

Gross–Llewellyn Smith sum rule with an analytic coupling

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We investigate the Gross–Llewellyn Smith sum rule within the framework of analytic quantum chromodynamics (QCD). A comparison is performed between experimental data, lattice calculations, and perturbative QCD based on conventional and analytic versions of perturbation theory with different parametrizations for the higher twist contributions. We show that conventional perturbation theory fails to reproduce the data, while the analytic version demonstrates good agreement with experiment and lattice calculations. We also discuss the relation between the Gross–Llewellyn Smith sum rule and the Bjorken sum rule.

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I. INTRODUCTION

The parity-odd structure function F_3 , which contributes to deep inelastic neutrino scattering, plays an important role in understanding processes related to both quantum chromodynamics (QCD) and electroweak physics. Its first Mellin moment determines the Gross–Llewellyn-Smith (GLS) sum rule [1]

$$C_{GLS}^{p+n}(Q^2) = \frac{1}{2} \int_0^1 [F_3^{\bar{\nu}p}(x, Q^2) + F_3^{\nu n}(x, Q^2)] dx, \quad (1)$$

which in the quark-parton model counts the number of valence quarks in the proton. Perturbatively computed radiative corrections serve as an important test of the suitability of QCD as a theory of strong interactions.

Since F_3 is a nonsinglet quantity, the GLS sum rule has a vanishing anomalous dimension at leading order (LO) of perturbation theory (PT). The GLS sum rule is unique in that the nucleon matrix element of the leading twist is known analytically, and the Wilson coefficient is known up to next-to-next-to-next-to-leading order (N³LO) [2,3], making it, along with the Bjorken sum rule (BSR) [4], an ideal

laboratory for determining the strong coupling constant α_s at the boundary of the nonperturbative regime. However, the available experimental data, unlike the BSR case (see the recent paper [5] and references therein), are rather sparse. In light of this, recent lattice QCD calculations [6] have indeed made valuable contributions to the study of the GLS sum rule.

It is also worth mentioning the contributions of a purely nonperturbative origin—the higher twists. Their structure is dictated by the OPE, whereas the corresponding numerical coefficients can, at present, only be extracted by fitting experimental data. The inverse-power dependence on Q^2 makes higher-twist terms indispensable ingredients of QCD sum rules in the IR regime [7]; omitting them makes any purely perturbative expansion unreliable. Consequently, both the commonly used parametrizations of higher twists and their later modifications [8,9] require careful check. Finally, we should mention the widely discussed relation between the two (both nonsinglet) sum rules considered above: the BSR and the GLS sum rule [10,11] their twist-2 parts, the perturbative coefficient functions (in the $\overline{MS}$ scheme) differ only by an overall normalization factor up to $O(\alpha_s^3)$, with the first differences arising from specific light-by-light-type diagrams. Moreover, the authors of [11] have demonstrated that the twist-four coefficients for both BSR and GLS have the same sign and are close in magnitude. Unfortunately, the accuracy with which these theoretical results can be tested is limited both by the scarcity of available

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experimental data and by issues related to the perturbative running coupling, which we will discuss below.

The rest of the paper is organized as follows: in Secs. II and III we present the basic formulas for the GLS and Bjorken sum rules and discuss their similarities. We present and discuss our results in Sec. IV, and provide a summary and concluding remarks in Sec. V. The Appendix provides a brief overview of the analytic QCD coupling.

II. GROSS-LLEWELLYN SMITH SUM RULE

The GLS sum rule is defined by the integral (1). The quantity $C_{GLS}^{p+n}(Q^2)$ can be expressed in the operator product expansion (OPE) form

$$C_{GLS}^{p+n}(Q^2) = 3(1 - D(Q^2)) + \sum_{n=2} \frac{\mu_{2n}}{Q^{2n-2}}, \quad (2)$$

where $(1 - D(Q^2))$ is the leading twist (or twist-two) contribution, and the coefficients μ_{2n} of the higher-twist terms are free parameters that must be determined from experimental data.

