

Direct algebraic pathway to hadronic observables in the contact model

Jiayin Kang[✉], Zanbin Xing^{✉,*}, and Lei Chang[†]

School of Physics, Nankai University, Tianjin 300071, China



(Received 15 January 2026; accepted 28 May 2026; published 22 June 2026)

We present a novel algebraic framework for computing hadron properties directly within the contact interaction model. Utilizing Fierz transformations, the method recasts the Bethe-Salpeter dynamics into equations for a minimal set of *projected amplitudes* for bound-state static properties and form factors, bypassing the conventional need for the meson wave function. This approach is fully demonstrated for the vector meson, enabling the direct extraction of its decay constants, form factors, and the parton distribution functions. The formalism provides a more efficient and unified pathway to hadron observables, with potential for extension to baryons and more sophisticated interactions.

DOI: [10.1103/t3zw-dh3w](https://doi.org/10.1103/t3zw-dh3w)

I. INTRODUCTION

Understanding the structure and properties of hadrons from the first principles of quantum chromodynamics (QCD) remains a central challenge in nuclear and particle physics [1]. In the nonperturbative regime, frameworks such as that of the Dyson-Schwinger equations (DSEs) [2–7] have proven fruitful. The contact interaction (CI) model, first systematically introduced in the DSEs context in Ref. [8] and subsequently developed in Refs. [9–30], serving as a tractable infrared approximation, is widely adopted within DSE studies both for pedagogical purposes and for quantitative hadronic calculations using a symmetry-preserving regularization scheme [22].

The conventional paradigm in DSE studies follows a well-established sequence [2,27,31–33]: a symmetry-preserving truncation, such as the rainbow-ladder (RL) approximation, is first employed to derive the quark propagator; subsequently, the bound-state Bethe-Salpeter or Faddeev equations are solved to obtain wave functions; finally, these wave functions are used to compute observable quantities such as decay constants and form factors within the symmetry preserving framework.

Although robust, this multistep procedure requires the intermediate determination of wave functions, which are not themselves direct physical observables. In this work, we propose and demonstrate an alternative, streamlined paradigm that circumvents this conventional sequence.

By making judicious use of Fierz transformations, we derive closed algebraic equations that directly govern the projected amplitudes for physical quantities—bypassing the explicit calculation of bound-state wave functions altogether. This approach offers a more efficient and conceptually direct pathway from the model interaction to measurable hadronic properties.

To illustrate the efficacy and simplicity of this new scheme, we present a detailed calculation for the vector meson as a benchmark case [20,27]. The procedure, however, is general and can be readily applied to other meson systems as well as baryons. Beyond computational efficiency, this algebraic reformulation offers a novel and unified perspective on the hadron structure, directly linking model interactions to a broad class of observables without intermediary wave functions.

The remainder of this paper is organized as follows. In Sec. II, we present the general formalism: starting from the Fierz-transformed contact interaction, we derive the homogeneous and inhomogeneous algebraic equations that directly govern the projected amplitudes for bound-state properties and form factors, respectively. Section III applies this formalism specifically to the vector meson, detailing the tensor structures of its projected amplitudes and the complete basis of form factors. The corresponding numerical results are also presented and discussed in this section. Finally, Sec. IV summarizes our findings and outlines future perspectives. Useful algebraic details are compiled in the Appendixes.

II. EQUATIONS FOR PHYSICAL OBSERVABLES

In this section, we derive the closed-form equations satisfied directly by the projected amplitudes of physical observables within the CI model. This is achieved by applying Fierz transformations to the standard Bethe-Salpeter equation (BSE), followed by appropriate Dirac

*Contact author: xingzb@mail.nankai.edu.cn

†Contact author: leichang@nankai.edu.cn

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

projections, thereby eliminating the need to first solve for the full Bethe-Salpeter amplitudes (BSAs).

The starting point is the Fierz reordering identity:

$$\gamma_\alpha^{ij} \gamma_\alpha^{kl} - \xi \tilde{\Gamma}_j^{ij} \tilde{\Gamma}_j^{kl} = \mathcal{C}_{aa'} \Lambda_{il}^a \Lambda_{kj}^{a'}, \quad (1)$$

where the second interaction channel [represents the anomalous magnetic moment effect [20]] is defined by $\tilde{\Gamma}_j = \{I_4, \gamma_5, i\sigma_{\alpha\beta}/\sqrt{6}\}$ with a strength parameter ξ . The coefficient matrix is $\mathcal{C}_{aa'} = \text{diag}\{1, -1, -1/2, -1/2, -\xi/3\}$, and the Dirac basis is $\{\Lambda^a\} = \Lambda_{(\mu_a \nu_a)}^{\{S, P, V, A, T\}} = \{I, \gamma_5, \gamma_{\mu_a}, \gamma_5 \gamma_{\mu_a}, \sigma_{\mu_a \nu_a}\}$. Herein we use the shorthand a, b, c to denote indices running over the set $\{S, P, V, A, T\}$ and suppress the possible Lorentz indices for brevity. Repeated indices imply summation.

A. Homogeneous equation: Masses and decay constants

To derive the closed-form equations, we start from the meson BSE in the CI model:

$$\begin{aligned} \Gamma_H(Q) = & -D_G \int_q \gamma_\alpha \chi_H(q, Q) \gamma_\alpha \\ & - D_G \xi \int_q \tilde{\Gamma}_j \chi_H(q, Q) \tilde{\Gamma}_j, \end{aligned} \quad (2)$$

where $\int_q = \int \frac{d^4 q}{(2\pi)^4}$, $D_G = 4/(3m_G^2)$, $\Gamma_H(Q)$ is the Bethe-Salpeter amplitude of meson type H , and the corresponding Bethe-Salpeter wave function is $\chi_H(q, Q) = S(q_+) \Gamma_H(Q) S(q_-)$ with $X_\pm = X \pm Q/2$. The quark propagator satisfies the corresponding DSE¹

$$S^{-1}(p) = i\not{p} + m + D_G \int_q \gamma_\alpha S(q) \gamma_\alpha, \quad (3)$$

and it can be parametrized as $S^{-1}(p) = i\not{p} + M$; we apply the symmetry preserving regularization scheme proposed in [22] and relevant parameters therein, which are presented in Table I.

Applying the Fierz identity Eq. (1) transforms the BSE into an algebraic form:

$$\Gamma_H(Q) = \Lambda_a \mathcal{D}_{aa'} g_a^H(Q), \quad (4)$$

where $\mathcal{D}_{aa'} \equiv -D_G \mathcal{C}_{aa'}$. Here, we have introduced the key projected amplitudes $g_a^H(Q)$, defined as

$$g_a^H(Q) = \text{tr} \int_q \Lambda_a \chi_H(q, Q), \quad (5)$$

¹The quark propagator is actually a diagonal matrix in flavor space denoted by $S = \text{diag}\{S_u, S_d, \dots\}$. For brevity we focus on the two lightest quarks and work in the isospin limit: $S_u(p) = S_d(p) \equiv S(p)$.

