeta^\mu] &= \frac{1}{2} [\zeta_{\mu;\nu} w^\nu \hat{R}^\mu - \zeta_{\nu;\mu} w^\nu \hat{R}^\mu] \\ &= \frac{1}{2} [-\zeta_\mu \hat{R}^\mu_{;\nu} w^\nu + \zeta_\nu w^\nu_{;\mu} \hat{R}^\mu] \\ &= \frac{1}{2} \zeta_\mu (w^\mu_{;\nu} \hat{R}^\nu - \hat{R}^\mu_{;\nu} w^\nu) \\ &= \frac{1}{2} \zeta_\mu [\hat{R}, w]^\mu = \frac{1}{2} \zeta_\mu \mathcal{L}_{\hat{R}} w^\mu, \end{aligned} \quad (18)$$

where we have used $0 = (\zeta_\mu \hat{R}^\mu)_{;\nu} \Rightarrow \zeta_{\mu;\nu} \hat{R}^\mu = -\zeta_\mu \hat{R}^\mu_{;\nu}$ and similarly for w^ν .

We relate this to the full normal flow by expanding $\zeta_\mu \mathcal{L}_{\hat{R}}(\Phi w^\mu) = \zeta_\mu (\Phi \mathcal{L}_{\hat{R}} w^\mu + w^\mu \hat{R}^\nu \Phi_{,\nu})$. Because $\zeta_\mu w^\mu = 0$, the second term vanishes, leaving $\zeta_\mu \mathcal{L}_{\hat{R}}(\Phi w^\mu) = \Phi \zeta_\mu \mathcal{L}_{\hat{R}} w^\mu$. Thus

$$\kappa_{\text{Noether}}[\zeta^\mu] = \frac{1}{2\Phi} \zeta_\mu \mathcal{L}_{\hat{R}}(\Phi w^\mu). \quad (19)$$

When we apply the Gaussian normal extension $\mathcal{L}_{\hat{R}}(\Phi w^\mu) = 0$ (see the Appendix), the charge contribution of the tangential drift identically vanishes. Therefore, the

total charge completely collapses to

$$\kappa_{\text{Noether}}[\xi^\mu] \equiv \kappa_{\text{Noether}}[\Phi w^\mu]. \quad (20)$$

We note that the Noether charge surface density is independent of the purely tangential drift parameters β and s^μ . The local surface gravity is therefore determined entirely by the normal worldtube-hugging velocity w^μ and the worldtube geometry (via $K_{\mu\nu}^{\mathcal{W}}$), and is insensitive to any tangential drift of the observer congruence along $\mathcal{S}$. Thus κ_{Noether} is an invariant geometric property of the worldtube's extrinsic embedding, evaluated relative to the chosen spatial foliation Σ_t .

We have now derived the naturally defined surface gravity for a family of observers and shown that it is the Noether charge density, κ_{Noether} . The construction of the Noether charge from Eq. (4) is not limited to stationary spacetimes, nor does this theorem require the existence of a Killing vector. The generality of this result and its physical interpretation both derive from the scenario we considered: on the one hand, a general timelike worldtube, and on the other, an observer on the 2-surface of that worldtube, who therefore feels a physical surface gravity. Together, the generality and physical grounding of our result provide the in-principle possibility of direct observational access to a gravitational Noether charge in dynamical spacetimes.

III. SURFACE GRAVITY AND GEOMETRIC CORRESPONDENCE

A. Surface gravity for null horizons

While Theorem 1 establishes κ_{Noether} as a local property of arbitrary dynamical worldtubes, justifying its physical interpretation as a generalized “surface gravity” requires a clear correspondence principle. Specifically, any proposed dynamical generalization must recover the standard, well-understood definition in the familiar limit of a stationary black hole. Providing this geometric validation serves as the critical logical bridge connecting our general Noether-charge formalism to established black hole thermodynamics.

By choosing differing worldtubes one can address different physical questions. To evaluate this correspondence principle, we transition our focus to horizons, where the choice of worldtube is unambiguous. A trapping (apparent) horizon whose worldtube is timelike, as occurs for an evaporating black hole, is a special case of the timelike worldtubes considered above, and may be treated in the same way. To derive the surface gravity, one need only choose the worldtube $\mathcal{W}$ as the horizon worldtube and apply the expression for κ_{Noether} . The case of null horizons is considered next (the case of a trapping horizon whose worldtube is spacelike, as occurs for an accreting black hole, is not considered in this paper).

For worldtubes of null horizons, one may take the local surface gravity to be the limit as the congruence of observers approaches the null worldtube on the hypersurface of interest. As we now explicitly demonstrate, evaluating this limit for our Noether charge density on a stationary background precisely reproduces the established classical textbook result. Because of this fundamental geometric correspondence, we will hereafter refer to κ_{Noether} interchangeably as the local Noether charge density and the generalized Noether surface gravity.

Consider a nondegenerate Killing horizon with Killing vector

$$K^\mu = (\partial_t)^\mu + \Omega_H (\partial_\phi)^\mu, \quad (21)$$

where t is the coordinate time at spatial infinity, ϕ is the azimuthal angular coordinate, and the constant Ω_H is the angular velocity of the horizon for a Kerr black hole. According to the conventional definition, the surface gravity, κ_{Killing} , for such a Killing horizon satisfies [21]

$$K^\mu{}_{;\nu} K^\nu = \kappa_{\text{Killing}} K^\mu, \quad (22)$$

on the horizon.

Proposition 1. The Noether surface gravity, κ_{Noether} , reduces to the standard Killing surface gravity κ_{Killing} on any nondegenerate Killing horizon.

Proof. We start by noting that the definition of Eq. (22) is not suitable to be treated as a limit, since the two sides of this equation can only be parallel for K^μ null. Instead, we use an alternative characterization of this surface gravity [22]. In order to formulate this alternative, consider a timelike “Killing observer” with 4-velocity

$$v^\mu \equiv K^\mu / \sqrt{-K^\mu K_\mu}, \quad (23)$$

situated outside the horizon. After a proper time, $\delta\tau$, such an observer will have moved by

$$\delta x^\mu = v^\mu \delta\tau. \quad (24)$$

Taking the inner product of both sides of this equation with $\nabla_\mu t$ yields

$$\delta t = \delta\tau / \sqrt{-K^\mu K_\mu}. \quad (25)$$

Thus the redshift factor associated with this observer satisfies $\Phi^2 = -K^\mu K_\mu$. We are now in a position to give the alternate formulation for the surface gravity as [22]

$$\kappa_{\text{Killing}} = \lim \Phi a, \quad (26)$$

where $a^2 = a^\mu a_\mu$ is the square of the 4-length of the 4-acceleration $a^\mu = v^\mu{}_{;\nu} v^\nu$ and the limit corresponds to considering Killing observers ever closer to the horizon.

We note that in the limit of approaching the horizon K^μ is tangent to the Killing horizon worldtube, so the Killing observers studied above may be considered as a suitable congruence of observers for the purposes of our Noether charge surface gravity of Eq. (8). Recalling that $\Phi^2 = -K^\mu K_\mu = 0$ on the Killing horizon, it follows that derivatives of any nontrivial function of Φ must therefore be normal to the horizon worldtube, along $\hat{R}^\mu$. We may now write

$$\begin{aligned} -(K^\mu K_\mu)_{;\nu} &\propto \hat{R}_\nu \\ \Rightarrow -K_{\mu;\nu} K^\mu &\propto \hat{R}_\nu \\ \Rightarrow K_{\nu;\mu} K^\mu &\propto \hat{R}_\nu \\ \Rightarrow a_\nu &\propto \hat{R}_\nu, \end{aligned} \quad (27)$$

where the Killing condition $K_{\mu;\nu} = -K_{\nu;\mu}$ is used in the third line. It follows that $a = a_\mu \hat{R}^\mu$ and we find that for nondegenerate Killing horizons that

$$\kappa_{\text{Noether}} = \kappa_{\text{Killing}}. \quad (28)$$

Remark. For bifurcate Killing horizons, the Killing vector vanishes at the bifurcation surface, rendering Eq. (22) degenerate ($0 = \kappa_{\text{Killing}} \cdot 0$) and unable to determine κ . The surface gravity must instead be obtained from the limit formula Eq. (26) or from the gradient of $|K|^2$. Our definition of κ_{Noether} via Eq. (8) does not require the Killing vector to be nonvanishing and is therefore well defined at bifurcation surfaces. For degenerate (extremal) horizons, $\kappa_{\text{Killing}} = 0$ by all standard characterizations; the Noether charge surface gravity likewise vanishes when the redshift factor Φ has a higher-order zero at the horizon.

B. Comparison with the Kodama-Hayward surface gravity at the apparent horizon

To explicitly demonstrate what new physical dynamics our covariant Noether framework captures without slicing-dependent coordinate features obfuscating local observer kinematics (see Sec. IV E for a detailed analysis), we evaluate κ_{Noether} on the exact, general Vaidya spacetime, before specializing to the self-similar case. By the self-similar case we mean the linear mass function $M(u) = \mu u$ with constant evaporation rate $\mu < 0$, for which the spacetime admits a homothetic Killing vector and every dimensionless quantity depends on r/M alone. We adopt it only where closed forms are wanted: it renders the apparent horizon tracking velocity $\dot{r}_{AH} = 2\mu$ constant, so that $\ddot{r}_{AH} \equiv 0$, and it admits an exact event horizon $r_{EH} = \rho_1 M$ with ρ_1 constant. The general tracking formula below is derived for arbitrary $M(u)$, and Theorem 3 establishes the local-to-asymptotic gap without any self-similarity assumption, so no generality is lost.

In Eddington-Finkelstein coordinates, the exact Vaidya metric is

$$ds^2 = -f(u, r) du^2 + 2du dr, \quad f(u, r) = 1 - \frac{2M(u)}{r}. \quad (29)$$

We define the dynamically evolving apparent horizon as the continuous worldtube $\mathcal{W}_{AH}$ given by $H(u, r) = r - r_{AH}(u) = 0$, where $r_{AH}(u) = 2M(u)$.

The outward tracking normal covector is $n_\mu = (-\dot{r}_{AH}, 1, 0, 0)$. Using the inverse metric ($g^{uu} = 0, g^{ur} = 1, g^{rr} = f$), the contravariant tracking normal reduces exactly to $n^\mu = (1, f - \dot{r}_{AH}, 0, 0)$.

Its norm-squared, representing the kinematic 4-velocity of the boundary itself, evaluates to

$$\Phi^2(u, r) = g^{\mu\nu} n_\mu n_\nu = f(u, r) - 2\dot{r}_{AH}(u). \quad (30)$$

We pause to explicitly clarify a notational nuance: in Theorem 1, Φ was defined as the redshift factor of the tracking observer. Here, we evaluate the proper norm of the unnormalized worldtube tracking normal, defining $\Phi \equiv \sqrt{n^\mu n_\mu}$. These two descriptions reconcile. Recognizing that the unit outward normal is exactly $\hat{R}^\mu = n^\mu / \Phi$, the operational formula for the Noether surface gravity evaluated below, $(n^\mu \partial_\mu \Phi^2) / (2\Phi^2)$, simplifies identically to our exact original geometric definition, $\hat{R}^\mu \partial_\mu \Phi$. Thus, substituting $\Phi = \sqrt{n^\mu n_\mu}$ connects the abstract worldtube geometry to the explicit metric evaluations below, averting any spurious sign clashes.

On the apparent horizon, by definition $f = 0$, yielding exactly

$$\Phi_{AH}^2 = -2\dot{r}_{AH}. \quad (31)$$

We note a physical consequence here: for the worldtube to be strictly timelike, as required by Theorem 1 for physical observer trajectories, we must geometrically enforce $\Phi_{AH}^2 > 0$. Because the black hole is evaporating (shrinking inward), $\dot{r}_{AH} < 0$, guaranteeing the outward tracking normal is indeed spacelike. Thus, our geometric framework enforces dynamic evaporation.

Evaluating our exact covariant formula, we compute the directional derivative at the horizon $N = n^\mu \partial_\mu \Phi^2$. Taking the formal partial derivative yields $\partial_u \Phi^2 = \partial_u f(u, r) - 2\ddot{r}_{AH}(u)$. Evaluated exactly at the apparent horizon ($f = 0$), this yields

$$\begin{aligned} N|_{AH} &= (\partial_u f - 2\ddot{r}_{AH}) + (-\dot{r}_{AH}) \partial_r f|_{AH} \\ &= \partial_u f - 2\ddot{r}_{AH} - \dot{r}_{AH} \partial_r f. \end{aligned} \quad (32)$$

Because the dynamic boundary $r_{AH}(u)$ is an exact root for all u , taking its total convective derivative along the horizon yields the exact identity

$$\frac{d}{du}f(u, r_{AH}(u)) = \partial_u f + \dot{r}_{AH} \partial_r f = 0. \quad (33)$$

This geometric cancellation mandates $\partial_u f = -\dot{r}_{AH} \partial_r f$. Substituting this condition into the gradient cancels the velocity-dependent dynamic temporal term, yielding

$$\begin{aligned} N|_{AH} &= (-\dot{r}_{AH} \partial_r f) - 2\ddot{r}_{AH} - \dot{r}_{AH} \partial_r f \\ &= -2\dot{r}_{AH} \partial_r f - 2\ddot{r}_{AH}. \end{aligned} \quad (34)$$

Dividing by $2\Phi_{AH}^2 = -4\dot{r}_{AH}$, we obtain the general geometric tracking formula for an arbitrary apparent horizon

$$\begin{aligned} \kappa_{\text{Noether}}^{(AH)} &= \frac{n^\mu \partial_\mu \Phi^2}{2\Phi_{AH}^2} = \frac{-2\dot{r}_{AH} \partial_r f - 2\ddot{r}_{AH}}{-4\dot{r}_{AH}} \\ &= \frac{1}{2} \partial_r f \Big|_{AH} + \frac{\ddot{r}_{AH}}{2\dot{r}_{AH}}. \end{aligned} \quad (35)$$

Because we evaluate this on the exact self-similar Vaidya spacetime ($M(u) = \mu u$), the apparent horizon tracking velocity $\dot{r}_{AH}$ is constant. Consequently, the boundary acceleration inherently vanishes ($\ddot{r}_{AH} \equiv 0$), and the explicit evaporation rate terms drop out:

$$\kappa_{\text{Noether}}^{(AH)} = \frac{1}{2} \partial_r f \Big|_{AH} = \frac{1}{4M}. \quad (36)$$

For this profile $\kappa_{\text{Noether}}^{(AH)}$ coincides with the Kodama-Hayward surface gravity $\kappa_{KH} \equiv \frac{1}{2} \square r|_{AH} = 1/4M$, with $\square$ the d'Alembertian of the normal two-space, as in Hayward's definition (the four-dimensional operator differs by $2f/r$ and so agrees on the horizon, where the expression is evaluated, but not off it) [6,7,23]. The agreement is not automatic. The Kodama-Hayward value is $1/4M$ for arbitrary $M(u)$, so for a generic profile the two constructions differ by the tracking term $\ddot{r}_{AH}/2\dot{r}_{AH} = \dot{M}/2\dot{M}$. Such a discrepancy is expected rather than anomalous: the surface gravity of a non-Killing horizon is not unique, and the several natural definitions agree in the stationary limit while differing at first order in the rate of evolution [11–13]. Here the difference has a clear origin, in that the two constructions answer different questions. The scalar $\square r$ is built from the areal radius alone and is insensitive to the acceleration of the horizon worldtube, whereas κ_{Noether} is constructed from the Noether charge on that worldtube and therefore registers its extrinsic curvature. Theorem 2 (below) shows that it is κ_{Noether} , and not $\frac{1}{2} \square r$, which appears in the tunneling exponent, so the former is the quantity governing local emission. The Clausius balance of Sec. VIA selects the other way: it returns $\frac{1}{2} \square r$ exactly. Neither is the correct surface gravity to the exclusion of the other. The tunneling exponent selects ours, the first law selects Hayward's, and they differ by precisely the tracking

term because they answer different questions. The two selections are not on the same footing, and the difference favors the first. The tunneling selection is unconditional: given the mode equation and Lemma 1 the residue is what it is, and nothing further is chosen. The Clausius selection is conditional on two choices, of entropy functional and of surface, and the candidates return different answers, as Sec. VIB records.

We note finally that the tracking term requires $\dot{r}_{AH} \neq 0$. At a turning point of the apparent horizon, where $\dot{r}_{AH}$ passes through zero with $\ddot{r}_{AH} \neq 0$, the worldtube is momentarily null, $\Phi_{AH}^2 \rightarrow 0$, and the timelike premise of Theorem 1 fails. This degeneracy is not of the same type as that at the event horizon, and it is not removed by the limit of Eq. (43): there the null condition holds identically along the worldtube, so that numerator and denominator vanish together and L'Hôpital's rule applies, whereas at a turning point only the denominator vanishes and the expression genuinely diverges. The apparent horizon formula is therefore to be read for monotonic evolution, $\dot{r}_{AH} \neq 0$, which is the regime of interest for evaporation.

