

Role of electromagnetic corrections in the $\pi\pi$ distributions of $\psi' \rightarrow J/\psi\pi\pi$

Zhao-Sai Jia^{1,2}, Gang Li^{1,*}, and Zhen-Hua Zhang^{3,†}

¹*College of Physics and Engineering, Qufu Normal University, Qufu 273165, China*

²*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

³*Center for High Energy Physics, Peking University, Beijing 100871, China*



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The cusp structure at the $\pi^+\pi^-$ threshold in the $\pi^0\pi^0$ invariant mass spectrum serves as a sensitive probe for extracting the S -wave $\pi\pi$ scattering lengths in processes where an S -wave $\pi^0\pi^0$ pair is produced in the final states. Within the framework of nonrelativistic effective field theory with coupled channels $\pi^0\pi^0$ and $\pi^+\pi^-$, we revisit the near-threshold structures in the $\pi^0\pi^0$ spectrum of $\psi' \rightarrow J/\psi\pi\pi$. Our analysis incorporates the $\pi\pi$ final-state rescattering, including both strong and Coulomb interactions. It turns out that the cusp near the $\pi^+\pi^-$ threshold becomes more prominent when Coulomb interactions are included. The electromagnetic corrections are found to alter the magnitude of the threshold cusp by about 2%–3%, underscoring the necessity of including these effects in precision determinations of the $\pi\pi$ scattering lengths. The coupled-channel amplitude constructed in this work provides a ready-to-use theoretical framework for experimental analyses of fine structures near $\pi\pi$ thresholds.

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I. INTRODUCTION

As the lightest quantum chromodynamics (QCD) bound states and the pseudo-Nambu-Goldstone bosons of spontaneous chiral symmetry breaking, pions play an essential role in understanding low-energy strong interaction dynamics. At low energies, the $\pi\pi$ scattering lengths are key to quantifying the strength of the $\pi\pi S$ -wave interaction, which can shed light on essential information about chiral symmetry breaking. The $\pi\pi$ scattering lengths have been extensively studied both theoretically and experimentally. Theoretically, the isospin $I = 0, 2$ scattering lengths and their difference were predicted with percent-level accuracy in units of the inverse pion mass M_π^{-1} [1,2], $a_0 M_{\pi^+} = 0.220 \pm 0.005$, $a_2 M_{\pi^+} = -0.0444 \pm 0.0010$, and $(a_0 - a_2) M_{\pi^+} = 0.265 \pm 0.004$, by combining chiral perturbation theory (ChPT) at the two-loop level [3,4] and the Roy equations [5], consistent with dispersion relation approaches without input from ChPT [6].

Experimentally, the $\pi\pi$ scattering lengths is extracted mainly through three approaches. First, using Watson's theorem and numerical solutions of the Roy equations [7,8], it is possible to determine the $\pi\pi$ scattering lengths

from the angular distributions of K_{e4} decay. Experiments along these lines have been carried out in, e.g., Refs. [9–12]. Second, the lifetime of pionium (the $\pi^+\pi^-$ hadronic atom) that decays into two neutral pions is related to $|a_0 - a_2|$ [13]. The experimental results reported by the DIRAC Collaboration are well consistent with the ChPT prediction [14]. Third, the cusp effect in decay processes with at least two neutral pions in the final states, such as $K^\pm \rightarrow \pi^0\pi^0\pi^\pm$ [15–23], $K_L \rightarrow 3\pi$ [24], $\eta \rightarrow 3\pi$ [25], and $\eta' \rightarrow \eta\pi\pi$ [26], has emerged as a precise tool for extracting $\pi\pi$ scattering lengths, which results from the charge exchange rescattering $\pi^+\pi^- \rightarrow \pi^0\pi^0$ [27]. The cusp effect is prominently manifested in the $\pi^0\pi^0$ invariant mass spectrum of the decay $K^\pm \rightarrow \pi^0\pi^0\pi^\pm$. The branching ratio difference $\mathcal{B}[K^\pm \rightarrow \pi^\pm\pi^\mp\pi^\pm]/\mathcal{B}[K^\pm \rightarrow \pi^0\pi^0\pi^\pm] = (5.583 \pm 0.024)\%/(1.760 \pm 0.023)\%$ [28] requires significant charge exchange rescattering contributions. The complexity of final-state interactions in the aforementioned decay processes, together with the reconstruction of the η , which decays into 2γ or $3\pi^0$, in η decay processes, constitute challenges for studying the cusp effect in η' decays. Combining both the K_{e4} [12] and $K \rightarrow 3\pi$ [20] measurements, the NA48/2 experiment reported $a_0^0 M_{\pi^+} = 0.2210 \pm 0.0047_{\text{stat}} \pm 0.0040_{\text{syst}}$, $a_2^0 M_{\pi^+} = -0.0429 \pm 0.0044_{\text{stat}} \pm 0.0028_{\text{syst}}$, and $(a_0^0 - a_2^0) M_{\pi^+} = 0.2639 \pm 0.0020_{\text{stat}} \pm 0.0015_{\text{syst}}$ [12].

In this regard, heavy quarkonium dipion transitions have also been considered for extracting the $\pi\pi S$ -wave phase shifts [29–31] and scattering lengths [32]. In these processes, the interaction between the final-state quarkonium

*Contact author: gli@qfnu.edu.cn

†Contact author: zhhzhang@pku.edu.cn

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and pions is highly suppressed by the Okubo-Zweig-Iizuka rule, and, thus, their contributions can be neglected, as indicated by the lattice result for the J/ψ scattering length $a_{J/\psi\pi} = (-0.01 \pm 0.01)$ fm [33]. The dipion transitions also exhibit sizable branching fractions of heavy quarkonium decays. For example, the branching fraction of $\psi' \rightarrow J/\psi\pi^0\pi^0$ is $(18.2 \pm 0.5)\%$ [28]. One also has $\mathcal{B}(\psi' \rightarrow J/\psi\pi^+\pi^-)/\mathcal{B}(\psi' \rightarrow J/\psi\pi^0\pi^0) \approx 2$ [28], due to the approximate isospin symmetry. The possibility of extracting the $\pi\pi$ scattering lengths using the cusp effect in the heavy quarkonium dipion transition $\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0$ was studied in Ref. [32] within the nonrelativistic effective field theory (NREFT) framework [22,26]. Using Monte Carlo (MC) simulation, the statistical precision of $a_0 - a_2$ was estimated to reach 2% and 1.5% with 2×10^7 and 4×10^7 events of $\Upsilon(3S) \rightarrow \Upsilon(2S)\pi^0\pi^0$, respectively. However, the Coulomb interaction was not included in the framework of Ref. [32]. The Coulomb interactions between π^+ and π^- have been incorporated in the $K^\pm \rightarrow \pi^0\pi^0\pi^\pm$ process by the NA48/2 Collaboration [20,34], where the $\pi\pi$ scattering lengths were defined to be proportional to the $\pi\pi$ scattering amplitude at the threshold in the isospin limit. Radiative corrections can reduce $a_0 - a_2$ and a_0 by about 9% and 6%, respectively. Therefore, the Coulomb interactions could also give non-negligible contributions in heavy quarkonium dipion transitions.