TABLE I. Constituent quark mass and static properties of the ρ meson, mass units in GeV. The magnetic and quadrupole moments are defined as $\mu_\rho \equiv G_M(0)$ and $\mathcal{Q}_\rho \equiv G_Q(0)$. In the CI framework, the chosen parameters are $m_G = 0.132$ GeV, $\tau_{ir} = 1/0.24$ GeV⁻¹, $\tau_{uv} = 1/0.905$ GeV⁻¹, and $m = 0.007$ GeV.

ξ	M	M_ρ	f_ρ	$f_\rho^\perp$	μ_ρ	$\mathcal{Q}_\rho$
0	0.368	0.929	0.183	0.189	2.084	-0.837
0.151	0.368	0.879	0.182	0.225	2.220	-0.629

with tr denoting the Dirac trace. These amplitudes carry the remaining color and flavor indices but encapsulate the essential dynamical information. Equation (4) shows that the full BSA in the CI model is completely determined by a finite set of these projected amplitudes.²

To obtain an equation governing $g_a^H(Q)$, we multiply both sides of the original BSE by the projection operator $S(p_-) \Lambda_b S(p_+)$, take the Dirac trace, and integrate over the relative quark momentum p . This procedure yields

$$g_b^H(Q) = \Pi_{ba}(Q) \mathcal{D}_{aa'} g_a^H(Q), \quad (6)$$

where the kernel is

$$\Pi_{ba}(Q) = \text{tr} \int_p S(p_-) \Lambda_b S(p_+) \Lambda_a. \quad (7)$$

Equation (6) is a homogeneous equation for the projected amplitudes. It can be recast as an eigenvalue problem,

$$\lambda(Q^2) g_b^H(Q) = \Pi_{ba}(Q) \mathcal{D}_{aa'} g_a^H(Q), \quad (8)$$

with the physical bound-state mass m_H identified by the condition $\lambda(Q^2 = -m_H^2) = 1$.

The canonical normalization condition for the BSA can also be expressed solely in terms of $g_a^H(Q)$. Starting from the standard normalization condition [31,34], and using Eq. (4), one finds

$$\begin{aligned} 1 = & \frac{\partial \ln \lambda(Q^2)}{\partial Q^2} \mathcal{D}_{aa'} \text{Tr}[\bar{g}_a^H(-Q) g_a^H(Q)]_{Q^2 = -m_H^2} \\ = & \frac{\partial \ln \lambda(Q^2)}{\partial Q^2} \text{Tr}[\bar{g}_a^H(-Q) \Pi_{ab}^{-1}(Q) g_b^H(Q)]_{Q^2 = -m_H^2}, \end{aligned} \quad (9)$$

where Tr denotes color and flavor traces, and $\bar{g}_a^H(-Q)$ is the charge-conjugate amplitude, related to $g_a^H(Q)$ via $\bar{\chi}_H(q, -Q) = C \chi_H^\top(-q, -Q) C^{-1}$ in Eq. (5). The first row is a simple formula that could be applied in the CI model, while the second row presents the normalization in the

²For the general momentum-dependent interaction model the number of projected amplitudes is infinite.

hadronic level since all partonic information is integrated out and is not limited for CI.

B. Inhomogeneous equation: Form factors and transition amplitudes

For processes involving an external current, we consider the generalized quark-quark correlation $\mathcal{M}_H(q, P, Q)$, where $P = (P_i + P_f)/2$ and $Q = P_f - P_i$ are the average and transferred momentum of the initial and final state mesons, satisfying $P^2 = -m_H^2 - Q^2/4$ and $P \cdot Q = 0$ for on-shell mesons. Its amputated version $F(q, P, Q)$ satisfies an inhomogeneous BSE [23,35]:

$$F_H(p, P, Q) = F_H^{ih}(p, P, Q) - D_G \int_q \gamma_\alpha \mathcal{M}_H(q, P, Q) \gamma_\alpha - D_G \xi \int_q \tilde{\Gamma}_j \mathcal{M}_H(q, P, Q) \tilde{\Gamma}_j, \quad (10)$$

with $\mathcal{M}_H(q, P, Q) = S(q_-)F_H(q, P, Q)S(q_+)$, and the inhomogeneous term $F_H^{ih}(p, P, Q) = \Gamma_H(P_i)S(p - P)\tilde{\Gamma}_H(-P_f)$.

Following the same Fierz transformation and projection technique, we obtain

$$F_H(p, P, Q) = F_H^{ih}(p, P, Q) + \Lambda_a \mathcal{D}_{aa'} \mathcal{A}_a^H(P, Q), \quad (11)$$

where the *invariant amplitudes* $\mathcal{A}_a^H(P, Q)$ are defined as

$$\mathcal{A}_a^H(P, Q) = \text{tr} \int_q \Lambda_a \mathcal{M}_H(q, P, Q). \quad (12)$$

Projecting with $S(p_+)\Lambda_c S(p_-)$ leads to the governing equation for these amplitudes:

$$\mathcal{A}_c^H(P, Q) = \mathcal{A}_c^{H,ih}(P, Q) + \Pi_{ca}(-Q) \mathcal{D}_{aa'} \mathcal{A}_a^H(P, Q). \quad (13)$$

The inhomogeneous term $\mathcal{A}_c^{H,ih}(P, Q)$ is computed from F_H^{ih} and can be expressed by using the previously defined amplitudes g_a^H :

$$\mathcal{A}_c^{H,ih}(P, Q) = \mathcal{D}_{aa'} \mathcal{D}_{bb'} g_a^H(P_i) \tilde{g}_b^H(-P_f) \mathcal{T}_{cab}(P, Q), \quad (14)$$

where the universal triangle diagram function is

$$\mathcal{T}_{cab}(P, Q) = \text{tr} \int_p \Lambda_c S(p_-) \Lambda_a S(p - P) \Lambda_b S(p_+). \quad (15)$$

We note that in Ref. [23], the treatment of the contact interaction inevitably introduced self-energy terms into the scattering amplitude. Our present approach avoids this complication, rendering the calculation more direct and generalizable.

C. Structure and universality of the projected amplitudes

The formalism developed above exhibits a clear, factorized structure:

- (i) Two types of basis amplitudes: The dynamics of a meson H are encoded in two sets of projected amplitudes:

$$\chi_H(q, Q) \leftrightarrow \{g_a^H(Q)\}, \quad (16)$$

$$\mathcal{M}_H(q, P, Q) \leftrightarrow \{\mathcal{A}_a^H(P, Q)\}, \quad (17)$$

governed by the homogeneous Eq. (6) and inhomogeneous Eq. (13), respectively. The normalization condition Eq. (9) and the inhomogeneous term in Eq. (13) depend solely on g_a^H .

