

High-precision scale setting with the Ω -baryon mass and gradient flow

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The gradient-flow scale w_0 in lattice QCD is determined using the mass of the Ω^- baryon to set the physical scale. Nine ensembles using the highly improved staggered quark (HISQ) action with lattice spacings of 0.15 fm down to 0.04 fm are used, seven of which have nearly physical light-quark masses. Electromagnetic corrections to the Ω^- mass are defined in order to compute a pure-QCD Ω mass. The final result is $w_0 = 0.17187(68)$ fm, corresponding to a relative uncertainty of 0.40% and a central value in good agreement with previous calculations in the literature.

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I. INTRODUCTION

Quantum chromodynamics (QCD)—the theory of the strong force governing quark and gluon interactions—provides a powerful explanation of the behavior of hadrons over a wide range of energy scales. At low energy, where the strong coupling α_s becomes large and the theory confines, nonperturbative methods become necessary. Lattice QCD gives a model-independent method for solving the equations governing strong dynamics by computing numerically the Euclidean path integral on a discretized spacetime grid [1].

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The inputs to a lattice calculation are the bare masses of the quarks $\{m_q\}$ and the bare gauge coupling. The relationship between the bare quark masses and the experimentally accessible hadron masses is not known *a priori*, but the bare quark masses can be tuned to reproduce the hadron spectrum. Via dimensional transmutation, the bare gauge coupling can be related to the lattice spacing a , whose inverse serves as an ultraviolet regulator for the theory. As a practical matter, since all outputs of a lattice-QCD calculation are dimensionless quantities (e.g., a times a hadron mass), the lattice spacing is needed to compare simulation results with experimental observations.

Scale setting—the determination of the lattice spacing—is therefore essential to most lattice-QCD calculations. For percent- and subpercent-level computations, such as the anomalous magnetic moment ($g - 2$) of the muon [2] and Cabibbo-Kobayashi-Maskawa (CKM) matrix elements [3], scale setting can contribute substantially to the overall uncertainty budget of the calculation, necessitating a high-precision determination of the scale. Hadronic contributions to the muon $g - 2$ are particularly sensitive to the scale setting due to the quadratic dependence of the integration kernel on the muon mass [4], a dimensionful nonhadronic scale.

Gradient flow [5], a procedure for smoothing the gluon field, offers a high-precision and computationally cheap method for relating the lattice spacing to a dimensionful *flow scale* w_0 , but this flow scale cannot be measured in an experiment and must be further matched to a different experimentally accessible observable. Therefore, an additional determination of a dimensionful scale from lattice QCD is necessary to relate w_0 , and thus a , to experimental data. Such a scale must be well known experimentally and also be computable to high precision in lattice QCD.

A commonly used scale is the pion decay constant f_π [6,7], which is related to the well-measured pion muonic decay rate and suffers from no signal-to-noise problems in lattice calculations. For subpercent calculations, however, electromagnetic corrections to the pion decay constant become important, requiring information about pion structure beyond the decay constant [8]. More seriously, the muonic decay rate yields the product $|V_{ud}|f_\pi$, where the CKM matrix element $|V_{ud}|$ must be obtained from other sources. Most commonly used is the value obtained from nuclear beta decay [9], but it would be unfortunate to rely on nuclear physics for a fundamental ingredient of lattice QCD.

Hadron masses provide other dimensionful scales. Typically, the pseudoscalar meson masses (M_π , M_K , M_{D_s}), which are computed with the highest precision, are used for quark-mass tuning, thus requiring some other choice for determining the scale. Vector mesons suffer from signal-to-noise problems and are usually resonances, so their masses cannot be determined precisely or unambiguously enough. Baryon masses provide an alternative. Nucleon masses have been determined experimentally to

more than ten significant digits [10], but the light quark propagators needed to reconstruct them in lattice QCD are computationally expensive, and the signal-to-noise problem is exacerbated at lighter masses [11,12]. In contrast, the Ω^- baryon (sss) sits in a mass regime where quark propagators are cheap, making it an attractive candidate for scale setting, and its mass has been measured experimentally to better than 0.02% precision [10]. This work presents a lattice-QCD computation of the mass of the Ω^- baryon at a precision of 0.4%, allowing a determination of w_0 to the same precision.

This paper is organized as follows: Section II contains theoretical background on the gradient flow (Sec. II A) and the construction of baryon operators with staggered quarks (Sec. II B). The numerical implementation of the calculations is described in Sec. III. Section IV covers all aspects of the data analysis: computing w_0/a (Sec. IV A), correlator fits to obtain aM_Ω (Sec. IV B), the electromagnetic contribution to M_Ω to obtain a value in pure QCD (Sec. IV C), and the continuum limit of w_0M_Ω (Sec. IV D). In Sec. V, we offer some concluding remarks. The appendixes cover Bayesian model averaging (Appendix A), uncertainty propagation for the retuned Ω baryon mass (Appendix B), raw results for aM_Ω , aM_π , aM_K , and w_0/a (Appendix C), and the continuum-limit results for w_0M_Ω in all flow implementations studied here (Appendix D). The material in Appendixes C and D should be useful to researchers using the $(2 + 1 + 1)$ -flavor HISQ ensembles.

II. THEORETICAL BACKGROUND

A. Gradient flow

Gradient flow is a continuous transformation of the gauge field, introduced in Ref. [5], that drives the field toward stationary points of the flow action S_{flow} —i.e., toward smooth solutions that satisfy the classical equations of motion of S_{flow} . In this section, we review the theoretical details of gradient flow and its relation to scale setting in lattice QCD. In this work, we apply the gradient flow only to the gauge field, so we restrict the discussion accordingly.

1. Defining equations

Consider a gauge-field configuration $A_\mu(x)$ (for now, in the continuum) that serves as an initial condition for the following gradient-flow equation:

$$\frac{\partial B_\mu}{\partial t} = -\frac{\delta S_{\text{flow}}}{\delta B_\mu}, \quad B_\mu(x, 0) = A_\mu(x), \quad (2.1)$$

where t is the fictitious flow time and x is a spacetime vector. If the flow action S_{flow} is simply the Yang-Mills action

$$S_{\text{flow}}[A] = -\int d^4x \frac{1}{2} \text{tr}[G_{\mu\nu} G_{\mu\nu}], \quad (2.2)$$

the flow equation takes the form

$$\frac{\partial B_\mu}{\partial t} = D_\nu G_{\nu\mu}, \quad (2.3)$$

where

$$D_\nu G_{\nu\mu} = \partial_\nu G_{\nu\mu} + g_0[B_\nu, G_{\nu\mu}], \quad (2.4)$$

$$G_{\nu\mu} = \partial_\nu B_\mu - \partial_\mu B_\nu + g_0[B_\nu, B_\mu]. \quad (2.5)$$

Note that the flow time is of mass dimension -2 .

In lattice gauge theory, the rescaled fields $g_0 A_\mu(x)$ are replaced by the gauge-link variables $U_{x,\mu}$, and the flow equation takes the form

$$\frac{\partial V_{x,\mu}(t)}{\partial t} = -g_0^2[\partial_{x,\mu} S_{\text{flow}}[V]]V_{x,\mu}(t), \quad (2.6)$$

$$V_{x,\mu}(0) = U_{x,\mu}, \quad (2.7)$$

where $\partial_{x,\mu} S_{\text{flow}}[V]$ is the appropriate SU(3)-valued derivative with respect to the link $V_{x,\mu}$.

With this choice of the flow action, the flow equation is first order in flow time and second order in spacetime variables. This partial differential equation thus describes a diffusion process that, as flow time proceeds, smooths out short-distance fluctuations of the original gauge field $U_{x,\mu}$.

For the discrete version of the flow action S_{flow} , this paper employs the Wilson plaquette action or the tree-level Symanzik-improved (plaquette and rectangle) action. The corresponding gradient flow is then referred to as the Wilson or Symanzik flow, respectively.

2. Gradient-flow scale w_0

The gradient flow can also provide a reference scale if there is a flow-time-dependent dimensionless quantity that is finite in the continuum limit. It has been shown in Ref. [13] that the action density evaluated on the flowed gauge-field configurations is finite to all orders in α_s . In other words, the flowed action density defines an observable. As the discretization of the observable does not need to be the same as the one for generating the flow, we denote it as $S_{\text{obs}}[V]$. The appropriate dimensionless quantity is then $t^2 \langle S[V] \rangle$, where the averaging is with respect to the gauge-field ensemble. The reference scale t_0 is then defined as a point in flow time where the observable reaches a pre-defined value k :

$$t^2 \langle S_{\text{obs}}[V(t)] \rangle|_{t=t_0} = k. \quad (2.8)$$

The most commonly used value for k is the originally proposed $k = 0.3$ [5].

The flow of duration t introduces a smearing radius $\sqrt{8t}$. Ideally, in order to avoid large-cutoff or finite-volume

effects, the flow should be used in the regime $a \ll \sqrt{8t} \ll a \min(N_s, N_t)$, where N_s (N_t) is the number of lattice points in the spatial (temporal) direction. The choice $k = 0.3$ allows the t_0 scale to satisfy this constraint for commonly used lattice spacings and volumes.

At finite lattice spacing, the observable $t^2 \langle S_{\text{obs}}[V(t)] \rangle$ is subject to discretization effects. Empirical evidence suggests [14] that the slope of $t^2 \langle S_{\text{obs}}[V(t)] \rangle$ is less affected by discretization effects than the quantity itself. A different condition for the reference scale w_0 was proposed in Ref. [14]:

$$W_o(t) \equiv t \frac{d}{dt} [t^2 \langle S_{\text{obs}}[V(t)] \rangle]|_{t=w_0^2} = k, \quad (2.9)$$

with the same value $k = 0.3$. The subscript “o” denotes “original” and will be contrasted with a corrected quantity below.

The lattice artifacts affecting the $\sqrt{t_0}$ or w_0 scale have three sources: the action S_{MC} used in the Markov chain Monte Carlo process to generate the ensemble of gauge-field configurations, the action that generates the flow S_{flow} , and the action for the observable S_{obs} . A perturbative analysis of the discretization effects at tree level in the finite or infinite volume was carried out in Ref. [15]. The leading discretization effects in the observable $t^2 \langle S_{\text{obs}}(t) \rangle$ can be represented as an expansion in a^2/t :

$$t^2 \langle S_{\text{obs}}(t) \rangle = \frac{3(N^2 - 1)g_0^2}{128\pi^2} (C(a^2/t) + O(g_0^2)), \quad (2.10)$$

where N is the number of colors and

$$C(a^2/t) = 1 + \sum_{m=1}^{\infty} C_{2m} \frac{a^{2m}}{t^m}. \quad (2.11)$$

Alternatively, $C(a^2/t)$ can be evaluated numerically, effectively summing all orders in a^2/t . Since this procedure would need to be carried out for every flow time t considered, it is more convenient to use the expansion up to and including the fourth order in a^2/t . The first four coefficients— C_2 , C_4 , C_6 , and C_8 —have been explicitly computed in the infinite-volume limit [15] for various gauge-flow-observable combinations.

B. Ω baryon with staggered quarks

The systematic construction of staggered baryon operators was first introduced in Ref. [16] and later expanded in Ref. [17] to include additional flavor symmetries. In the following subsections, we provide a high-level summary of the group theory needed to construct lattice operators for Ω . In the following, we reserve Ω^- for the physical baryon (with QED included) and Ω for pure QCD, while introducing further similar notations for staggered baryons that

have Ω properties in the continuum limit. Interested readers should refer to Refs. [16,17] for more details.