In this paper, we particularly focus on rather low Q^2 values, so it is possible to use the so-called “massive” (which is stemming from what may be called “spectral”) twist-four representation (see Refs. [8,9]), we have

$$\bar{C}_{GLS}^{p+n}(Q^2) = 3(1 - D(Q^2)) + \frac{\hat{\mu}_{2n} M^2}{Q^2 + M^2}. \quad (3)$$

The spectral representation comes [8] from the parametrization of higher twist terms in (2)

$$\mu_{2n} = (-)^n \hat{M}^{2n-2} \int \rho(x) x^{n-2} dx, \quad (4)$$

where $\hat{M}$ is an arbitrary mass scale of order of the hadron one, and $\rho(x)$ is the dimensionless spectral function of the dimensionless auxiliary integration parameter whose physical meaning will be discussed later, as well as integration limits. Substituting (4) into (2) and interchanging the order of summation and integration one gets

$$\sum_{n=2} \frac{\mu_{2n}}{Q^{2n-2}} = \int dx \rho(x) \frac{\hat{M}^2}{Q^2 + x \hat{M}^2}. \quad (5)$$

The resulting expression has the correct analytical properties [the cut at *negative* Q^2 , which was the reason to introduce the sign factor in (4)] provided spectral function is positive, and integration is performed in the region of positive x . Such analytic properties were the main motivation of [8]. It seems natural to combine the introduced spectral representation of higher twists with APT to guarantee the correct analytic properties of the whole result.

To clarify the physical meaning of the variable x it is instructive to introduce the new variable $s = x \hat{M}^2$, so that

$$\sum_{n=2} \frac{\mu_{2n}}{Q^{2n-2}} = \int ds \frac{\rho\left(\frac{s}{\hat{M}^2}\right)}{Q^2 + s}, \quad (6)$$

which has now the manifestly spectral form with s being the invariant mass in virtual photon channel. This suggests the simplest ansatz which in fact was used in [9]

$$\rho(x) \rightarrow \text{const} \delta(x - x_0); \quad x_0 = \frac{M^2}{\hat{M}^2}, \quad (7)$$

where M is a mass of single resonance (typically, ρ -meson). This justifies the use of “massive” high-twist [see Eq. (3)].

Note that, as in the case of (2), the values of M and the high-twist terms $\hat{\mu}_4$ are free parameters that must be fitted from experimental data.

Up to the k th order of PT (hereafter we use $\overline{MS}$ -scheme), the twist-two parts have the form

$$D^{(1)}(Q^2) = \frac{4}{\beta_0} a_s^{(1)},$$

$$D^{(k \geq 2)}(Q^2) = \frac{4}{\beta_0} a_s^{(k)} \left(1 + \sum_{m=1}^{k-1} d_m (a_s^{(k)})^m \right), \quad (8)$$

where

$$a_s^{(k)}(Q^2) = \frac{\beta_0 \alpha_s^{(k)}(Q^2)}{4\pi} = \beta_0 \bar{a}_s^{(k)}(Q^2) \quad (9)$$

and d_1 , d_2 , and d_3 are exactly known (see Ref. [3] and references therein). Hereafter, β_0 is the first coefficient of the QCD β -function

$$\beta(\bar{a}_s^{(k)}) \equiv \frac{d\bar{a}_s^{(k)}(Q^2)}{dL} = -(\bar{a}_s^{(k)})^2 \left(\beta_0 + \sum_{i=1}^k \beta_i (\bar{a}_s^{(k)})^i \right), \quad (10)$$

where β_i are known up to $k = 4$ [12].

Following [13], we introduce and utilize the derivatives (at the k th order of PT)

$$\tilde{a}_{n+1}^{(k)}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n a_s^{(k)}(Q^2)}{(dL)^n}, \quad L = \ln \frac{Q^2}{\Lambda^2}, \quad (11)$$

which play a key role in the construction of analytic QCD (see, e.g., Ref. [14]).

The series of derivatives $\tilde{a}_n(Q^2)$ can be used instead of the series of a_s -powers. Indeed, although each derivative reduces the a_s power, it also introduces an additional β -function and consequently an additional a_s^2 factor. By definition (11), at LO, the expressions for $\tilde{a}_n$ and a_s^n coincide exactly. Beyond LO, there is a one-to-one

correspondence between $\tilde{a}_n$ and a_s^n , established in [13,15] and extended to the fractional case in [16].

Converting the powers of the coupling constant into its derivatives, we obtain

$$\begin{aligned} D^{(1)}(Q^2) &= \frac{4}{\beta_0} \tilde{a}_s^{(1)}, \\ D^{(k \geq 2)}(Q^2) &= \frac{4}{\beta_0} \tilde{a}_s^{(k)} \left(1 + \sum_{m=1}^{k-1} \tilde{d}_m (\tilde{a}_s^{(k)})^m \right), \end{aligned} \quad (12)$$

where

$$\begin{aligned} \tilde{d}_1 &= d_1, \quad \tilde{d}_2 = d_2 - b_1 d_1, \\ \tilde{d}_3 &= d_3 - \frac{5}{2} b_1 d_2 - \left(b_2 - \frac{5}{2} b_1^2 \right) d_1 \end{aligned} \quad (13)$$

and $b_i = \beta_i / \beta_0^{i+1}$.