Huge data samples have already been accumulated at the BESIII experiment in the $\psi' \rightarrow J/\psi\pi^0\pi^0$ channel. In particular, a sample of 2.7×10^9 ψ' events has been collected at the BESIII experiment [35], and 6.4×10^{11} ψ' events can be collected with one year of running at the Super Tau-Charm Facility (STCF) [36] under consideration. It is, therefore, of great interest to have an accurate framework for analyzing the near-threshold $\pi\pi$ invariant mass distributions of $\psi' \rightarrow J/\psi\pi\pi$. This is the goal of this work, with an emphasis on investigating the cusp effect in the $\pi^0\pi^0$ invariant mass distribution of $\psi' \rightarrow J/\psi\pi^0\pi^0$ within the NREFT framework. The $\pi\pi$ final-state interactions (FSIs) are included along with the Coulomb interactions between the $\pi^+\pi^-$ pair, while the $J/\psi\pi$ FSI will be neglected. Using inputs from the $\pi^0\pi^0$ scattering lengths, the energy level and lifetime of the ground-state pionium, and the BESII and ATLAS data [37,38] of the $\pi^+\pi^-$ spectrum for $\psi' \rightarrow J/\psi\pi^0\pi^0$, we predict the near-threshold line shapes of the $\pi^0\pi^0$ and $\pi^+\pi^-$ invariant mass distributions. We also perform a MC simulation to provide a quantitative estimation of the effects of electromagnetic corrections on the experimental extraction of the $\pi\pi$ scattering lengths.

The structure of the paper is as follows: In Sec. II, we derive expressions for the $\psi' \rightarrow J/\psi\pi\pi$ partial decay width within the coupled-channel NREFT. In Sec. IV, we generate synthetic data using MC simulation to estimate the precision required for the extraction of scattering lengths when

electromagnetic corrections should be considered. In Sec. III, the near-threshold line shapes of the $\pi\pi$ invariant mass distribution are predicted for the $\psi' \rightarrow J/\psi\pi\pi$ process. A brief summary is given in Sec. V.

II. NONRELATIVISTIC EFFECTIVE FIELD THEORY

In this section, we present the amplitudes and differential decay widths of $\psi' \rightarrow J/\psi\pi\pi$ near the $\pi^+\pi^-$ threshold. In the following, the $\pi^0\pi^0$ and $\pi^+\pi^-$ channels are designated as channels 1 and 2, respectively.

A. $\pi\pi$ S-wave scattering amplitude

The energy dependence of the differential decay widths near the threshold is predominantly determined by the $\pi\pi$ final-state interaction amplitudes, which receive contributions from both Coulomb and strong interactions. In this work, we focus on S -wave $\pi\pi$ scattering, since the near-threshold D -wave amplitude is suppressed by a factor of v_π^2 relative to the S -wave amplitude, where v_π is the pion velocity in the center-of-mass (c.m.) frame. The S -wave two-channel scattering T matrix can be expressed in the two-potential formalism as [39–41]

$$T(E) = \begin{pmatrix} 0 & 0 \\ 0 & T_C(E) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & W_C(E) \end{pmatrix} \times T_{SC}(E) \begin{pmatrix} 1 & 0 \\ 0 & W_C(E) \end{pmatrix}, \quad (1)$$

where $E = \sqrt{s}$ is the total energy in the $\pi\pi$ c.m. frame,

$$T_C(E) = \frac{i\pi}{\mu_c k_c} \left(\frac{\Gamma(1 - ix)}{\Gamma(1 + ix)} - 1 \right) \quad (2)$$

is the $\pi^+\pi^-$ Coulomb scattering amplitude including infinite Coulomb photon exchanges as shown in Fig. 1, with $\mu_c = M_{\pi^+}/2$ the $\pi^+\pi^-$ reduced mass, $k_c = \sqrt{2\mu_c(E - 2M_{\pi^+})}$ the c.m. momentum of π^+ , $x = \alpha\mu_c/k_c$, and $\alpha = 1/137$ the fine-structure constant; and

$$W_C(E) = \left(\frac{2\pi x}{1 - e^{-2\pi x}} \frac{\Gamma(1 - ix)}{\Gamma(1 + ix)} \right)^{1/2} \quad (3)$$

accounts for the Coulomb photon exchanges between the $\pi^+\pi^-$ pair in the initial or final states. T_{SC} is the strong-Coulomb scattering amplitude. For the S -wave $\pi\pi$ near-threshold scattering, at leading order (LO) in NREFT, there are only constant contact terms for the strong interaction-induced pion-pion scattering potential, and T_{SC} satisfies the Lippmann-Schwinger equation (LSE):

$$T_{SC}(E) = [I - V_S(\Lambda)G_C(E, \Lambda)]^{-1}V_S(\Lambda), \quad (4)$$

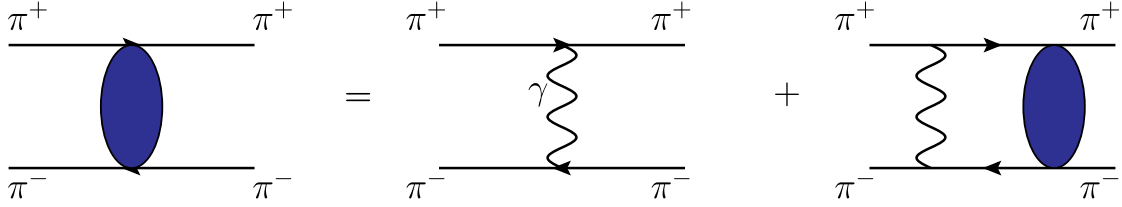


FIG. 1. Resummation in the Coulomb T matrix for the $\pi^+\pi^-$ scattering. The wavy line represents a photon, and the blue blob represents resummation of the Coulomb photon exchanges.