In stationary spacetimes with Killing vectors, the definition of surface gravity is closely related to Hawking temperature. It is natural, therefore, to investigate the connection between our Noether-charge-based surface gravity for dynamical scenarios and the thermal radiation perceived by accelerating observers.

IV. SEMICLASSICAL THERMODYNAMICS AND QUANTUM TUNNELING

A. The Parikh-Wilczek tunneling method

Because the argument below adapts the tunneling picture of Hawking radiation to a fully dynamical setting, we first summarize that method in the form in which we shall use it. Parikh and Wilczek [20] treat Hawking emission as a quantum tunneling process: the emitted quantum traverses a classically forbidden region bounded by the horizon, and the emission probability is governed by the imaginary part of the classical action accumulated across that region.

In the WKB approximation the outgoing mode is written as $\psi \propto \exp(iS)$, with S the action evaluated along the emission path, so that the emission probability is

$$\Gamma \sim |\psi|^2 \sim \exp(-2 \text{Im } S). \quad (37)$$

The imaginary part arises entirely from the integration path crossing a pole of the integrand at the horizon. Deforming the contour into a semicircle about that pole contributes $i\pi$ times its residue, so that $\text{Im } S$ is fixed by the residue alone and is insensitive to the regular part of the integrand.

The temperature then follows by comparison with the Boltzmann form $\Gamma \sim \exp(-\hbar\omega/k_B T)$, in which ω denotes the frequency of the emitted quantum as measured by the family of observers with respect to which the mode is

defined; its precise normalization with respect to that congruence is fixed within the proof of Theorem 2 below. A tunneling calculation therefore reduces to two steps, locating the pole and evaluating its residue, and it is in these two steps that a dynamical horizon differs from a stationary one. We take them in turn.

B. The tunneling barrier and the event horizon surface gravity

Locating the pole is a question about the geometry and causal structure alone, and we settle it before performing the tunneling calculation itself. It has two ingredients of different character: a local condition on the causal character of the worldtube, and a global condition on the accumulation of retarded time along its future. The temperature which the barrier then carries is, by contrast, fixed entirely locally. The following lemma is stated for an arbitrary worldtube and makes no reference to the tunneling amplitude, which is constructed only in the proof of Theorem 2.

Lemma 1. Let the worldtube $\mathcal{W}$ be the level set $H(x^\mu) = 0$ of a smooth function H with $\partial_\mu H \neq 0$ on $\mathcal{W}$, and let $n_\mu \propto \partial_\mu H$ be its tracking normal, with $\Phi^2 \equiv g^{\mu\nu} n_\mu n_\nu$. Then $\mathcal{W}$ supports a tunneling barrier for modes anchored to the congruence clock if and only if

(a) $\mathcal{W}$ is a null worldtube,

$$\Phi^2 = 0 \quad \text{identically along } \mathcal{W}, \quad (38)$$

a condition independent of the normalization chosen for n_μ ; and

(b) the retarded time of the congruence accumulates on $\mathcal{W}$: the exact eikonal acquires a logarithmic singularity there, so that a neighboring outgoing ray launched at proper distance x from $\mathcal{W}$ leaves its neighborhood only after a congruence time diverging as $x \rightarrow 0$. This is a statement about causal structure. In practice it is verified through the equivalent criterion that the exponential peeling of outgoing rays from $\mathcal{W}$, at the local rate κ_{peeling} , the transverse gradient of the outgoing null flow at $\mathcal{W}$, which for the spherically symmetric metrics evaluated below is $\frac{1}{2}\partial_r f$ (Sec. VA), is sustained along the entire future of $\mathcal{W}$, that is that $\int^\infty \kappa_{\text{peeling}} du'$ diverges.

When (a) and (b) hold, the transverse limit

$$\kappa = \lim_{\mathcal{W}' \rightarrow \mathcal{W}} \Phi_{;\nu} \hat{R}^\nu, \quad (39)$$

taken over the same approaching timelike tracking family, and with the same tracking convention $H = r - r_H(u)$, as in the definition of $\kappa_{\text{Noether}}^{(\text{EH})}$ below, Eq. (43), is finite and nonzero, and it alone fixes the residue: the function $1/\Phi(\eta)$ has a simple pole at $\eta = 0$ of residue $1/\kappa$. Clause (b) locates the barrier; the transverse limit values its residue.

Proof. We take the two clauses in turn.

For (a), the causal character of $\mathcal{W}$ is fixed by the norm of its normal. If $\Phi^2 > 0$ the normal is spacelike and $\mathcal{W}$ is timelike; an outgoing null ray then crosses $\mathcal{W}$ along a classically permitted trajectory, so that no classically forbidden region is bounded and no barrier is present. If $\Phi^2 = 0$ the normal is null and lies along the generator of $\mathcal{W}$, so that $\mathcal{W}$ is ruled by outgoing null rays; no outgoing ray crosses it, and $\mathcal{W}$ is the boundary of the region from which outgoing rays cannot escape. Worldtubes with $\Phi^2 < 0$ are spacelike and are not considered in this paper.

It is essential that (a) is imposed *identically along* $\mathcal{W}$, and not merely at an instant. A worldtube of constant areal radius $r = c$ has $\Phi^2 = f(u, c)$ and therefore $\partial_u \Phi^2 = -2\dot{M}/c$, which is nonzero whenever the spacetime is dynamical. Such a tube is spacelike within the apparent horizon, instantaneously null as the horizon sweeps past it, and timelike thereafter, and outgoing rays cross it inward where $f < 0$ and outward where $f > 0$. It is not a one-way membrane, and the argument above never applies to it. A fixed-slice evaluation which locates a pole on such a tube is detecting a momentary degeneracy of the tube, not a null hypersurface.

For (b), nullity alone is insufficient: every outgoing null cone $r = \rho(u)$ with $\dot{\rho} = f/2$ has $\Phi^2 = 0$ identically, yet a generic such cone carries no barrier. The discriminating property is not the transverse limit (39), which is generically nonzero on any outgoing cone: applying the same L'Hôpital evaluation to the tracking family of the cone at radius ρ gives, in Schwarzschild, $\kappa = 2M/\rho^2 - 1/(2\rho)$, which correctly reproduces $1/(4M)$ at $\rho = 2M$ but vanishes only at the isolated radius $\rho = 4M$. What discriminates is instead whether the exponential separation of neighboring rays is sustained.

Along an outgoing ray near $\mathcal{W}$ the proper distance obeys $\dot{\eta} = \frac{1}{2}\kappa_{\text{peeling}}\eta$, with $\kappa_{\text{peeling}} = \frac{1}{2}\partial_r f$ evaluated on $\mathcal{W}$. The factor of one half is a consequence of η being a *proper* distance: near a null worldtube $\eta \propto \sqrt{r - r_H}$, so proper separations grow at half the rate of coordinate separations. A ray launched at $\eta = x$ therefore escapes only after a retarded time $\sim (2/\kappa_{\text{peeling}}) \ln(1/x)$.

For a generic exterior cone the tube expands into the asymptotic region, $\int \kappa_{\text{peeling}} du'$ converges, the total expansion factor is bounded, and neighboring rays reach infinity at retarded-time offsets which vanish smoothly with the launch separation. The eikonal phase is regular across the tube, and the apparent pole of the tracking integrand there is a frame artifact of a congruence that degenerates without an accompanying barrier, the inertial-family light cone of the first remark below in curved dress. On the event horizon, by contrast, the peeling integral diverges, the retarded time of the congruence accumulates, and the eikonal acquires the logarithmic singularity that produces the simple pole. Its residue is then computed transversally: expanding $\Phi \simeq \kappa\eta$ with κ the limit (39) gives residue $1/\kappa$.

The location of the barrier is thus governed by the sustained peeling rate, and its residue by the transverse tracking limit. The two coincide in the stationary limit and separate at first order in the rate of evolution, in exact correspondence with the two surface gravities of Theorem 3. ■

Three remarks. First, clause (b) is congruence relative, and appropriately so: a Minkowski light cone fails it for an inertial tracking family, for which there is no accumulation and no barrier, but satisfies it for a uniformly accelerated family with $\kappa = a$, which is the content of Corollary 1 below. Horizon thermality is a property of a surface together with an observer family, not of the surface alone.

Second, both clauses are conformally invariant. For (a), with $n_\mu = \partial_\mu H$ a gradient, $\Phi^2 \rightarrow \Omega^{-2}\Phi^2$ under $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, so the set of null worldtubes is preserved exactly, as it must be since nullity is a statement about causal structure. For (b), the accumulation of retarded time is likewise a property of the causal structure, which conformal rescalings preserve; we note that the practical criterion $\int^\infty \kappa_{\text{peeling}} du'$ is not written in manifestly invariant form, since κ_{peeling} is not itself invariant, the invariant content being the logarithmic singularity of the eikonal. Both match the conformal invariance of the Hamilton-Jacobi equation itself. We claim invariance for the barrier conditions only, and not for the value of κ_{Noether} . Jacobson and Kang showed that the surface gravity of a stationary black hole is conformally invariant only under restricted circumstances, for transformations that are the identity at infinity and with Ω regular on the horizon, and only for a particular definition adapted to conformal Killing horizons; the naive surface gravity, the force required at infinity to hold a test particle, is not invariant, and neither is the horizon area [24]. The marginally trapped condition enjoys no such invariance: the outgoing expansion of $\Omega^2 g$ at $r = 2M$ is $2\ell^\mu \partial_\mu \ln \Omega$, generically nonzero, so a conformal rescaling displaces the trapping horizon while leaving the Hamilton-Jacobi problem unchanged [25].

Third, clause (b) is teleological: it refers to the entire future of $\mathcal{W}$, and we state this openly rather than treat it as a defect. It is the same future dependence already carried by κ_{peeling} and by the location of the event horizon itself, and it is exactly what the causal-character argument of Sec. VB requires. What an infalling detector registers is fixed by the transverse limit, which is instantaneous; which worldtube carries the barrier is a global question about causal structure. The lemma thus separates a local quantity, the residue, from a global one, the location, and it is that separation which the remainder of the paper quantifies. We note also that the criterion survives complete evaporation: for the self-similar profile $M(u) = \mu u$ terminating at $u = 0$, $\int \kappa_{\text{peeling}} du'$ diverges logarithmically, even though outgoing rays are ultimately released at the endpoint and no ray is trapped forever.

Example (Vaidya). Lemma 1 settles the long-standing question of whether dynamical Hawking radiation originates at the apparent horizon or at the event horizon, and we evaluate the barrier condition explicitly on the exact Vaidya spacetime.

Applying the condition to the dynamically shrinking apparent horizon, defined by $H_{AH}(u, r) = r - 2M(u) = 0$, its exact norm-squared on the horizon evaluates to

$$\Phi_{AH}^2 = -2\dot{r}_{AH}. \quad (40)$$

Because the black hole is evaporating (shrinking inward), $\dot{r}_{AH} < 0$, and therefore $\Phi_{AH}^2 > 0$. The outward tracking normal is spacelike, so the shrinking apparent horizon is a timelike worldtube. Because it shrinks inward over time, an outgoing light ray can classically outrun it, so that particles cross the apparent horizon classically. By Lemma 1 there is no classically forbidden region there, and hence no pole at the apparent horizon.

Consider instead the event horizon, defined as the continuous worldtube $H_{EH}(u, r) = r - r_{EH}(u) = 0$ generated by outgoing null rays. Its tracking normal $n_\mu = (-\dot{r}_{EH}, 1, 0, 0)$ evaluates on the boundary to

$$\Phi_{EH}^2 = f(u, r_{EH}) - 2\dot{r}_{EH}. \quad (41)$$

Because the event horizon is constrained by the global geometry to track the outgoing null flow, it evolves at the boundary velocity $\dot{r}_{EH} = f(u, r_{EH})/2$, and we obtain

$$\Phi_{EH}^2 = 0. \quad (42)$$

The event horizon is thus a null worldtube, satisfying clause (a).

Clause (b) must also be checked, since it is what excludes the generic outgoing cone. On the event horizon the peeling rate is $\kappa_{\text{peeling}} = M(u)/r_{EH}^2$, and the tube remains at $r_{EH} \sim 2M$ rather than expanding into the asymptotic region, so that $\int^\infty \kappa_{\text{peeling}} du'$ diverges: for a horizon which settles to a final mass M_1 the integrand tends to the constant $1/4M_1$, and for the self-similar profile terminating at $u = 0$ it diverges logarithmically. A generic exterior cone, by contrast, escapes with $\rho \sim u/2$, so that $\kappa_{\text{peeling}} \sim 4M/u^2$ and the integral converges. Both clauses therefore hold on the event horizon and clause (b) fails on every other outgoing null cone, so by Lemma 1 the event horizon is the unique locus of the tunneling barrier. Evaluating tunneling on a moving apparent horizon along a surface of constant coordinate time yields a slicing-dependent pole there [23, 26–28] (as detailed in Sec. IV E), whereas the covariant treatment places the pole at the event horizon.

Remark (The teleological objection). A common conceptual objection to this conclusion is that the event horizon is globally “teleological,” requiring knowledge of the

infinite future, and that a local tunneling particle therefore cannot “know” its location. However, this objection confounds two distinct questions, that of where the barrier lies and that of what temperature it carries. The location is indeed a global matter, as clause (b) of Lemma 1 makes explicit. The WKB particle nonetheless reacts to the local geometry at the moment of emission, the barrier being null there ($\Phi^2 = 0$). That this local null barrier traces the global event horizon is a consequence of spacetime causality, not a violation of local physical predictability. The residue, and hence the local temperature, is fixed by the instantaneous transverse limit (39); what is teleological is only the selection of the worldtube on which the peeling is sustained, the same future dependence already carried by κ_{peeling} and $\kappa_{\text{Noether}}^{(\text{EH})}$ themselves.

Definition (Event horizon Noether surface gravity). Lemma 1 places the barrier on a null worldtube, and this introduces a geometric subtlety if Theorem 1 is applied there naively. On a null worldtube the unit normal $\hat{R}^\mu = n^\mu/\Phi$ diverges, and the expression for κ_{Noether} in Eq. (8) collapses to an indeterminate 0/0 form. This is not a failure of the construction but a requirement that the surface gravity of a null boundary be defined through a limit. Accordingly we define

$$\kappa_{\text{Noether}}^{(\text{EH})} \equiv \lim_{\mathcal{W} \rightarrow \mathcal{W}_{\text{EH}}} \Phi_{,\mu} \hat{R}^\mu, \quad (43)$$

the limit being taken over a sequence of timelike worldtubes $\mathcal{W}$ approaching the event horizon worldtube $\mathcal{W}_{\text{EH}}$. The indeterminacy resolves under L'Hôpital's rule, and the limit is evaluated in closed form in Sec. VA.

Unlike the vanishing of Φ^2 , which is insensitive to the normalization of n_μ , the limit (43) does depend on the level-set function H through more than its zero set. Replacing H by $He^{\lambda u}$ sends $\Phi^2 \rightarrow e^{2\lambda u}(\Phi^2 + 2\lambda H)$, and the limit is not invariant under it. In the stationary limit one finds $\kappa \rightarrow e^{\lambda u}(\kappa + \frac{3}{2}\lambda)$ exactly; on a dynamical horizon the transformation is not even affine in λ . We therefore fix throughout the tracking convention $H = r - r_H(u)$ with unit radial gradient. The residual freedom is precisely the synchronization gauge $t \rightarrow h(t)$ of Sec. II B, under which the locally measured $T_{\text{local}} = T/\Phi$ is invariant.

C. Hawking temperature from quantum tunneling

Theorem 2. Let $\mathcal{W}_{\text{EH}}$ be the event horizon worldtube of a dynamical black hole and let $\kappa_{\text{Noether}}^{(\text{EH})}$ be defined by Eq. (43). Then, at leading semiclassical order and subject to the standard adiabatic condition that the horizon's dynamical timescale be long compared with the mode's oscillation period, the Parikh-Wilczek tunneling temperature [20] measured locally on $\mathcal{W}_{\text{EH}}$, conjugate to the frequency referred to the drift-free normal flow Φw^μ , is

$$T_{\text{tunneling}} = \frac{\kappa_{\text{Noether}}^{(\text{EH})}}{2\pi}. \quad (44)$$

Proof. We compute the Hawking temperature directly using the quantum tunneling formalism developed by Parikh and Wilczek [20], reviewed in Sec. IV A. We explicitly adapt this method to a strictly dynamical scenario where the horizon lacks both general radial and temporal symmetry. (For extensive related work developing covariant Hamilton-Jacobi tunneling methodologies for dynamical horizons, see Refs. [23,26–28].) In a general dynamical spacetime, the relevant boundary is not constrained to a constant-time hypersurface. Instead, it dynamically evolves as a continuous worldtube $\mathcal{W}$ locally defined by $H(x^\mu) = r - r_H(t, \theta, \varphi) = 0$. This lack of symmetry allows for the possibility that the local tunneling temperature varies dynamically across both time and the horizon's spatial surface.