For the case of 3 active quark flavors ($f = 3$), which is adopted in this work, the results are¹

$$\begin{aligned} d_1 &= \tilde{d}_1 = 1.59, \quad d_2 = 3.75 (\tilde{d}_2 = 2.51), \\ d_3 &= 16.77 (\tilde{d}_3 = 10.44), \end{aligned} \quad (14)$$

i.e., the coefficients in the derivative series are slightly smaller.

A. Heavy quark contribution

In addition to the light flavors, the GLS sum rule receives contributions from heavy quarks (HQs).

Taking the HQ effects into account (see Ref. [19]), one obtains

$$C_{GLS}^{p+n}(Q^2) = 3 \left(1 + \frac{|V_{cd}|}{1 + \xi_c} - D(Q^2, \xi) \right) + \sum_{n=2} \frac{\mu_{2n}}{Q^{2n-2}}, \quad (15)$$

$$\bar{C}_{GLS}^{p+n}(Q^2) = 3 \left(1 + \frac{|V_{cd}|}{1 + \xi_c} - D(Q^2, \xi) \right) + \frac{\hat{\mu}_4 M^2}{Q^2 + M^2}, \quad (16)$$

where

$$\xi_i = \frac{Q^2}{m_i^2} \quad (i = c, b, t), \quad (17)$$

and $m_i = m_i(Q^2 = m_i^2)$ in the $\overline{MS}$ -scheme, with $m_c = 1.27$ GeV, $m_b = 4.18$ GeV, and $m_t = 172.76$ GeV (see Ref. [20]).

¹The coefficients β_i ($i \geq 0$) of the β function (10) and hence the coupling constant $\alpha_s(Q^2)$ itself depend on the number f , and each new quark enters/leaves the game at a certain threshold Q_f^2 according to [17]. The corresponding parameters $\Lambda^{(f)}$ in NⁱLO PT can be found in [18].

Up to the k th order of PT, the twist-two parts have the form, similar to (8),

$$\begin{aligned} D^{(1)}(Q^2, \xi) &= \frac{4}{\beta_0} \tilde{a}_s^{(1)} \left(1 - \frac{|V_{cd}|}{6} \frac{2 \ln(1 + \xi_c) + 1}{1 + \xi_c} \right), \\ D^{(k \geq 2)}(Q^2, \xi) &= \frac{4}{\beta_0} \tilde{a}_s^{(k)} \left(1 - \frac{|V_{cd}|}{6} \frac{2 \ln(1 + \xi_c) + 1}{1 + \xi_c} \right. \\ &\quad \left. + \tilde{d}_1(\xi) \tilde{a}_s^{(k)} + \sum_{m=2}^{k-1} \tilde{d}_m (\tilde{a}_s^{(k)})^m \right), \end{aligned} \quad (18)$$

where $\tilde{d}_1(\xi)$ is obtained from d_1 by the following redefinition:

$$d_1 \rightarrow d_1 - \frac{1}{3} \sum_{i=c,b,t} C_1(\xi_i), \quad \tilde{d}_1 \rightarrow \tilde{d}_1 - \frac{1}{3} \sum_{i=c,b,t} C_1(\xi_i), \quad (19)$$

with $C_1(\xi)$ having the following form:

$$\begin{aligned} C_1(\xi) &= \frac{8}{3\beta_0} \left\{ \frac{6\xi^2 + 2735\xi + 11724}{5040\xi} \right. \\ &\quad - \frac{3\xi^3 + 106\xi^2 + 1054\xi + 4812}{2520\xi} L(\xi) \\ &\quad \left. - \frac{5}{3\xi(\xi + 4)} L^2(\xi) + \frac{3\xi^2 + 112\xi + 1260}{5040} \ln(\xi) \right\} \end{aligned} \quad (20)$$

and

$$L(\xi) = \frac{1}{2\delta} \ln \left(\frac{1 + \delta}{1 - \delta} \right), \quad \delta^2 = \frac{\xi}{4 + \xi}. \quad (21)$$

The CKM-matrix element $|V_{cd}| = 0.22486$ in (15), (16), and (18) is rather small. Actually, there are also corresponding contributions from b and t -quarks, but they are multiplied by CKM-matrix coefficients $|V_{bu}|^2$ and $|V_{td}|^2$, which have negligible values: $|V_{bu}| = 0.00369$ and $|V_{td}| = 0.00857$. Consequently, such contributions $\sim |V_{bu}|$ and $\sim |V_{td}|$ in (15), (16), and (18) can be safely ignored.