where V_S is the strong potential and $\mathbf{G}_C(E, \Lambda) = \text{diag}(G_{C11}, G_{C22})$ is the Green's function regularized by a sharp cutoff Λ , with nonvanishing diagonal matrix elements [39,42,43]

$$G_{C11}(E, \Lambda) = -\frac{\mu_n \Lambda}{\pi^2} - i \frac{\mu_n}{2\pi} k_n(E), \quad (5)$$

$$G_{C22}(E, \Lambda) = -\frac{\mu_c \Lambda}{\pi^2} - \frac{\alpha \mu_c^2}{\pi} \left(\ln \frac{\Lambda}{\alpha \mu_c} - \gamma_E \right) - \frac{\mu_c}{2\pi} \kappa_c(E). \quad (6)$$

Here, $\mu_n = M_{\pi^0}/2$ is the $\pi^0\pi^0$ reduced mass, γ_E is the Euler constant, $k_n = \sqrt{2\mu_n(E - 2M_{\pi^0})}$ is the c.m. momentum of π^0 , and $\kappa_c = 2\alpha\mu_c[\ln(ix) + 1/(2ix) - \psi(-ix)]$, with $\psi(x)$ being the digamma function.

Since the strong-Coulomb scattering amplitude $T_{SC}(E)$ is independent of the cutoff Λ , the strong potential V_S in the LSE must be Λ dependent to absorb the Λ dependence in $\mathbf{G}_C(E, \Lambda)$. After renormalization, $T_{SC}(E)$ can be expressed in terms of cutoff-free parameters [41,42]:

$$\begin{aligned} T_{SC}^{-1}(E) &= (\mathbf{V}_S^R)^{-1} - \mathbf{G}_C^R(E) \\ &= (\mathbf{T}_{SC}^{\text{thr}})^{-1} - \tilde{\mathbf{G}}_C^R(E), \end{aligned} \quad (7)$$

where

$$\mathbf{G}_C^R(E) = \begin{pmatrix} \frac{-i\mu_n}{2\pi} k_n(E) & 0 \\ 0 & \frac{-\mu_c}{2\pi} \kappa_c(E) \end{pmatrix} \quad (8)$$

is the renormalized Green's function, $\tilde{\mathbf{G}}_C^R = \mathbf{G}_C^R(E) - \mathbf{G}_C^R(E = 2M_{\pi^0})$, and

$$\mathbf{T}_{SC}^{\text{thr}} = -2\pi\mu^{-\frac{1}{2}} \mathbf{a}_{NN,\text{eff}} \mu^{-\frac{1}{2}} \quad (9)$$

represents the T_{SC} at the $\pi^0\pi^0$ threshold expressed in terms of the $\pi\pi$ scattering lengths [44] with $\boldsymbol{\mu} = \text{diag}(\mu_n, \mu_c)$. Here,

$$\mathbf{a}_{NN,\text{eff}} = \begin{pmatrix} a_{11,\text{eff}} & a_{12,\text{eff}} \\ a_{12,\text{eff}} & a_{22,\text{eff}} \end{pmatrix} \quad (10)$$

denotes the scattering length matrix, and its matrix elements can be written in terms of the isospin $I = 0$ and $I = 2$ S -wave $\pi\pi$ scattering lengths a_0 and a_2 as

$$a_{11,\text{eff}} = \frac{2}{3M_{\pi^0}} (a_0 + 2a_2) M_{\pi^+} (1 - \eta), \quad (11)$$

$$a_{12,\text{eff}} = \frac{2}{3M_{\pi^0}} (a_0 - a_2) M_{\pi^+} \left(1 + \frac{\eta}{3} \right),$$

$$a_{22,\text{eff}} = \frac{1}{3M_{\pi^+}} (2a_0 + a_2) M_{\pi^+} (1 + \eta), \quad (12)$$

where $\eta = (M_{\pi^+}^2 - M_{\pi^0}^2)/M_{\pi^+}^2$ accounts for the isospin-breaking effect at LO in ChPT [22,32] and $(a_0 - a_2)M_{\pi^+} = 0.2640 \pm 0.004$, $a_2 M_{\pi^+} = -0.0444 \pm 0.0010$ [1,2,45].

B. $\pi\pi$ invariant mass distribution in $\psi' \rightarrow J/\psi\pi\pi$

The diagrams illustrating the $\psi' \rightarrow J/\psi\pi\pi$ processes are shown in Fig. 2. In the near-threshold region, the short-distance decay vertices for $\psi' \rightarrow J/\psi\pi\pi$ can be approximated as constants, and the decay amplitudes can be written as

$$\begin{aligned} \mathcal{M}[\psi' \rightarrow J/\psi\pi^0\pi^0] &= \mathbf{V}_n^\Lambda + \mathbf{V}_n^\Lambda \times \frac{1}{2} G_{C11}^\Lambda \times T_{SC11} \\ &\quad + \mathbf{V}_c^\Lambda \times G_{C22}^\Lambda \times T_{SC21}, \end{aligned} \quad (13)$$

$$\begin{aligned} \mathcal{M}[\psi' \rightarrow J/\psi\pi^+\pi^-] &= \sqrt{2} \left(\mathbf{V}_c^\Lambda + \mathbf{V}_c^\Lambda \times G_{C22}^\Lambda \times T_{SC22} \right. \\ &\quad \left. + \mathbf{V}_n^\Lambda \times \frac{1}{2} G_{C11}^\Lambda \times T_{SC12} \right) \times W_C, \end{aligned} \quad (14)$$

where $\mathbf{V}_n^\Lambda$ and $\mathbf{V}_c^\Lambda$ are the cutoff-dependent short-distance vertices for $\psi' \rightarrow J/\psi\pi^0\pi^0$ and $\psi' \rightarrow J/\psi\pi^+\pi^-$, with the final-state $\pi\pi$ pair in the S wave. The $1/2$ factors in the equations account for the effects of identical neutral pions in the intermediate loops, and the effect of identical pions in the $J/\psi\pi^0\pi^0$ final state is accounted for by the $\sqrt{2}$ factor in Eq. (14) relative to Eq. (13). The Λ dependence in the intermediate Green's functions in Eqs. (13) and (14) can be absorbed by the short-distance vertices $\mathbf{V}_n^\Lambda$ and $\mathbf{V}_c^\Lambda$ through multiplicative renormalization [44,46]. The cutoff-independent amplitudes obtained by keeping only the LO terms in the Green's functions are