Consider a *massless* field mode in the WKB approximation whose amplitude is slowly varying

$$\psi_\omega(x^\mu) \propto \exp\left(i\omega \int^{x^\mu} \ell_\nu dx^\nu\right), \quad (45)$$

where, following Parikh and Wilczek, the Hamilton-Jacobi equation reduces to $g^{\mu\nu} \ell_\mu \ell_\nu = 0$. In other words, ℓ^μ is an outgoing null vector.

In the Parikh-Wilczek approach, the temperature stems from the integral inside the exponent of Eq. (45) crossing a pole at the horizon. To properly evaluate this across a dynamically evolving horizon, the spatial integration may be performed along a path locally orthogonal to the horizon worldtube $\mathcal{W}$. Let $\hat{R}^\mu \propto \partial^\mu H$ be the outward spacelike unit normal to this dynamical worldtube. The differential path element across the horizon is thus $dx^\mu = \hat{R}^\mu d\eta$, where η is the proper spatial distance along the normal. By framing the tunneling trajectory with respect to the worldtube normal $\hat{R}^\mu$, the time-dependence $-\partial_t r_H$ is covariantly absorbed into the path, bypassing the need for a constant-time hypersurface approximation (a choice that demands careful treatment to respect the exactness of the 1-form in WKB theory, as explicitly detailed below in Sec. IV E).

To evaluate the integrand we must fix ℓ_μ , and we do so in three steps. The Hamilton-Jacobi equation makes ℓ_μ null. In the local basis spanned by the worldtube-hugging observer w^μ , the worldtube unit normal $\hat{R}^\mu$, and the 2-surface $\mathcal{S}$, a general outgoing null vector may be written $\ell^\mu = a w^\mu + b \hat{R}^\mu + t^\mu$, with t^μ tangent to $\mathcal{S}$. Since $w_\mu w^\mu = -1$, $\hat{R}_\mu \hat{R}^\mu = 1$, $w_\mu \hat{R}^\mu = 0$, and t^μ is orthogonal to both, nullity requires $b = (a^2 - t_\mu t^\mu)^{1/2}$. Nullity alone therefore leaves a two-parameter family of directions rather than a single one.

The scale is fixed by the frequency normalization. For ω in Eq. (45) to be the frequency measured by the observer

congruence, rather than an arbitrary rescaling of it, ℓ_μ must satisfy $\ell_\mu \xi^\mu = -1$, where $\xi^\mu = \Phi w^\mu + \Psi \beta s^\mu$ is the congruence generator (constructed so that $\xi^0 = 1$). Since the drift direction s^μ lies in $\mathcal{S}$ and is therefore orthogonal to both w^μ and $\hat{R}^\mu$, this gives $-a\Phi + \Psi\beta(t \cdot s) = -1$, that is

$$a = \frac{C}{\Phi}, \quad C \equiv 1 + \Psi\beta(t \cdot s). \quad (46)$$

The redshift factor therefore enters not as an assumption but as the normalization forced by measuring frequency with respect to the congruence. For a drift-free congruence, $\beta = 0$, one has $C = 1$ and $\ell_\mu = (w_\mu + \hat{R}_\mu)/\Phi$ exactly.

We retain t^μ throughout, since it does not affect the pole. The tunneling integrand along the normal path is $\ell_\mu \hat{R}^\mu = b$, and expanding about the barrier at $\Phi = 0$ using Eq. (46) gives

$$\ell_\mu \hat{R}^\mu = \frac{C}{\Phi} - \frac{\Phi t_\mu t^\mu}{2C} + O(\Phi^2). \quad (47)$$

The magnitude of the tangential momentum enters only at $O(\Phi)$, which is regular and therefore leaves the residue unchanged: as $\Phi \rightarrow 0$ the normal component of any null vector diverges while a fixed tangential component does not. This is an exact statement rather than a restriction to a lowest partial wave, which matters here because the spacetimes of interest possess neither radial nor temporal symmetry and a decomposition into spherical harmonics is not available. For the drift-free normal anchoring, $C = 1$, this reduces to the residue $1/\kappa$ quoted in Lemma 1.

By Lemma 1 the pole of the integrand (47) lies on the null worldtube, that is at the event horizon, where $\Phi = 0$, and the residue is $C/\kappa_{\text{Noether}}^{(\text{EH})}$.

The factor C is not a modification of the temperature but a change of frequency frame. Writing the mode momentum as $p_\mu = \omega \ell_\mu$ and referring the frequency to the drift-free normal flow Φw^μ , which by Theorem 1 is the object carrying the entire Noether charge,

$$\tilde{\omega} \equiv -p_\mu(\Phi w^\mu) = \omega a \Phi = \omega C, \quad (48)$$

an exact identity. The Boltzmann exponent $-2\pi\omega C/\kappa_{\text{Noether}}^{(\text{EH})}$ is therefore $-2\pi\tilde{\omega}/\kappa_{\text{Noether}}^{(\text{EH})}$: the spectrum is exactly thermal at $T = \kappa_{\text{Noether}}^{(\text{EH})}/2\pi$ in the frequency conjugate to the normal flow, for arbitrary drift β and arbitrary tangential momentum. The drift never touches the temperature; it enters as a chemical potential. Writing $\xi^\mu = \Phi w^\mu + \zeta^\mu$ and $\omega = -p_\mu \xi^\mu$, and defining

$$\mu_{\text{drift}} \equiv -\zeta^\mu p_\mu = -\Psi\beta s^\mu p_\mu, \quad (49)$$

the identity (48) reads $\tilde{\omega} = \omega - \mu_{\text{drift}}$, so that

$$\Gamma \sim \exp \left[-\frac{2\pi(\omega - \mu_{\text{drift}})}{\kappa_{\text{Noether}}^{(\text{EH})}} \right], \quad (50)$$

which is the standard grand-canonical form (the subscript distinguishes it from the evaporation rate μ of Sec. III B). We note that μ_{drift} is in general mode dependent, through $s^\mu p_\mu$; only when the worldtube possesses an axial symmetry, so that the associated momentum component is conserved, does it collapse to the familiar product of an angular velocity with a quantum number.

Thus, the Boltzmann factor reduces to

$$\begin{aligned} \exp \left(\frac{-\hbar\omega}{k_B T} \right) &= \left| \exp \left[i\omega \int \ell_\nu \hat{R}^\nu d\eta \right] \right|^2 \\ &= \exp \left[-2\pi\omega \text{Res}_{(\eta=0)} \left(\frac{1}{\Phi} \right) \right], \end{aligned} \quad (51)$$

where the integration contour is deformed into a semicircle in the upper half η -plane, picking up $i\pi$ times the residue (and the factor of 2 arises from taking the modulus squared), and $\text{Res}_{(\eta=0)}$ denotes the residue of the integrand at the pole at the event horizon worldtube (i.e., at $\Phi(\eta) = 0$). (For clarity, we include $\hbar$ and k_B in this section, but they are set to unity in our final step).

Given that the redshift factor Φ vanishes at the event horizon, the integrand $1/\Phi$ yields a simple pole. Expanding Φ in a Taylor series locally along the normal path η gives $\Phi(\eta) \approx \eta \partial_\eta \Phi = \eta(\Phi_{,\nu} \hat{R}^\nu)$. The reciprocal of the residue of $1/\Phi(\eta)$ at $\eta = 0$ is therefore uniquely extracted by taking the directional derivative of Φ strictly along the path, yielding

$$\frac{1}{\text{Res}_{(\eta=0)} \left(\frac{1}{\Phi} \right)} = \frac{d\Phi}{d\eta} = \lim_{\eta \rightarrow 0} \Phi_{,\nu} \hat{R}^\nu, \quad (52)$$

and hence the local temperature evaluated at the dynamical horizon is given by

$$\begin{aligned} T(t, \theta, \varphi) &= \frac{\hbar}{2\pi k_B} \frac{1}{\text{Res}_{(\eta=0)} \left(\frac{1}{\Phi} \right)} \Big|_{r=r^{(\text{EH})}(t, \theta, \varphi)} \\ &= \frac{\hbar}{2\pi k_B} \lim_{r \rightarrow r^{(\text{EH})}} \Phi_{,\nu} \hat{R}^\nu \\ &= \frac{\hbar \kappa_{\text{Noether}}^{(\text{EH})}}{2\pi k_B} \Big|_{r=r^{(\text{EH})}(t, \theta, \varphi)}. \end{aligned} \quad (53)$$

The final identification of the local tunneling temperature as the Noether-charge-based surface gravity $\kappa_{\text{Noether}}^{(\text{EH})}$ follows directly from Theorem 1 and the limit defining $\kappa_{\text{Noether}}^{(\text{EH})}$ in Eq. (43). By covariantizing the integration using the proper distance η intrinsically along the continuous worldtube normal $\hat{R}^\mu$, this derivation holds at leading semiclassical order for slowly evolving dynamical black holes without

requiring spherical symmetry, subject to the standard adiabatic condition. ■

We note that in a general dynamical spacetime, the frequency ω defined by $\ell_\mu \xi^\mu = -1$ is not conserved along null geodesics, since ξ^μ is not a Killing vector. However, the tunneling temperature is determined by the residue of the integrand evaluated at the horizon, which depends only on the leading Taylor coefficient $d\Phi/d\eta|_{\eta=0} = \kappa_{\text{Noether}}^{(\text{EH})}$. Corrections from the nonconservation of ω and the dynamical evolution of the horizon are smooth functions of η in a neighborhood of the horizon, and therefore do not contribute to the pole. The result $T_{\text{tunneling}} = \kappa_{\text{Noether}}^{(\text{EH})}/(2\pi)$ thus holds at leading semiclassical order whenever the standard adiabatic condition is satisfied: the horizon's dynamical timescale must be long compared to the mode's oscillation period.

Proposition 2 (Frame invariance of the exponent). At fixed foliation of $\mathcal{W}$ into two-surfaces $\mathcal{S}$, the normal-frame frequency $\tilde{\omega} = -p_\mu(\Phi w^\mu)$ is invariant under the tangential gauge transformations $\zeta^\mu \rightarrow \zeta'^\mu$ of Sec. II C, while $\omega = -p_\mu \xi^\mu$ is not. Under a genuine change of foliation neither $\tilde{\omega}$ nor $\kappa_{\text{Noether}}^{(\text{EH})}$ is invariant, but their ratio is.

Proof. At fixed $\mathcal{S}$ the objects $\mathcal{S}$, w^μ and Φ are constructed without reference to ζ^μ , so $\tilde{\omega}$ cannot depend on it; the invariance group of $\tilde{\omega}$ is therefore identically that of the Noether charge in Sec. II C. For the second statement, the Boltzmann exponent may be written

$$\frac{2\pi\tilde{\omega}}{\kappa_{\text{Noether}}^{(\text{EH})}} = 2\pi \text{Res}_{(\eta=0)}(p_\mu \hat{R}^\mu), \quad (54)$$

whose right-hand side is assembled from the mode covector p_μ , the worldtube normal $\hat{R}^\mu$ and the normal geodesic parameter η , none of which refers to the foliation of $\mathcal{W}$. The foliation dependences of numerator and denominator therefore cancel identically. ■

This is the exact semiclassical mirror of the tangential gauge invariance of the Noether charge: the charge is drift invariant, and so is the temperature, provided frequency is referred to the normal frame. Equivalently, since $\tilde{\omega} = \Phi\omega_{\text{loc}}$ with $\omega_{\text{loc}} = -p_\mu w^\mu$ and $T_{\text{local}} = \kappa_{\text{Noether}}^{(\text{EH})}/2\pi\Phi$, the quantity (54) is $\omega_{\text{loc}}/T_{\text{local}}$, the same Tolman-invariant combination identified in Sec. II B. That $\tilde{\omega}$ itself is foliation dependent is not a defect: it is a measured frequency, and $\kappa_{\text{Noether}}^{(\text{EH})}$ is foliation dependent in exactly the same way, as already noted at Eq. (43).

Remark (Several black holes). The construction above is local to a single event horizon worldtube, and nothing in it refers to the global configuration. For a spacetime containing N dynamical black holes the argument therefore applies unchanged to each horizon separately, assigning to the j th hole the temperature $T_j = \kappa_{\text{Noether}}^{(\text{EH})}|_j/2\pi$ evaluated on its own worldtube.

D. The Kerr limit: Corotation, chemical potential and superradiance

The frequency-frame identity (48) is not merely book-keeping, and the stationary rotating case exhibits it exactly.

Take Boyer-Lindquist coordinates and the worldtube $r = \text{const}$, with $\mathcal{S}$ the constant- t two-surfaces. The normal congruence is then forced: a unit timelike vector tangent to $\mathcal{W}$ and orthogonal to ∂_θ and ∂_ϕ is unique, and the condition $u_\phi = 0$ identifies it as the zero-angular-momentum observer, so that

$$\Phi w^\mu = \partial_t + \omega_Z \partial_\phi, \quad \omega_Z = -\frac{g_{t\phi}}{g_{\phi\phi}}. \quad (55)$$

At the horizon $\omega_Z \rightarrow \Omega_H = a/(r_+^2 + a^2)$, and it becomes independent of θ there although it is not outside, which is the rigidity result. The normal flow therefore limits to the horizon generator itself, and without that observation the correspondence below would appear coincidental.

Now choose instead the congruence $\xi^\mu = \partial_t$ of asymptotic-time observers. Relative to the zero-angular-momentum observers this carries drift $\Psi\beta s^\mu = -\omega_Z \partial_\phi$, so β is emphatically nonzero at the horizon. For a mode with $p_t = -\omega_\infty$ and $p_\phi = m$, Eq. (48) gives

$$\tilde{\omega}|_{r_+} = -(p_t + \omega_Z p_\phi)|_{r_+} = \omega_\infty - m\Omega_H, \quad (56)$$

exactly, so that the chemical potential (49) is $\mu_{\text{drift}} = \omega_Z m \rightarrow m\Omega_H$ and the exponent is $\exp[-2\pi(\omega_\infty - m\Omega_H)/\kappa]$. The textbook rotating spectrum follows from the present formalism with no additional input, and μ_{drift} takes the familiar angular-velocity-times-quantum-number form because the axial symmetry makes p_ϕ conserved.

The sign is also correct. Superradiance is the regime $\omega_\infty < m\Omega_H$, that is $\omega < \mu_{\text{drift}}$, for which $C < 0$, the exponent changes sign and $\Gamma > 1$. Reproducing the thermal factor alone might be read as a calibration; recovering the superradiant regime from the same factor is a test the construction could have failed.

Rigidity and the independence of the foliation. The same configuration exhibits the mechanism behind Proposition 2. Retain the worldtube $r = \text{const}$ but tilt the slicing, $t' = t + g(\phi)$. The normal congruence becomes $w' \sim \partial_t + \Omega' \partial_\phi$ with

$$\Omega' = \frac{g'_{t\phi} - g_{t\phi}}{g_{\phi\phi} - g'_{\phi\phi}}, \quad (57)$$

and one finds that $\Omega'(r_+) = \Omega_H$ for *arbitrary* tilt slope g' , with $\partial\Omega'/\partial g' = 0$ there. The mechanism is horizon rigidity: the induced metric on $\mathcal{W}$ degenerates at the horizon, $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\Delta \sin^2 \theta \rightarrow 0$, so its kernel is one-dimensional and every foliation's normal congruence is forced onto that unique direction, the null generator. The

direction of w' is therefore rigidly fixed at the horizon and only its normalization remains free, with $\Phi^2/\Phi_{\text{ZAMO}}^2 \rightarrow (1 + g'\Omega_H)^{-2}$. Consequently $\tilde{\omega}$ and $\kappa_{\text{Noether}}^{(\text{EH})}$ both acquire the same factor $(1 + g'\Omega_H)^{-1}$ and their ratio is exactly invariant, as Proposition 2 requires. (The tilt must of course keep the slices spacelike, $1 + g'\Omega_H \neq 0$.) This is the same rigidity responsible for ω_Z becoming independent of θ at r_+ although it is not outside.

Two caveats. This validation is stationary, where the apparent and event horizons coincide; it confirms the frequency-frame structure and the normalization of ω , and says nothing about the location of the pole in the dynamical case, which is the content of Lemma 1. That separation is by construction: the residue identity (54), the frequency-frame identity (48) and Proposition 2 all attach to the pole of $p_\mu \hat{R}^\mu$ on whichever null worldtube satisfies the barrier conditions of Lemma 1, so the frame structure established here would transfer unchanged were the accumulation criterion to select a different null worldtube. Only the identification of that worldtube with the event horizon rests on Lemma 1 and the eikonal analysis of Sec. V C.

E. WKB exactness and path-dependence in slicing

It is crucial at this juncture to precisely interpret what our covariant contour integral physically evaluates, and explicitly demonstrate how evaluating the integral on fixed spatial slices [23,26–28] can mathematically yield path-dependent results and produce an apparent coordinate pole at the apparent horizon.