1. Lattice irreducible representations for Ω

In this work, we use rooted staggered fermions for sea quarks by taking the fourth root of quark determinants. The rooting procedure is not applicable for solving quark propagators, so each of the valence quarks still retains four taste copies, and the simulation is partially quenched. In the continuum limit, the four tastes become degenerate and form the $SU(4)_T$ taste-symmetry group for each quark type. When the taste symmetry is combined with the $SU(2)_I$ isospin-symmetry group, the internal symmetry of the continuum action becomes $SU(8)_{ud} \times SU(4)_s$, where $SU(8)_{ud} \supset SU(2)_I \times SU(4)_T$ is the taste and isospin rotation of up and down quarks, and $SU(4)_s$ is the taste rotation of the strange quark. To distinguish the staggered tastes from the physical states, we refer to them as N -like, Ω -like, etc. [18].

Despite the presence of many superfluous states, operators constructed with all quarks assigned to the same taste must have the same spin and flavor structure as baryons in tasteless QCD in order to satisfy antisymmetrization requirements. Therefore, observables computed using valence quarks of a single taste must have the same continuum-limit masses as their QCD counterparts [17]. Because of the enlarged isospin-taste symmetry group of the continuum action, many non-single-taste baryons become degenerate with the single-taste ones in the continuum limit, and, as discussed below, they can then be used to extract properties of the desired baryon.

Operators at finite lattice spacings may be decomposed into irreducible representations (irreps) of the lattice-symmetry group and the flavor-symmetry group. For staggered fermions, the lattice-symmetry group is called the geometrical time slice group (GTS) [16]. The GTS mixes the spacetime symmetry with the taste symmetry and has three fermionic irreps: 8, $8'$, and 16, where the number denotes the dimension. The basis elements of the fermionic representations correspond to the eight corners of equally spaced three-dimensional unit cubes distributed at every other site across an entire spatial time slice. The representations also encode additional behavior under rotations, acquiring an additional $+1$ (-1) factor in the case of the 8 ($8'$) representation, or transforming with an additional two-dimensional representation structure in the case of the 16 representation.

The number of Ω -like states interpolated by each lattice irrep can be obtained by comparing the subduced continuum single-flavor baryon irrep with the totally symmetric $SU(4)$ taste representation:

$$SU(2)_S \times SU(4)_T \rightarrow \text{GTS},$$

$$\left(\frac{3}{2}, 20_S\right) \rightarrow 2 \cdot 8 \oplus 2 \cdot 8' \oplus 3 \cdot 16, \quad (2.12)$$

where the notation $2 \cdot 8'$ means that the $8'$ irrep appears with multiplicity 2. Numbers are used to denote the dimensions of irreps in all groups except the $SU(2)_S$ spin group, where the usual spin notation is used. Subscripts on the irrep dimensions indicate whether the irrep is symmetric (S) or mixed symmetry (M) as appropriate. The 20_S is the symmetric irrep of $SU(4)_T$ obtained from the tensor product of three fundamental representations, which contains single-taste Ω -like states with physical masses. The multiplicities obtained from subduction of the continuum-limit symmetry group [Eq. (2.12)] show that the 8 and $8'$ both interpolate two Ω -like multiplets, and the 16 interpolates three. These Ω -like multiplets have degenerate masses in the continuum limit but have taste splittings at finite lattice spacing.

The 8 and 16 representations also both interpolate a fictitious N_s baryon multiplet, as seen from the subduction of a totally symmetric spin- $\frac{1}{2}$ representation:

$$SU(2)_S \times SU(4)_T \rightarrow \text{GTS},$$

$$\left(\frac{1}{2}, 20_M\right) \rightarrow 3 \cdot 8 \oplus 0 \cdot 8' \oplus 1 \cdot 16. \quad (2.13)$$

The N_s is a spin- $\frac{1}{2}$ N -like state formed from two distinct flavors, both of mass equal to that of the strange quark. It has a lower mass than the Ω -like states. The N_s states are only possible because the mixed-symmetric taste symmetry (in the 20_M representation) allows for a mixed-symmetric spin representation (in the spin- $\frac{1}{2}$ representation) that satisfies overall wave function symmetrization. Without taste symmetry, the spin- $\frac{1}{2}$ representation would vanish due to the overall wave function antisymmetrization required by fermion statistics, which is why these states are not present in tasteless QCD. The existence of the N_s state motivates our choice of analysis strategy: we use only the $8'$ irrep in this work, since it does not interpolate N_s .¹

The relevant subduction for the Ω -like states with $SU(3)$ flavor symmetry is

$$SU(2)_S \times SU(12)_{FT} \rightarrow SU(3)_F \times \text{GTS},$$

$$\left(\frac{3}{2}, 364_S\right) \rightarrow 2 \cdot (10_S, 8) \oplus 2 \cdot (10_S, 8')$$

$$\oplus 3 \cdot (10_S, 16) \oplus (8_M, 8') \oplus \dots, \quad (2.14)$$

where $SU(12)_{FT}$ denotes the flavor-taste-symmetry group, and 364_S is the symmetric irrep thereof that contains the single-taste Ω -like states with three strange quarks of identical flavor and taste, so all states within the irrep are degenerate with the physical Ω baryon in the continuum

¹See Refs. [19,20] for information on the 8 and 16 irreps.

and $SU(3)$ -flavor-symmetric limits. On the right side of the equation, the first three irreps correspond to those obtained in Eq. (2.12). The last irrep, $(8_M, 8')$, is another consequence of the additional taste symmetry with no corresponding representation in tasteless QCD. This irrep interpolates a single Ω -like multiplet and no N_s multiplets. Other omitted irreps, indicated by the ellipsis in Eq. (2.14), interpolate both N_s and Ω -like states and are not used in this work.

We can construct more interpolators of Ω -like states by taking advantage of the freedom to adjust quark masses in partially quenched simulations. In particular, we can use the operators in the irrep $(8_M, 8')$ of Eq. (2.14) with all three quark masses set to the physical strange quark mass. This construction requires at least two flavors of valence strange quarks to satisfy antisymmetrization, which leads to a valence-sector symmetry group identical in construction to the usual isospin symmetry with light quarks. We refer to this valence-sector symmetry containing two valence strange quarks as strange isospin to distinguish it from the isospin of light quarks and denote the corresponding flavor-symmetry group by $SU(2)_{I_s}$. We distinguish the Ω -like states produced from strange-isospin interpolating operators by denoting them as Ω_2 states, in contrast to Ω_1 states produced from a valence sector containing only a single strange quark flavor.

The subduction of the two-strange-valence-flavor theory produces the mapping

$$\begin{aligned} SU(3)_F &\rightarrow SU(2)_{I_s}, \\ 10_S &\rightarrow 4_S \oplus \cdots \\ 8_M &\rightarrow 2_M \oplus \cdots, \end{aligned} \quad (2.15)$$

where only irreps containing interpolating operators constructed exclusively from the two valence strange quarks are listed. The continuum symmetry group of the strange-isospin valence sector, including the spin group, then becomes $SU(2)_S \times SU(8)_{I_s T}$, where $SU(8)_{I_s T} \supset SU(2)_{I_s} \times SU(4)_T$ is the strange-isospin-taste group. The relevant subduction for the Ω_2 states is

$$\begin{aligned} SU(2)_S \times SU(8)_{I_s T} &\rightarrow SU(2)_{I_s} \times \text{GTS}, \\ \left(\frac{3}{2}, 120_S\right) &\rightarrow 2 \cdot (4_S, 8) \oplus 2 \cdot (4_S, 8') \\ &\oplus 3 \cdot (4_S, 16) \oplus (2_M, 8') \oplus \cdots, \end{aligned} \quad (2.16)$$

which are in a one-to-one mapping with the irreps in Eq. (2.14). The last irrep, $(2_M, 8')$, is the only strange-isospin interpolating operator used in this work. The relevant subduction for the Ω_1 states, which are a subset

of the interpolating operators in the $(4_S, 8')$ irrep of Eq. (2.16), are the same as those in the $8'$ irrep of Eq. (2.12).

In summary, by subducing irreps of the continuum-symmetry group down to irreps of the lattice-symmetry group, we arrive at two irreps that will be used in this work to compute the Ω mass. These are

- (1) The flavor-symmetric $8'$ irrep that interpolates two taste-split Ω_1 multiplets as the lowest-energy multiplets.
- (2) The strange-isospin $8'$ mixed-symmetry irrep that interpolates to one Ω_2 multiplet as the ground state.

Explicit expressions for operators transforming in these irreps are presented in Sec. III C below. In the continuum limit, all three Ω -like multiplets should have the same masses. Other operators that interpolate to Ω -like states have more complicated spectra including lower-lying N_s states, and we do not consider them here.

III. NUMERICAL IMPLEMENTATION

A. Gauge-field ensembles

The calculations described in this work were carried out using nine $(2 + 1 + 1)$ -flavor ensembles generated primarily by the MILC Collaboration [7,21], listed in Table I. One of the ensembles, 0.09C, was generated by the combined efforts of the CalLat [22], MILC, and PNDME Collaborations. The highly improved staggered quark (HISQ) action [23] was used for both sea and valence quarks. The sea-quark masses for these ensembles have been tuned to reproduce the physical π -, K -, and D_s -meson masses to within few-percent accuracy (except for 0.12H and 0.09H, which have a heavier-than-physical am_l^{sea} to study sea-quark mistuning), and the lattice spacings span from 0.15 fm down to about 0.04 fm.

B. Gradient-flow observables

In this work, we consider Wilson (plaquette), Symanzik (plaquette and rectangle), and clover discretizations of the observable S_{obs} . Together with the two flows, it gives us six combinations that are labeled as gauge-flow-observable: SWW, SWS, SWC, SSW, SSS, SSC. The first letter, S, is the same, as it corresponds to the action used to generate the MILC ensembles that are used in this study: the one-loop Symanzik-improved gauge action [26–32] and the highly improved staggered quark (HISQ) action [23,33,34]. The second and third letters denote the actions used for flowing the gauge field and measuring the observable, respectively. The tree-level corrections of Ref. [15] can be applied to the flowed observable S_{obs} to cancel partially the discretization effects. This leads to the following corrected scale conditions:

$$\frac{t^2 \langle S_{\text{obs}}(t) \rangle}{1 + \sum_{m=1}^4 C_{2m}(a^2/t)^m} \Big|_{t=t_0} = k, \quad (3.1)$$

TABLE I. List of ensembles used in this work. The three near-physical ensembles at $a \approx 0.09$ fm are the original, slightly mistuned (M) ensemble; a corrected (C) ensemble with closer-to-physical meson masses; and a large-volume (L) ensemble designed to study finite-volume effects. Ensembles labeled with H represent a heavier-than-physical pion mass of about 220 MeV and were used to study pion-mass mistuning. On four of the ensembles, note that two valence strange quark masses were used for studying the effects of quark-mass mistuning. Values for meson masses on the physical-point ensembles (0.15, 0.12, 0.09M, 0.09C, 0.06, and 0.04 fm) are taken from Ref. [24] and references therein. Values on the heavier-than-physical ensembles 0.12H and 0.09H are taken from Refs. [7,25], respectively. The value on the large-volume ensemble 0.09L is new. In all cases, values are determined from multi-exponential fits to the truncated spectral decomposition using methods and parameters described in Refs. [7,25]. The number of configurations, total number of sources, and number of independent samples after binning are also shown.