- (ii) Universal kernels: The functions $\Pi_{ba}(Q)$ and $\mathcal{T}_{cab}(P, Q)$ are *universal*; they depend only on the quark propagator and the Dirac basis, not on the specific meson quantum numbers. All meson-specific information is coded by the corresponding quantum numbers and contained within the color-flavor structure of g_a^H and $\mathcal{A}_a^H$.
- (iii) Basis of observables: The sets $\{g_a^H\}$ and $\{\mathcal{A}_a^H\}$ for a given ξ form a basis. Once the basis observables are known, the projected amplitudes g_Γ^H or $\mathcal{A}_\Gamma^H$ with arbitrary project operator Γ instead of the basis operators Λ can be expressed in terms of these basis amplitudes:

$$g_\Gamma^H(Q) = \Pi_{\Gamma a}(Q) \mathcal{D}_{aa'} g_a^H(Q), \quad (18)$$

$$\mathcal{A}_\Gamma^H(P, Q) = \mathcal{A}_\Gamma^{H,ih}(P, Q) + \Pi_{\Gamma a}(-Q) \mathcal{D}_{aa'} \mathcal{A}_a^H(P, Q). \quad (19)$$

For instance, in the RL truncation ($\xi = 0$), the tensor ($a = T$) amplitudes are not basis amplitudes but can be expressed in terms of the vector ($a = V$) basis amplitudes. The gravitational form factors can also be obtained using a similar method.

- (iv) Tensor structure: After factoring out color and flavor (denoted by a tilde, e.g., $\tilde{g}_a^H$), the Lorentz-Dirac structure of these amplitudes is simple. For the homogeneous case

$$\tilde{g}_a^H(Q) = \sum_i T_i^a(Q) \tilde{f}_{a,i}^H, \quad (20)$$

where $T_i^a(Q)$ are tensors formed from Q and the metric/antisymmetric tensors, and $\tilde{f}_{a,i}^H$ are Lorentz-invariant coefficients. The structure of $\tilde{\mathcal{A}}_a^H(P, Q)$ is analogous to a meson- a -meson vertex:

$$\tilde{\mathcal{A}}_a^H(P, Q) = \sum_i T_i^a(P, Q) \tilde{F}_{a,i}^H(Q^2), \quad (21)$$

where $T_i^a(P, Q)$ now depend on both P and Q , and $\tilde{F}_{a,i}^H(Q^2)$ are the sought-after form factors.

III. VECTOR MESON OBSERVABLES

We proceed to apply the general formalism of Sec. II to vector mesons, a key class of hadrons characterized by spin-1 and odd parity. Focusing on the ρ meson as a representative case, we detail the implementation of our approach: constructing the appropriate tensor structures, identifying the complete basis of form factors, and extracting physical observables such as decay constants, electromagnetic form factors. For convenience we suppress the explicit meson type in the projected amplitudes $g_a(P)$ and $\mathcal{A}_a(P, Q)$ in this section.

A. Basis projected amplitudes

For a ρ meson with momentum P and polarization vector ε , the Lorentz-Dirac structures of the homogeneous projected amplitudes, $\tilde{g}_a(P)$ (obtained after factoring out color and flavor), are constrained by parity, charge conjugation, and angular momentum. They take the following form on the mass shell³:

$$\tilde{g}_V = \varepsilon^\mu \tilde{f}_V, \quad (22)$$

$$\tilde{g}_T = (\varepsilon^\mu P^\nu - \varepsilon^\nu P^\mu) \tilde{f}_T, \quad (23)$$

where $\tilde{f}_V$ and $\tilde{f}_T$ are the independent Lorentz-invariant amplitudes to be determined from Eq. (6). Substituting this structure into Eq. (6) reduces it to a coupled eigenvalue problem for the eigenvector $\{\tilde{f}_V, \tilde{f}_T\}$.

The corresponding structures for the invariant amplitudes $\tilde{\mathcal{A}}_a(P, Q)$ associated with the $\rho - a - \rho$ transition are more numerous. They are constructed from the available vectors P_μ, Q_μ ; the polarization vectors $\varepsilon, \varepsilon'^*$; and the metric tensor. A complete basis can be chosen as

$$\tilde{\mathcal{A}}_S = s \tilde{F}_{S,1} + s_{PP} \tilde{F}_{S,2}, \quad (24)$$

$$\tilde{\mathcal{A}}_P = \epsilon_{PQ\epsilon\epsilon'^*} \tilde{F}_P, \quad (25)$$

$$\tilde{\mathcal{A}}_V = P_\mu s \tilde{F}_{V,1} + P_\mu s_{PP} \tilde{F}_{V,2} + 2s_{\mu P} \tilde{F}_{V,3}, \quad (26)$$

$$\tilde{\mathcal{A}}_A = \epsilon_{\mu P \epsilon \epsilon'^*} \tilde{F}_{A,1} + Q_\mu \epsilon_{PQ \epsilon \epsilon'^*} \tilde{F}_{A,2} + 2s_{\nu P} \epsilon_{\mu \nu P Q} \tilde{F}_{A,3}, \quad (27)$$

$$\begin{aligned} \tilde{\mathcal{A}}_T = & s(P_\mu Q_\nu - Q_\mu P_\nu) \tilde{F}_{T,1} + s_{PP}(P_\mu Q_\nu - Q_\mu P_\nu) \tilde{F}_{T,2} \\ & + 2(Q_\mu s_{\nu P} - Q_\nu s_{\mu P}) \tilde{F}_{T,3} \\ & + 2(P_\mu a_{\nu P} - P_\nu a_{\mu P}) \tilde{F}_{T,4} + 2a_{\nu \mu} \tilde{F}_{T,5}. \end{aligned} \quad (28)$$

³We employ a Euclidean metric with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$; $\gamma_\mu^\dagger = \gamma_\mu$; $\gamma_5 = \gamma_4 \gamma_1 \gamma_2 \gamma_3$ so that $\text{tr}(\gamma_5 \gamma_a \gamma_b \gamma_c \gamma_d) = -4\epsilon_{abcd}$; and $a \cdot b = \sum_i^4 a_i b_i$.