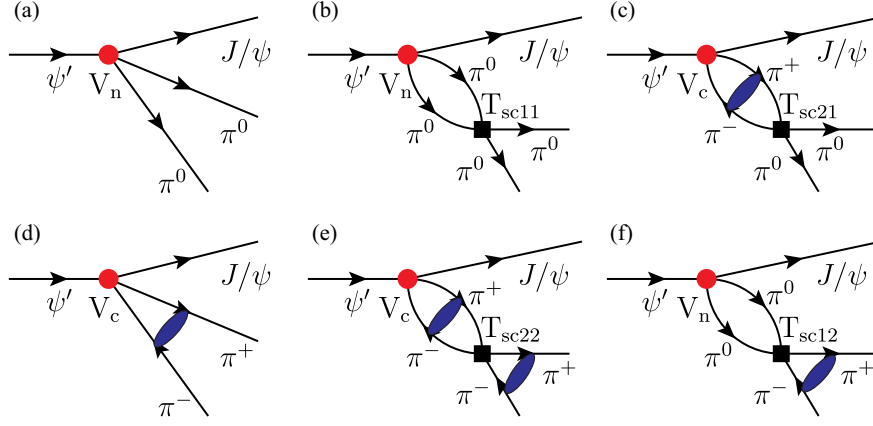


FIG. 2. Diagrams for $\psi' \rightarrow J/\psi \pi^0 \pi^0$ (a-c) and $\psi' \rightarrow J/\psi \pi^+ \pi^-$ (d-f). The red filled circles represent the pointlike production sources V_n and V_c for $J/\psi \pi^0 \pi^0$ and $J/\psi \pi^+ \pi^-$ production, respectively. The black filled squares represent the $\pi\pi$ interactions described by the solution of the LSE in Eq. (7). The blue blob represents infinite Coulomb-photon exchanges between π^+ and π^- as shown in Fig. 1. The Coulomb-photon resummation between the final-state π^+ and π^- is accounted for by the W_C factor in the amplitude for $\psi' \rightarrow J/\psi \pi^+ \pi^-$ given in Eq. (14). The amplitude for $\psi' \rightarrow J/\psi \pi^0 \pi^0$ is given in Eq. (13).

$$\mathcal{M}[\psi' \rightarrow J/\psi \pi^0 \pi^0] = V_n \times \left(\frac{1}{2} T_{\text{SC11}} + T_{\text{SC21}} \right), \quad (15)$$

$$\mathcal{M}[\psi' \rightarrow J/\psi \pi^+ \pi^-] = \sqrt{2} V_n \times \left(T_{\text{SC22}} + \frac{1}{2} T_{\text{SC12}} \right) \times W_C, \quad (16)$$

where V_n is the renormalized Λ -independent short-distance vertex. We have taken the isospin limit in the renormalization so that $V_c \simeq V_n$ and $\mu_n \simeq \mu_c$ in the linearly divergent terms of the Green's functions. It is worth to note that our formalism is similar as the treatment in Ref. [21] at one loop for $K \rightarrow 3\pi$, where the impact of the pionium is also considered carefully by the infinite Coulomb photon exchange in the Green's function. Both approaches are equivalent in treating the $\pi\pi$ final-state interactions through the Lippmann-Schwinger equation with coupled channels $\pi^0 \pi^0$ and $\pi^+ \pi^-$ and the inclusion of Coulomb effects near the charged-pion threshold. However, there is an important difference that, in the $K \rightarrow 3\pi$ case studied in Ref. [21], the spectator pion has strong $\pi\pi$ rescattering with the other two pions; while in $\psi' \rightarrow J/\psi \pi\pi$, the corresponding spectator interaction is $J/\psi \pi$, which is Okubo-Zweig-Iizuka suppressed. Quantitatively, lattice QCD gives $a_{J/\psi \pi} = (-0.01 \pm 0.01)$ fm, which is much smaller than the relevant $\pi\pi$ scattering lengths. Therefore, $J/\psi \pi$ rescattering is beyond the accuracy of the present analysis and is consistently neglected in our power counting. Moreover, since the J/ψ is electrically neutral, there is no Coulomb photon exchange between J/ψ and a charged pion at leading order. Electromagnetic interactions involving the J/ψ can arise only through higher multipole operators, such as polarizability-type couplings to the electromagnetic field. These effects are short-ranged and smooth across the $\pi^+ \pi^-$

threshold; they cannot generate the Coulomb singularity or pionium poles associated with the charged-pion pair. We, therefore, absorb such effects, if present, into higher-order smooth production operators and neglect them in the present threshold analysis.

In Refs. [47,48], the short-distance vertex for $\psi' \rightarrow J/\psi \pi\pi$ was derived using ChPT [49], and the parameters in the vertex were fixed by fitting to the BESII and ATLAS data [37,38] for the $\pi^+ \pi^-$ spectrum and the helicity angular distribution, with the $\pi\pi$ FSIs included through the Omnès function. By matching the $\psi' \rightarrow J/\psi \pi\pi$ vertex from the LO NREFT to that from ChPT in Refs. [47,48] at the $\pi\pi$ threshold, one obtains

$$V_n = -\frac{\sqrt{6m_{\psi'} m_{J/\psi}} M_\pi \alpha^{21}}{\beta^0} \times \left\{ (9\kappa - 10) M_\pi^2 + 3\kappa \left[\frac{\lambda(m_{\psi'}^2, 4M_\pi^2, m_{J/\psi}^2)}{4m_{\psi'}^2} \right] \right\}, \quad (17)$$

where $m_{\psi'}$ and $m_{J/\psi}$ are the masses of ψ' and J/ψ , respectively, $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ is the Källén function, $\beta^0 = (33 - 2N_f)/3$ is the first coefficient of the QCD beta function [50,51] with N_f being the number of light-quark flavors, $|\alpha^{21}| = (1.18 \pm 0.01 \pm 0.05) \text{ GeV}^{-3}$, and $\kappa = 0.26 \pm 0.01 \pm 0.01$ [48].