$\approx a/\text{fm}$	$\approx L/\text{fm}$	$N_s^3 \times N_t$	am_l^{sea}	am_s^{sea}	am_c^{sea}	am_s^{val}	am_π	am_K	N_{cfg}	$N_{\text{src}}/10^3$	N_{bin}
0.15H	4.8	$32^3 \times 48$	0.002426	0.0673	0.8447	0.0673 0.067973	0.103414(11)	0.37847(11)	9936	238	331
0.12H	4.8	$40^3 \times 64$	0.00507	0.0507	0.628	0.0507	0.13426(8)	0.30806(12)	1030	33	51
0.12H	5.8	$48^3 \times 64$	0.001907	0.05252	0.6382	0.05252 0.0530452	0.0830651(63)	0.303949(77)	9995	320	333
0.09H	4.3	$48^3 \times 96$	0.00363	0.0363	0.432	0.0363	0.09860(13)	0.22706(15)	979	94	48
0.09M	5.8	$64^3 \times 96$	0.00120	0.0363	0.432	0.0363	0.057184(30)	0.219482(70)	5333	256	266
0.09C	5.8	$64^3 \times 96$	0.001326	0.03636	0.4313	0.03636 0.0367236	0.060069(26)	0.220231(40)	4692	196	234
0.09L	11.5	$128^3 \times 96$	0.001326	0.03636	0.4313	0.03636	0.0600523(72)	0.220225(13)	1051	50	52
0.06H	5.8	$96^3 \times 192$	0.0008	0.022	0.260	0.022	0.038842(29)	0.142607(51)	4061	390	135
0.04H	5.8	$144^3 \times 288$	0.000569	0.01555	0.1827	0.01555	0.028981(18)	0.106297(39)	1007	145	50

$$W_c(t) \equiv t \frac{d}{dt} \frac{t^2 \langle S_{\text{obs}}(t) \rangle}{1 + \sum_{m=1}^4 C_{2m} (a^2/t)^m} \Big|_{t=w_0^2} = k. \quad (3.2)$$

For the scales determined from the original definitions (2.8) and (2.9), the letter “o” (original) is added to the gauge-flow-observable combinations; for scales determined from the tree-level corrected definitions (3.1) and (3.2), the letter “c” (corrected) is added. The nomenclature for the scales determined from various gauge-flow-observable discretizations with or without the tree-level corrections applied is then $S[W,S][W,S,C][o,c]$. Previous work from MILC on the scales $\sqrt{t_0}$ and w_0 [35] included only the SSCo and SSCc combinations.

To integrate the gradient flow, we use a fourth-order integrator described in Refs. [36,37] with the coefficient scheme of Ref. [38]. To control the systematic step-size errors in the flow integration, we perform all measurements with two step sizes $\Delta t = 1/20$ and $\Delta t = 1/40$. The integration is stable on all ensembles, and the systematic errors coming from the flow integration are always below the statistical errors by at least 2 orders of magnitude.

C. Ω -baryon correlators

Using the notation of Refs. [16,17], the basic building block of staggered baryon operators with zero momenta is the quark trilinear

$$_{ijk} B_{A,B,C} \equiv \sum_{x_i \in \text{even}} \frac{1}{6} \epsilon_{abc} D_A \chi_i^a(x) D_B \chi_j^b(x) D_C \chi_k^c(x), \quad (3.3)$$

$$D_A \chi_i^a(x) \equiv \frac{1}{2} (\chi_i^a(x - \hat{A}) + \chi_i^a(x + \hat{A})). \quad (3.4)$$

Here, a, b, c are color indices, $i, j, k \in \{s, s'\}$ are the strange-isospin indices for two valence strange quarks s and s' , and $A, B, C \in \{0, 1, 2, 3, 12, 13, 23, 123\}$ denote the eight corners of a three-dimensional spatial cube with side length a , and $\hat{A}$ is the corresponding shift vector. To get a zero-momentum state, we sum over all the even spatial indices as dictated by the symmetry of the staggered action. Because the shifts in Eq. (3.4) are not accompanied by gauge links, the quark trilinear is not gauge invariant; suitable gauge fixing is described below.

For flavor-symmetric $8'$ irreps coupling to Ω_1 states, $i = j = k = s$, and there are two different classes of operators [16] belonging to the same lattice irrep. These are

$$S_4 \equiv_{sss} B_{1,12,13} -_{sss} B_{2,21,23} +_{sss} B_{3,31,32}, \quad (3.5)$$

$$S_7 \equiv_{sss} B_{1,2,3}, \quad (3.6)$$

where S denotes that the irreps are symmetric and the subscript denotes the class of the operators as enumerated in Ref. [17]. On the other hand, there is a single class-4, strange-isospin, $8'$ operator coupling to the Ω_2 state formed from the mixed-symmetric combination of flavor indices:

$$\mathcal{M}_4 \equiv {}_{sss'}M_{1,12,13} - {}_{sss'}M_{2,21,23} + {}_{sss'}M_{3,31,32} + {}_{ss's}M_{1,12,13} - {}_{ss's}M_{2,21,23} + {}_{ss's}M_{3,31,32}, \quad (3.7)$$

$${}_{ijk}M_{A,B,C} \equiv \frac{1}{3}(2_{ijk}B_{ABC} - {}_{jki}B_{ABC} + {}_{kij}B_{ABC}). \quad (3.8)$$

Reference [39] uses the same set of Ω -like operators for computing the Ω baryon mass. In particular, their Ω_{VI} , Ω_{XI} , and Ω_{Ba} operators are the same as our $\mathcal{S}_4$, $\mathcal{S}_7$, and $\mathcal{M}_4$ operators, respectively, up to normalization factors.

The quark propagators in this work were computed between a corner wall source and a point sink. The corner wall source is defined to be 1 on one of the eight corners of each of the three-dimensional spatial cubes on a time slice (specified by A , B , C above) and zero elsewhere. Since none of the Ω -baryon interpolating operators described in the previous section make use of the 0 or 123 corners of the hypercube, only six quark propagators must be computed for each source timeslice. The various shifts required by Eq. (3.4) are included at the sink. Since the operations at both source and sink are not gauge invariant, the configurations were fixed to the Coulomb gauge.

For the four most computationally intensive ensembles—0.09M, 0.09L, 0.06, and 0.04 fm—the truncated solver method [40,41] was used to reduce the computational cost of propagator solves. Sloppy solves were performed with precision $\varepsilon_{\text{sloppy}} = 10^{-4}$, and a single fine propagator solve with a precision of $\varepsilon_{\text{fine}} = 10^{-12}$ was performed on each configuration for bias correction. For the coarser ensembles, where quark propagators are computationally cheap, all solves were performed at the fine precision. The total number of sources used for each ensemble is given in Table I.

D. Electromagnetism

Although the electromagnetic contribution to the Ω^- baryon mass is expected to be small, the precision sought here requires care. Our approach follows the electroquenched approximation [42], in which the charges of the sea quarks are neglected and the valence quarks interact with free photon fields. The MILC Collaboration has previously used this approach to compute electromagnetic effects for mesons [43,44]. For the Abelian U(1) group, we choose the noncompact formulation.

Thus, the electromagnetic gauge action is

$$S_{\text{em}} = \frac{1}{4} \sum_{n,\mu,\nu} [\partial_\mu A_\nu(n) - \partial_\nu A_\mu(n)]^2, \quad (3.9)$$

with Coulomb gauge-fixing $\partial_i A_i(n) = 0$. The momentum-space distribution is Gaussian, so the $A_\mu(k)$ can be independently and trivially generated (with no need for a

Markov chain) and converted to position space via a fast Fourier transform. To obtain a convergent path integral, it is necessary to drop zero modes. We choose the QED_{TL} formulation [42,45]—namely,

$$\begin{aligned} A_0(k_0, \mathbf{k} = \mathbf{0}) &= 0, \quad \forall k_0 \\ \mathbf{A}(k_0 = 0, \mathbf{k} = \mathbf{0}) &= \mathbf{0}, \\ \mathbf{k} \cdot \mathbf{A}(k_0, \mathbf{k}) &= 0, \quad \forall k_0, \mathbf{k}, \end{aligned} \quad (3.10)$$

with the last condition enforcing the Coulomb gauge. More details can be found in Ref. [44].

The Coulomb gauge A_μ field is promoted to a U(1) link variable $U_\mu^{\text{em}}(n) = e^{ieqA_\mu(n)}$, with q being the electric charge of the quark in units of the positron charge e . The gauge-fixed SU(3) links are multiplied by the U(1) link phases before HISQ smearing is carried out and the solver is called. A different U(1) field is generated for each source time slice used.

IV. DATA ANALYSIS

A. Computing w_0/a

To extract w_0/a on each ensemble, we compute the ensemble average and the uncertainty for the quantity $W(t)$ on the left side of Eq. (2.9) or Eq. (3.2) at all flow times (which are on a grid with spacing Δt). The derivative with respect to t is approximated with a finite difference computed using a five-point stencil that is valid up to order $(\Delta t)^4$. On the grid of points where $W(t)$ is computed, let $\tilde{t}$ be the one where $W(t)$ is closest to the number k appearing on the right-hand side of Eq. (2.9) or Eq. (3.2). We then fit $W(t)$ in the range $[\tilde{t} - n\Delta t, \tilde{t} + n\Delta t]$ with a fourth-order polynomial $p_4(t)$, where the number n ranges from 3 to 6 depending on the ensemble. Solving $p_4(t) = k$ for $t = (w_0/a)^2$ (which generically lies in between the original grid points) then provides $(w_0/a)^2$ to fourth-order accuracy in Δt .

To propagate the statistical uncertainty on w_0/a , we use a jackknife binning procedure. We use 10–200 jackknife bins, depending on the length of the ensemble, and extrapolate the uncertainty to infinite bin size. We also directly measure the integrated autocorrelation time for the action density S_{obs} at the flow time close to $(w_0/a)^2$ on various ensembles, as has been reported earlier [46]. We observe that the extrapolation to infinite bin size provides

the most conservative uncertainty estimate, and we report it as the final statistical uncertainty on w_0/a in Table V.

B. Correlator fits

The correlation functions $C(t)$ for the Ω baryon at source-sink separation² t are fit to the spectral decomposition in the form

$$f(t) = \sum_{i=1}^{N_+} c_i^{(+)} e^{-E_i^{(+)} t} + \sum_{i=1}^{N_-} c_i^{(-)} e^{-E_i^{(-)} t} (-1)^t, \quad (4.1)$$

which has N_+ decaying (nonoscillating) states and N_- oscillating states, which correspond to negative-parity staggered baryons. In this analysis, all fits considered have equal numbers of oscillating and nonoscillating states—that is, $N_+ = N_-$. We define a χ^2 function

$$\chi^2 = \sum_{t,t'} [f(t) - C(t)] \Sigma_{tt'}^{-1} [f(t') - C(t')], \quad (4.2)$$

where $C(t)$ denotes correlator data; Σ is the corresponding sample covariance matrix; and t, t' range over the time slices fitted.

Correlation functions at various source times are first averaged on each gauge configuration. Significant autocorrelations are observed on the source-averaged correlators, corresponding to an integrated autocorrelation time τ_{int} of about 5–10 configurations. To correct for this, correlators are binned by 20–25 configurations depending on the ensemble, after which autocorrelations are negligible in the binned stream. The statistical uncertainty in the binned correlators is observed to be stable with additional binning.

However, some ensembles contain only about 1000 configurations, so only $N_{\text{bin}} \approx 50$ independent samples remain after binning by 20 (as shown in Table I), an insufficient number to estimate the covariance matrix Σ to a sufficient precision to compute its inverse reliably. This can be ameliorated by thinning the data to limit the number of time slices N_{fit} in the fit, thereby ensuring $N_{\text{fit}}^2/2 \lesssim N_{\text{bin}}$. Due to the oscillating states in Eq. (4.1), the fits become unstable if every other time slice is removed (as the oscillating and nonoscillating states become impossible to distinguish). However, removing every third time slice, or two out of every three time slices, preserves the oscillating structure while decreasing N_{fit} below $\sim \sqrt{2N_{\text{bin}}}$.

Priors are added to the fit by augmenting the χ^2 function in Eq. (4.2) with Bayesian priors,

²In this subsection, t refers to distances in the Euclidean time direction of the spacetime lattice, rather than the gradient flow time underlying the observables of Eqs. (3.1) and (3.2).

$$\chi_{\text{aug}}^2 = \chi_{\text{data}}^2 + \sum_i \frac{(c_i - c_i^{(0)})^2}{(\Delta c_i)^2} + \sum_i \frac{(E_i - E_i^{(0)})^2}{(\Delta E_i)^2}, \quad (4.3)$$

where $c_i^{(0)}$ and $E_i^{(0)}$ are prior central values, and Δc_i and ΔE_i are prior widths. In the `corrfit` fitting framework[47–49], these priors are implemented effectively as additional data points—that is, the covariance matrix Σ is augmented to include the prior contribution as well. For numerical stability, a singular-value decomposition cut of 10^{-12} is imposed on the covariance matrix of each ensemble.