Here, we have defined the following symmetric and antisymmetric tensors:

$$s_{\mu\nu} = \frac{1}{2}(\varepsilon_\mu \varepsilon_\nu'^* + \varepsilon_\nu \varepsilon_\mu'^*), \quad a_{\mu\nu} = \frac{1}{2}(\varepsilon_\mu \varepsilon_\nu'^* - \varepsilon_\nu \varepsilon_\mu'^*), \quad (29)$$

$$s = s_{\mu\mu}, \quad s_{PP} = s_{\mu\nu} P_\mu P_\nu, \quad s_{\mu P} = s_{\mu\nu} P_\nu, \quad a_{\mu P} = a_{\mu\nu} P_\nu. \quad (30)$$

Upon substituting these structures into the inhomogeneous equation (13), the system decouples into several subsets. Under nonzero ξ , the form factors $\tilde{F}_{S,1}, \tilde{F}_{S,2}, \tilde{F}_{A,1}, \tilde{F}_{A,3}, \tilde{F}_{T,4}$, and $\tilde{F}_{T,5}$ satisfy simple, uncoupled equations. In contrast, $\tilde{F}_{V,i}$ couples to $\tilde{F}_{T,i}$ for $i = 1, 2, 3$, and $\tilde{F}_P$ couples to $\tilde{F}_{A,2}$, and then the axial form factor $\tilde{F}_{A,2}$ exhibits a pseudoscalar pole while the tensor form factors $\tilde{F}_{T,\{1,2,3\}}$ exhibits a vector pole. Furthermore, the form factors $\tilde{F}_{T,i+2}$ and $\tilde{F}_{A,1}$ enter the respective coupled equations for $\{\tilde{F}_{V,i}, \tilde{F}_{T,i}\}$ and $\{\tilde{F}_P, \tilde{F}_{A,2}\}$ for $i = 2, 3$ as inhomogeneous driving terms.

The standard physical observables are obtained from these invariant amplitudes after including color and flavor factors ($N_c = 3$). The decay constant f_ρ and its transverse counterpart $f_\rho^\perp$ are

$$m_\rho f_\rho = N_c \tilde{f}_V, \quad (31)$$

$$f_\rho^\perp = N_c \tilde{f}_T. \quad (32)$$

The dimensionless electromagnetic and transition form factors are conventionally defined as

$$F_{S,1} = N_c m_\rho^{-1} \tilde{F}_{S,1}, \quad F_{S,2} = N_c m_\rho \tilde{F}_{S,2}/2, \quad (33)$$

$$F_P = -N_c m_\rho \tilde{F}_P, \quad F_{V,1} = i N_c \tilde{F}_{V,1}/2, \quad (34)$$

$$F_{V,2} = -i N_c m_\rho^2 \tilde{F}_{V,2}/4, \quad F_{V,3} = -i N_c \tilde{F}_{V,3}/2, \quad (35)$$

$$F_{A,1} = i N_c \tilde{F}_{A,1}, \quad F_{A,2} = i N_c m_\rho^2 \tilde{F}_{A,2}, \quad (36)$$

$$F_{A,3} = i N_c m_\rho^2 \tilde{F}_{A,3}, \quad F_{T,1} = i N_c m_\rho \tilde{F}_{T,1}, \quad (37)$$

$$F_{T,2} = i N_c m_\rho^3 \tilde{F}_{T,2}, \quad F_{T,3} = i N_c m_\rho \tilde{F}_{T,3}/2, \quad (38)$$

$$F_{T,4} = i N_c m_\rho \tilde{F}_{T,4}, \quad F_{T,5} = i N_c m_\rho^{-1} \tilde{F}_{T,5}. \quad (39)$$

The electric-monopole, magnetic-dipole, and electric-quadrupole form factors are associated with a set of irreducible tensors basis [36], and their relations with the vector form factors defined herein are

$$\begin{aligned}
G_E(Q^2) &= F_{V,1}(Q^2) + \frac{2}{3} \frac{Q^2}{4m_\rho^2} G_Q(Q^2), \\
G_M(Q^2) &= F_{V,3}(Q^2), \\
G_Q(Q^2) &= F_{V,1}(Q^2) - F_{V,3}(Q^2) + F_{V,2}(Q^2) \left[1 + \frac{Q^2}{4m_\rho^2} \right].
\end{aligned} \tag{40}$$

The axial vector and tensor form factors could be similarly interpreted in analogy with the electromagnetic case.

B. Parton distribution functions

In addition to computing other form factors through Eq. (19), the proposed framework is also able to establish more explicit connections between the projected amplitudes and quantities such as the parton distribution functions (PDFs), thereby unifying the description of elastic and inelastic hadron structure within a single formulation.

The hadronic scale PDFs of the ρ meson are accessible through its generalized quark-quark correlation at zero momentum transfer

$$\Phi_\Gamma(x) = \frac{1}{2} N_c \text{tr} \int_l \delta(x, l) \Gamma \mathcal{M}(l, P, Q=0); \tag{41}$$

here P is the total momentum of ρ meson, $\delta(x, l) = n \cdot l - xn \cdot P$ with n being a lightlike vector $n^2 = 0$, and Γ is a Dirac matrix that determines different types of PDFs. Projecting Eq. (11) with $\delta(x, p) S(p) \Gamma S(p)$ and integrating over p , the PDF is given as follows:

$$N_c^{-1} \Phi_\Gamma(x) = \Phi_\Gamma^{ih}(x) + \Pi_{\Gamma a}(x) \mathcal{D}_{aa'} \tilde{\mathcal{A}}_{a'}, \tag{42}$$

where

$$\Phi_\Gamma^{ih}(x) = \mathcal{D}_{aa'} \mathcal{D}_{bb'} \tilde{g}_{a'}(P) \tilde{g}_{b'}(-P) \mathcal{T}_{\Gamma ab}(x), \tag{43}$$

with

$$\mathcal{T}_{\Gamma ab}(x) = \frac{1}{2} \text{tr} \int_p \delta(x, p) \Gamma S(p) \Lambda_a S(p-P) \Lambda_b S(p) \tag{44}$$

and

$$\Pi_{\Gamma a}(x) = \frac{1}{2} \text{tr} \int_p \delta(x, p) \Gamma S(p) \Lambda_a S(p). \tag{45}$$

Herein we focus on two types of unpolarized PDFs, the twist-2 $f(x)$ and twist-3 $e(x)$,

$$f(x) = \frac{1}{3} \sum_{\text{spin}} \Phi_{in}(x), \tag{46}$$

$$e(x) = \frac{1}{3} \sum_{\text{spin}} \frac{n \cdot P}{m_\rho} \Phi_I(x), \tag{47}$$

where the polarization tensor satisfies

$$\sum_{\text{spin}} \epsilon_\mu^*(P) \epsilon_\nu(P) = \delta_{\mu\nu} + \frac{P_\mu P_\nu}{m_\rho^2}. \tag{48}$$

Among the rich physical contents behind PDFs, it is beneficial to draw insights from their Mellin moments

$$\langle x^m \rangle^{e,f} = \int_0^1 x^m e(x), f(x) dx, \tag{49}$$

with $m = \{0, 1, 2, \dots\}$. After regularization, the moments are given as follows:

$$\begin{aligned}
\langle x^m \rangle^{e,f} &= \langle x^m \rangle_0^{e,f} + \theta(m-1) \langle x^m \rangle_1^{e,f} + \theta(m-2) \langle x^m \rangle_2^{e,f} \\
&\quad + \langle x^m \rangle_{\text{pole}}^{e,f}.
\end{aligned} \tag{50}$$