In the geometric WKB approximation, the scalar field amplitude is given by $\psi \propto \exp(iI/\hbar)$. The four-momentum of the emitted quantum is the precise gradient of the classical action I , meaning $p_\mu = \partial_\mu I$. For the action I to be mathematically well defined as a true, continuous state function, the momentum 1-form $p_\mu dx^\mu$ must be exact (closed). By the symmetry of mixed partial derivatives in Eddington-Finkelstein coordinates, this mathematically demands

$$\partial_u p_r - \partial_r p_u = 0. \quad (58)$$

In the fixed-slice Hamilton-Jacobi approach [23,26–28], the energy ω is taken as a constant of motion, fixing the temporal momentum component to $p_u = -\omega$. The radial momentum p_r is then derived directly from the exact null condition $g^{\mu\nu} p_\mu p_\nu = 0$. For an outgoing mode in the Vaidya spacetime ($ds^2 = -f(u, r)du^2 + 2dudr$), this dictates

$$p_r(f p_r + 2p_u) = 0 \Rightarrow p_r = \frac{2\omega}{f(u, r)}. \quad (59)$$

We test the exactness condition of Eq. (58). The radial derivative of the temporal momentum is trivially zero

$$\partial_r p_u = \partial_r(-\omega) = 0. \quad (60)$$

However, because the black hole is dynamically radiating ($\partial_u f \neq 0$), the temporal derivative of the radial momentum evaluates to

$$\partial_u p_r = \partial_u \left(\frac{2\omega}{f(u, r)} \right) = -\frac{2\omega \partial_u f}{f^2}. \quad (61)$$

Because $\partial_u f \neq 0$ for a true evaporating black hole, we establish that

$$\partial_u p_r \neq \partial_r p_u. \quad (62)$$

Because the momentum 1-form is not strictly closed in this coordinate system, the WKB action $I = \int p_\mu dx^\mu$ requires careful consideration to exist globally as a proper state function. The spatial integral therefore depends on the chosen path, so that the tunneling probability is fixed only once a definite geometric trajectory is specified. Mathematical approaches evaluating the integral strictly along lines of constant time u [23,26–28] correspond to evaluating the metric on a fixed coordinate slice, piercing the evolving boundary at an arbitrary, slicing-dependent angle. In contrast, tracking the spatial normal respects the dynamic continuity of the evolving worldtube.

The apparent pole located at the apparent horizon ($f = 0$) found in fixed-slicing WKB models is therefore a consequence of integrating a nonexact 1-form over a restricted spatial coordinate constraint. When the contour integral is parametrized intrinsically along the normal vector evaluating the dynamically evolving worldtube (as mathematically mandated by WKB steepest-descent paths crossing a physical barrier), this apparent coordinate feature completely vanishes. As established in the previous subsection, the true, structurally invariant WKB pole emerges at the null event horizon.

Furthermore, it is crucial to precisely interpret what our covariant contour integral physically evaluates, which directly addresses why the local tunneling temperature separates from the asymptotic Hawking spectrum. This specific contour evaluation takes place in the local proper Gaussian normal tracking frame (t, η) defined by our foliation. In this local frame, the time flow is exactly $\xi = \partial_t$. By enforcing the mode frequency to match this local tracking observer's frame ($\mathcal{L}_\mu \xi^\mu = -1$), we fix the temporal momentum to $p_t = -\omega$, yielding $\partial_\eta p_t = 0$. However, because the black hole boundary is dynamically evolving, the normal spatial momentum $p_\eta = \omega/\Phi(t, \eta)$ depends explicitly on time ($\partial_t p_\eta \neq 0$). Consequently,

$$\partial_\eta p_t - \partial_t p_\eta \neq 0. \quad (63)$$

Because the 1-form is not exact in this local tracking frame, the local WKB integral is inherently path-dependent.

By explicitly parametrizing the contour along the spatial tracking normal vector ($d\eta$), our integration path respects the dynamic continuity of the boundary, forcing the residue theorem to extract the directional derivative $\partial_\eta \Phi = \kappa_{\text{Noether}}^{(\text{EH})}$. This computes the local proper tracking temperature, the thermodynamic kinematic cost of crossing the dynamic barrier from the perspective of an observer tracking it.

This calculation does not capture the full asymptotic spectrum. A true asymptotic calculation must solve the eikonal equation globally along the outgoing null ray parametrized by the asymptotic retarded time (U).

F. Relation to the earlier tunneling literature

The tunneling picture of Hawking emission has a long and, in places, contentious history, and since the present construction touches several of the open threads it is worth setting out where it sits.

Origins and variants. The picture originates with the observation that emission may be treated as quantum tunneling through a barrier furnished by the horizon itself, with the backreaction of the emitted quantum incorporated by allowing the horizon to recede [20]. Two computational routes developed in parallel: the null-geodesic method of Parikh and Wilczek, and the Hamilton-Jacobi method in which one solves for the classical action directly and reads the emission rate from the imaginary part accumulated at the horizon [23,26–28]. The construction of Sec. IV A is of the second kind, differing in that the action is integrated along the normal to a covariantly defined worldtube rather than on a coordinate slice.

The factor of two. It was pointed out [29] that the tunneling rate computed from the spatial part of the action alone is not invariant under canonical transformations, since $\int p_r dr$ is not, whereas $\oint p_r dr$ is; and that evaluating the latter in Schwarzschild coordinates returns a temperature twice the accepted value. Three resolutions were proposed. Mitra [30] observed that the ingoing solution also carries an imaginary contribution, and that the physically meaningful quantity is the ratio of emission to absorption probabilities, which is the detailed-balance relation and returns the standard temperature; the invariant is $\text{Im } S_{\text{out}} - \text{Im } S_{\text{in}}$, which is the same in Schwarzschild and Painlevé coordinates. Akhmedov *et al.* [31] identified an imaginary contribution from the temporal part of the action, arising because the Schwarzschild time coordinate is discontinuous across the horizon, and showed that adding it restores the accepted value. Di Criscienzo *et al.* [27] then established that the temporal contribution is gauge dependent, being present in some coordinate systems and absent in others, and traced the entire controversy to the use of coordinates singular on the horizon, for which the temporal contribution is not well defined.

The present calculation is not exposed to this. We work throughout in coordinates regular on the horizon, in which the temporal contribution is absent [27], and our contour is

closed rather than open. That the contour closes is not obvious from its presentation: because Φ vanishes linearly in the proper distance η while Φ^2 vanishes linearly in $r - r_h$, we have $\eta \propto \sqrt{r - r_h}$, so a semicircle in η is a complete circuit in the areal radius, and

$$\pi \text{Res}_\eta(1/\Phi) = 2\pi \text{Res}_r(1/f) = \pi/\kappa. \quad (64)$$

The same square-root transformation of the contour is noted in [29]. Our prescription is therefore the canonically invariant one, which is why it returns $\kappa/2\pi$ rather than the doubled value.

Which surface gravity. A separate thread concerns the nonuniqueness of surface gravity for horizons that are not Killing. Several inequivalent definitions coincide in the stationary limit and differ at first order in the rate of evolution: the geometric Kodama-Hayward $\frac{1}{2}\square r$ [6,7,23], the definition of Nielsen and Visser [12], which carries a $\partial_v m$ term, and others cataloged and compared in [11,13,32]. The degeneracy among them is lifted precisely when a horizon is dynamical, and no purely geometric criterion selects one. Our κ_{Noether} is a further member of this family, differing from the Kodama-Hayward value by the tracking term of Sec. III B. What distinguishes it is not a geometric preference but Theorem 2: it is κ_{Noether} , and not $\frac{1}{2}\square r$, that appears in the tunneling exponent, so the physical process supplies the selection principle that geometry alone does not.

Which horizon. Finally, there is an unresolved question of whether the emission should be associated with the trapping horizon or the event horizon. Working on a fixed slice, Hayward *et al.* [23] and Di Criscienzo *et al.* [27] locate the pole where $C = 1 - 2m/r$ vanishes, that is at the trapping horizon, and conclude that the method yields a positive temperature if and only if a future outer trapping horizon is present. Nielsen [25] argues against that identification on grounds of conformal invariance: the Hamilton-Jacobi equation is conformally invariant whereas the trapping horizon is not, so the association is at best frame dependent. Lemma 1 takes the latter view and sharpens it. The fixed-slice pole arises on a worldtube that is *momentarily* null, as a constant-areal-radius tube must be when the apparent horizon sweeps past it, whereas the barrier condition requires nullity to hold identically along the worldtube. Both clauses of Lemma 1 are conformally invariant (accumulation is a property of the causal structure), and the event horizon is the unique null worldtube on which the retarded time accumulates, hence the unique invariant singular locus of the exact eikonal phase of Sec. V C.

We adopt without reservation two methodological principles urged in [27]: that the tunneling rate is an observable and must therefore be constructed from invariantly defined quantities, and that coordinate systems failing to cover the horizon are inadmissible.

V. ASYMPTOTIC DYNAMICS AND THE UNIVERSAL TRACKING GAP

A. Exact evaluation at the event horizon

Because the event horizon is a strictly null boundary ($\Phi_{\text{EH}}^2 = 0$), the denominator $2\Phi^2$ identically vanishes. Furthermore, the numerator $N = n^\mu \partial_\mu \Phi^2 = \partial_u f - 2\ddot{r}_{\text{EH}} + (f - \dot{r}_{\text{EH}})\partial_r f$ evaluated at the exact event horizon ($f = 2\dot{r}_{\text{EH}}$) reduces to $\partial_u f - 2\ddot{r}_{\text{EH}} + \dot{r}_{\text{EH}}\partial_r f$. This identically vanishes due to the exact total convective derivative of the null boundary condition ($2\ddot{r}_{\text{EH}} = \partial_u f + \dot{r}_{\text{EH}}\partial_r f$). Thus, evaluating Theorem 1 directly yields an indeterminate $0/0$ form. To derive the proper surface gravity $\kappa_{\text{Noether}}^{(\text{EH})}$, we must evaluate the limit of a sequence of timelike worldtubes approaching the event horizon using L'Hôpital's rule

$$\kappa_{\text{Noether}}^{(\text{EH})} = \lim_{r \rightarrow r_{\text{EH}}} \frac{\partial_r (n^\mu \partial_\mu \Phi^2)}{\partial_r (2\Phi^2)}. \quad (65)$$

We evaluate this on the exact self-similar Vaidya spacetime ($M(u) = \mu u$). The exact event horizon is the outgoing null ray $r_{\text{EH}}(u) = \rho_1 M(u)$, where ρ_1 is a constant dimensionless parameter. The exact boundary velocity is constrained by the null flow to $\dot{r}_{\text{EH}} = f(u, r_{\text{EH}})/2 = \frac{\rho_1 - 2}{2\rho_1}$, while self-similarity gives $\dot{r}_{\text{EH}} = \rho_1 \mu$. Consistency fixes ρ_1 as the root of $2\mu\rho_1^2 - \rho_1 + 2 = 0$ with $0 < \rho_1 < 2$, namely $\rho_1 = [1 - \sqrt{1 - 16\mu}]/(4\mu)$, which exists for every evaporation rate $\mu < 0$. As $\mu \rightarrow 0^-$ one has $\rho_1 = 2 + 8\mu + O(\mu^2)$, whence $\kappa_{\text{Noether}}^{(\text{EH})} \rightarrow (1 - 10\mu)/4M$, $\kappa_{\text{peeling}} \rightarrow (1 - 8\mu)/4M$ and the gap $\rightarrow |\mu|/2M$: both surface gravities recover $1/4M$ in the stationary limit and separate at first order in the evaporation rate, as claimed.

The denominator limit evaluates to

$$\partial_r (2\Phi^2)|_{\text{EH}} = 2\partial_r f|_{\text{EH}} = \frac{4M}{r_{\text{EH}}^2} = \frac{4}{\rho_1^2 M}. \quad (66)$$

The numerator limit requires $\partial_r N = \partial_r \partial_u f + (\partial_r f)^2 + \dot{r}_{\text{EH}} \partial_r^2 f$. Evaluating these three terms exactly at the event horizon yields

$$\begin{aligned} \partial_r \partial_u f|_{\text{EH}} &= \frac{2\mu}{r_{\text{EH}}^2} = \frac{\rho_1 - 2}{\rho_1^4 M^2}, \\ (\partial_r f)^2|_{\text{EH}} &= \left(\frac{2M}{r_{\text{EH}}^2}\right)^2 = \frac{4}{\rho_1^4 M^2}, \\ \dot{r}_{\text{EH}} \partial_r^2 f|_{\text{EH}} &= \left(\frac{\rho_1 - 2}{2\rho_1}\right) \left(-\frac{4M}{r_{\text{EH}}^3}\right) = \frac{4 - 2\rho_1}{\rho_1^4 M^2}. \end{aligned} \quad (67)$$

Summing these components exactly yields the numerator limit $\frac{6 - \rho_1}{\rho_1^4 M^2}$. Dividing by the denominator produces the geometric tracking acceleration

$$\kappa_{\text{Noether}}^{(\text{EH})} = \frac{6 - \rho_1}{4\rho_1^2 M}. \quad (68)$$

Crucially, we must compare this to the exact asymptotic Hawking flux measured by a distant observer, which is governed purely by the optical “peeling” surface gravity κ_{peeling} (the rate at which neighboring outgoing rays separate from the exact horizon generator). Attempts to extract this peeling invariant locally by adapting the Noether charge directly to the null generators encounter geometric limitations, yielding instead the parameter-dependent generator inaffinity (as explicitly analyzed in Sec. VI C). On the Vaidya spacetime, the peeling rate evaluates to

$$\kappa_{\text{peeling}} = \frac{1}{2} \partial_r f \Big|_{\text{EH}} = \frac{1}{\rho_1^2 M}. \quad (69)$$

Subtracting the two parameters isolates the physical tracking penalty

$$\kappa_{\text{Noether}}^{(\text{EH})} - \kappa_{\text{peeling}} = \frac{6 - \rho_1}{4\rho_1^2 M} - \frac{4}{4\rho_1^2 M} = \frac{2 - \rho_1}{4\rho_1^2 M}. \quad (70)$$

The physical origin of this gap can be understood by evaluating the purely kinematic boundary tracking term, $-\frac{\dot{r}_{\text{EH}}}{2r_{\text{EH}}}$. Physically, this term captures the instantaneous inertial penalty imposed on an observer attempting to locally track the advecting boundary. Because the black hole is dynamically evaporating, the event horizon continuously shrinks inward ($\dot{r}_{\text{EH}} < 0$). To keep their worldtube strictly locked onto this receding surface, a quasilocal tracking observer must effectively “chase” the horizon, geometrically necessitating an additional outward proper acceleration that manifests as this positive tracking penalty. Mathematically, we can explicitly verify this convergence by recalling the horizon's exact instantaneous radius, $r_{\text{EH}} = \rho_1 M$, and substituting its constrained boundary velocity, $\dot{r}_{\text{EH}} = \frac{\rho_1 - 2}{2\rho_1}$. Unpacking the nested fraction yields

$$-\frac{\dot{r}_{\text{EH}}}{2r_{\text{EH}}} = -\left(\frac{1}{2\rho_1 M}\right) \left(\frac{\rho_1 - 2}{2\rho_1}\right) = -\frac{\rho_1 - 2}{4\rho_1^2 M} = \frac{2 - \rho_1}{4\rho_1^2 M}. \quad (71)$$

This convergence validates the exact dynamical gap equation on the exact self-similar radiation solution. A closed form of the same character exists for the dynamical entropy on this solution; since the entropy functional it evaluates is introduced only in Sec. VI B, we state and derive it there, Eq. (101). To prove this beyond the self-similar assumption, we formally derive the local kinematics across a generic Vaidya evaporation profile $M(u)$.

Theorem 3 (Universal kinematic tracking gap). For a dynamically evaporating event horizon in a general Vaidya spacetime, the exact mathematical separation between the

proper local tracking kinematics ($\kappa_{\text{Noether}}^{(\text{EH})}$) and the asymptotic peeling surface gravity ($\kappa_{\text{peeling}} = \frac{1}{2}\partial_r f|_{\text{EH}}$) obeys the dynamical gap equation

$$\kappa_{\text{Noether}}^{(\text{EH})} - \kappa_{\text{peeling}} = -\frac{\dot{r}_{\text{EH}}}{2r_{\text{EH}}} - \frac{\ddot{r}_{\text{EH}}}{1 - 2\dot{r}_{\text{EH}}}. \quad (72)$$

Proof. We impose the exact null-flow condition strictly characterizing the event horizon, $f(u, r_{\text{EH}}) = 2\dot{r}_{\text{EH}}$. From the general metric $f = 1 - 2M(u)/r$ we extract the structural gradients: $\partial_r f = (1 - f)/r$, $\partial_r^2 f = -2(1 - f)/r^2 = -2\partial_r f/r$, and $\partial_r \partial_u f = -\partial_u f/r$.

Taking the total convective derivative of the null boundary condition strictly along the horizon yields $2\ddot{r}_{\text{EH}} = \partial_u f + \dot{r}_{\text{EH}}\partial_r f$, allowing us to analytically isolate the time derivative as $\partial_u f = 2\ddot{r}_{\text{EH}} - \dot{r}_{\text{EH}}\partial_r f$. Substituting this back into the mixed derivative gives $\partial_r \partial_u f = -2\ddot{r}_{\text{EH}}/r_{\text{EH}} + \dot{r}_{\text{EH}}\partial_r f/r_{\text{EH}}$.