The $\pi\pi$ invariant mass distribution can be written as

$$\frac{d\Gamma}{dm_{\pi\pi}} = \frac{P_{\pi^-} P_{J/\psi}}{96\pi^3 m_{\psi'}^2} \sum_{\text{spin}} |\mathcal{M}|^2, \quad (18)$$

where

$$p_{J/\psi} = \frac{\sqrt{\lambda(m_{\psi'}^2, E^2, m_{J/\psi}^2)}}{2m_{\psi'}}, \quad p_{\pi} = \frac{\sqrt{\lambda(E^2, M_{\pi}^2, M_{\pi}^2)}}{2E} \quad (19)$$

are the 3-momenta of the J/ψ in the ψ' rest frame and the final-state π^- in the $\pi\pi$ rest frame, respectively, $\sum_{\text{spin}}$ corresponds to the sum over the polarizations of the ψ' and J/ψ , and the amplitude $\mathcal{M}$ is given by Eq. (15) for $\psi' \rightarrow J/\psi \pi^0 \pi^0$ and Eq. (16) for $\psi' \rightarrow J/\psi \pi^+ \pi^-$.

III. NEAR-THRESHOLD LINE SHAPES

With the parameters in the amplitude of $\psi' \rightarrow J/\psi \pi\pi$ settled, we give predictions of the $\pi\pi$ invariant mass distributions.

In Fig. 3, we compare the line shapes of the $\pi^0 \pi^0$ distribution in the near-threshold region with $[(d\Gamma/dm_{\pi\pi})_{\text{wc}}$, represented by the blue solid line] and without $[(d\Gamma/dm_{\pi\pi})_{\text{woc}}$, obtained by setting $\kappa_c \rightarrow ik_c$ [19,23] in Eq. (18), represented by the red solid line] Coulomb interactions. The cusp at the $\pi^+ \pi^-$ threshold becomes more prominent when Coulomb interactions are included. The inset magnifies the line shapes in the immediate vicinity of the $\pi^+ \pi^-$ threshold and reveals numerous peaks from the ground and excited states of pionium. It should be noted that the uncertainties shown in Fig. 3 mainly originate from the parameters α^{21} and κ through the overall strength of the production vertex given in Eq. (17). These uncertainties rescale only the $\pi^0 \pi^0$ distribution and do not change the nonanalytic cusp shape generated by the $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$ rescattering amplitude. Therefore, they do not hinder the extraction of scattering lengths from the cusp shape. Figure 4 shows the ratio of

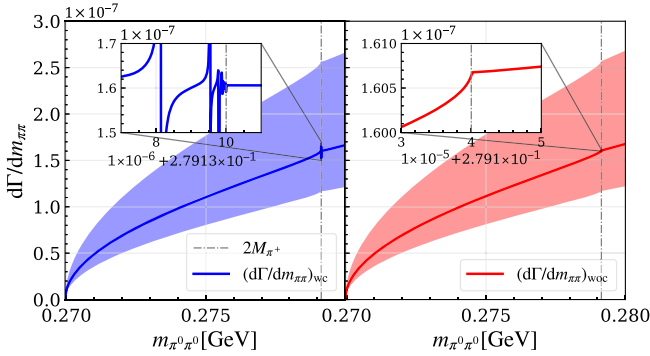


FIG. 3. The invariant mass distribution of the $\pi^0 \pi^0$ pair for $\psi' \rightarrow J/\psi \pi^0 \pi^0$. The blue and red curves represent the distributions with and without the Coulomb effect, respectively, using the central values of $(a_0 - a_2)M_{\pi^+}$ and $a_2 M_{\pi^+}$, α^{21} , and κ . The error bands are dominated by the errors of α^{21} and κ , which change only the overall normalization of the spectrum and do not affect the shape of the cusp near the $\pi^+ \pi^-$ threshold. The insets enlarge the line shape with and without Coulomb effects in the very near-threshold region. The gray vertical dot-dashed line denotes the $\pi^+ \pi^-$ threshold.

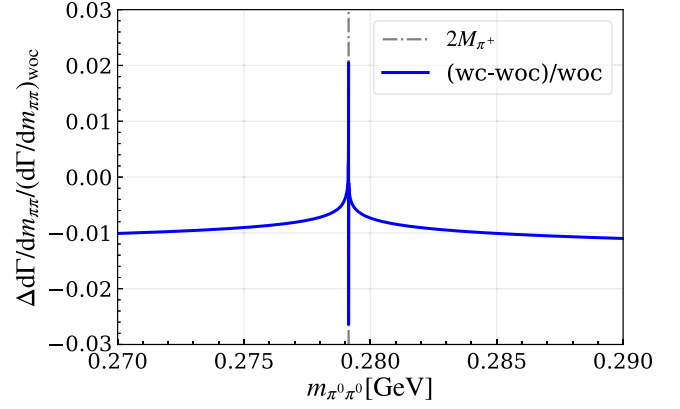


FIG. 4. Ratio of the difference $\Delta d\Gamma/dm_{\pi\pi}$ between the $\pi^0 \pi^0$ invariant mass distributions with and without Coulomb interactions (denoted as “wc” and “woc,” respectively) to the invariant mass distribution without Coulomb interactions.

the difference $\Delta d\Gamma/dm_{\pi\pi} = (d\Gamma/dm_{\pi\pi})_{\text{wc}} - (d\Gamma/dm_{\pi\pi})_{\text{woc}}$ between the central values of the $\pi^0 \pi^0$ spectrum with and without Coulomb interactions to the central value of the $\pi^0 \pi^0$ spectrum $(d\Gamma/dm_{\pi\pi})_{\text{woc}}$ without Coulomb interactions. The Coulomb corrections can change the magnitude of the cusp at the $\pi^+ \pi^-$ threshold by approximately 2%–3%, which is consistent with the $K \rightarrow 3\pi$ cases discussed in Ref. [21]. Therefore, electromagnetic corrections should be included for precise extraction of the $\pi\pi$ scattering lengths in charmonium dipion transitions.