The priors are chosen to improve the numerical stability of the fit without biasing the Ω -baryon fit results. Thus, the prior widths are chosen to be much wider than the posterior widths. The central value for the ground-state-energy prior is chosen based on the time-averaged effective mass in the plateau region, where time averaging damps the contribution of the oscillating states [50]. Priors for gaps to excited-state energies are estimated from positive- and negative-parity excitations of the Ω baryon in the Particle Data Group (PDG) tables [10] similarly to Ref. [39]. The overlap factors in the spectral decomposition [i.e., the coefficients of the exponentials in Eq. (4.1)]—nuisance parameters from the standpoint of energy determination—are given very wide priors with widths exceeding 100% of the central value. Figure 1 shows an example fit with the much larger prior for comparison.

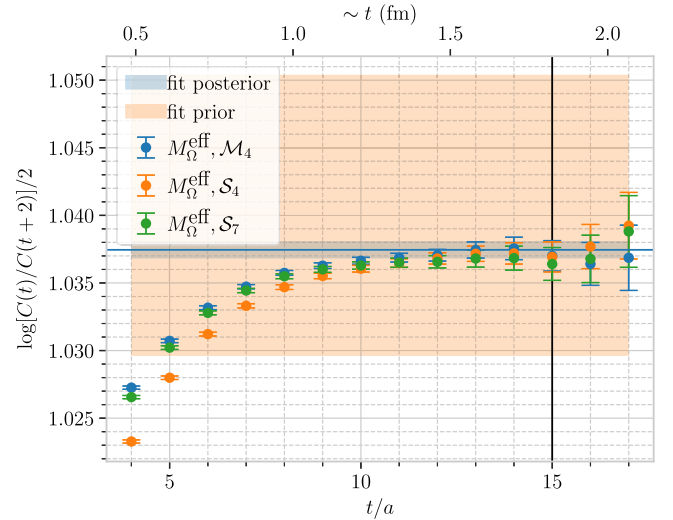


FIG. 1. The three tastes of the Ω baryon (the two Ω_1 multiplets, interpolated by $\mathcal{S}_4$ and $\mathcal{S}_7$, and the single Ω_2 multiplet, interpolated by $\mathcal{M}_4$) on the $a = 0.12$ fm ensemble. Effective masses have been averaged in time via the method in Ref. [50] to suppress oscillatory states. The blue horizontal band shows the Bayesian-model-averaged fit result to the Ω_2 taste. The prior width (yellow) is an order of magnitude larger than the posterior fit width, indicating that the prior is nonconstraining. The vertical line indicates the largest value of t used in the fit.

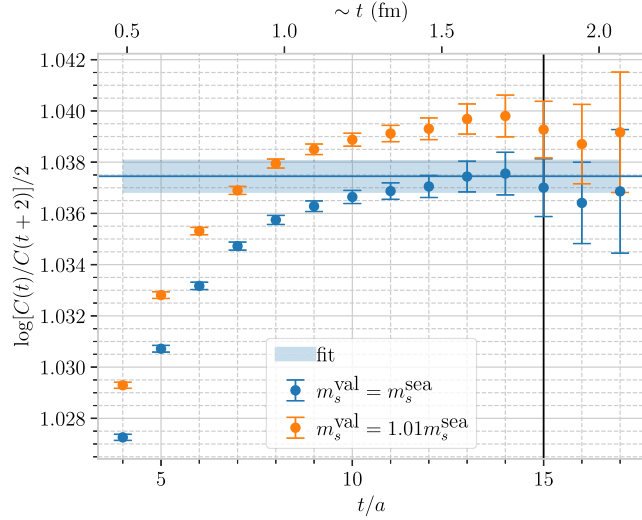


FIG. 2. The variation in the Ω baryon mass with m_s^{val} . Shown are the unitary point $m_s^{\text{val}} = m_s^{\text{sea}}$ and a slightly heavier valence mass $m_s^{\text{val}} = 1.01 m_s^{\text{sea}}$. While the uncertainty on a single-mass fit is comparable to the difference between the two Ω -baryon masses, the two correlation functions are highly correlated (and effective masses fluctuate in tandem), so the correlated difference between the Ω -baryon masses can be resolved to few-percent accuracy.

The maximum time t_{max} in the fit ranges used was fixed at about 1.8 fm, corresponding to a statistical uncertainty in $C(t_{\text{max}})$ of about 0.4% on most ensembles. We then performed a Bayesian model average (BMA) [51,52] over both the number of states in the fit (comparing 1 + 1– and 2 + 2–state fits) and the minimum time t_{min} (varied from about 0.5 fm to $t_{\text{max}} - 5a$, so all fits have at least six data points) using the procedure discussed in Appendix A. All fit parameters are treated as free parameters that count against the total number of degrees of freedom, since the prior widths are large enough to be unconstraining. The results of this procedure are tabulated in Appendix C and used in subsequent fits. As a cross-check, fits with additional states (3 + 3– and 4 + 4–state fits) have also been performed; these are consistent with the 1 + 1– and 2 + 2–state fits on coarser ensembles but numerically unstable on finer ensembles (with fewer configurations) and are therefore excluded from the final averages.

On ensembles with multiple valence strange-quark masses used to study the quark-mass dependence of M_Ω (as shown in Fig. 2), we perform a joint fit to the pair of correlation functions. The results of the joint fits are combined using a Bayesian model average with the same prescription as single fits (albeit with twice as many degrees of freedom from the two correlation functions). This procedure furnishes a model-averaged correlated difference between the masses. Strong correlations between the input hadronic correlation functions result in precise values for the correlated differences, and therefore also the

slope of M_Ω versus m_s . As the individual correlation functions are highly correlated, this correlated difference (and therefore the slope of M_Ω versus m_s) is very precise.

As a cross-check, we have also computed positive-definite correlation functions with identically smeared source and sink. The masses obtained are consistent with those from point sources, but with statistical uncertainties too large to be meaningfully constraining.

Lanczos/Rayleigh-Ritz methods [53–58] that have recently been developed provide further cross-checks of our correlator fits. When this approach is applied to $C(t)$ with $t \in [t_{\text{min}}, t_{\text{min}} + 2m - 1]$, algebraic energy estimators $E_i = -\ln \lambda_i$ are first obtained from Ritz values λ_i , which can be computed by solving generalized eigenvalue problems $(H^{(0)}, H^{(1)})$ of Hankel matrices $H_{ij}^{(p)} \equiv C(t_{\text{min}} + i + j + p)$, as in the generalized-pencil-of-functions method [59–62]. The Ritz values and associated overlap factors c_i provide an exact m -state decomposition of the correlator $f(t) = \sum_{i=1}^m c_i \lambda_i^t$ for this time range. Subsequently, spurious-state filtering is applied to remove spurious Ritz values arising from statistical noise using either the Cullum-Willoughby (CW) test [63,64] or the Z-factor Cullum-Willoughby (ZCW) test [57].

The first step, computing Ritz values, does not require any modifications for staggered fermions [53]. However, ZCW filtering does require slight modification due to negative Ritz values and squared overlaps arising from oscillating states. In particular, the cutoff ϵ_{ZCW} for the minimum overlap of a nonspurious state can be set based on the last iteration where all Ritz values and overlaps are in their physical ranges [57]. For staggered fermions, one should instead consider the last iteration where at least half the states satisfy all physical constraints, as the other half are oscillating states corresponding to negative eigenvalues of the transfer matrix. We then set ϵ_{ZCW} to the minimum normalized overlap product $|c_i/C(t_{\text{min}})|$ from this iteration divided by a hyperparameter F_{ZCW} . After filtering, the ground-state energy estimator is defined as the negative logarithm of the largest Ritz value. To avoid outliers arising from spurious state misidentification, we use bootstrap median estimators whose uncertainties are computed with a nested bootstrap approach [53–57].

We apply this method to all correlator results with $t \in [2, N_t/2]$ and quote the median energy estimator from the last three iterations as a final result (large-iteration results are highly correlated). Filtered Lanczos/Rayleigh-Ritz results using the ZCW test with $F_{\text{ZCW}} = 10$ are statistically consistent with the correlator fits described above. Similar uncertainties are obtained by filtered Lanczos/Rayleigh-Ritz and correlator fits. Consistent results with similar uncertainties are also obtained using the CW test with the cutoff ϵ_{CW} chosen as described in Ref. [53], with default hyperparameters described therein. While these Lanczos fits were not included in subsequent-stage fits, they are in good agreement

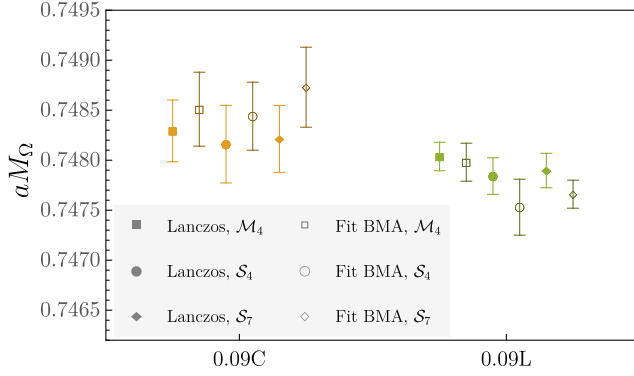


FIG. 3. Comparison of filtered Lanczos/Rayleigh-Ritz and BMA correlator fit results for the two retuned $a = 0.09$ fm ensembles, which differ only in their physical volume, and all three Ω baryon tastes.

with BMA correlator results, as shown in Fig. 3 for the $a \approx 0.09$ fm ensembles.

C. Electromagnetic corrections

Let $M_\Omega(m_v, q_v)$ be the mass of the spin-3/2 baryon made from three identical quarks of bare valence mass m_v and electric charge eq_v , where e is the charge of the positron. We are interested in the electromagnetic contribution to $M_\Omega(m_v, q_v)$,

$$(M_\Omega)^\gamma \equiv M_\Omega\left(\zeta m_s, -\frac{1}{3}\right) - M_\Omega(m_s, 0). \quad (4.4)$$

When the electric charge is nonzero, the additional ultra-violet divergences of QED require a different tuning of the valence mass, and ζm_s is the bare mass corresponding to the strange quark in QCD + QED, while m_s denotes the bare mass corresponding to the strange quark in pure QCD. The precise definition of ζ is connected to the scheme for matching pure (electrically neutral) QCD to the real world of QCD + QED.

In chiral perturbation theory (χ PT) combined with QED, the pseudoscalar meson masses satisfy

$$M_{xy}^2 = B_0(m_x + m_y) + e^2 \Delta_{\text{EM}}(q_x - q_y)^2 + \dots, \quad (4.5)$$

where x and y label the flavors of the valence quark and antiquark, and the ellipsis denotes higher orders in χ PT + QED. This motivates the definition

$$\begin{aligned} v(m_v, q_v) &\equiv M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 \\ &= 2B_0 m_v + 2e^2 \Delta_{\text{EM}}(q_v - q_u)(q_v - q_d) + \dots \\ &= 2B_0 m_v + \dots \end{aligned} \quad (4.6)$$

at valence strange quark mass m_v , which does not depend on m_u or m_d until next-to-leading order (NLO). Since

$q_s = q_d$, the value v is independent of the charges until NLO as well.