Returning to the exact L'Hôpital limit for the Noether surface gravity, the numerator explicitly expands to

$$\begin{aligned} \partial_r(n^\mu \partial_\mu \Phi^2)|_{\text{EH}} &= \partial_r \partial_u f + (\partial_r f)^2 + \dot{r}_{\text{EH}} \partial_r^2 f \\ &= \left(-\frac{2\ddot{r}_{\text{EH}}}{r_{\text{EH}}} + \frac{\dot{r}_{\text{EH}} \partial_r f}{r_{\text{EH}}} \right) \\ &\quad + (\partial_r f)^2 - \frac{2\dot{r}_{\text{EH}} \partial_r f}{r_{\text{EH}}} \\ &= -\frac{2\ddot{r}_{\text{EH}}}{r_{\text{EH}}} - \frac{\dot{r}_{\text{EH}} \partial_r f}{r_{\text{EH}}} + (\partial_r f)^2. \end{aligned} \quad (73)$$

Dividing by the denominator $\partial_r(2\Phi^2) = 2\partial_r f$, we extract the geometric tracking acceleration

$$\begin{aligned} \kappa_{\text{Noether}}^{(\text{EH})} &= \frac{-\frac{2\ddot{r}_{\text{EH}}}{r_{\text{EH}}} - \frac{\dot{r}_{\text{EH}} \partial_r f}{r_{\text{EH}}} + (\partial_r f)^2}{2\partial_r f} \\ &= -\frac{\ddot{r}_{\text{EH}}}{r_{\text{EH}} \partial_r f} - \frac{\dot{r}_{\text{EH}}}{2r_{\text{EH}}} + \frac{1}{2}\partial_r f. \end{aligned} \quad (74)$$

Recognizing that the macroscopic optical peeling mathematically evaluates to $\kappa_{\text{peeling}} \equiv \frac{1}{2}\partial_r f|_{\text{EH}}$, and substituting $r_{\text{EH}}\partial_r f = 1 - f|_{\text{EH}} = 1 - 2\dot{r}_{\text{EH}}$, we isolate the tracking gap equation (72). ■

We note that the explicit acceleration penalty inherently vanishes ($\ddot{r}_{\text{EH}} = 0$) on the self-similar Vaidya background, recovering our previous explicit evaluation. This proves that the separation between proper boundary WKB tracking kinematics ($\kappa_{\text{Noether}}^{(\text{EH})}$) and the distant macroscopic Bogoliubov spectrum (κ_{peeling}) holds for arbitrary evaporation profiles, not only the self-similar case. We emphasize that while this exact L'Hôpital evaluation captures the transient nonequilibrium $\dot{r}_{\text{EH}}$ kinematics, the global eikonal exactness proof mapping this to spatial infinity (detailed in Sec. V C) specifically evaluates the macroscopic adiabatic limit. By treating the peeling rate as

slowly varying, the eikonal proof corroborates the macroscopic thermodynamic separation by suppressing these higher-order tracking kinematics from the eventual distant Bogoliubov spectrum.

B. Causal character of the dynamical temperatures

The instantaneous tracking surface gravity $\kappa_{\text{Noether}}^{(\text{AH})}(u)$ is a function of the instantaneous data $\{M(u), \dot{M}(u), \ddot{M}(u)\}$, whereas both the event-horizon tunneling temperature $\kappa_{\text{Noether}}^{(\text{EH})}(u)$ and the peeling surface gravity $\kappa_{\text{peeling}}(u)$ are functionals of the causal future $\{M(u') : u' \geq u\}$.

Proof. The apparent horizon lies at $r_{\text{AH}} = 2M(u)$, and the tracking formula obtained above expresses $\kappa_{\text{Noether}}^{(\text{AH})}$ through r_{AH} and its first two time derivatives, hence through $M(u), \dot{M}(u), \ddot{M}(u)$ alone. In contrast, the peeling rate is $\kappa_{\text{peeling}} = \frac{1}{2}\partial_r f|_{\text{EH}} = M(u)/r_{\text{EH}}(u)^2$, which depends on the event-horizon radius $r_{\text{EH}}(u)$, as does $\kappa_{\text{Noether}}^{(\text{EH})}$. The event horizon is the outgoing null generator that asymptotes to the final stationary state, and $r_{\text{EH}}(u)$ is fixed by integrating the null condition $\dot{r}_{\text{EH}} = \frac{1}{2}f(u, r_{\text{EH}})$ backward from a boundary condition imposed at late times; it is therefore determined by the mass profile at all $u' \geq u$. ■

This dichotomy is not merely formal. Consider two evaporation histories that coincide for $u \leq u_0$ and differ only thereafter. The instantaneous purely classical tracking acceleration $\kappa_{\text{Noether}}^{(\text{AH})}(u_0)$ is identical for the two, provided it is evaluated with left derivatives at u_0 , since it is then fixed by the shared past data, yet both $\kappa_{\text{Noether}}^{(\text{EH})}(u_0)$ and $\kappa_{\text{peeling}}(u_0)$ differ, because the event horizon at u_0 already registers the divergent futures, contracting in anticipation of whichever mass loss is to come. Deferring the onset of subsequent evaporation to progressively later times returns $\kappa_{\text{peeling}}(u_0)$ and $\kappa_{\text{Noether}}^{(\text{EH})}(u_0)$ continuously to their static values, while $\kappa_{\text{Noether}}^{(\text{AH})}(u_0)$ never moves.

The consequence is a sharp statement about what is locally knowable. Any purely classical quantity built from the metric and a finite number of its derivatives at a point, such as the local field equations, the curvature at the horizon, or the proper acceleration required to hover a test body at fixed radius, is a functional of the instantaneous data $\{M, \dot{M}, \ddot{M}\}$, and hence expressible through quantities such as $\kappa_{\text{Noether}}^{(\text{AH})}$ and κ_{KH} , and cannot depend on κ_{peeling} or $\kappa_{\text{Noether}}^{(\text{EH})}$ without an acausal reference to the future. The asymptotic peeling temperature and the exact event horizon WKB barrier are physical, but inherently depend on the global causal structure.

This furnishes the precise operational content of the three surfaces. The apparent horizon, defined instant by instant, carries the classical local dynamics through $\kappa_{\text{Noether}}^{(\text{AH})}$, fixed by the instantaneous data $\{M, \dot{M}, \ddot{M}\}$, and the balanced quasilocal first law through κ_{KH} , fixed by $\{M, \dot{M}\}$, the two

coinciding when $\dot{r}_{AH} = 0$. That one surface carries two quantities is not a redundancy: $\kappa_{\text{Noether}}^{(AH)}$ is the kinematic cost of tracking the apparent-horizon worldtube, that is, the Noether charge per unit area which Theorem 1 assigns to it, whereas κ_{KH} is what balances the Clausius relation on the same surface, and they differ by exactly the tracking term. No thermal reading is claimed for the former: Theorem 2 supplies a temperature only on a null worldtube satisfying Lemma 1, and the apparent-horizon worldtube is timelike. The event horizon carries the tunneling barrier: its location is a global question, answered teleologically, while the residue at the barrier, $\kappa_{\text{Noether}}^{(EH)}$, is what a local measurement of the emission temperature returns (Theorem 2). Spatial infinity carries the asymptotic Bogoliubov spectrum through κ_{peeling} . Four questions, namely what a tracking observer registers, what balances locally, what is emitted at the barrier, and what survives to infinity, are answered on three surfaces of three causal characters, the apparent horizon carrying two, and local dynamical laws couple only to the first two.

C. Global eikonal exactness and the asymptotic peeling temperature

To validate the claim that the true asymptotic spectrum registered by a distant observer is exclusively governed by the optical peeling surface gravity (κ_{peeling}), we must formulate the exact global eikonal action. This provides analytical closure demonstrating that when WKB exactness is fully restored, the local kinematic tracking terms decouple from the contour.

In the geometric eikonal (WKB) approximation, the scalar field amplitude is $\psi \propto \exp(iI/\hbar)$. The four-momentum of the emitted quantum is exactly the gradient of the classical action, $p_\mu = \partial_\mu I$.

For thermal quanta measured by an asymptotic observer at spatial infinity, the phase must be parametrized by the global macroscopic retarded time $U(u, r)$. To ensure the frequency measured by this distant observer is ω , the classical action must be strictly proportional to U

$$I(u, r) = -\omega U(u, r). \quad (75)$$

Because the momentum is given by the pure gradient of a global scalar, $p_\mu = -\omega \partial_\mu U$, the momentum 1-form is guaranteed by the symmetry of mixed partial derivatives to be exactly closed

$$\begin{aligned} \partial_u p_r - \partial_r p_u &= \partial_u(-\omega \partial_r U) - \partial_r(-\omega \partial_u U) \\ &= -\omega(\partial_u \partial_r U - \partial_r \partial_u U) \equiv 0. \end{aligned} \quad (76)$$

This mathematical exactness guarantees that the spatial WKB integral $\int p_\mu dx^\mu$ is path-independent, bypassing the coordinate-dependent features of fixed spatial slicing.

We evaluate the global phase using the exact Vaidya metric

$$ds^2 = -f(u, r)du^2 + 2dudr. \quad (77)$$

For a massless field, the eikonal equation $g^{\mu\nu} \partial_\mu U \partial_\nu U = 0$ dictates that the level sets of U are null hypersurfaces. For outgoing rays ($ds^2 = 0$, $du > 0$), this reduces to the local geometric flow

$$\frac{dr}{du} = \frac{1}{2}f(u, r) \equiv F(u, r). \quad (78)$$

Because the asymptotic retarded time $U(u, r)$ labels specific outgoing null rays, its phase must remain constant along any given ray. Taking its total convective derivative along the flow yields the exact linear partial differential equation

$$\frac{dU}{du} = \partial_u U + \frac{dr}{du} \partial_r U = 0 \Rightarrow \partial_u U + F(u, r) \partial_r U = 0. \quad (79)$$

To isolate the quantum tunneling pole, we Taylor-expand the geometric flow $F(u, r)$ around the dynamically evolving event horizon $r_{EH}(u)$

$$F(u, r) \approx F(u, r_{EH}) + \partial_r F(u, r)|_{EH}(r - r_{EH}). \quad (80)$$

By definition, the event horizon is uniquely generated by outgoing null rays, meaning its intrinsic macroscopic boundary expansion identically matches the flow evaluated at the boundary: $\dot{r}_{EH} = F(u, r_{EH})$. Furthermore, as established in the main text, the optical peeling surface gravity is exactly the radial gradient of the flow on the horizon: $\kappa_{\text{peeling}} \equiv \partial_r F|_{EH}$. Therefore

$$F(u, r) \approx \dot{r}_{EH} + \kappa_{\text{peeling}}(r - r_{EH}). \quad (81)$$

Substituting this into the eikonal PDE for $U(u, r)$ gives

$$\partial_u U + [\dot{r}_{EH} + \kappa_{\text{peeling}}(r - r_{EH})] \partial_r U = 0. \quad (82)$$

To solve this, we transition to a locally comoving coordinate strictly tracking the moving boundary: $x = r - r_{EH}(u)$. The partial derivatives transform via the chain rule as $\partial_r = \partial_x$ and $\partial_u|_r = \partial_u|_x - \dot{r}_{EH} \partial_x$. Substituting these into the PDE aligns the kinematic terms

$$\begin{aligned} (\partial_u - \dot{r}_{EH} \partial_x)U + [\dot{r}_{EH} + \kappa_{\text{peeling}}x] \partial_x U &= 0 \\ \partial_u U - \dot{r}_{EH} \partial_x U + \dot{r}_{EH} \partial_x U + \kappa_{\text{peeling}}x \partial_x U &= 0 \\ \partial_u U + \kappa_{\text{peeling}}x \partial_x U &= 0. \end{aligned} \quad (83)$$

The tracking boundary velocity $\dot{r}_{EH}$ mathematically cancels identically. This cancellation proves that when evaluating the true global null phase mapped to infinity,

the local kinematic acceleration of the boundary universally drops out of the equation.

Because this eikonal derivation explicitly treats the boundary evolution as a slowly varying adiabatic process tracking a steady macroscopic flow, the parameter κ_{peeling} evaluated here represents the adiabatic (instantaneous) peeling rate. This structurally isolates the first-derivative flow and naturally accounts for the absence of higher-order acceleration terms (such as $\ddot{r}_{\text{EH}}$) in the strict asymptotic limit. Consequently, this asymptotic eikonal proof captures the leading-order thermodynamic decoupling of the advection velocity, while operating at an approximation order that structurally bypasses the transient $\ddot{r}_{\text{EH}}$ kinematics formally captured in the exact quasilocal L'Hôpital limit [Eq. (72)].

To correctly match the proper time clock $\partial_u U \sim 1$ at spatial infinity, the unique analytic solution must take the form

$$U(u, x) \approx u - \frac{1}{\kappa_{\text{peeling}}} \ln(x). \quad (84)$$

Mapping back to standard coordinates, the exact WKB action near the horizon is therefore

$$I(u, r) \approx -\omega u + \frac{\omega}{\kappa_{\text{peeling}}} \ln(r - r_{\text{EH}}(u)). \quad (85)$$

Taking the spatial gradient extracts the exact radial WKB momentum p_r

$$p_r = \partial_r I = \frac{\omega}{\kappa_{\text{peeling}}(r - r_{\text{EH}})}. \quad (86)$$

Because p_μ is a globally exact 1-form, the path-independent integral $\int p_r dr$ across the classically forbidden spatial barrier possesses a simple pole at the exact event horizon. Smoothly deforming the spatial contour into the complex lower-half r -plane to analytically continue the logarithm across the boundary (the standard Feynman $i\epsilon$ prescription), the residue yields the imaginary part

$$\text{Im}(I) = \text{Im} \int p_r dr = \pi \times \text{Res}_{(r=r_{\text{EH}})}(p_r) = \frac{\pi\omega}{\kappa_{\text{peeling}}}. \quad (87)$$

The macroscopic quantum tunneling probability Γ for the wave packet to escape to spatial infinity is governed by the square modulus of the amplitude

$$\Gamma \propto |e^{iI/\hbar}|^2 = \exp(-2 \text{Im}(I)) = \exp\left(-\frac{2\pi\omega}{\kappa_{\text{peeling}}}\right). \quad (88)$$

By equating this directly to the canonical macroscopic Boltzmann distribution factor $\Gamma = \exp(-\omega/T_{\text{asym}})$, we extract the asymptotic temperature

$$T_{\text{asym}} = \frac{\kappa_{\text{peeling}}}{2\pi}. \quad (89)$$

This provides a proof that while path-dependent local WKB integrals correctly extract the quasilocal tracking kinematics required to penetrate the boundary ($\kappa_{\text{Noether}}^{(\text{EH})}$), restoring WKB exactness by mapping the state function to future null infinity suppresses these local dynamics, isolating the asymptotic peeling surface gravity.

D. Recovering the Unruh effect

Corollary 1. When applied to a uniformly accelerating worldtube in flat spacetime, the Noether-charge framework naturally recovers the Unruh effect. The local physical temperature, interpreted as the thermal flux of Noether charge across the Rindler horizon, is exactly the Unruh temperature.

Proof. Consider flat Minkowski spacetime from the perspective of a uniformly accelerating observer. In standard Rindler coordinates (t, η, y, z) , the line element is [33]

$$ds^2 = -(a\eta)^2 dt^2 + d\eta^2 + dy^2 + dz^2, \quad (90)$$

where η is the proper spatial distance to the Rindler horizon (located at $\eta = 0$) and a is a reference acceleration parameter.

We define the timelike worldtube $\mathcal{W}$ as a surface of constant proper distance η . The outward spacelike unit normal to the worldtube is $\hat{R}^\mu = (0, 1, 0, 0) = (\partial/\partial\eta)^\mu$. The congruence of worldtube-hugging observers has a time flow generated by the time-translating vector field $\xi^\mu = (1, 0, 0, 0) = (\partial/\partial t)^\mu$. Normalizing the 4-velocity such that $w_\mu w^\mu = -1$, we obtain $w^\mu = \frac{1}{a\eta}(1, 0, 0, 0)$. According to our definition $\xi^\mu = \Phi w^\mu$, the redshift factor is strictly proportional to the proper distance: $\Phi = a\eta$.