Now, using the derivatives of the coupling constant introduced above (11), the results (2) and (3) can be written in APT as

$$C_{A, GLS}^{p+n}(Q^2) = 3(1 - D_A(Q^2)) + \sum_{n=2} \frac{\mu_{A, 2n}}{Q^{2n-2}}, \quad (22)$$

$$\bar{C}_{A, GLS}^{p+n}(Q^2) = 3(1 - D_A(Q^2)) + \frac{\hat{\mu}_{4,A} M^2}{Q^2 + M^2}, \quad (23)$$

where the perturbative part $D_A(Q^2)$ takes the same forms as (12), but the derivatives $\tilde{a}_v^{(k)}$ are replaced by the

corresponding analytical derivatives $\tilde{A}_\nu^{(k)}$ (the corresponding expressions for the analytic coupling constants $\tilde{A}_\nu^{(k)}$ can be found in the Appendix)

$$D_A^{(1)}(Q^2) = \frac{4}{\beta_0} A^{(1)},$$

$$D_A^{k \geq 2}(Q^2) = \frac{4}{\beta_0} \left(A^{(k)} + \sum_{m=2}^k \tilde{d}_{m-1} \tilde{A}_{\nu=m}^{(k)} \right). \quad (24)$$

In the case of HQ contributions, we have a modification of (15) and (16)

$$C_{A, \text{GLS}}^{p+n}(Q^2) = 3 \left(1 + \frac{|V_{cd}|}{1 + \xi_c} - D_A(Q^2, \xi) \right) + \sum_{n=2} \frac{\mu_{A,2n}}{Q^{2n-2}}, \quad (25)$$

$$\bar{C}_{A, \text{GLS}}^{p+n, (2)}(Q^2) = 3 \left(1 + \frac{|V_{cd}|}{1 + \xi_c} - D_A(Q^2, \xi) \right) + \frac{\hat{\mu}_{4,A} M^2}{Q^2 + M^2}, \quad (26)$$

where $D_A(Q^2, \xi)$ is obtained from $D(Q^2, \xi)$ in (18) by replacing the derivatives $\tilde{a}_\nu^{(k)}$ with the corresponding analytical derivatives $\tilde{A}_\nu^{(k)}$.

III. BJORKEN SUM RULE

BSR [4] provides another classical observable in deep inelastic scattering, closely related to the GLS sum rule. Both sum rules admit an OPE with perturbative coefficients and higher twist corrections, and both can therefore be treated consistently within APT.

Indeed, BSR is defined by the integral

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 [g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)] dx. \quad (27)$$

Within the OPE, the quantity $\Gamma_1^{p-n}(Q^2)$ can be written in the form

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6} (1 - D_{\text{BS}}(Q^2)) + \sum_{n=2} \frac{\mu_{2n}^{BS}}{Q^{2n-2}}, \quad (28)$$

similar to (2) for the GLS sum rule. Here, $g_A = 1.2762 \pm 0.0005$ [20] is the axial charge of the nucleon, $(1 - D_{\text{BS}}(Q^2))$ is the leading twist contribution, and the coefficients μ_{2n}^{BS} of the higher-twist terms are free parameters that must be determined from experimental data.

Using the massive twist-four representation as in (3), we have

$$\bar{\Gamma}_1^{p-n}(Q^2) = \frac{g_A}{6} (1 - D_{\text{BS}}(Q^2)) + \frac{\hat{\mu}_{4,A} M_{\text{BS}}^2}{Q^2 + M_{\text{BS}}^2}, \quad (29)$$

where the coefficients $\hat{\mu}_{2n}^{BS}$ and $M_{\text{BS}, 2n}^2$ of the massive twist-four term are free parameters that must be determined from experimental data.

Up to the k th order of PT, the twist-two parts have the form,² similar to (8)

$$D_{\text{BS}}^{(1)}(Q^2) = \frac{4}{\beta_0} a_s^{(1)},$$

$$D_{\text{BS}}^{(k \geq 2)}(Q^2) = \frac{4}{\beta_0} a_s^{(k)} \left(1 + \sum_{m=1}^{k-1} d_m^{BS} (a_s^{(k)})^m \right), \quad (30)$$

where $a_s^{(k)}(Q^2)$ is defined in (9) and d_1^{BS} , d_2^{BS} , and d_3^{BS} are exactly known (see Ref. [3] and references therein).