The factor ζ can now be fixed by adjusting the quantities $v(\zeta m_s, -\frac{1}{3})$ and $v(m_s, 0)$, respectively, to the experimental QCD + QED value in the Particle Data Group [10] and the pure-QCD FLAG average value [3]:

$$v\left(\zeta m_s, -\frac{1}{3}\right) = v_{\text{PDG}} = 4.7186(2) \times 10^5 \text{ MeV}^2, \quad (4.7)$$

$$v(m_s, 0) = v_{\text{FLAG}} = 4.7103332 \times 10^5 \text{ MeV}^2, \quad (4.8)$$

where the difference between these quantities is indicative of NLO electromagnetic effects in Eq. (4.6). The dependence on the valence strange quark mass can be approximated as

$$v(\zeta m_s, q_v) \approx \zeta v(m_s, q_v), \quad (4.9)$$

so

$$\begin{aligned} \zeta &= \frac{v(\zeta m_s, -\frac{1}{3})}{v(m_s, -\frac{1}{3})} \\ &= \frac{v(m_s, 0)}{v(m_s, -\frac{1}{3})} \frac{v(\zeta m_s, -\frac{1}{3})}{v(m_s, 0)} \\ &= \frac{a^2 v(m_s, 0)}{a^2 v(m_s, -\frac{1}{3})} \frac{v_{\text{PDG}}}{v_{\text{FLAG}}}, \end{aligned} \quad (4.10)$$

where the lattice spacing has been inserted into the first factor as a reminder that it is a ratio of lattice calculations, while the second factor is the ratio of strange-quark-mass tuning conditions. Since the factors of a^2 cancel (as w_0/a and w_0 are independent of valence quark charges), ζ can be obtained without any prescription for the conversion from lattice units to physical units.

It is convenient to abbreviate the mass shift at fixed bare quark mass:

$$\Delta M_\Omega(m_v) \equiv M_\Omega\left(m_v, -\frac{1}{3}\right) - M_\Omega(m_v, 0), \quad (4.11)$$

and we can take the correlated difference between data at the sea strange-quark mass $m_v = m'_s \equiv m_s^{\text{sea}}$. This was computed using strange-quark propagators on $\text{SU}(3) \times \text{U}(1)$ configurations, as discussed in Sec. III D, with $\text{U}(1)$ charge $q = \pm \frac{1}{3}$. After averaging over the positive and negative charges and over the source locations, we obtain the difference in Eq. (4.11) above.

Quark-mass mistuning is accommodated by expanding Eq. (4.4) around m'_s :

$$\begin{aligned}
(M_\Omega)^\gamma &= \Delta M_\Omega(m'_s) + (\zeta m_s - m'_s) \frac{M_\Omega}{m'_s} \frac{\partial \ln M_\Omega}{\partial \ln m'_s} \\
&\quad - (m_s - m'_s) \frac{M_\Omega}{m'_s} \frac{\partial \ln M_\Omega}{\partial \ln m'_s} \\
&= \Delta M_\Omega(m'_s) + (\zeta - 1) M_\Omega \frac{\partial \ln M_\Omega}{\partial \ln m'_s}, \quad (4.12)
\end{aligned}$$

neglecting higher-order terms.³ Now, $\zeta - 1 \sim q_s^2 \alpha \ln am'_s$ grows as $a \rightarrow 0$, so eventually this approximation fails, but even on the finest lattice, where $a \approx 0.04$ fm, $-\ln am'_s \approx 4$ is not yet too big. For taking the continuum limit of $(M_\Omega)^\gamma$, it is natural to work with the relative contribution

$$\frac{(M_\Omega)^\gamma}{M_\Omega} = \frac{\Delta M_\Omega(m'_s)}{M_\Omega} + (\zeta - 1) \frac{\partial \ln M_\Omega}{\partial \ln m'_s}, \quad (4.13)$$

with ζ from Eq. (4.10) and

$$\frac{\partial \ln M_\Omega}{\partial \ln m'_s} \approx \frac{\partial \ln M_\Omega}{\partial \ln v} = 0.24(1) \quad (4.14)$$

taken from pure-QCD global fits below as discussed in Appendix B.

When computing $M_\Omega(m_v, q_v)$ for nonzero q_v , there are power-law (not exponentially) suppressed finite-volume effects that have to be corrected for in both the Ω -baryon and meson masses before combining data for Eq. (4.4) into a continuum limit. Finite-volume corrections $\delta_{\text{EM}}^{\text{FV}} M$ to the mass M of any charged hadron can be estimated using the point-particle approximation [45] truncated at N²LO, before structure-dependent corrections enter—namely,

$$\frac{\delta_{\text{EM}}^{\text{FV}} M}{M} = \frac{\kappa q^2 \alpha_{\text{EM}}}{2ML} \left[1 + \frac{2}{ML} \left(1 - \frac{\pi T}{2\kappa L} \right) \right], \quad (4.15)$$

with $\kappa \approx 2.837297$ defined via an integral expression in Ref. [45] and q being the hadron charge. Note that unlike the finite-volume corrections to the pure-QCD mass, which are exponentially suppressed and subdominant, corrections to ΔM are only power-law suppressed and cannot be altogether neglected. The small difference between NLO and N²LO finite-volume corrections is taken to be a systematic uncertainty on $\delta_{\text{EM}}^{\text{FV}} M$.

The relative correction $(M_\Omega)^\gamma/M_\Omega$ was computed on the three coarsest ensembles in this work (0.15, 0.12, and 0.09C) and then extrapolated to the continuum using a fit function linear in $a^2 \alpha_s$, the leading-order contribution to discretization effects for hadron masses measured using the HISQ action. The extrapolation for the mixed-symmetry,

³The distinction between $M_\Omega(q=0)$ and $M_\Omega(q=-1/3)$ in the term multiplying $(\zeta - 1)$ is higher order in α and the strange mass mistuning $m_s - m'_s$ and is negligible at the precision of this calculation.

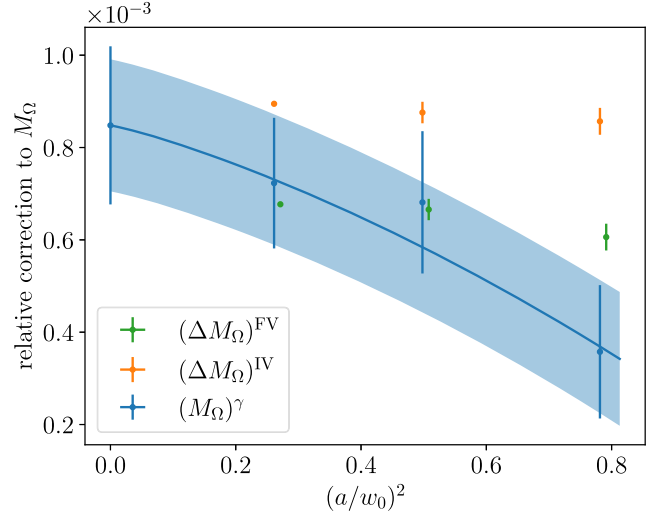


FIG. 4. Relative size of the electromagnetic contribution to the Ω^- mass. The green points represent the difference in the Ω^- mass in QCD + QED versus pure QCD at fixed bare quark masses $\Delta M_\Omega(m'_s)$ with no corrections applied (but a small horizontal offset for clarity). The orange points include finite-volume corrections to electromagnetic effects from Eq. (4.15). The blue points additionally include the effects of quark-mass retuning from Eq. (4.10), including the uncertainty from electroquenching in the meson sector. The blue curve is the continuum extrapolation of $(M_\Omega)^\gamma/M_\Omega$ with all corrections applied, and the continuum limit includes an additional uncertainty from electroquenching in the baryon sector. The case of the Ω_2 state with the SSCc definition of w_0/a is shown; the flavor-symmetric Ω_1 states and other definitions of w_0/a give results consistent within a small fraction of the electroquenching error.

strange-isospin Ω_2 baryon is shown in Fig. 4, giving a continuum-limit value of

$$\frac{(M_\Omega)^\gamma}{M_\Omega} = 8.5(1.7) \times 10^{-4}. \quad (4.16)$$

The result in Eq. (4.16) has been obtained in the electroquenched approximation, in which sea-quark charges are neglected. Errors from this approximation enter both indirectly through the meson masses (and in turn the renormalization factor ζ) as well as directly in sea-quark diagrams contributing to M_Ω . The size of the missing contributions can be estimated in the $1/N_c$ approximation [65].

An important subtlety is that the lattice spacing, as measured by w_0/a , also differs between electroquenched and electrodynamical theories, so inserting factors of a^2 in the final line of Eq. (4.10) is no longer justified. Rather, the arguments above should be thought of as applying to hadron masses with the lattice spacing fixed between the theories—or, more precisely, to the products $w_0 M_\pi$, $w_0 M_K$, and $w_0 M_\Omega$. For the most important product $w_0 M_\Omega$, the electrodynamical effects were explicitly computed in

Ref. [39] and are $O(10^{-4})$, in line with the power-counting estimates above.

For mesons, the ratio $v(\zeta m_s, -\frac{1}{3})/v(m_s, 0)$ differs from its quenched counterpart by effects that one expects to be approximately the valence contribution (i.e., the difference from 1) divided by N_c . Thus, one can approximate the uncertainty here by $[v(\zeta m_s, -\frac{1}{3})/v(m_s, 0) - 1]/N_c$, which we add as a systematic error on ζ .

For baryons, the leading-order QED effects of sea quarks enter through sea-sea diagrams (with the photon attached to two sea quarks) and sea-valence diagrams (with one photon end attached to a sea quark and the other attached to a valence quark). Since the baryon-to-meson electric charge ratio grows like N_c , sea-sea electromagnetic corrections to baryon masses are suppressed by two powers of N_c . Sea-valence diagrams are suppressed by only one power of N_c but are zero in the SU(3) flavor-symmetric limit (since $q_u + q_d + q_s = 0$), so they are also suppressed by meson mass differences that scale as $(M_K^2 - M_\pi^2)/(4\pi f_\pi)^2 \sim 0.2$. One can then take $\max(1/N_c^2, 0.2/N_c) = 1/N_c^2 = 1/9$ as an additional systematic uncertainty in $(M_\Omega)^\gamma$.

Electroquenching is the dominant systematic uncertainty in the value of $(M_\Omega)^\gamma$ in Eq. (4.16), and electrodynamical calculations in subsequent work would be beneficial for removing this systematic. However, since electromagnetism is a small effect in absolute terms, this is a small fraction of the total error budget in the final determination of w_0 below.

Subtracting $(M_\Omega)^\gamma$ from the PDG average value of the experimental (i.e., QCD + QED) Ω^- mass⁴ of 1672.43(32) MeV [10] gives a pure-QCD Ω mass of

$$M_\Omega = 1671.01(43) \text{ MeV}. \quad (4.17)$$

Since the Ω baryon has isospin 0, no correction for isospin breaking is necessary until second order, which is beyond the precision needed for Eq. (4.17). Therefore, this can be used on equal footing with the isospin-symmetric pure-QCD meson masses in Ref. [3],

$$M_\pi = 135.0 \text{ MeV}, \quad (4.18)$$

$$M_K = 494.6 \text{ MeV}, \quad (4.19)$$

to define a pure-QCD world that serves as inputs for the determination of w_0 in pure QCD. This scheme is equivalent to the FLAG scheme [3] (sometimes called the

⁴The PDG lists an average of 1672.43(32) MeV for all measurements of the Ω^- baryon and an average of 1672.5(7) MeV for measurements of its antiparticle $\bar{\Omega}^+$, giving a global fit of 1672.45(29) MeV assuming CPT symmetry. This work uses the former value, averaging only Ω^- data, as in Ref. [22]. The difference is negligible at the current precision, corresponding to a change of about 0.02σ in the extracted w_0 value.

Edinburgh consensus), except that M_Ω is used for scale setting instead of the pion decay constant f_π .