Applying Theorem 1, we can compute the local Noether charge surface density κ_{Noether} . Using the directional derivative of the redshift factor, we find

$$\kappa_{\text{Noether}} = \Phi_{;\mu} \hat{R}^\mu = \partial_\eta(a\eta) = a. \quad (91)$$

We may verify this identically using the normal component of the 4-acceleration $a_\eta^w = w_{\eta;t} w^t = -\Gamma_{\eta t}^t w_t w^t$. With $\Gamma_{\eta t}^t = \frac{1}{2} g^{tt} \partial_\eta g_{tt} = 1/\eta$ and $w_t = g_{tt} w^t = -a\eta$, the proper 4-acceleration magnitude is $a_\eta^w = (-1/\eta)(-a\eta)(1/(a\eta)) = 1/\eta \equiv \alpha$. Thus, $\kappa_{\text{Noether}} = \Phi a_\mu^w \hat{R}^\mu = (a\eta)(1/\eta) = a$. Alternatively, utilizing the extrinsic curvature $K_{tt}^{\mathcal{W}} = \hat{R}_{t;t} = -\Gamma_{tt}^\eta \hat{R}_\eta = -a^2\eta$, we find $\kappa_{\text{Noether}} = -\Phi K_{\mu\nu}^{\mathcal{W}} w^\mu w^\nu = -(a\eta)(-a^2\eta)(1/(a\eta))^2 = a$. All three equalities converge to a .

Invoking Theorem 2, the global tunneling temperature evaluated with respect to the reference time t is exactly dictated by this constant Noether surface gravity

$$T_{\text{tunneling}} = \frac{\kappa_{\text{Noether}}}{2\pi} = \frac{a}{2\pi}. \quad (92)$$

To determine the physical temperature T_{local} measured locally by the accelerating observer on their specific worldtube at η , we apply the standard Tolman-Ehrenfest redshift relation [34] $T_{\text{local}} = T_{\text{tunneling}}/\Phi$. This yields

$$T_{\text{local}} = \frac{a/(2\pi)}{a\eta} = \frac{1}{2\pi\eta}. \quad (93)$$

Recognizing that the local proper acceleration of the observer is precisely $\alpha = 1/\eta$, we recover the exact Unruh temperature [33] $T_{\text{local}} = \alpha/(2\pi)$. This demonstrates that the Unruh effect is a direct consequence of a local flux of gravitational Noether charge across the Rindler horizon. ■

VI. THERMODYNAMIC IMPLICATIONS

A. The dynamical first law and the Clausius balance

The explicit separation of the local proper tracking temperature $T_{\text{local}} = \kappa_{\text{Noether}}^{(\text{EH})}/2\pi$ from the asymptotic peeling temperature $T_{\text{asym}} = \kappa_{\text{peeling}}/2\pi$ necessitates a careful consideration of the first law of black hole thermodynamics in dynamical spacetimes. Given these two distinct thermal parameters, which temperature correctly governs the fundamental thermodynamic identity $dE = TdS + \text{work terms}$?

The answer is neither of them. The first law is a quasilocal identity, balancing the instantaneous change of horizon entropy against the energy flux crossing the boundary, and both of its members should therefore be fixed by instantaneous data. Neither $\kappa_{\text{Noether}}^{(\text{EH})}$ nor κ_{peeling} is, since both are evaluated on the event horizon and inherit its future dependence, as established in Sec. V B. The surface gravity entering the first law is instead the Kodama-Hayward value on the apparent horizon, fixed by M and $\dot{M}$ at the instant in question, and with which $\kappa_{\text{Noether}}^{(\text{AH})}$ coincides whenever $\dot{r}_{\text{AH}} = 0$. The two event-horizon quantities answer different questions: $\kappa_{\text{Noether}}^{(\text{EH})}$ is the tunneling temperature of local emission through Theorem 2, a spectral statement about where the barrier lies and what residue it carries, and κ_{peeling} governs the spectrum that survives to infinity. Four quantities on three surfaces, one causal principle.

Furthermore, Wald famously established that black hole entropy is defined as a Noether charge associated with the diffeomorphism invariance of gravity [1], an insight subsequently generalized to dynamical spacetimes by Iyer and Wald [2]. That the entropy is a Noether charge does not by itself fix the conjugate intensive variable, since the surface gravity of a non-Killing horizon is not unique. What it does fix is that the conjugate must be determined locally. Attempting to formulate a local first law using the

asymptotic peeling temperature introduces a tension with causality: it would imply that the instantaneous boundary area responds to the integrated global redshift which the emitted quanta will experience only in the distant future. Which local surface gravity is selected is then settled by computation rather than by assumption, and we turn to that next.

To mathematically justify this, we examine the exact local geometric heat flux. The most generalized form of the first law for dynamical spacetimes relates the variation of the quasilocal Misner-Sharp-Hawking mass E enclosed by the horizon to the dynamic change in the horizon's entropy S and the physical work W performed by matter fluxes

$$dE = TdS + dW. \quad (94)$$

By defining the thermodynamic heat flux localized to the boundary as $\delta Q = dE - dW$, the first law enforces the exact Clausius relation at the horizon

$$\delta Q = TdS. \quad (95)$$

For a dynamically evolving event horizon with area A , the standard Bekenstein-Hawking entropy is given by $S = A/4$. This identification is not innocent away from stationarity: $A/4$ is the entropy selected by the linearized second law, whereas the first law selects a different functional, and the two differ on a dynamical horizon. We return to this in Sec. VI B, where we note that the comparison drawn below is insensitive to the choice, since the same δS appears throughout. A fundamental geometric consequence of general relativity dictates that the physical heat flux traversing a null boundary is inherently proportional to the boundary's defining geometric surface gravity κ and its differential area change δA , specifically $\delta Q = \frac{\kappa}{8\pi} \delta A$.

It is important not to read the geometric identity $\delta Q = (\kappa/8\pi)\delta A$ as selecting a surface gravity. It holds for whatever κ is inserted, since it defines δQ in terms of that κ , and is therefore an algebraic identity rather than a balance. To determine which surface gravity the first law actually selects, the heat must be supplied independently of κ . On Vaidya it is. Using the inverse metric components already displayed, $\nabla^a r \nabla_a r = g^{rr} = f$, so that the Misner-Sharp energy

$$E = \frac{r}{2} (1 - \nabla^a r \nabla_a r) = \frac{r}{2} (1 - f) = M(u) \quad (96)$$

at every radius; and since $T_{ab} \propto \delta_a^u \delta_b^u$ while $g^{uu} = 0$, the work density $w = -\frac{1}{2} h^{ab} T_{ab} = -\frac{1}{2} g^{uu} T_{uu}$ vanishes identically. Hence $\delta Q = dE - dW = Mdu$ exactly, across the apparent horizon, the event horizon or any tracking worldtube. Imposing $\delta Q = (\kappa/2\pi)\delta S$ then determines κ uniquely for each entropy candidate. Taking $S = A_{\text{AH}}/4$ returns $\kappa = 1/4M$ exactly, for arbitrary $M(u)$, which is

Hayward's unified first law, the exact statement here belonging to the apparent-horizon entropy rather than to the first-law-selected one; taking the dynamical entropy S_{dyn} of Sec. [VIB](#) returns $1/[\rho_1(4 - \rho_1)M] = (1/4M)(1 + 16\mu^2) + O(\mu^3)$. Both are the apparent-horizon Kodama-Hayward value κ_{KH} , the second to first order in the evaporation rate. The balance therefore selects κ_{KH} rather than $\kappa_{\text{Noether}}^{(\text{AH})}$, the two differing by the tracking term $\dot{M}/2\dot{M}$ off self-similarity. That the tunneling exponent selects the one and the Clausius balance the other is not a tension but the content of the assignment: they answer different questions. Pairing $\kappa_{\text{Noether}}^{(\text{EH})}$ with either misses the balance at first order, by a relative factor $1 - 10\mu$ on the self-similar profile, which is the same order at which the temperatures of this paper are distinguished. The balanced local first law is therefore fully quasilocal, both of its members being instantaneous and nonteleological.

Taking $S = A_{\text{EH}}/4$ instead returns κ_{peeling} exactly on the self-similar profile. That coincidence is an artifact of self-similarity rather than a general relation: under self-similar flow the entire future is rigidly determined by instantaneous data, so a future functional such as ρ_1 can masquerade as a local constant. The condition is exact: for a tracked surface $r_s(u)$ with $S = A/4$ the balance returns $\dot{M}/(r_s \dot{r}_s)$ against the local geometric value M/r_s^2 , and the two agree precisely when $r_s \propto M$, which holds identically at the apparent horizon and only under self-similarity at the event horizon. Splicing two histories in the manner of Sec. [VB](#) breaks the event-horizon relation while leaving the apparent-horizon one exact for arbitrary $M(u)$. That the profile-independent Clausius relation lives at the apparent horizon, on locally determined data, reinforces rather than weakens the causal thesis of this paper.

We now mathematically examine the discrepancy that occurs if one substitutes the asymptotic peeling temperature $T_{\text{asym}} = \kappa_{\text{peeling}}/2\pi$ into this local boundary balance. Utilizing the exact dynamical gap equation established in Eq. [\(72\)](#), we recall

$$\kappa_{\text{peeling}} = \kappa_{\text{Noether}}^{(\text{EH})} + \frac{\dot{r}_{\text{EH}}}{2r_{\text{EH}}} + \frac{\ddot{r}_{\text{EH}}}{1 - 2\dot{r}_{\text{EH}}}. \quad (97)$$

Multiplying this relation by $\delta S/2\pi$ allows us to evaluate the pseudoheat flux associated with the peeling temperature

$$T_{\text{asym}}\delta S = T_{\text{local}}\delta S + \frac{1}{2\pi} \left[\frac{\dot{r}_{\text{EH}}}{2r_{\text{EH}}} + \frac{\ddot{r}_{\text{EH}}}{1 - 2\dot{r}_{\text{EH}}} \right] \delta S. \quad (98)$$

Since a dynamically evaporating black hole requires $\dot{r}_{\text{EH}} \neq 0$, the two differ at first order in the evaporation rate. Physically, κ_{peeling} tracks the macroscopic retarded time u at spatial infinity, encoding the integrated bulk work done by the collapsing spacetime on the radiation *after* it has tunneled past the boundary. It is therefore a functional

of the future, as Sec. [VB](#) establishes, and a local balance struck at T_{asym} would make the instantaneous boundary area respond to nonlocal data. No local balance can therefore be maintained at T_{asym} . Combined with the Clausius analysis above, which exhibits the balance that does hold, exactly and for arbitrary $M(u)$, between δQ and κ_{KH} , the conclusion is that the quasilocal first law operates at the apparent horizon, while $\kappa_{\text{Noether}}^{(\text{EH})}$ retains its distinct role as the tunneling temperature of Theorem [2](#).

B. Relation to the quasilocal and dynamical-entropy literature

Two ingredients of the preceding argument, the identification $S = A/4$ and the choice of surface on which the first law is evaluated, have been the subject of substantial recent work. Our conclusions should be positioned against it.

Dynamical entropy is neither $A/4$ nor unique. For a stationary black hole Wald's construction identifies the entropy as a Noether charge [\[1,2\]](#), and Jacobson *et al.* established that the resulting expression is unaffected by the ambiguities inherent in that construction and returns the same value on an arbitrary cross section of a Killing horizon, not merely at the bifurcation surface [\[35\]](#). Away from stationarity neither statement survives, and several inequivalent dynamical entropies have been proposed, each selected by demanding a different law or a different symplectic potential [\[36\]](#). Requiring a linearized second law for arbitrary higher-curvature theories selects the entropy of Wall [\[37\]](#), which agrees to linear order with the holographic entanglement entropy computed by Dong [\[38\]](#) and is accordingly referred to as the Dong-Wall entropy, although general agreement between the two has not been proven [\[39,40\]](#). Requiring instead a local physical-process version of the first law selects the entropy of Hollands *et al.* [\[39\]](#),

$$S[\mathcal{C}] = \frac{A[\mathcal{C}]}{4} - \frac{1}{4} \int_{\mathcal{C}} v \vartheta, \quad (99)$$

evaluated on a cross section $\mathcal{C}$ of the horizon, with v the affine parameter along the null generators, vanishing at the bifurcation surface, and ϑ their expansion. The two do not agree. Visser and Yan recast this as an improved Noether charge, $S_{\text{dyn}} = (1 - v\partial_v)S_{\text{BH}}$ in general relativity and $(1 - v\partial_v)S_{\text{Wall}}$ in general diffeomorphism-covariant theories, show it is unambiguous at first order, and treat $f(\text{Riemann})$ theories explicitly [\[40\]](#); a gauge-invariant extension to charged configurations fixes the residual ambiguity [\[41\]](#). The several proposals coincide at the bifurcation surface, where all reduce to the Wald entropy, and their domains of validity differ: the Dong-Wall entropy applies to nonstationary geometries but is derived between the bifurcation surface and future infinity, whereas the

Hollands-Wald-Zhang entropy is constructed for arbitrary cross sections [40]. We should add that the two laws do not partition the candidates cleanly. The Hollands-Wald-Zhang entropy is constructed from the first law, but satisfies the second law as an immediate corollary for first-order perturbations sourced by matter obeying the null energy condition [39]. What distinguishes the proposals is the requirement from which each is derived, not the set of laws each happens to obey.

The distinction bears directly on the identification made above. In general relativity the Wall entropy is the Bekenstein-Hawking entropy $A/4$, whereas the first-law entropy S_{dyn} is not [40]. Writing $\delta S = \delta(A/4)$, as we did in deriving the Clausius relation, therefore amounts to using the entropy selected by the second law within an argument about the first. We note that the comparison which that section actually draws, between T_{local} and T_{asym} and the residual flux separating them, is unaffected, since the same δS appears on both sides and cancels from the difference; what is at stake is only the identification of $\delta(A/4)$ with the first-law entropy variation.

The structural situation is the one this paper describes for the temperature. No purely geometric criterion selects among the candidates, and the selection is instead made by demanding that some physical process be correctly described. Our identification of κ_{Noether} through the tunneling residue is an argument of the same kind applied to the conjugate variable.

Which surface carries the entropy. Hollands *et al.* find that to leading perturbative order in general relativity their dynamical entropy equals one quarter the area of the apparent horizon rather than that of the event horizon [39]. Ashtekar, Paraizo and Shu reach the same conclusion without a perturbative restriction, constructing a first law for horizons arbitrarily far from equilibrium on quasilocal dynamical horizons and identifying the entropy with the area of marginally trapped surfaces [42]. This is the division of labor argued for here, reached independently from the entropy side of $T\delta S$, and we regard it as corroboration rather than competition.

The convergence extends to the reasoning and not merely to the conclusion. Hollands *et al.* motivate their construction by an argument of exactly the form given in Sec. VB above, applied to the entropy rather than to the temperature. Because the event horizon is teleological, its area responds to what will happen rather than to what has happened: if one resolves to throw a baseball into a black hole, the Bekenstein-Hawking entropy $A/4$ of the horizon has already increased while the baseball is still in one's hand, since the horizon began to expand in anticipation of matter it has not yet received. Their entropy does not behave this way. The area term still increases in advance, but the expansion term in their formula contributes negatively, and the two cancel at first order, so that the entropy changes only when the baseball actually crosses the

horizon. As they put it, the increase in entropy then has an entirely local cause [39].

This is the same objection we raise against using κ_{peeling} in a local relation, and it is answered in the same way: a quantity entering a local law must be a functional of local data, and a teleologically defined one is not. That the argument has been made independently on the conjugate variable, and resolved by an improved Noether charge, we take as substantial support for the position adopted here.

We should be explicit about how far the agreement extends, since endorsing this reasoning might appear to concede the case against event horizons altogether. It does not. Hollands *et al.* use the teleology of the event horizon to move the entropy off the naive area $A/4$, retaining the event horizon as the surface on which their entropy is defined. Our position is the same in structure: Lemma 1 retains the teleologically defined event horizon for the location of the barrier, which is a global question, while confining locality to the residue, which is what a local measurement returns. In neither case is the event horizon abandoned; in both, what is removed from it is the claim to carry a locally measurable quantity directly.

The three candidates are not variants of one proposal but three separate identifications: the apparent-horizon area is the entropy of Ashtekar *et al.* [42], the event-horizon area is the Dong-Wall entropy in general relativity [37,38], and S_{dyn} is that of Hollands *et al.* [39]. Requiring the surface gravity which balances the Clausius relation to be a local functional therefore discriminates among them, accepting the first and third and rejecting the second: off self-similarity the second returns $\dot{M}/(r_{\text{EH}}\dot{r}_{\text{EH}})$, and r_{EH} is fixed by integrating the null condition backward from a late-time boundary condition, hence by the entire future. That this is the verdict reached independently in [39,42], from the teleology of the area rather than from the conjugate variable, is worth recording. We note that on our side it is a consistency requirement rather than independent evidence, since locality is the premise of the argument.