The first row corresponds to the intrinsic quark contribution $\Phi_\Gamma^{ih}(x)$, and the step function $\theta(x) = 1$ when $x \geq 0$; otherwise, $\theta(x) = 0$. The second row corresponds to the pole contribution and directly connects to the projected amplitudes, in particular the basis form factors. It is noted that $f(x)$ does not receive any pole contribution while $e(x)$ receives pole contributions from its first two moments, i.e.,

$$\langle x^m \rangle_{\text{pole}}^f = 0, \tag{51}$$

$$\langle x^m \rangle_{\text{pole}}^e = \delta_{m0} \langle x^0 \rangle_S^e + \delta_{m1} \langle x^1 \rangle_V^e. \tag{52}$$

The vanishing of pole contributions for $f(x)$ is a consequence of the CI, similar to the cases of the gravitational mass form factor [23], since no pole that corresponds to particles with higher spin can be produced in this model. The scalar form factor $F_{S,1}(0)$ and the vector form factor $F_{V,1}(0)$ contribute to the first and second moments of $e(x)$, establishing connections between elastic and inelastic processes. The explicit forms of the moments are given in Appendix B, with which the PDFs can be reconstructed (see, e.g., [24,37]).

C. Numerical results

Given the scope of this paper, we provide only a broad overview of the calculations, without delving into technical details.

The static properties obtained from the homogeneous equation as well as the magnetic and quadrupole moments of the ρ meson are listed in Table I. The parameter $\xi = 0$ recovers the original RL truncation, while the choice $\xi = 0.151$ is fixed under present parameters to effectively compensate the chiral anomaly loss due to the finite cutoff effect [25,28]. As shown in Table I, the parameter ξ also lowers the mass and enhances $f_\rho^\perp$, while f_ρ remains nearly unchanged. It is noted that the results are partially different

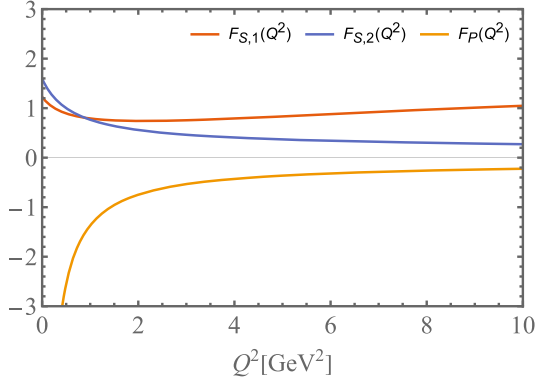


FIG. 1. Scalar and pseudoscalar form factors. $F_{S,\{1,2\}}(Q^2)$ possesses a scalar meson pole while $F_P(Q^2)$ possesses a pseudoscalar meson pole.

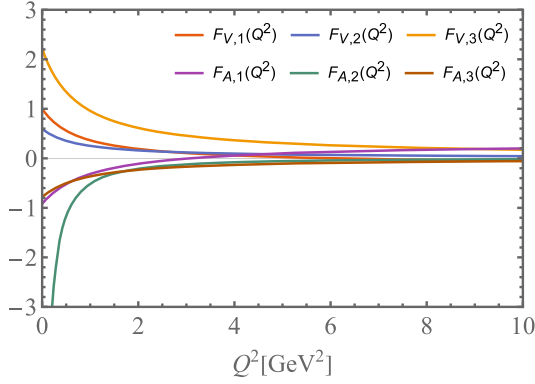


FIG. 2. Vector and axial vector form factors. $F_{V,\{1,2,3\}}(Q^2)$ possesses a vector meson pole while $F_{A,\{1,2,3\}}(Q^2)$ possesses an axial vector meson pole. The form factor $F_{A,2}(Q^2)$ possesses an additional pseudoscalar meson pole.

from that in [20] (up to a factor of $\sqrt{2}$ for $f_\rho^\perp$ and f_ρ due to definition). The reasons lie in two aspects: 1. the present work employs a new regularization procedure that automatically fulfills symmetry constraints [22], and thus some regularized expressions are different from that in [20]; 2. the parameter ξ is chosen to reproduce the experimental value $m_\rho = 0.77$ GeV in [20], and the same mass can be produced with $\xi = 0.38$ under the current regularization procedure and parameters.

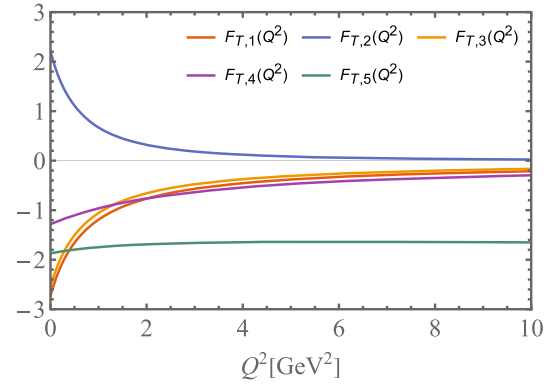


FIG. 3. Tensor form factors. $F_{T,\{1,2,3\}}(Q^2)$ possesses vector meson poles while $F_{T,\{4,5\}}(Q^2)$ does not possess any meson pole.

For the form factors, we choose to present them uniformly under $\xi = 0.151$ because of the anomaly related pseudoscalar form factor and axial vector form factor. The computed scalar, pseudoscalar, vector, axial vector, and tensor form factors, which are obtained by solving the self-consistent, closed system of equations, are displayed in Figs. 1 (scalar and pseudoscalar), 2 (vector/electromagnetic and axial vector), and 3 (tensor), respectively. For clarity of the figure, we choose not to show the timelike region but instead describe the associate poles with corresponding form factors in the captions of figures, which could essentially affect the slope of the form factors at $Q^2 = 0$. While in the ultraviolet region, the form factors demonstrate relative stiffness, which are typical features of the CI model. Note that the scalar form factor $F_{S,1}(Q^2)$ exhibits nonmonotonic behavior; similar behavior is also found in the pion scalar form factors. These features—relative stiffness in the ultraviolet and meson poles in the timelike region—are characteristic of the contact interaction model and are consistent with earlier studies of vector mesons within similar frameworks [20,27]. The behavior underscores the model's effective description of low-momentum dynamics while retaining a simple analytic structure. Finally, a few moments are presented in Table II. The zeroth moments of $f(x)$ and $e(x)$ equal $F_{V,1}(0)$ and $F_{S,1}(0)/2$, respectively, consistent with their definitions. While $\langle x^1 \rangle^f = 0.5$ is consistent with the picture under isospin symmetry, the constituent quark and antiquark each carries half the ρ meson total momentum. Further investigation of these PDFs and their moments,

TABLE II. First few moments of the ρ meson PDFs under $\xi = 0$ (upper) and $\xi = 0.151$ (lower).

m	0	1	2	3	4	5	6	7
$\langle x^m \rangle^f$	1	0.5	0.290	0.185	0.127	0.092	0.070	0.055
$\langle x^m \rangle^e$	0.716	-0.018	0.066	0.025	0.005	-0.004	-0.010	-0.013
$\langle x^m \rangle^f$	1	0.5	0.294	0.190	0.133	0.098	0.075	0.060
$\langle x^m \rangle^e$	0.613	-0.100	0.008	-0.024	-0.038	-0.045	-0.048	-0.049

such as the appearance of negative moments for $e(x)$ and the analysis of $e(x)$ in terms of QCD equations of motion, is beyond the scope of the current work and will be studied in the future [38,39].