D. Chiral-continuum fits

In a pure-QCD simulation, extrapolation to the physical point requires meson masses to be matched to their electromagnetism-subtracted values given in Eqs. (4.17)–(4.19). The charm quark mass is set using the charm-to-strange mass ratio $m_c/m_s = 11.773(49)$ in Ref. [66]; systematic uncertainties from this mass ratio or from the effects of a mistuned strange quark mass on m_c are negligible at the precision of this work.⁵ With the global scale not known *a priori* in this calculation, one must solve self-consistently for both meson masses and the overall scale from computed values of aM_K , aM_π , aM_Ω in lattice units. Following Ref. [22], we form the dimensionless ratios M_π/M_Ω and M_K/M_Ω and adjust the quark masses to obtain the ratios implied by Eqs. (4.17)–(4.19). To avoid analyst bias, the pure-QCD Ω baryon mass in Eq. (4.17) was multiplied by a blinding factor chosen from a Gaussian with width 0.01 until the analysis was frozen.

To leading order in χ PT, the linear combinations of meson masses proportional to the strange valence mass and the sum of the sea-quark masses are⁶

$$v_\Omega \equiv \frac{2(M_K^{\text{val}})^2 - (M_\pi^{\text{val}})^2}{M_\Omega^2}, \quad (4.20)$$

$$s_\Omega \equiv \frac{2(M_K^{\text{sea}})^2 + (M_\pi^{\text{sea}})^2}{M_\Omega^2}. \quad (4.21)$$

Inserting Eqs. (4.17)–(4.19) yields the physical values

$$v_\Omega^{\text{phys}} = 0.16869(9), \quad (4.22)$$

$$s_\Omega^{\text{phys}} = 0.18175(9), \quad (4.23)$$

where v_Ω is the linear combination of meson masses v in Eq. (4.6) divided by M_Ω^2 .

About the quark-mass-corrected point $(w_0 M_\Omega)^{\text{corr}}$, one can expand $w_0 M_\Omega$ through linear order in the deviations of quark masses from their physical values as

⁵The discrepancy in the computed value of w_0 from this $2+1+1$ -flavor analysis and $2+1$ -flavor calculations [3,14,67] is at most 1%, so the effect of percent-level charm-mass mistunings is $O(10^{-4})$, well below the working precision here.

⁶Note that this work uses s_Ω for the sum of the sea masses rather than the strange mass as in Ref. [22], and also that this work does not square v_Ω and s_Ω in the definitions.

$$(w_0 M_\Omega)^{\text{latt}} = (w_0 M_\Omega)^{\text{cont}} \left[1 + \gamma_v \frac{v_\Omega^{\text{latt}} - v_\Omega^{\text{phys}}}{v_\Omega^{\text{phys}}} + \gamma_s \frac{s_\Omega^{\text{latt}} - s_\Omega^{\text{phys}}}{s_\Omega^{\text{phys}}} + \dots \right] \quad (4.24)$$

for some quark-mass-independent coefficients γ_v, γ_s . Both γ_v and γ_s are nonzero at leading order in χ PT, although the results of this fit find γ_v numerically larger than γ_s by a factor of about 5.

The mass-corrected values of $w_0 M_\Omega$ are related to their continuum values via the expansion

$$(w_0 M_\Omega)(a) = (w_0 M_\Omega)^{\text{cont}} [1 + c_1 (a/w_0)^2 \alpha_s(a)^{\{0,1\}} + c_2 (a/w_0)^4 + c_3 (a/w_0)^6 + \dots], \quad (4.25)$$

defining the lattice spacing explicitly via (one of the definitions of) w_0/a . Only even powers of a appear, because staggered fermions obey a taste nonsinglet chiral symmetry.

The improvement to staggered fermions from the Naik term suppresses the a^2 term in aM_Ω by a factor of α_s , which approaches zero logarithmically in the continuum limit. Discretization effects in the flow scale w_0/a , however, begin at $O(a^2)$ with no additional factors of $\alpha_s(a)$. As a result, fits with and without this power of α_s are included in the BMA.⁷

At finite lattice spacing, the quark-mass corrections can also depend on the lattice spacing, so one can write, for example,

$$\gamma_v(a) = \gamma_v^{(0)} [1 + \gamma_v^{(1)} (a/w_0)^2 \alpha_s(a)^{\{0,1\}} + \gamma_v^{(2)} (a/w_0)^4 + \dots], \quad (4.26)$$

and similarly for γ_s . However, since γ_s is numerically smaller than γ_v , and the computed pion-mass dependence of M_Ω at $a = 0.12$ fm and $a = 0.09$ fm are consistent (within $\sim 50\%$ uncertainties), discretization effects in γ_s are neglected,⁸ and this work sets $\gamma_s(a) = \gamma_s^{(0)}$. Assembling all the pieces, one has

$$(w_0 M_\Omega)^{\text{latt}}(a, m_{\text{val}}, m_{\text{sea}}) = (w_0 M_\Omega)^{\text{cont}} \left\{ 1 + c_1 (a/w_0)^2 \alpha_s(a)^{\{0,1\}} + c_2 (a/w_0)^4 + c_3 (a/w_0)^6 + \gamma_v^{(0)} [1 + \gamma_v^{(1)} (a/w_0)^2 \alpha_s(a)^{\{0,1\}} + \gamma_v^{(2)} (a/w_0)^4] \frac{v_\Omega^{\text{latt}} - v_\Omega^{\text{phys}}}{v_\Omega^{\text{phys}}} + \gamma_s^{(0)} \frac{s_\Omega^{\text{latt}} - s_\Omega^{\text{phys}}}{s_\Omega^{\text{phys}}} \right\}. \quad (4.27)$$

The lattice spacing a appearing as the argument of α_s must also be made explicit, and we choose

$$a \equiv \frac{(w_0 M_\Omega)^{\text{cont}}}{M_\Omega^{\text{phys}}} \frac{a}{w_0} \quad (4.28)$$

so that the right side of Eq. (4.27) is a (nonlinear) function of the fit coefficients $(w_0 M_\Omega)^{\text{cont}}, c_1, c_2, c_3, \dots$, the calculated values of w_0/a on the various ensembles, and the quark-mass corrections. The value of α_s as a function of lattice spacing was obtained by running from $\alpha_s(5 \text{ GeV}) = 0.2530$ [69] to the scale of $2/a$ at four-loop order using the package QCDEVOL [70].

As with the electromagnetic correction, we apply the BMA procedure to obtain a final result for $(w_0 M_\Omega)^{\text{cont}}$. To

assess the impact of discretization effects, we perform the following variations in the fit function and dataset:

- (1) Order of the discretization effects to the continuum extrapolation: $\{c_1\}, \{c_1, c_2\}, \{c_1, c_2, c_3\}$.
- (2) Order of the discretization effects to the valence-mass correction: $\{\gamma_v^{(0)}\}, \{\gamma_v^{(0)}, \gamma_v^{(1)}\}, \{\gamma_v^{(0)}, \gamma_v^{(1)}, \gamma_v^{(2)}\}$.
- (3) Range of data included: $[0.042, 0.15] \text{ fm}, [0.042, 0.12] \text{ fm}, [0.042, 0.09] \text{ fm}$.

Because the maximum sizes of the $\{c_i\}$ and $\{\gamma_v^{(i)}\}$ sets are reduced when excluding coarser lattices, the variations above correspond to a total of 14 physical-point fit variations. As above, we also repeat the physical-point extrapolation for the 12 definitions of w_0/a and three tastes of interpolating operators used for aM_Ω . In total, this gives 36 possible datasets times 14 fit functions that are averaged to the final continuum-limit result⁹ for $w_0 M_\Omega$.

Some representative fits are shown in Fig. 5, with the full set of these variations, as well as the changes in fit model,

⁷At the time of unblinding, we did not appreciate the fact that w_0/a does not include a factor of $\alpha_s(a)$, even in schemes where the flow is improved. Until then, only fits including the factor of α_s were included in the BMA. In post-unblinding discussions, this oversight was pointed out [68], after which we began to incorporate fits without α_s into the BMA. This change shifted the original result for w_0 by about 0.1σ and increased the uncertainty by about 10%.

⁸As a cross-check, we also performed the BMA including fits with the term $\gamma_s^{(1)} a^2 \alpha_s$; this shifted the central value by 0.02σ and left the uncertainty essentially unchanged.

⁹The Jupyter notebook implementing these fits and the BMA over them can be found at <https://github.com/agrebe/omega-scale-setting>.

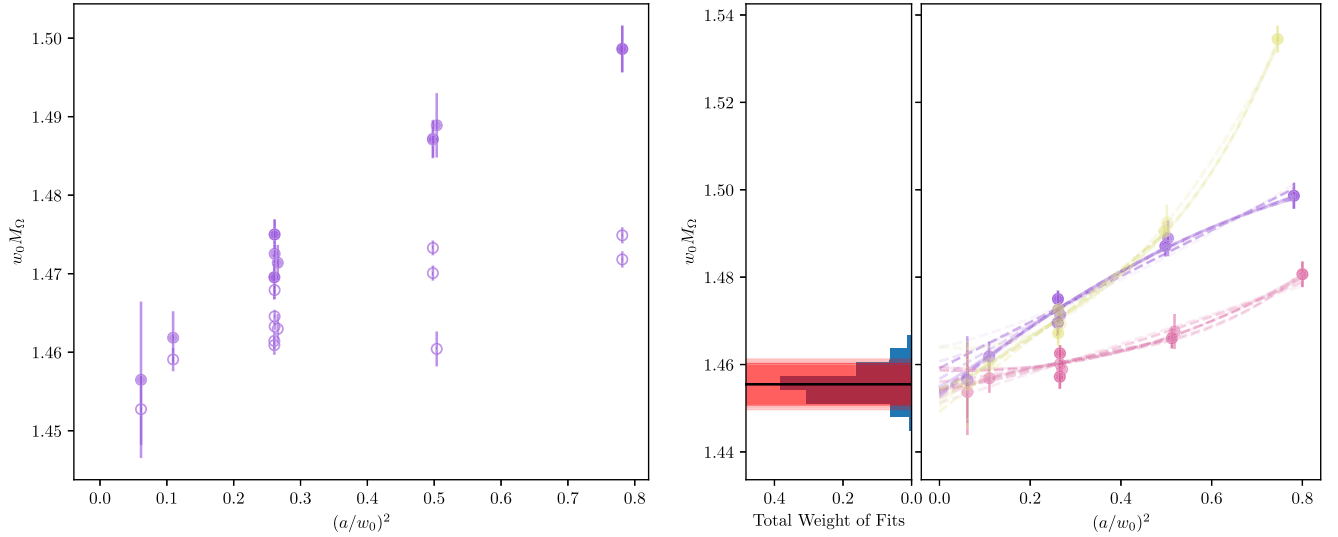


FIG. 5. Left: The impact of corrections for quark-mass retuning on M_Ω . The uncorrected values of aM_Ω (using the Ω_2 interpolating operators), multiplied by the corresponding w_0/a values (in one of the 12 schemes for concreteness), are shown in open circles on the nine ensembles. The filled circles show quark-mass-corrected data points, with the corrections applied using the highest-weight fit. Note that at a given lattice spacing, applying the quark-mass corrections significantly improves the agreement among various sea and valence masses but also inflates the uncertainties, as discussed in Appendix B. Right: The various extrapolations of $w_0 M_\Omega$ to the continuum. The left panel shows a histogram in blue of the continuum-limit values of $w_0 M_\Omega$, weighted by the expression in Eq. (A2) applied to the fit generating that value. Overlaid on the histogram are a black line representing the central value of $(w_0 M_\Omega)^{\text{cont}}$ and red and pink bands showing its statistical and total uncertainties. The right panel shows representative fit curves from three w_0/a definitions (colored to match the w_0/a definitions listed in Fig. 6) and the 14 physical-point extrapolation models shaded proportionally to their model weights, with the model with the highest overall weight (quadratic fit in a^2 with $v_\Omega a^2$ mass corrections to all lattice spacings) for each w_0/a definition shown with a solid line.

shown in Fig. 6. For each class of submodel (namely, variations on the details of the continuum extrapolation, the mass corrections, the w_0/a discretization, the Ω baryon taste, and the power of α_s), a set of results is shown for the various choices within that submodel with Bayesian model averaging over the remaining submodels. The full space of choices within the different submodels weighted by the corresponding subset probabilities is used in the global BMA [71]. Variation among different models and datasets is relatively mild, and the uncertainty budget is dominated by the statistical uncertainty in the data.