It requires care in the pairing, though less than might first appear. The Hollands-Wald-Zhang entropy is defined on a cross section of the event horizon; the apparent-horizon expression is a derived equality holding at leading order, not an alternative definition. There is therefore no need to move the entropy off the event horizon in order to use it. Ashtekar, Paraizo and Shu make the relation explicit for adiabatic Vaidya collapse, establishing that the change in $(1 - v\partial_v)\delta A_{\text{EH}}$ across two cross sections equals the change in the marginally trapped surface area of the slowly evolving dynamical horizon [42]. What our Clausius relation should employ is therefore

$$\delta S = \delta \left[\frac{1}{4} \left(A_{\text{EH}} - v \frac{dA_{\text{EH}}}{dv} \right) \right], \quad (100)$$

with v the affine parameter along the generators, normalized to vanish in the asymptotic past. That qualification is essential and not a technicality: read instead with v the Eddington coordinate u , self-similarity gives $A_{\text{EH}} \propto u^2$ and hence $S = -A_{\text{EH}}/4$, a negative entropy of the same magnitude. The nonaffinity of the u parametrization is an $O(1)$ effect, not a correction of order μ . That particular value moreover presumes the coordinate origin at the evaporation endpoint $u = 0$, and shifting the origin returns yet other values, so the nonaffine reading is not merely wrong at $O(1)$ but normalization dependent. By contrast $(1 - v\partial_v)$ with v affine and anchored in the asymptotic past is invariant under the residual rescaling $v \rightarrow av$.

Evaluated on the exact self-similar solution the construction closes in elementary form. The generator $\ell = \partial_u + \frac{1}{2}f\partial_r$ has inaffinity M/r^2 exactly, for arbitrary $M(u)$ and everywhere, not only on the horizon. Integrating $dv/du \propto \exp \int \kappa du$ along the self-similar event horizon gives $v \propto (-u)^{\rho_1/(\rho_1-2)}$, which vanishes in the asymptotic past and increases toward the future, so that normalization is the only consistent one. Hollands *et al.* anchor v at the bifurcation surface; a hole formed by collapse and evaporating has none, and the asymptotic past is the analog forced here. With that fixed,

$$S_{\text{dyn}} = \frac{1}{4} \left(A_{\text{EH}} - v \frac{dA_{\text{EH}}}{dv} \right) = \pi \rho_1 (4 - \rho_1) M^2, \quad (101)$$

whence

$$\begin{aligned} \frac{S_{\text{dyn}}}{A_{\text{AH}}/4} &= \frac{\rho_1(4 - \rho_1)}{4} = 1 - \frac{(\rho_1 - 2)^2}{4} = 1 - \mu^2 \rho_1^4 \\ &= 1 - 16\mu^2 - 256\mu^3 + O(\mu^4), \end{aligned} \quad (102)$$

the last two forms following from the self-similar constraint $2\mu\rho_1^2 = \rho_1 - 2$. The terms of order μ cancel identically, and structurally rather than by accident: the deviation from unity is a perfect square in the constraint variable, so no odd term can appear. An entropy defined on cross sections of the event horizon therefore reproduces one quarter of the apparent horizon area through first order in the evaporation rate, with computable second-order deviation, which is an independent confirmation of the Hollands-Wald-Zhang identification and of the Ashtekar-Paraizo-Shu equality on a nonperturbative solution. It reduces to $\delta(A_{\text{AH}}/4)$ at leading order in Vaidya, rather than $\delta(A_{\text{EH}}/4)$ as written above. The equality is leading order and adiabatic, and we do not claim more. We note, however, that the comparison actually drawn in that argument, between T_{local} and T_{asym} , is insensitive to the choice, since the same δS appears on both sides and cancels from the difference. The correction is one of framing rather than of result.

Which surface gravity. The quasilocal program supplies a first law but not, in the relevant sense, a temperature.

Ashtekar, Beetle and Lewandowski record that their analysis provides no specific choice of the surface gravity function, that any choice satisfying their integrability condition yields a Hamiltonian and a first law, and that the canonical choice is made by appeal to the no-hair theorems, that is by matching to Kerr-Newman; the permissible time translations at the horizon form an infinite dimensional family, and there is in general no canonical choice among them and hence no canonical notion of horizon energy [43]. The recent review restates the position, noting that there are as many first laws as there are permissible evolution fields, each providing its own surface gravity, horizon angular velocity and horizon energy [44]. Ashtekar and Krishnan are explicit that the resulting quantity is only an effective surface gravity, that it lacks the geometrical interpretation of a surface gravity and can be read only as a two-sphere average of one, and that in highly dynamical situations it should not be given a direct interpretation as an instantaneous physical temperature [45]. Booth and Fairhurst, constructing the first law for slowly evolving horizons, define a genuine pointwise surface gravity κ_V which retains its interpretation as the acceleration of the evolution field along the horizon, decompose it as $\kappa_V = \kappa^{(0)} + \epsilon\kappa^{(1)}$ with $\kappa^{(0)}$ constant, and fix $\kappa^{(0)}$ by requiring that it take its Kerr value at the same area and angular momentum, the stated motivation being that the left-hand side of the first law may then be integrated [10]. Their law, which holds at $O(\epsilon^2)$, contains $\kappa^{(0)}$ alone.

The common feature of these constructions is that each obtains its first law by integrating a balance law, and the Kerr matching is what makes that integration possible: because $\kappa^{(0)}$ and Ω are thereby functions of the horizon area and angular momentum alone, the left-hand side integrates to an energy, which is the Kerr mass [10]. The requirement is one of integrability rather than of physics.

A caution on terminology is due here. Ashtekar, Paraizo and Shu object that the phrase “entropy is a Noether charge” is misleading, on the grounds that the entropy is $A/4$, which is not a horizon Noether charge associated with any symmetry, that the two carry different dimensions and differ by a factor $\hat{\kappa}/2\pi$, and that far from equilibrium, where $\hat{\kappa}$ is not constant, they are distinct objects [42]. We agree, and note that the claim of the present paper is about the flux of Noether charge across a worldtube rather than the identification of entropy with a charge. We would add that their objection locates the difficulty precisely in the non-constancy of κ away from equilibrium, and that supplying a κ defined without reference to a stationary background is what the present construction does.

Hayward’s unified first law is of a different character, and illustrates the point rather than contradicting it. Its dynamic surface gravity is defined geometrically, by substituting the Kodama vector and the trapping horizon for the Killing vector and the Killing horizon in the static definition [7], and no Kerr matching is required precisely

because the Misner-Sharp energy is supplied independently by the geometry rather than obtained by integrating the law. In spherical symmetry it agrees with the quasilocal effective value, both reducing to $1/2R$ on the trapping horizon. With $R = r_{AH} = 2M$ that is $1/4M$, which is the value the Clausius balance of Sec. VI A returns once the heat is supplied independently of κ . The number these constructions impose by convention is therefore the number that balance derives. We note that $\kappa_{\text{Noether}}^{(AH)}$ reduces to the same value whenever $\dot{r}_{AH} = 0$, which includes the self-similar model used throughout this paper. The first two agree identically in spherical symmetry, and ours joins them whenever $\dot{r}_{AH} = 0$, differing otherwise by the tracking term alone.

Event horizons and locality. A forceful objection to any use of the event horizon in a dynamical context is that it is teleological. Hayward puts it sharply: the event horizon cannot be located by local observers, its position is unknown until the universe has ended, and it therefore carries no local dynamical significance, so that one might be passing through the reader at this moment without any physical effect [7]. The quasilocal program is built upon that objection [8,42,45], and Ashtekar and Krishnan sharpen it for precisely the setting of interest here: once Hawking radiation is included black holes are no longer black, so that the *raison d'être* of the event horizon, capturing the idea that nothing escapes, disappears [44].

We accept it, and the framework presented here is arranged to respect it. Nothing local is attributed to the event horizon. The local thermodynamics is carried on the apparent horizon, which is locally defined: the tracking rate registered by a worldtube-hugging observer, through $\kappa_{\text{Noether}}^{(AH)}$, and the surface gravity entering the quasilocal first law, through κ_{KH} , with which our value agrees for the self-similar profile used here. The event horizon enters only in determining what reaches null infinity. Clause (b) of Lemma 1 is explicitly a statement about the entire future, and the peeling temperature it selects governs the asymptotic Bogoliubov spectrum, which is a global quantity by construction. That a global question should be answered by a globally defined surface is not a failure of local predictability but a consequence of it. We would add that the objection in its quantum form, that a radiating hole is not black, tells against using the event horizon to delimit a region from which nothing escapes, which is not the use made of it here: what Lemma 1 requires is the accumulation of retarded time along a null worldtube, a condition on the propagation of modes rather than on the confinement of matter, and one which survives complete evaporation.

A remark on the relation to the slowly evolving construction. On the Vaidya apparent horizon the Booth-Fairhurst evolution vector is tangent when its expansion coefficient equals $2\dot{M}$, and their normalization then gives $e^2 = 4|\dot{M}|$: their expansion parameter is the square root of ours, so the two gradings are incommensurate and no

term-by-term matching is available. More decisively, computing their κ_V directly on Vaidya, the n contribution vanishes identically because $n = -\partial_r$ is affinely geodesic, and the ℓ contribution equals $1/4M$ exactly for arbitrary $M(v)$. Since their $\kappa^{(0)}$ is also $1/4M$, the correction $e\kappa^{(1)}$ which they define and then discard is identically zero on Vaidya, whereas our tracking term $\dot{M}/2\dot{M}$ is nonzero for every profile that is not self-similar. Booth's review works this same Vaidya example and records the expansion coefficient $2\dot{m}$, $e^2 = 4\dot{m}$ and $\kappa_V = 1/4m$ directly [46]; his v is advanced time and his $m(v)$ our $M(u)$, the slowly evolving framework being native to the accreting branch. On the only solution where both are computable in closed form theirs vanishes identically and ours does not, so they are not the same object. We do not claim their correction vanishes generally, since it is nonzero for nonspherical or non-null-dust matter; what is special here is that their ingoing normal is affinely geodesic and the generator inaffinity is exactly M/r^2 . One counterexample suffices to break the identification. What they share is structural: each is the piece which the corresponding integrated first law leaves out.

C. Null-adapted charges and generator inaffinity

To establish whether any purely geometric Noether formulation could directly capture the asymptotic Hawking flux, we address a natural geometric challenge: what if the Komar charge is evaluated on the null generators of the event horizon rather than a timelike tracking worldtube?

Consider a null-adapted Komar surface integral evaluated on the spatial cross section $\mathcal{S}$ of the event horizon. The binormal to the strictly null surface is $d\Sigma_{\mu\nu} = 2\ell_{[\mu}\bar{\ell}_{\nu]}dA$, where ℓ^μ is the exact outgoing null generator of the event horizon and $\bar{\ell}^\mu$ is the auxiliary ingoing null normal ($\ell_\mu\bar{\ell}^\mu = -1$). Evaluating the Komar charge density for the null vector field $\xi^\mu = \ell^\mu$ yields

$$\begin{aligned}\kappa_{\text{null-adapted}} &= \frac{1}{2}\nabla^\mu\ell^\nu(\ell_\mu\bar{\ell}_\nu - \ell_\nu\bar{\ell}_\mu) \\ &= \frac{1}{2}[(\ell^\mu\nabla_\mu\ell^\nu)\bar{\ell}_\nu - (\ell_\nu\nabla^\mu\ell^\nu)\bar{\ell}_\mu].\end{aligned}\quad (103)$$

Because ℓ^μ is a null vector field, its integral curves obey the geodesic equation $\ell^\mu\nabla_\mu\ell^\nu = \kappa_{\text{inaff}}\ell^\nu$, where κ_{inaff} is the generator inaffinity. Additionally, the null condition $\ell_\nu\ell^\nu = 0$ implies $\ell_\nu\nabla^\mu\ell^\nu = \frac{1}{2}\nabla^\mu(\ell_\nu\ell^\nu) = 0$. Substituting these identities

$$\begin{aligned}\kappa_{\text{null-adapted}} &= \frac{1}{2}[(\kappa_{\text{inaff}}\ell^\nu)\bar{\ell}_\nu - 0] \\ &= \frac{1}{2}\kappa_{\text{inaff}}(\ell^\nu\bar{\ell}_\nu) = -\frac{1}{2}\kappa_{\text{inaff}}.\end{aligned}\quad (104)$$

Up to an overall sign dictated by orientation, and the factor $\frac{1}{2}$ inherited from the normalization of the charge in Eq. (2),

the null-adapted Noether charge evaluates to the generator inaffinity.

As established previously, for an exact dynamically evolving self-similar Vaidya spacetime, both the inaffinity and the true optical peeling rate evaluate to exactly $1/(\rho_1^2 M)$. Thus, evaluating the null-adapted Noether charge returns $\kappa_{\text{null-adapted}} = -\frac{1}{2}\kappa_{\text{inaff}} = -\frac{1}{2}\kappa_{\text{peeling}}$ (up to orientation and normalization).

This reduction shows that anchoring the Noether construction to the null generators recovers the local generator inaffinity. However, the generator inaffinity is not a geometric invariant; it is gauge-dependent, relying entirely on the arbitrary parametrization of the local null vector ℓ^μ . Under a generic rescaling $\ell^\mu \rightarrow e^\lambda \ell^\mu$, the inaffinity transforms as $\kappa_{\text{inaff}} \rightarrow e^\lambda (\kappa_{\text{inaff}} + \ell^\mu \partial_\mu \lambda)$.

By contrast, the true optical peeling rate κ_{peeling} governing the asymptotic thermal spectrum is a strict global physical invariant, uniquely fixed by the exponential separation of rays mapped to the exact retarded time U at spatial infinity. While one can impose a fine-tuned parametrization of ℓ^μ to locally force κ_{inaff} to mathematically match the peeling rate (as happens to naturally occur in the standard coordinate parametrization by the retarded time u of the Vaidya metric where both evaluate to $1/(\rho_1^2 M)$), they are distinct mathematical quantities. It is structurally challenging for an inherently parametrization-dependent local quantity to universally capture the globally integrated optical redshift required to yield the distant Bogoliubov spectrum.

VII. DISCUSSION AND CONCLUSIONS

In summary, the sequence of results presented here provides a unified framework linking classical spacetime geometry to quantum black hole thermodynamics beyond the stationary regime. Four components form the logical arc. Theorem 1 establishes κ_{Noether} as a local property of arbitrary worldtubes, generalizing the Komar charge to dynamical settings. Proposition 1 supplies the correspondence principle, confirming that this generalized charge density recovers the standard surface gravity on stationary Killing horizons. Theorem 2 carries the classically anchored quantity into the semiclassical regime, demonstrating that the Parikh-Wilczek tunneling temperature of a dynamically evolving black hole is dictated by $\kappa_{\text{Noether}}^{(\text{EH})}$ and Theorem 3 gives, in closed form, its separation from the optical peeling rate that governs the asymptotic Bogoliubov spectrum. Finally, the Clausius balance of Sec. VI A, with the heat supplied independently of any surface gravity, shows that the quasilocal first law at the apparent horizon is balanced by the Kodama-Hayward value: the balance derives the surface gravity which the quasilocal program imposes by convention, while the tunneling exponent selects a different member, one that misses the balance at first order in the evaporation rate and that no integrated

first law can reach. Four surface gravities on three surfaces answer four distinct questions. This derive-plus-differ pairing is what sharpens the organizing claim of the paper: which member appears is fixed by the physical process, the family being degenerate only in the stationary limit.

It is instructive to situate this framework within the broader literature on dynamical surface gravity. Historically, extending black hole thermodynamics beyond the stationary regime has required either restricted spacetime symmetries or specialized quasilocal geometric definitions, largely due to the absence of a unique time-translating Killing vector.

In spherically symmetric dynamical spacetimes, a geometrically preferred time direction can be identified using the Kodama vector [6]. Because the Kodama vector acts as a natural analog to a stationary Killing vector, it allows for a well defined dynamical surface gravity and a corresponding unified first law of thermodynamics [7]. However, a significant limitation of the Kodama approach is its strict reliance on spherical symmetry; attempts to generalize a unique, divergence-free Kodama vector to arbitrary asymmetric or rotating dynamical spacetimes have proven largely elusive.

To address scenarios lacking spherical symmetry, successful geometric frameworks have been developed relying on the expansion of outgoing null geodesics. Foremost among these are Hayward's trapping horizons, [8] which define a dynamical surface gravity using the dual-null foliation of marginally trapped surfaces. Similarly, the Ashtekar-Krishnan framework of isolated and dynamical horizons [9,47] provides a basis for analyzing evolving black holes. Building on this, Booth and Fairhurst introduced the concept of "slowly evolving horizons," [10] where a dynamical surface gravity is defined by constructing an approximate, quasistationary time evolution vector field on the horizon, constrained by the requirement that the horizon is expanding very slowly.

The Noether-charge approach developed in this paper offers a distinctly different perspective that complements these established methodologies. Rather than relying on null expansions, dual-null foliations, or attempts to construct an approximate global symmetry, our formulation extracts the surface gravity directly from the extrinsic geometry of the worldtube ($K_{\mu\nu}^\mathcal{V}$) and the kinematics of locally defined worldtube-hugging observers. It bypasses the need for global spatial symmetries, making it applicable to arbitrary dynamical worldtubes. While our explicit comparison verified the equivalence with Hayward's covariant framework in spherical symmetry as a strong proof-of-concept, evaluating this framework on fully nonspherically symmetric spacetimes offers an avenue for future investigation.