Figure 5 presents a comparison of the line shapes for the $\pi^+ \pi^-$ spectrum with (blue solid line) and without (red solid line) Coulomb interactions. The distribution with Coulomb interactions is nonzero at the threshold due to contributions from the pionium excited state poles in $W_C(E)$, as shown in the inset in Fig. 3.

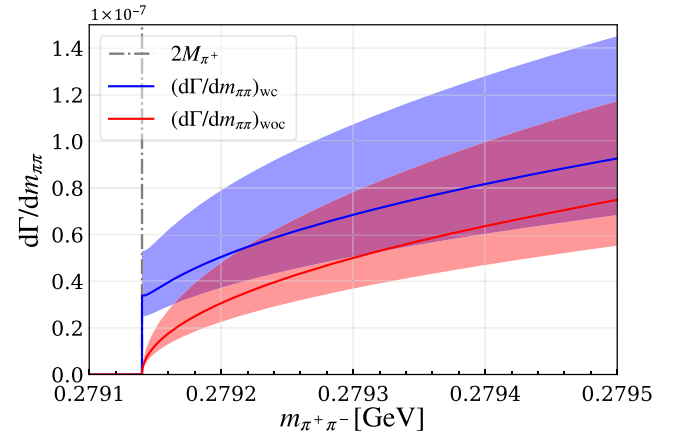


FIG. 5. The invariant mass distribution of the $\pi^+ \pi^-$ pair for $\psi' \rightarrow J/\psi \pi^+ \pi^-$. The blue and red line shapes represent the distributions with and without the Coulomb interactions, respectively.

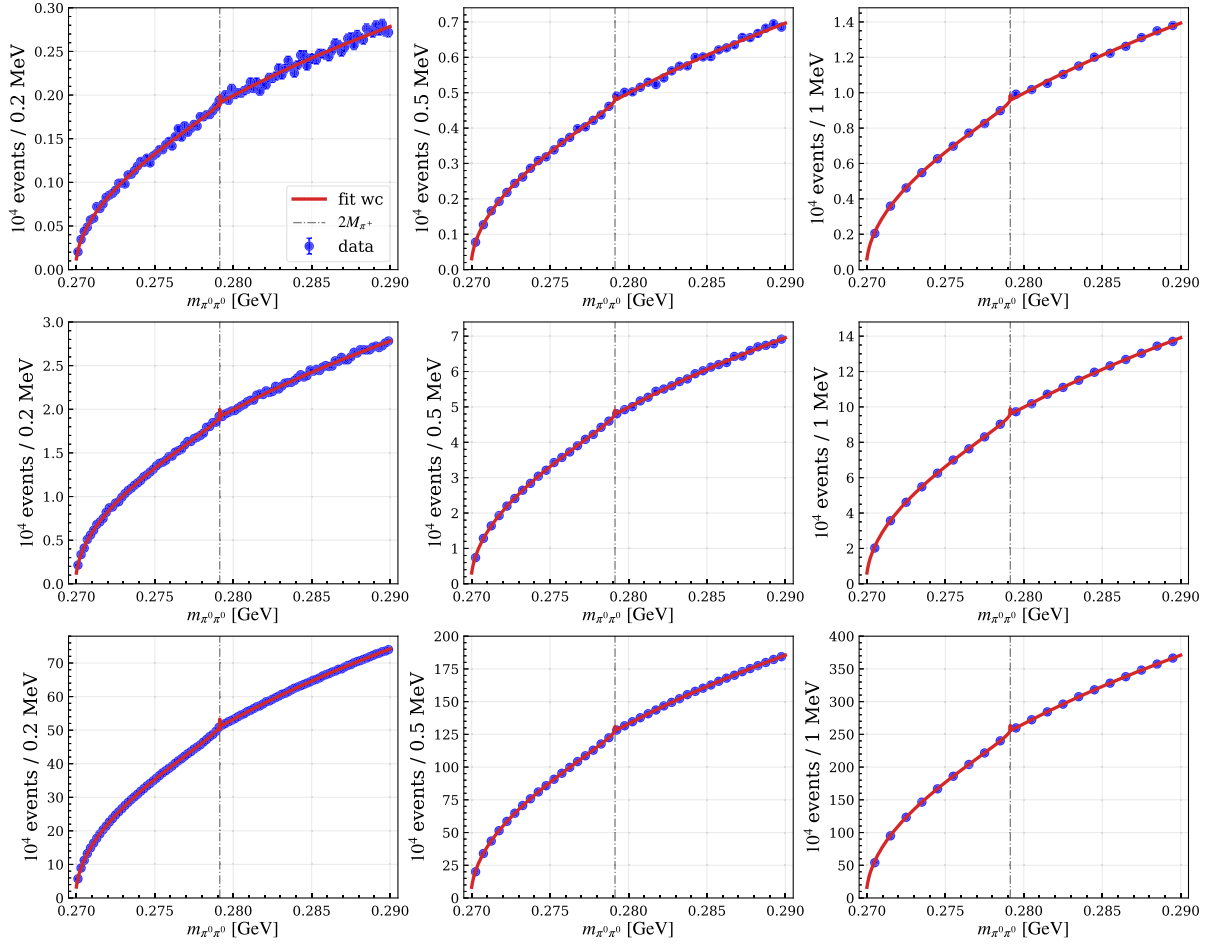


FIG. 6. Various sets of MC events that conform to the normalized $\pi^0\pi^0$ invariant mass distribution with Coulomb corrections and the best fits using the model with Coulomb interactions. The event numbers in the range $[0.270, 0.290]$ GeV for the first, second, and third rows are about 1.5×10^5 , 1.5×10^6 , and 4×10^7 , respectively. The bin widths for the first, second, and third columns are 0.2, 0.5, and 1 MeV, respectively. The gray vertical dot-dashed line indicates the $\pi^+\pi^-$ threshold.

IV. MONTE CARLO SIMULATIONS

In this section, we perform MC simulations to quantitatively estimate the precision with which electromagnetic corrections can influence the experimental extraction of $\pi\pi$ scattering lengths. We use the von Neumann rejection method to generate random data points that conform to the normalized $\pi^0\pi^0$ distribution within the energy spectrum ranging from 0.270 to 0.290 GeV, as predicted by the NREFT in Eq. (18) with Coulomb interactions included. These data are generated using the central values $(a_0 - a_2)M_{\pi^+} = 0.2640$, $a_2M_{\pi^+} = -0.0444$ [1,2,45], $|\alpha^{21}| = 1.18$, and $\kappa = 0.26$ [48] and are subdivided into bins, with statistical uncertainties given by the square root of the event number in each bin. By varying the MC event numbers and data binning¹ influence the extent to which

¹In principle, one also needs to consider the energy resolution by convolving the $\pi\pi$ invariant mass distribution with a Gaussian function.

electromagnetic corrections affect the precision of scattering length extraction.