E. Uncertainty budget

In this section, we present estimates of the contributions of various statistical and systematic effects to the final quoted uncertainty of 0.4% in w_0 in Eq. (5.2). These are summarized in Table II. Global fit systematics reflect the deviations of the various submodels from the model average, as prescribed in Ref. [51]. The other uncertainties can be approximated by turning off the corresponding uncertainty in the input to the final-stage fit, keeping all submodel weights fixed, and determining the change in

variance that would arise from eliminating that source of uncertainty. As a cross-check, the contributions from the various components sum in quadrature to about 0.39%, in good agreement with the total 0.40% uncertainty.

The dominant uncertainties are statistical errors in the fits to correlation functions and systematic errors in the extrapolation to the physical point. These can be further reduced by additional data, especially on the finest $a = 0.04$ fm ensemble.

QCD finite-volume corrections on M_Ω are negligible: the contribution from kaon loops is proportional to $\exp(-M_K L) \lesssim \exp(-12) < 10^{-5}$, and contributions from pion loops are proportional to $(M_\pi/4\pi f_\pi)^4 e^{-M_\pi L}/(M_\pi L)^{3/2} \approx 10^{-6}$ [22]. Statistical consistency between the QCD contributions on the two volumes used at 0.09 fm confirms this prediction of negligible finite-volume corrections, albeit at the level of 10^{-3} , which can be seen comparing the two volumes shown in Fig. 3. Finite-volume corrections to electromagnetism are larger but are well controlled by Eq. (4.15); truncation errors from this approximation are included in the electromagnetism uncertainty but are much smaller than the electroquenching error. Strong-isospin breaking effects are also negligible, as the Ω

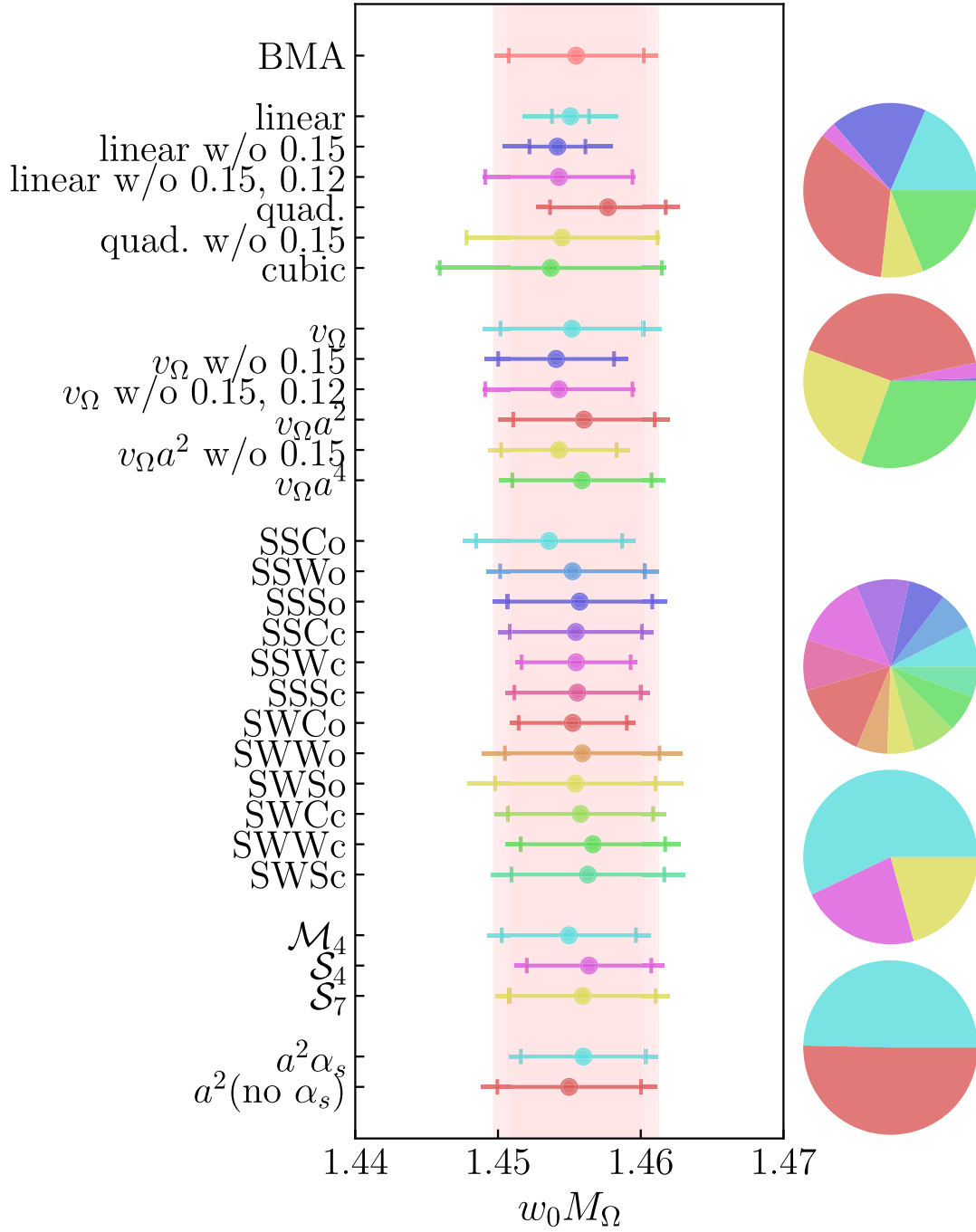


FIG. 6. Summary of model variations in the fits used to extract $w_0 M_\Omega$ at the physical point. The different groupings represent the order of the continuum extrapolation, the order of discretization effects included in the valence-mass correction based on v_Ω , the gradient flow used (Symanzik or Wilson for the flow action; improved or unimproved; and clover, Wilson, or Symanzik for the flow observable), the representation of the Ω baryon used (mixed representation, or subgroups 4 or 7 of the symmetric representation), and the power of α_s in the leading term in the continuum extrapolation. Inner error bars represent statistical uncertainties, and outer bars also include systematic uncertainties from varying other submodels. The pie charts on the right represent the relative weight of each submodel within the corresponding class of variations.

TABLE II. Estimates of the approximate impact of various uncertainties on the total uncertainty budget. Quantities are shown as a percentage of the final quoted value of w_0 (fm).

Source	Uncertainty (%)
Correlator fits (stat. + sys.)	0.32
Gradient flow (stat.)	0.04
Experimental hadron masses	0.02
Electroquenching	0.02
Global fit systematics	0.22
Total	0.40

baryon is isospin-0, so effects only enter at second order in $(m_u - m_d)/\Lambda_{\text{QCD}}$ and are therefore estimated to be $\approx 10^{-4}$.

V. CONCLUSIONS AND OUTLOOK

We obtain from the analysis of the previous section the following QCD value for the Ω baryon mass in units of the gradient-flow scale:

$$w_0 M_\Omega = 1.4555(47)(32)[57]. \quad (5.1)$$

The error in the first set of parentheses is from correlator fits (statistics and fit systematics), while the error in the second set of parentheses is from all other systematics. The final error in brackets is their sum in quadrature and constitutes the total error from the BMA over chiral-continuum fit variations. Combining Eq. (5.1) with our QCD value of $M_\Omega = 1671.01(43)$ MeV [Eq. (4.17)], we obtain for the physical value of the gradient-flow scale

$$w_0 = 0.17187(56)(38)[68] \text{ fm}, \quad (5.2)$$

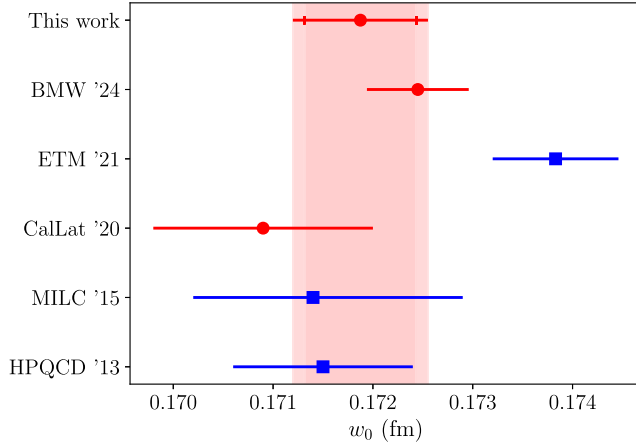


FIG. 7. Comparison of the result in this work with those of previous calculations of w_0 with $2 + 1 + 1$ flavors. The inner and outer bands depict statistical and total uncertainties. Calculations defined via $M_\Omega (f_\pi)$ are shown in red (blue). Previous results are BMW '24 [72], ETM '21 [73], CalLat '20 [22], MILC '15 [35], and HPQCD '13 [74].

with the errors in parentheses and brackets defined analogously as before. This is our result in both QCD + QED and pure QCD. The electroquenched approximation used here is insensitive to the difference, and the uncertainty covers the error from electroquenching.

We compare our result for the gradient-flow scale in Eq. (5.2) with those of previous calculations in Fig. 7. This work achieves uncertainties competitive with the most precise determinations by BMW [39,72] and ETM [73]. Our final w_0 value agrees well with the most recent $(2 + 1 + 1)$ -flavor results from BMW [72] and CalLat [22] using the Ω^- baryon, as well as older results from MILC [35] and HPQCD [74] using the pion decay constant, f_π . (Superseded calculations by BMW [39] and ETM [75] are also consistent.) There is some tension, however, between the most recent ETM result [73], determined using f_π scale setting, and all other scale-setting results, including the one reported here.

En route to Eq. (5.2), we determined a pure-QCD value of the Ω baryon mass, consistent with the Edinburgh consensus of FLAG [3]. In ongoing work, we will determine the continuum limit of f_π/M_Ω in pure QCD and in QCD + QED. In future work, we plan to compute the electromagnetic corrections stemming from valence-sea and sea-sea photon exchange, thereby removing the electroquenching uncertainties in M_Ω . We will also improve statistics, especially on the finest ($a \approx 0.04$ fm) ensemble, to improve the precision of the continuum limit.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: BAYESIAN MODEL AVERAGING

Various correlator fits are combined using the Bayesian model-averaging (BMA) procedure introduced in Refs. [45,51,52,80,81]. Specifically, for a given model i , one defines the Bayesian Akaike information criterion (BAIC), a modified Akaike information criterion to account for data excluded from the fit range [52,82], as

$$\text{BAIC}_i = \chi^2_{\text{data}}(\mathbf{a}^*) + 2k + 2N_{\text{cut}}, \quad (\text{A1})$$

where $\chi^2_{\text{data}}(\mathbf{a}^*)$ is the contribution to the χ^2 function from the data (as opposed to the priors) evaluated at the minimum $\mathbf{a}^*$ of the augmented χ^2 function (including priors) [83], k is the number of fit parameters, and N_{cut} is the number of data points removed from the fit range. The weight of the model is defined as its probability given the data, computed as

$$w_i \propto \exp \left[-\frac{1}{2} \text{BAIC}_i \right], \quad (\text{A2})$$

normalized such that $\sum_i w_i = 1$. The model average of the ground-state energies E_i measured on the various models is given by

$$\langle E \rangle = \sum_i w_i E_i \quad (\text{A3})$$

with statistical and systematic uncertainties given by

$$(\sigma_E^2)_{\text{stat}} = \sum_i w_i (\sigma_E^2)_i, \quad (\text{A4})$$

$$(\sigma_E^2)_{\text{sys}} = \sum_i w_i (E_i - \langle E \rangle)^2, \quad (\text{A5})$$

which are added in quadrature to give the full uncertainty σ_E^2 .