By covariantly absorbing the time-dependence of the boundary into the spatial normal path of the WKB tunneling trajectory, Theorem 2 demonstrates that the tunneling

temperature can be evaluated locally and exactly at leading semiclassical order. Furthermore, Corollary 1 solidifies this framework by recovering the Unruh effect purely from classical worldtube kinematics.

Finally, evaluating our methodology covariantly clarifies the conceptual tension regarding the origin of Hawking radiation in dynamical spacetimes. The fixed-slice Hamilton-Jacobi approach [23,26–28], by integrating the momentum 1-form over a surface of constant coordinate time, locates the tunneling pole at the dynamically shrinking apparent horizon, which geometrically represents a classically traversable timelike surface. As we show in Sec. IV E, this result depends on the choice of slicing, since the 1-form is not closed in a dynamical spacetime. A fully covariant evaluation removes that dependence, and by enforcing exact geometric covariance tracking the worldtube normal, our formalism physically isolates the tracking kinematics of the timelike apparent horizon from the strict null barrier constraint of the true event horizon. We prove that the requisite geometric tunneling pole ($\Phi \rightarrow 0$) mathematically vanishes entirely at the apparent horizon and emerges exclusively at the null event horizon. By resolving the indeterminate geometry of the null surface via a physical spatial tracking limit, this yields an exact semiclassical proof that dynamical Hawking radiation uniquely physically originates from the true event horizon.

This covariant framework brings into relief a physical subtlety characteristic of dynamical radiating environments: the explicit divergence of local horizon thermodynamics from the asymptotic flux. We have established that the local tunneling mechanics tracking the evolving boundary uniquely evaluates the physical geometric quasilocal tracking acceleration $\kappa_{\text{Noether}}^{(\text{EH})}$. However, as we explicitly demonstrated using the exact Vaidya spacetime, this exact local dynamical value structurally analytically separates from the optical “peeling” surface gravity that exclusively governs the precise Bogoliubov thermal spectrum registered by a distant observer. This exact structural separation is not a failure of the semiclassical method, but a consequence of phase exactness in the adiabatic regime. As proven by our global eikonal exactness theorem (see Sec. V C), while the path-dependent normal integration explicitly captures the physical quasilocal tracking penalty, mapping the state function to future null infinity completely structurally cancels these local boundary kinematics out of the continuous eikonal phase. The Vaidya evaluation confirms the gap equation: subtracting the asymptotic peeling rate from the local Noether surface gravity reproduces the tracking kinematics. This indicates that in the presence of dynamic evaporation, the geometric quasilocal horizon tracking temperature and the asymptotic thermal Hawking spectrum represent well-separated, mathematically decoupled geometric physical phenomena. By explicitly demonstrating that the unclosed local WKB contour captures the full boundary tracking mechanics (including

acceleration) while the exact global eikonal PDE cancels the dominant adiabatic tracking kinematics to recover the peeling flux, we clarify the role of each surface gravity in the dynamical Hawking temperature debate.

Two results follow, and the positive one comes first. The Clausius balance, with the heat supplied independently of any surface gravity, returns $\kappa_{\text{KH}} = 1/4M$. That is not merely Hayward’s value: it is the number the quasilocal program imposes by hand, whether as the Kerr-matched $\kappa^{(0)} = 1/2r_H$ of Booth and Fairhurst [10], the effective $1/2R$ of Ashtekar and Krishnan [45], or the $J \rightarrow 0$ limit of the Ashtekar-Paraizo-Shu $\hat{\kappa}$ [42]. Our balance derives what they assume, converting an integrability convention into a result, and answers on its own terms the reservation of Ashtekar and Krishnan that their effective surface gravity carries no reading as an instantaneous physical temperature. The second result is that the tunneling exponent selects a different member of the family, one which no integrated first law can reach, since the integrability requirement fixing $\kappa^{(0)}$ as a function of horizon area and angular momentum alone is precisely what excludes the tracking term. Neither is therefore *the* surface gravity of a dynamical horizon: which member appears is fixed by the process, and the family is degenerate only in the stationary limit. The analyses of Secs. VI A and V B make this precise: the Vaidya geometry supplies the heat independently of κ , since the Misner-Sharp energy is $M(u)$ at every radius and the null-dust work density vanishes, the balance holds exactly and for arbitrary $M(u)$ with both members of the law fixed by instantaneous data, and $\kappa_{\text{Noether}}^{(\text{AH})}$ differs from the balancing value by the tracking term $\dot{M}/2M$, vanishing on the self-similar profile. That the Clausius balance and the tunneling exponent select different members of the family is the expected outcome given that the surface gravity of a non-Killing horizon is not unique. The event horizon carries the tunneling barrier, whose location is a global question but whose residue $\kappa_{\text{Noether}}^{(\text{EH})}$ is what a local measurement of the emission temperature returns. Spatial infinity carries the Bogoliubov spectrum through κ_{peeling} . The exact balance of $A_{\text{EH}}/4$ against κ_{peeling} on the self-similar profile is an artifact of self-similarity, under which the entire future is determined by instantaneous data and a future functional can masquerade as a local constant. The exact condition, derived in Sec. VI A, is that the balance and the local geometric value agree precisely when the tracked radius is proportional to M , which holds identically at the apparent horizon and only under self-similarity at the event horizon; splicing two histories destroys the latter while leaving the apparent-horizon relation exact.

This assignment agrees with an independent line of work approaching the same question from the entropy side of $T\delta S$. Hollands *et al.* argue that the Bekenstein-Hawking area of the event horizon responds to what will happen rather than to

what has happened, and construct a dynamical entropy free of that defect which, at leading order in general relativity, equals one quarter of the apparent-horizon area [39]; Ashtekar *et al.* reach the same identification without a perturbative restriction [42]. That argument is ours applied to the conjugate variable, and the convergence of the two is stronger evidence for the position than either alone. What the present work adds is the temperature: the quasilocal program supplies a first law but fixes its surface gravity by matching to Kerr-Newman, an integrability requirement rather than a physical one [43,44], and Theorem 2 supplies the physical process that selects a member of that family which no integrated law can reach.

A natural physical question arises: could a geometrically modified Noether framework recover the asymptotic peeling flux by anchoring the charge to the exact null generators rather than a timelike tracking worldtube? As we demonstrated in Sec. VIC by explicitly evaluating a null-adapted Noether charge, this approach encounters geometric limitations in capturing local barrier thermodynamics. While a null-adapted construction isolates the generator inaffinity, the inaffinity depends on the choice of affine parameter and does not reproduce the peeling invariant. This indicates a structural feature: the integrated Noether charge is a quasilocal geometric property. No purely local boundary evaluation can capture the globally integrated optical redshift that defines the distant Bogoliubov spectrum. Rather than a flaw, this gap indicates that the kinematic cost of boundary tracking (measured by $\kappa_{\text{Noether}}^{(\text{EH})}$) and the asymptotic Hawking flux are two distinct phenomena that permanently decouple once stationarity is broken.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: GAUSSIAN NORMAL CONSTRUCTION

In the proof of Theorem 1, evaluating the Noether charge surface density on the worldtube $\mathcal{W}$ naturally produces a term proportional to the Lie derivative $\mathcal{L}_{\hat{R}}(\Phi_{\mathcal{W}}^\mu)$, associated with the tangential drift. To justify setting this term to zero, we have used the Gaussian normal construction, which we briefly introduce here.

By definition, our congruence of observers ξ^μ and its worldtube-hugging normal component $\Phi_{\mathcal{W}}^\mu$ are vector fields defined strictly on the 3-dimensional worldtube boundary $\mathcal{W}$. However, computing the normal gradient $\xi_{[\mu;\nu]} \hat{R}^\nu$ requires evaluating the covariant derivative of ξ^μ along the outward normal direction $\hat{R}^\nu$. Geometrically, this represents a directional derivative pointing infinitesimally off the worldtube and into the 4-dimensional bulk spacetime. Because the vector field is initially defined only on

the boundary, its transverse derivative is mathematically undefined until we explicitly specify a rule for how the vector field extends into the bulk.

The standard and most geometrically natural procedure is to construct a local Gaussian normal coordinate system (also known as synchronous coordinates) extending off the hypersurface [21]. We begin on the worldtube $\mathcal{W}$ by parametrizing the integral curves of the normal worldtube-hugging observers by the synchronization coordinate t of the foliation, so that the time-evolution vector field on $\mathcal{W}$ is $\Phi_{\mathcal{W}}^\mu = (\partial/\partial t)^\mu$. From every point on $\mathcal{W}$, we then construct a congruence of spacelike geodesics shooting orthogonally outward into the bulk spacetime. These normal geodesics are parametrized by their proper spatial distance, η , which serves exactly as the affine parameter along the curves. As long as the geodesics do not intersect, η acts as a spatial coordinate rather than a path-dependent distance. By construction, the tangent vector to these geodesics is precisely our spacelike unit normal vector, $\hat{R}^\mu = (\partial/\partial \eta)^\mu$.

To define the vector fields off the worldtube, we extend the time coordinate t (along with any spatial coordinates on the tube, such as angular coordinates θ^A) into the bulk by demanding that they remain constant along the normal geodesics. Physically, this corresponds to extending the vector field $\Phi_{\mathcal{W}}^\mu$ by “Lie-dragging” it strictly along the normal vector $\hat{R}^\mu$ into the bulk.

This geometric construction provides a unique extension into the bulk spacetime. Given the initial boundary $\mathcal{W}$ and its orthogonal outward unit normal vector field $\hat{R}^\mu$, the fundamental existence and uniqueness theorems for the geodesic equation guarantee that the congruence of spacelike geodesics is uniquely determined. Consequently, the extension of the boundary coordinates, and therefore the prolonged vector field $\Phi_{\mathcal{W}}^\mu$, is unambiguously fixed by the initial boundary conditions. This ensures that extending the fields into the bulk is not an arbitrary mathematical choice, but rather a canonical geometric extension dictated uniquely by the worldtube itself. Because the integrated Komar charge depends on the transverse extension of the vector field, this canonical geometric constraint naturally resolves the historical ambiguity and physically fixes the generalized Noether charge.

It is also important to emphasize that using the proper distance η as a coordinate variable is exact in this construction. Provided we restrict our analysis to a local “tubular neighborhood” around $\mathcal{W}$, specifically, halting before spacetime curvature causes the normal geodesics to intersect and form caustics (focal points), the parameters (η, t, θ^A) form a well defined, smooth coordinate chart [21,22,48,49]. In these Gaussian normal coordinates, the spacetime metric is mathematically guaranteed to take the form

$$ds^2 = d\eta^2 + \gamma_{ab}(\eta, y) dy^a dy^b, \quad (\text{A1})$$

where $y^a = (t, \theta^A)$ are the coordinates on $\mathcal{W}$ and γ_{ab} is the induced metric on the surfaces of constant η . The exact $g_{\eta\eta} = 1$ component and the absence of $g_{\eta a}$ cross-terms explicitly confirm that η measures proper orthogonal distance, and that the basis vectors $\hat{R}^\mu$ and $\Phi_{\mathcal{W}}^\mu$ are independent coordinate vector fields.

Because $\hat{R}^\mu$ and $\Phi_{\mathcal{W}}^\mu$ serve as exact coordinate basis vectors in this local chart, their Lie bracket (or commutator) identically vanishes due to the symmetry of mixed partial derivatives (Clairaut's theorem)

$$[\hat{R}, \Phi_{\mathcal{W}}]^\mu = \left(\frac{\partial}{\partial \eta} \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \frac{\partial}{\partial \eta} \right)^\mu = 0. \quad (\text{A2})$$

By definition, the Lie derivative of a vector field along another is given precisely by their Lie bracket. Therefore, Eq. (A2) immediately yields the exact kinematic constraint

$$\mathcal{L}_{\hat{R}}(\Phi_{\mathcal{W}}^\mu) = [\hat{R}, \Phi_{\mathcal{W}}]^\mu = 0. \quad (\text{A3})$$

By resolving the initial extension ambiguity through this natural, uniquely determined Gaussian normal foliation, we geometrically ensure that the tangential time flow and the normal vector commute. This geometric choice smoothly eliminates the Lie derivative term associated with the tangential drift.

-
- [1] R. M. Wald, *Phys. Rev. D* **48**, R3427 (1993).
 - [2] V. Iyer and R. M. Wald, *Phys. Rev. D* **50**, 846 (1994).
 - [3] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
 - [4] G. Penington, *J. High Energy Phys.* 09 (2020) 002.
 - [5] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, *J. High Energy Phys.* 12 (2019) 063.
 - [6] H. Kodama, *Prog. Theor. Phys.* **63**, 1217 (1980).
 - [7] S. A. Hayward, *Classical Quantum Gravity* **15**, 3147 (1998).
 - [8] S. A. Hayward, *Phys. Rev. D* **49**, 6467 (1994).
 - [9] A. Ashtekar and B. Krishnan, *Living Rev. Relativity* **7**, 10 (2004).
 - [10] I. Booth and S. Fairhurst, *Phys. Rev. Lett.* **92**, 011102 (2004).
 - [11] G. Fodor, K. Nakamura, Y. Oshiro, and A. Tomimatsu, *Phys. Rev. D* **54**, 3882 (1996).
 - [12] A. B. Nielsen and M. Visser, *Classical Quantum Gravity* **23**, 4637 (2006).
 - [13] A. B. Nielsen and J. H. Yoon, *Classical Quantum Gravity* **25**, 085010 (2008).
 - [14] E. Noether, *Nachr. König. Ges. Wiss. Göttingen Math.-Phys. Kl.* **1918**, 235 (1918).
 - [15] J. Baez, [arXiv:2006.14741](https://arxiv.org/abs/2006.14741).
 - [16] A. Komar, *Phys. Rev.* **113**, 934 (1959).
 - [17] D. Bak, D. Cangemi, and R. Jackiw, *Phys. Rev. D* **49**, 5173 (1994).
 - [18] J. G. Fletcher, *Rev. Mod. Phys.* **32**, 65 (1960).
 - [19] R. Beig, *Phys. Lett.* **69A**, 153 (1978).
 - [20] M. K. Parikh and F. Wilczek, *Phys. Rev. Lett.* **85**, 5042 (2000).
 - [21] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, England, 2004).
 - [22] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
 - [23] S. A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini, and S. Zerbini, *Classical Quantum Gravity* **26**, 062001 (2009).
 - [24] T. Jacobson and G. Kang, *Classical Quantum Gravity* **10**, L201 (1993).
 - [25] A. B. Nielsen, *Galaxies* **2**, 62 (2014).
 - [26] R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini, and G. Zoccatelli, *Phys. Lett. B* **657**, 107 (2007).
 - [27] R. Di Criscienzo, S. A. Hayward, M. Nadalini, L. Vanzo, and S. Zerbini, *Classical Quantum Gravity* **27**, 015006 (2010).
 - [28] L. Vanzo, G. Acquaviva, and R. Di Criscienzo, *Classical Quantum Gravity* **28**, 183001 (2011).
 - [29] E. T. Akhmedov, V. Akhmedova, and D. Singleton, *Phys. Lett. B* **642**, 124 (2006).
 - [30] P. Mitra, *Phys. Lett. B* **648**, 240 (2007).
 - [31] E. T. Akhmedov, T. Pilling, and D. Singleton, *Int. J. Mod. Phys. D* **17**, 2453 (2009).
 - [32] B. Cropp, S. Liberati, and M. Visser, *Classical Quantum Gravity* **30**, 125001 (2013).
 - [33] W. G. Unruh, *Phys. Rev. D* **14**, 870 (1976).
 - [34] R. C. Tolman, *Phys. Rev.* **35**, 904 (1930).
 - [35] T. Jacobson, G. Kang, and R. C. Myers, *Phys. Rev. D* **49**, 6587 (1994).
 - [36] A. Rignon-Bret, *Phys. Rev. D* **108**, 044069 (2023).
 - [37] A. C. Wall, *Int. J. Mod. Phys. D* **24**, 1544014 (2015).
 - [38] X. Dong, *J. High Energy Phys.* 01 (2014) 044.
 - [39] S. Hollands, R. M. Wald, and V. G. Zhang, *Phys. Rev. D* **110**, 024070 (2024).
 - [40] M. R. Visser and Z. Yan, *J. High Energy Phys.* 10 (2024) 029.
 - [41] M. R. Visser and Z. Yan, *J. High Energy Phys.* 02 (2026) 003.
 - [42] A. Ashtekar, D. E. Paraizo, and J. Shu, *Phys. Rev. Lett.* **136**, 251405 (2026).
 - [43] A. Ashtekar, C. Beetle, and J. Lewandowski, *Phys. Rev. D* **64**, 044016 (2001).

-
- [44] A. Ashtekar and B. Krishnan, [Living Rev. Relativity](#) **28**, 8 (2025).
- [45] A. Ashtekar and B. Krishnan, [Phys. Rev. D](#) **68**, 104030 (2003).
- [46] I. Booth, [arXiv:1202.5789](#).
- [47] B. Krishnan, Ph.D. thesis, The Pennsylvania State University, 2002.
- [48] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
- [49] J. D. Brown and J. W. York, [Phys. Rev. D](#) **47**, 1407 (1993).