Before proceeding to the fits, we comment on the treatment of the very narrow pionium bound states that appear below the $\pi^+\pi^-$ threshold (see the inset in Fig. 3). In realistic experiments, these states are not resolvable, because their natural width is much smaller than the typical energy resolution. Following the approach adopted in Ref. [21] for $K \rightarrow 3\pi$ decays, one can either exclude the immediate threshold region (e.g., the first few MeV above threshold) from the fit or use a bin width that naturally smears out the bound-state peaks. In our MC simulation, the bin widths are chosen to be 0.1–2 MeV, which are orders of magnitude larger than the spacing of pionium levels ($\sim$ keV). Therefore, our synthetic data effectively average over these resonances. The theoretical distribution used in our fitting is integrated over the same bins as the pseudodata, so that the fit is not sensitive to pointwise values of the extremely narrow pionium peaks. However, this procedure should be distinguished from a true

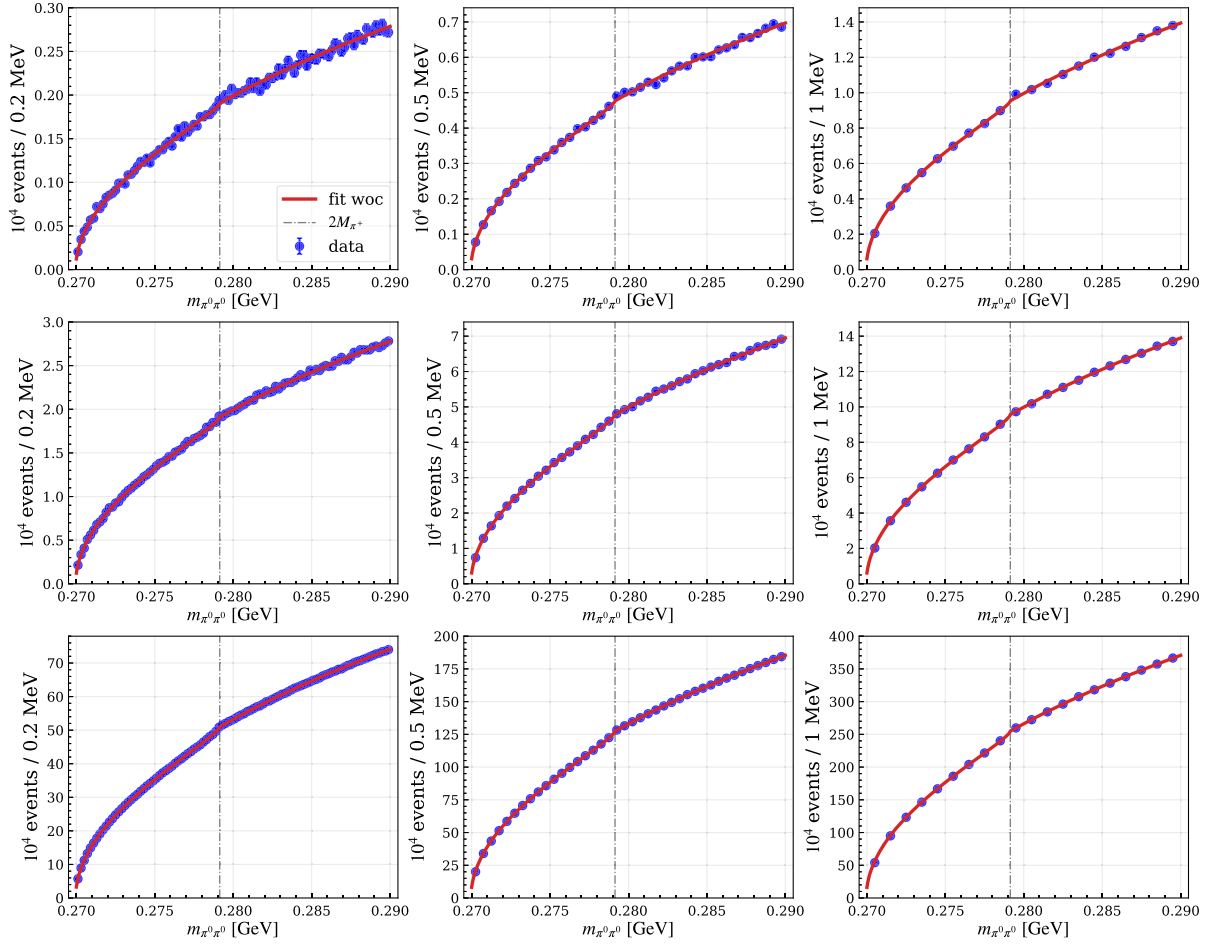


FIG. 7. The same datasets of MC events as in Fig. 6 and the best fits using the model without the Coulomb interactions. Other notations are the same as those in Fig. 6.

detector-resolution convolution, which was not included in the present exploratory study. In an experimental extraction, one should either convolve the theoretical line shape with the detector response or exclude the bins immediately affected by ponium poles, following the strategy of Ref. [21].

Using the amplitude with $\pi\pi$ FSIs from Ref. [48], one can estimate that the fraction of the $\pi^0\pi^0$ distribution below 0.29 GeV comprises about 3×10^{-4} of the total distribution over the entire phase space. Considering the branching fraction of $(18.2 \pm 0.5)\%$ for $\psi' \rightarrow J/\psi\pi^0\pi^0$ [28], the 2.7×10^9 ψ' events collected at the BESIII experiment [35] correspond to about 1.5×10^5 $\psi' \rightarrow J/\psi\pi^0\pi^0$ events for $m_{\pi^0\pi^0} \leq 0.290$ GeV. We tested a number of different combinations of event numbers (1.5×10^5 , 1.5×10^6 , and 4×10^7)² and bin widths (0.1, 0.2, 0.5, 1, and 2 MeV) in the range of [0.270, 0.290] GeV, as shown in Figs. 6 and 7.