IV. SUMMARY AND OUTLOOK

In summary, we have developed and demonstrated a novel, to the best of our knowledge, algebraic paradigm for computing hadronic observables directly within the framework of the CI model and the Dyson-Schwinger equations. Moving beyond the conventional multistep procedure—which requires sequential solutions for the quark propagator, the bound-state wave function, and finally the observables—we have systematically employed Fierz transformations and recast the integral equations into a closed set of algebraic equations that govern a finite number of projected amplitudes, $g_a^H(P)$ and $\mathcal{A}_a^H(P, Q)$.

The core advantage of this method is that it completely bypasses the explicit calculation of the bound-state wave function, establishing a direct and efficient mapping from the model interaction to physical observables. Using the ρ meson as a detailed case study, we have illustrated the full application of this paradigm: deriving the specific tensor structures of the projected amplitudes, identifying the complete set of basis form factors, and demonstrating the extraction of physical quantities such as decay constants (f_ρ , $f_\rho^\perp$), various form factors. The advantage of the proposed framework is further illustrated through the calculation of ρ meson PDFs, where the relations between the basis form factors and corresponding PDFs are explicitly shown according to the natural decompositions of PDFs into intrinsic quark contributions and pole contributions.

Finally, apart from the accessibility to study meson systems in a CI model, this approach is also possible to be generalized to three-body bound states and noncontact interactions, but the generalization is not straightforward and additional efforts are required to achieve this goal. The former opens a promising route for the direct calculation of baryon properties—such as nucleon masses, form factors, and partonic distributions—within the same streamlined framework in CI. However, the description for baryons is much more complicated compared with that for mesons [40,41]. The generalization to the latter is even more difficult, as mentioned earlier in the manuscript, and one will encounter infinitely many basis projection amplitudes under realistic interactions due to gluon momentum dependence. In practice, it is impossible to apply this approach under such interactions unless reasonable approximations or truncations are made.

ACKNOWLEDGMENTS

This work is supported by National Natural Science Foundation of China, Grant No. 12135007; and

Postdoctoral Fellowship Program of CPSF under Grant No. GZC20240759.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: $\Pi_{ba}(Q)$ AND $\mathcal{T}_{cab}(P, Q)$ IN CI

When evaluating the function $\Pi_{ba}(Q)$ and $\mathcal{T}_{cab}(P, Q)$ in CI, a proper regularization procedure is required. We use the symmetry preserving regularization scheme proposed in [22]. The final expressions are expressed in terms of the so called irreducible loop integrals [42]

$$I_{-2\alpha}(s) := \int_q \frac{1}{(q^2 + s)^{\alpha+2}} \rightarrow \int_{\tau_{uv}}^{\tau_{ir}} d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+2)} \frac{e^{-\tau s}}{16\pi^2}. \quad (\text{A1})$$

The arrow denotes its form under proper time regularization. Special care should be taken for the regularization of chiral trace (trace that contains an odd number of γ_5), and the procedure is described in [28].

The nonzero $\Pi_{ba}(Q)$ is given as follows:

$$\Pi_{SS}(Q) = -4f_1(Q^2), \quad (\text{A2})$$

$$\Pi_{PP}(Q) = 4f_2(Q^2), \quad (\text{A3})$$

$$\Pi_{PA}(Q) = 4if_3(Q^2)M Q_{\mu_a}, \quad (\text{A4})$$

$$\Pi_{VV}(Q) = 8f_4(Q^2)(Q^2\delta_{\mu_a\mu_b} - Q_{\mu_a}Q_{\mu_b}), \quad (\text{A5})$$

$$\Pi_{VT}(Q) = 4f_3(Q^2)M(Q_{\mu_a}\delta_{\nu_a\mu_b} - Q_{\nu_a}\delta_{\mu_a\mu_b}), \quad (\text{A6})$$

$$\Pi_{AP}(Q) = -4if_3(Q^2)M Q_{\mu_b}, \quad (\text{A7})$$

$$\Pi_{AA}(Q) = 8f_5(Q^2)\delta_{\mu_a\mu_b} - 8f_4(Q^2)Q_{\mu_a}Q_{\mu_b}, \quad (\text{A8})$$

$$\Pi_{TV}(Q) = -4f_3(Q^2)M(Q_{\mu_b}\delta_{\nu_b\mu_a} - Q_{\nu_b}\delta_{\mu_b\mu_a}), \quad (\text{A9})$$

$$\begin{aligned} \Pi_{TT}(Q) = & 4f_6(Q^2)(\delta_{\mu_a\mu_b}\delta_{\nu_a\nu_b} - \delta_{\mu_a\nu_b}\delta_{\mu_b\nu_a}) \\ & + 8f_4(Q^2)(Q_{\mu_a}Q_{\mu_b}\delta_{\nu_a\nu_b} + Q_{\nu_a}Q_{\nu_b}\delta_{\mu_a\mu_b}) \\ & - 8f_4(Q^2)(Q_{\mu_a}Q_{\nu_b}\delta_{\mu_b\nu_a} + Q_{\mu_b}Q_{\nu_a}\delta_{\mu_a\nu_b}). \end{aligned} \quad (\text{A10})$$

The transverse projector is $\delta_{\mu\nu}^T = Q_\mu Q_\nu - \delta_{\mu\nu}Q^2$, where (with $\omega = M^2 + u\bar{u}Q^2$ and $\bar{u} = 1 - u$)

$$f_1(Q^2) = \int_0^1 I_2(\omega) - 2\omega I_0(\omega) du, \quad (\text{A11})$$

$$f_2(Q^2) = \int_0^1 I_2(\omega) - 2u\bar{u}Q^2 I_0(\omega) du, \quad (\text{A12})$$

$$f_3(Q^2) = \int_0^1 I_0(\omega) d\omega, \quad (\text{A13})$$

$$f_4(Q^2) = - \int_0^1 u \bar{u} I_0(\omega) d\omega, \quad (\text{A14})$$

$$f_5(Q^2) = - \int_0^1 \omega I_0(\omega) d\omega, \quad (\text{A15})$$

$$f_6(Q^2) = \int_0^1 I_2(\omega) + 2\omega I_0(\omega) d\omega. \quad (\text{A16})$$