APPENDIX B: ERROR AMPLIFICATION FROM QUARK-MASS CORRECTIONS

While γ_v and γ_s are nuisance parameters from the standpoint of scale setting, they are physical quantities that encode the variation of the Ω baryon mass and w_0 with quark masses. Performing a BMA over fits gives

$$\gamma_v^{(0)} = 0.55(5), \quad (\text{B1})$$

$$\gamma_s^{(0)} = 0.08(3). \quad (\text{B2})$$

The error bars contain both statistical and systematic uncertainties, and the values are of magnitudes similar to those computed from single-ensemble fits in Ref. [84]. The strange-quark mass dependence $d \ln M_\Omega / d \ln m_s \approx \gamma_v / (1 + 2\gamma_v + 2\gamma_s) = 0.24(1)$ can be computed from these quantities. Recall that this logarithmic derivative is needed to obtain the mass-matching factor ζ between QCD + QED and QCD. In practice, a few iterations of the whole procedure are needed until it stabilizes.

This substantial dependence of M_Ω on quark masses increases the uncertainty in quark-mass-corrected values of M_Ω . Suppose that aM_Ω has been determined on a given ensemble with relative uncertainty δM_Ω . Neglecting the (much smaller) uncertainties in meson masses, this induces a relative uncertainty of

$$\delta v_\Omega = \delta s_\Omega = -2\delta M_\Omega \quad (\text{B3})$$

in the proxies used for valence and sea-quark masses. Thus, the quark-mass-corrected Ω -baryon mass M_Ω^{corr} has an uncertainty given by

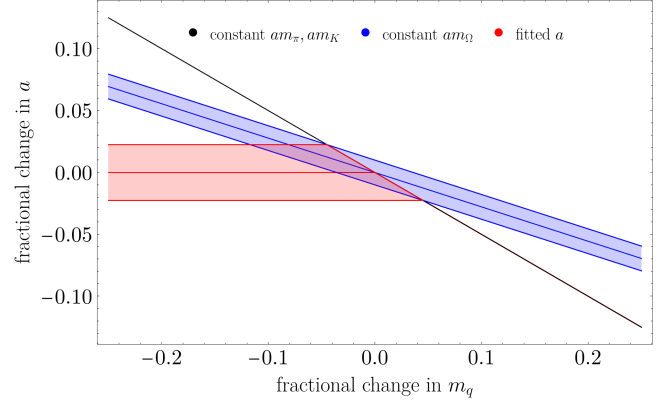


FIG. 8. An illustration of the error inflation inherent in self-consistently solving for the lattice spacing and quark masses using the Ω -baryon and pseudoscalar meson masses. The black and blue lines represent contours of constant aM_{PS} and aM_Ω , respectively, which have slopes $-1/2$ and $-(\gamma_v + \gamma_s)/(1 + 2\gamma_v + 2\gamma_s) \approx -0.28$ from leading-order χ PT when plotted in the $(\log m_q, \log a)$ plane. Assuming that the mesons can be measured precisely, the uncertainty in the quantity of interest (a) comes entirely from that in M_Ω , where the blue band represents a 1% relative uncertainty in the measurement. Propagating this uncertainty through the self-consistent solution of equations to extract a inflates this uncertainty by a factor $1 + 2\gamma_v + 2\gamma_s \approx 2.2$ indicated by the red band.

$$\delta M_\Omega^{\text{corr}} = \delta M_\Omega - \gamma_v \delta v_\Omega - \gamma_s \delta s_\Omega \quad (\text{B4})$$

$$= (1 + 2\gamma_v + 2\gamma_s) \delta M_\Omega; \quad (\text{B5})$$

that is, the original uncertainty has been amplified by the factor $1 + 2\gamma_v + 2\gamma_s = 2.2(1)$, leading to the larger error bars in the quark-mass-corrected values in the left panel of Fig. 5.

This error-inflation factor can be thought of as the Jacobian of the change of basis between the measured quantities $\{aM_\Omega, (am_{\text{val}})^2, (am_{\text{sea}})^2\}$ to the desired results $\{a, m_{\text{val}}, m_{\text{sea}}\}$. In particular, this factor is present even if γ_v and γ_s are known exactly, and even if the ensembles are well-enough tuned that the quark-mass corrections are consistent with zero. This uncertainty inflation is illustrated graphically in Fig. 8.

APPENDIX C: RESULTS FOR aM_Ω AND w_0/a

This appendix tabulates results for the Ω -baryon mass and gradient-flow scale on each ensemble. Results for aM_Ω from the correlator fits of Sec. IV B are given in Table III—that is, without strange-quark-mass retuning. To aid the retuning, meson masses (in lattice units) are provided in Table IV. Results for the gradient flow w_0/a for each of the 12 definitions, as described in Sec. II A, are listed in Table V.

TABLE III. Results for aM_Ω from the correlator fits of Sec. IV B for the three tastes of the Ω baryon. For ensembles with two valence masses, the result for the unitary valence mass is given first, followed by that for the heavier-than-unitary valence mass, and the correlation coefficient between the two results is given in brackets.

$\approx a/\text{fm}$	aM_Ω^{mixed}	aM_Ω^{44}	aM_Ω^{77}
0.15	$\left\{ \begin{array}{l} 1.30102(88) \\ 1.30374(87) \\ [0.999938] \end{array} \right.$	$\left\{ \begin{array}{l} 1.30017(97) \\ 1.30289(96) \\ [0.999946] \end{array} \right.$	$\left\{ \begin{array}{l} 1.29991(90) \\ 1.30264(89) \\ [0.999918] \end{array} \right.$
0.12H	1.0366(14)	1.0352(17)	1.0357(13)
0.12	$\left\{ \begin{array}{l} 1.03745(65) \\ 1.03971(63) \\ [0.999219] \end{array} \right.$	$\left\{ \begin{array}{l} 1.03718(46) \\ 1.03946(45) \\ [0.999527] \end{array} \right.$	$\left\{ \begin{array}{l} 1.03717(62) \\ 1.03945(62) \\ [0.999367] \end{array} \right.$
0.09H	0.75443(82)	0.75382(93)	0.75421(84)
0.09M	$\left\{ \begin{array}{l} 0.74613(60) \\ 0.74641(61) \\ [0.999917] \end{array} \right.$	$\left\{ \begin{array}{l} 0.74598(49) \\ 0.74626(49) \\ [0.999646] \end{array} \right.$	$\left\{ \begin{array}{l} 0.74647(40) \\ 0.74675(40) \\ [0.999952] \end{array} \right.$
0.09C	$\left\{ \begin{array}{l} 0.74851(37) \\ 0.75023(36) \\ [0.998037] \end{array} \right.$	$\left\{ \begin{array}{l} 0.74844(34) \\ 0.75014(34) \\ [0.999798] \end{array} \right.$	$\left\{ \begin{array}{l} 0.74873(40) \\ 0.75042(39) \\ [0.999240] \end{array} \right.$
0.09L	0.74798(19)	0.74753(28)	0.74766(14)
0.06	0.48257(48)	0.48249(70)	0.48266(70)
0.04	0.3599(11)	0.36040(64)	0.36014(96)

TABLE IV. Results for the sea-meson masses used for quark-mass tuning. Values on the physical-point ensembles (0.15, 0.12, 0.09M, 0.09C, 0.06, and 0.04 fm) are taken from Ref. [24] and references therein. Values on the heavier-than-physical ensembles 0.12H and 0.09H are taken from Refs. [25,7], respectively. The value on the large-volume ensemble 0.09L is new. In all cases, values are determined from multi-exponential fits to the truncated spectral decomposition using methods and parameters described in Refs. [7,25].

$\approx a/\text{fm}$	aM_π	aM_K
0.15	0.103414(11)	0.37847(11)
0.12H	0.13426(8)	0.30806(12)
0.12	0.0830651(63)	0.303949(77)
0.09H	0.09860(13)	0.22706(15)
0.09M	0.057184(30)	0.219482(70)
0.09C	0.060069(26)	0.220231(40)
0.09L	0.0600523(72)	0.220225(13)
0.06	0.038842(29)	0.142607(51)
0.04	0.028981(18)	0.106297(39)

TABLE V. Results for the gradient-flow scale w_0/a for each of the 12 combinations of flow, improvement, and operator.

$\approx a/\text{fm}$	Flow	Correction	Clover	Wilson	Symanzik
0.15	Symanzik	No	1.14272(18)	1.13067(18)	1.12784(19)
		Yes	1.13128(20)	1.12168(20)	1.11795(19)
	Wilson	No	1.13227(18)	1.15111(18)	1.15825(19)
		Yes	1.13359(21)	1.14303(19)	1.14421(19)
0.12H	Symanzik	No	1.40407(82)	1.39489(85)	1.39251(86)
		Yes	1.40891(87)	1.39402(87)	1.38916(87)
	Wilson	No	1.40302(83)	1.40936(86)	1.41230(86)
		Yes	1.41301(87)	1.40686(87)	1.40466(87)
0.12	Symanzik	No	1.41156(27)	1.40262(28)	1.40033(28)
		Yes	1.41701(29)	1.40198(29)	1.39707(29)
	Wilson	No	1.41060(28)	1.41729(29)	1.42035(29)
		Yes	1.42110(30)	1.41489(29)	1.41267(29)
0.09H	Symanzik	No	1.92854(130)	1.92405(132)	1.92285(133)
		Yes	1.93917(134)	1.92691(134)	1.92291(133)
	Wilson	No	1.93328(131)	1.93522(134)	1.93622(135)
		Yes	1.94453(135)	1.93568(135)	1.93279(134)
0.09M	Symanzik	No	1.94664(40)	1.94251(40)	1.94144(41)
		Yes	1.95796(42)	1.94563(41)	1.94161(41)
	Wilson	No	1.95148(41)	1.95386(42)	1.95501(42)
		Yes	1.96328(42)	1.95444(42)	1.95155(42)
0.09C	Symanzik	No	1.94537(56)	1.94121(57)	1.94012(58)
		Yes	1.95663(58)	1.94430(58)	1.94028(58)
	Wilson	No	1.95021(57)	1.95255(58)	1.95368(58)
		Yes	1.96195(59)	1.95312(58)	1.95023(58)
0.09L	Symanzik	No	1.94506(30)	1.94091(31)	1.93982(31)
		Yes	1.95632(31)	1.94400(31)	1.93999(31)
	Wilson	No	1.94989(31)	1.95223(32)	1.95336(32)
		Yes	1.96163(32)	1.95280(31)	1.94991(32)
0.06	Symanzik	No	3.01269(91)	3.01215(92)	3.01207(92)
		Yes	3.02358(92)	3.01587(92)	3.01333(92)
	Wilson	No	3.01838(92)	3.01978(92)	3.02034(92)
		Yes	3.02786(93)	3.02116(92)	3.01896(92)
0.04	Symanzik	No	4.02741(194)	4.02772(194)	4.02786(194)
		Yes	4.03663(195)	4.03099(195)	4.02912(195)
	Wilson	No	4.03242(195)	4.03358(196)	4.03402(196)
		Yes	4.04009(196)	4.03490(196)	4.03318(196)

APPENDIX D: CONTINUUM-EXTRAPOLATED $w_0 M_\Omega$ SCHEMES

While different definitions of w_0/a (characterized by different choices of the flow and observable) disagree at finite lattice spacing, they should all provide the same

continuum limit for $w_0 M_\Omega$. Nevertheless, to facilitate comparisons with previous work using a specific definition of w_0/a , we tabulate in Table VI values using the 12 different definitions. They are shown in Fig. 6, and their Bayesian model average gives the overall result in the main text.

TABLE VI. Continuum extrapolations of $w_0 M_\Omega$ for the 12 combinations of flow, improvement, and operator described in Sec. II A.

Flow	Correction	Clover	Wilson	Symanzik
Symanzik	No	1.4536(60)	1.4552(61)	1.4557(61)
	Yes	1.4555(55)	1.4555(43)	1.4556(51)
Wilson	No	1.4552(44)	1.4559(70)	1.4554(76)
	Yes	1.4558(60)	1.4566(61)	1.4563(68)

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