²The last event number corresponds to the expected value at STCF [36], which is more than the event number in the same region used in the NA48/2 analysis [20].

The $\pi^0\pi^0$ invariant mass distribution was then fitted to the same MC data sets using the equation

$$\text{events} = \mathcal{N} \times \frac{d\Gamma[\psi' \rightarrow J/\psi\pi^0\pi^0]}{dm_{\pi^0\pi^0}}, \quad (20)$$

with and without Coulomb interactions, where the free parameters include an overall normalization factor $\mathcal{N}$ and scattering lengths $(a_0 - a_2)M_{\pi^+}$ and $a_2M_{\pi^+}$.

The best fits using the model with Coulomb interactions are represented by the red lines in Fig. 6, and the red lines in Fig. 7 show the best fitted results obtained using the model without Coulomb interactions. The values of the scattering lengths $(a_0 - a_2)M_{\pi^+}$ extracted from the fits are collected in Table I. The values outside and inside parentheses in the table correspond to the presence or absence of Coulomb interactions in the fitting model, respectively. These fitted results are not guaranteed to be identical to the input $[(a_0 - a_2)M_{\pi^+} = 0.2640]$ due to random fluctuations during the data-generation process. One can see that the precision of the extracted value can reach (4–6)% for 1.5×10^5 , (2–3)% for 1.5×10^6 , and (0.3–0.6)% for

TABLE I. Results of fitting to various sets of MC data. The extracted scattering lengths are given in units of $M_{\pi^+}^{-1}$. The values outside and inside the parentheses correspond to the presence or absence of Coulomb interactions in the fitting models. The uncertainties reflect only the statistical errors in the fit.

Bin width	Events	1.5×10^5	1.5×10^6	4×10^7
0.1 MeV	$\chi^2/\text{d.o.f.}$	0.90(0.91)	1.10(1.10)	0.97(1.14)
	$a_0 - a_2$	$0.270^{+0.011}_{-0.012} (0.260 \pm 0.014)$	$0.265^{+0.004}_{-0.005} (0.268 \pm 0.005)$	$0.2646^{+0.0009}_{-0.0010} (0.2674 \pm 0.0016)$
0.2 MeV	$\chi^2/\text{d.o.f.}$	0.90(0.91)	1.13(1.15)	0.85(1.19)
	$a_0 - a_2$	$0.268^{+0.011}_{-0.010} (0.259^{+0.013}_{-0.012})$	$0.265 \pm 0.005 (0.268 \pm 0.008)$	$0.2645 \pm 0.0009 (0.2674^{+0.0015}_{-0.0016})$
0.5 MeV	$\chi^2/\text{d.o.f.}$	0.60(0.63)	0.93(0.93)	0.96(1.65)
	$a_0 - a_2$	$0.262^{+0.011}_{-0.012} (0.256 \pm 0.014)$	$0.265^{+0.005}_{-0.004} (0.268^{+0.008}_{-0.007})$	$0.2646^{+0.0008}_{-0.0009} (0.2673 \pm 0.0016)$
1 MeV	$\chi^2/\text{d.o.f.}$	0.54(0.59)	0.57(0.58)	0.91(2.33)
	$a_0 - a_2$	$0.264 \pm 0.010 (0.255 \pm 0.014)$	$0.266 \pm 0.005 (0.269 \pm 0.008)$	$0.2647^{+0.0008}_{-0.0011} (0.2674 \pm 0.0016)$
2 MeV	$\chi^2/\text{d.o.f.}$	0.50(0.63)	0.29(0.30)	1.48(3.31)
	$a_0 - a_2$	$0.266 \pm 0.010 (0.252^{+0.014}_{-0.015})$	$0.266 \pm 0.005 (0.269^{+0.007}_{-0.008})$	$0.2648^{+0.0010}_{-0.0009} (0.2671 \pm 0.0016)$

4×10^7 events in the range of [0.270, 0.290] GeV. For larger event numbers, the precision improves by a factor of approximately $\sqrt{N/N'}$, where N' and N represent the new and old event numbers, respectively, and the central values of the extracted scattering lengths are closer to the input value. For different bin widths, the extraction precision remains remarkably consistent, as found in Ref. [32].

The scattering lengths extracted using the two amplitudes with and without Coulomb interactions are similar; however, the central values exhibit a difference of 1%–5%. When the extraction precision approaches 1% [number of events $\sim \mathcal{O}(10^7)$ in our simulation], the input value of the scattering length $(a_0 - a_2)M_{\pi^+}$ could fall outside the error region of the result extracted using the amplitude without Coulomb interactions, as can be seen from Table I.

V. SUMMARY

In this work, we studied the near-threshold $\pi\pi$ invariant mass distributions in $\psi' \rightarrow J/\psi\pi\pi$ within the NREFT framework, incorporating effects from Coulomb interactions. Our results show that the cusp structure in the $\pi^0\pi^0$ invariant mass distribution at the $\pi^+\pi^-$ threshold becomes slightly more prominent when Coulomb interactions are included. The Coulomb corrections can change the magnitude of the cusp by about (2–3)%, indicating the necessity of including the electromagnetic corrections for accurately extracting the $\pi\pi$ scattering lengths from the threshold cusp. To evaluate the impact of electromagnetic interactions on the extraction of

scattering lengths, we generated artificial data using MC simulation and performed fits with $(a_0 - a_2)M_{\pi^+}$ and $a_2M_{\pi^+}$ as free parameters. We found that neglecting the Coulomb corrections can lead to an overestimation of the scattering lengths, especially for experiments with high energy resolution. This level of precision corresponds to about 20 million ψ' events in the considered region $m_{\pi\pi} \in [0.270, 0.290]$ GeV. For datasets with significantly fewer events, omitting electromagnetic corrections in the extraction of scattering lengths is likely acceptable. Our approach provides a reliable framework for the precise description of the near-threshold $\pi^0\pi^0$ invariant mass distribution in $\psi' \rightarrow J/\psi\pi^0\pi^0$, applicable to data from BESIII and future STCF experiments. Deviations from the predicted distribution may indicate the necessity of including additional effects, such as the exchange of isospin-1 Z_c structures [52] and the $J/\psi\pi$ FSI.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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