Regarding $\mathcal{T}_{cab}$, although it is not difficult to compute the full $5 \times 5 \times 5$ amplitudes, for a specific physical process, one does not need to compute the complete amplitudes. For example, consider the vector form factors of the ρ meson under RL truncation. Since $\xi = 0$, $\tilde{g}_T$ is no longer basis amplitudes, one only needs to consider the two point function Π_{VV} and three point function $\mathcal{T}_{VVV}$. The equation for the vector form factors decouples from the complete equation and reads

$$\tilde{\mathcal{A}}_V(P, Q) = \tilde{\mathcal{A}}_V^i(P, Q) + \Pi_{VV}(-Q) \mathcal{D}_{VV} \tilde{\mathcal{A}}_V(P, Q) \quad (\text{A17})$$

with the driving term

$$\tilde{\mathcal{A}}_V^{ih}(P, Q) = \mathcal{D}_{VV}^2 \tilde{g}_V(P_i) \tilde{g}_V(-P_f) \mathcal{T}_{VVV}(P, Q). \quad (\text{A18})$$

The general form of the driving term is the same as Eq. (26) with the replacement $\tilde{F}_{V,i} \rightarrow \tilde{F}_{V,i}^{ih}$. The equation is further decoupled according to the tensor structures, and one obtains three independent equations for the scalar functions

$$\tilde{F}_{V,i}(Q^2) = \tilde{F}_{V,i}^{ih}(Q^2) + 4D_G f_4(Q^2) Q^2 \tilde{F}_{V,i}, \quad (\text{A19})$$

where the regularized $\tilde{F}_{V,i}^{ih}$ is

$$\begin{aligned} \tilde{F}_{V,1}^{ih} = & i\tilde{f}_V^2 D_G^2 \int_{\Omega} \left(-2(u_1 + u_2 - 1)(u_1 + u_2) \right. \\ & \times (2u_1 + 2u_2 - 3) \text{I}_{-2}(\Delta) m_{\rho}^2 \\ & \left. - (u_1 + u_2 - 2)(4u_1 u_2 Q^2 \text{I}_{-2}(\Delta) - \text{I}_0(\Delta)) \right), \quad (\text{A20}) \end{aligned}$$

$$\tilde{F}_{V,2}^{ih} = i\tilde{f}_V^2 D_G^2 \int_{\Omega} -32u_1 u_2 (u_1 + u_2 - 1) \text{I}_{-2}(\Delta), \quad (\text{A21})$$

$$\begin{aligned} \tilde{F}_{V,3}^{ih} = & i\tilde{f}_V^2 D_G^2 \int_{\Omega} \left((u_1 + u_2 + 2) \text{I}_0(\Delta) \right. \\ & - 2((u_1 + u_2 - 1)(u_1 + u_2)(2u_1 + 2u_2 + 3) m_{\rho}^2 \\ & \left. + 2u_1 u_2 (u_1 + u_2 + 1) Q^2) \text{I}_{-2}(\Delta) \right), \quad (\text{A22}) \end{aligned}$$

where $\Delta = (u_1 + u_2 - 1)(u_1 + u_2) m_{\rho}^2 + M^2 + Q^2 u_1 u_2$ and $\int_{\Omega} = \int_0^1 du_1 \int_0^{1-u_1} du_2$.

APPENDIX B: PDF MOMENTS

The components $\langle x^m \rangle_{0,1,2}^{e,f}$ in Eq. (50) are given as follows: following the regularization and recalling that $n^2 = 0$, one obtains

$$2N_c^{-1} \langle x^m \rangle_i^{e,f} = D_G^2 \int_0^1 du \mathcal{V}_1 \cdot \mathcal{C}_i^{e,f} \cdot \mathcal{V}_2^T, \quad (\text{B1})$$

where T denotes transpose, $\mathcal{V}_1 = (\tilde{f}_V^2, \xi \tilde{f}_T \tilde{f}_V, \xi^2 \tilde{f}_T^2)$, $\mathcal{V}_2 = (\text{I}_{-2}(\hat{\omega}), \text{I}_0(\hat{\omega}), \text{I}_2(\hat{\omega}))$, and $\hat{\omega} = \omega|_{Q^2 \rightarrow -m_{\rho}^2}$. The explicit forms of the coefficient matrices $\mathcal{C}_i^{e,f}$ are given by

$$\mathcal{C}_0^f = u^m \begin{pmatrix} 2u(2u+1)\bar{u}^2 m_{\rho}^2 & (u+1)\bar{u} & 0 \\ \frac{16}{3} M(3-2u)u\bar{u} m_{\rho}^2 & \frac{16M\bar{u}}{3} & 0 \\ \frac{32}{9} u\bar{u} m_{\rho}^2 (\bar{u}(2\bar{u}-1)m_{\rho}^2 + 4M^2) & \frac{16}{9} \bar{u}^2 m_{\rho}^2 & 0 \end{pmatrix}, \quad (\text{B2})$$

$$\mathcal{C}_1^f = m u^{m-1} \begin{pmatrix} 0 & \frac{1}{3} u(u+1)\bar{u} & 0 \\ 0 & \frac{8}{3} M u \bar{u} & 0 \\ 0 & -\frac{16}{27} (u-2)u\bar{u} m_{\rho}^2 & 0 \end{pmatrix}, \quad (\text{B3})$$

$$\mathcal{C}_2^e = \frac{m(m-1)}{2} u^{m-2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{B4})$$

$$\mathcal{C}_0^e = u^m \begin{pmatrix} 2M(3-2u)u\bar{u} m_{\rho} & 0 & 0 \\ \frac{16}{3} \bar{u} m_{\rho} (u(2u-1)\bar{u} m_{\rho}^2 + 2M^2) & \frac{8}{3} (1-3\bar{u})\bar{u} m_{\rho} & 0 \\ \frac{32}{9} M \bar{u} m_{\rho} (u(2u-1)m_{\rho}^2 + 4M^2) & -\frac{32}{9} M \bar{u} m_{\rho} & 0 \end{pmatrix}, \quad (\text{B5})$$

$$\mathcal{C}_1^e = m u^{m-1} \begin{pmatrix} 0 & -\frac{M\bar{u}^2}{m_{\rho}} & 0 \\ 0 & \frac{4}{3} u(4u-3)\bar{u} m_{\rho} & -\frac{8\bar{u}}{3m_{\rho}} \\ 0 & -\frac{16}{9} M \bar{u}^2 m_{\rho} & 0 \end{pmatrix}, \quad (\text{B6})$$

$$C_2^e = \frac{m(m-1)}{2} u^{m-2} \begin{pmatrix} 0 & 0 & -\frac{2M\bar{u}}{3m_p^3} \\ 0 & 0 & -\frac{8u\bar{u}}{3m_p} \\ 0 & 0 & -\frac{64M\bar{u}}{27m_p} \end{pmatrix}. \quad (\text{B7})$$

The pole contributions to the twist-3 PDF moments are

$$2N_c^{-1} \langle x^0 \rangle_S^e = \frac{4D_G \tilde{F}_{S,1}(0)}{m_p} (I_2(M^2) - 2M^2 I_0(M^2)), \quad (\text{B8})$$

$$2N_c^{-1} \langle x^1 \rangle_V^e = -\frac{2iD_G \tilde{F}_{V,1}(0)M}{m_p} I_2(M^2). \quad (\text{B9})$$

-
- [1] F. Gross *et al.*, *Eur. Phys. J. C* **83**, 1125 (2023).
[2] C. D. Roberts and A. G. Williams, *Prog. Part. Nucl. Phys.* **33**, 477 (1994).
[3] R. Alkofer and L. von Smekal, *Phys. Rep.* **353**, 281 (2001).
[4] A. Bashir, L. Chang, I. C. Cloet, B. El-Bennich, Y.-X. Liu, C. D. Roberts, and P. C. Tandy, *Commun. Theor. Phys.* **58**, 79 (2012).
[5] C. S. Fischer, *Prog. Part. Nucl. Phys.* **105**, 1 (2019).
[6] C. D. Roberts, D. G. Richards, T. Horn, and L. Chang, *Prog. Part. Nucl. Phys.* **120**, 103883 (2021).
[7] G. Eichmann, *arXiv:2503.10397*.
[8] L. X. Gutierrez-Guerrero, A. Bashir, I. C. Cloet, and C. D. Roberts, *Phys. Rev. C* **81**, 065202 (2010).
[9] H. L. L. Roberts, C. D. Roberts, A. Bashir, L. X. Gutierrez-Guerrero, and P. C. Tandy, *Phys. Rev. C* **82**, 065202 (2010).
[10] H. L. L. Roberts, L. Chang, I. C. Cloet, and C. D. Roberts, *Few Body Syst.* **51**, 1 (2011).
[11] H. L. L. Roberts, A. Bashir, L. X. Gutierrez-Guerrero, C. D. Roberts, and D. J. Wilson, *Phys. Rev. C* **83**, 065206 (2011).
[12] S.-S. Xu, C. Chen, I. C. Cloet, C. D. Roberts, J. Segovia, and H.-S. Zong, *Phys. Rev. D* **92**, 114034 (2015).
[13] K. Raya, M. A. Bedolla, J. J. Cobos-Martínez, and A. Bashir, *Few Body Syst.* **59**, 133 (2018).
[14] F. E. Serna, B. El-Bennich, and G. Krein, *Phys. Rev. D* **96**, 014013 (2017).
[15] L. X. Gutiérrez-Guerrero, A. Bashir, M. A. Bedolla, and E. Santopinto, *Phys. Rev. D* **100**, 114032 (2019).
[16] P.-L. Yin, C. Chen, G. Krein, C. D. Roberts, J. Segovia, and S.-S. Xu, *Phys. Rev. D* **100**, 034008 (2019).
[17] J.-L. Zhang, Z.-F. Cui, J. Ping, and C. D. Roberts, *Eur. Phys. J. C* **81**, 6 (2021).
[18] Z.-N. Xu, Z.-F. Cui, C. D. Roberts, and C. Xu, *Eur. Phys. J. C* **81**, 1105 (2021).
[19] H.-Y. Xing, Z.-N. Xu, Z.-F. Cui, C. D. Roberts, and C. Xu, *Eur. Phys. J. C* **82**, 889 (2022).
[20] Z. Xing, K. Raya, and L. Chang, *Phys. Rev. D* **104**, 054038 (2021).
[21] X. Wang, Z. Xing, J. Kang, K. Raya, and L. Chang, *Phys. Rev. D* **106**, 054016 (2022).
[22] Z. Xing and L. Chang, *Phys. Rev. D* **107**, 014019 (2023).
[23] Z. Xing, M. Ding, and L. Chang, *Phys. Rev. D* **107**, L031502 (2023).
[24] Z. Xing, M. Ding, K. Raya, and L. Chang, *Eur. Phys. J. A* **60**, 33 (2024).
[25] H. Dang, Z. Xing, M. A. Sultan, K. Raya, and L. Chang, *Phys. Rev. D* **108**, 054031 (2023).
[26] G. Paredes-Torres, L. X. Gutiérrez-Guerrero, A. Bashir, and Á. S. Miramontes, *Phys. Rev. D* **109**, 114006 (2024).
[27] R. J. Hernández-Pinto, L. X. Gutiérrez-Guerrero, M. A. Bedolla, and A. Bashir, *Phys. Rev. D* **110**, 114015 (2024).
[28] Z. Xing, H. Dang, M. A. Sultan, K. Raya, and L. Chang, *Phys. Rev. D* **109**, 054028 (2024).
[29] M. A. Sultan, Z. Xing, K. Raya, A. Bashir, and L. Chang, *Phys. Rev. D* **110**, 054034 (2024).
[30] Y. Xu, M. A. Sultan, K. Raya, and L. Chang, *Phys. Rev. D* **110**, 094036 (2024).
[31] C. H. Llewellyn-Smith, *Ann. Phys. (N.Y.)* **53**, 521 (1969).
[32] H. J. Munczek and P. Jain, *Phys. Rev. D* **46**, 438 (1992).
[33] P. C. Tandy, *Prog. Part. Nucl. Phys.* **39**, 117 (1997).
[34] N. Nakanishi, *Prog. Theor. Phys. Suppl.* **43**, 1 (1969).
[35] S. R. Cotanch and P. Maris, *Phys. Rev. D* **66**, 116010 (2002).
[36] R. G. Arnold, C. E. Carlson, and F. Gross, *Phys. Rev. C* **21**, 1426 (1980).
[37] C. Mezrag, H. Moutarde, J. Rodríguez-Quintero, and F. Sabatié, *arXiv:1406.7425*.
[38] C. Shi, L. Lu, I. Cloët, W. Jia, and P. Tandy, *arXiv:2602.14164*.

-
- [39] A. V. Efremov and P. Schweitzer, *J. High Energy Phys.* **08** (2003) 006.
- [40] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, and C. S. Fischer, *Prog. Part. Nucl. Phys.* **91**, 1 (2016).
- [41] X.-Y. Bai, Y. Lu, Z.-Q. Yao, C.D. Roberts, and S.M. Schmidt, [arXiv:2602.02880](#).
- [42] Y.-L. Wu, *Int. J. Mod. Phys. A* **18**, 5363 (2003).