Converting the powers of the coupling constant into its derivatives (11), we have, similarly to (12)

$$D_{\text{BS}}^{(1)}(Q^2) = \frac{4}{\beta_0} \tilde{a}_1^{(1)},$$

$$D_{\text{BS}}^{(k \geq 2)}(Q^2) = \frac{4}{\beta_0} \left(\tilde{a}_1^{(k)} + \sum_{m=2}^k \tilde{d}_{m-1}^{BS} \tilde{a}_m^{(k)} \right), \quad (31)$$

where the results for $\tilde{d}_i^{BS}$ ($i = 1, 2, 3$) can be obtained in close analogy with (13).

For the case of three active quark flavors ($f = 3$), we have

$$d_1^{BS} = \tilde{d}_1^{BS} = 1.59, \quad d_2^{BS} = 3.99 (\tilde{d}_2^{BS} = 2.73),$$

$$d_3^{BS} = 15.42 (\tilde{d}_3^{BS} = 8.61), \quad (32)$$

i.e., the coefficients in the derivative series are slightly smaller, as was the case for the GLS sum rule [see Eq. (14)].

In APT, the results (28) and (29) can be written as

$$\Gamma_{A,1}^{p-n}(Q^2) = \frac{g_A}{6} (1 - D_{A, \text{BS}}(Q^2)) + \sum_{n=2} \frac{\mu_{A,2n}^{BS}}{Q^{2n-2}}, \quad (33)$$

$$\bar{\Gamma}_{A,1}^{p+n}(Q^2) = \frac{g_A}{6} (1 - D_{A, \text{BS}}(Q^2)) + x \frac{\hat{\mu}_{4,A} M_{\text{BS}}^2}{Q^2 + M_{\text{BS}}^2}, \quad (34)$$

where the perturbative part $D_{A, \text{BS}}(Q^2)$ takes the same forms as (31), but the derivatives $\tilde{a}_\nu^{(k)}$ are replaced by the corresponding analytical derivatives $\tilde{A}_\nu^{(k)}$

²Here we consider the so-called nonsinglet part in the BSR case. Note that the BSR singlet part is $\sim \sum_{i=1}^f e_i$ (see, e.g., Ref. [21]), where e_i is the charge of the i quark and f is the number of active quark flavors. Thus, in the case $f = 3$ used here, the singlet part is absent. In general, its contribution may be important (see Refs. [21,22] and discussions therein).

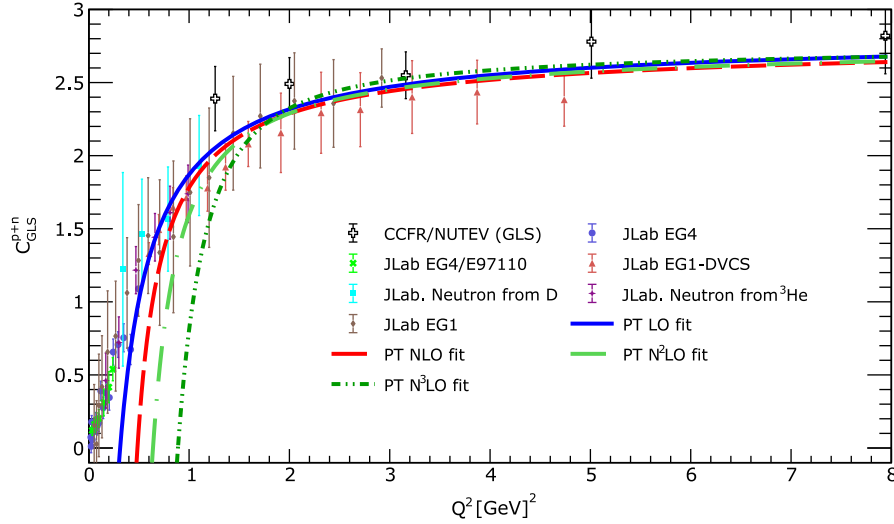


FIG. 1. The results (2) with $\mu_{2n} = 0$ for $n \geq 3$ in the first three orders of PT.

$$D_{A,BS}^{(1)}(Q^2) = \frac{4}{\beta_0} A^{(1)},$$

$$D_{A,BS}^{k \geq 2}(Q^2) = \frac{4}{\beta_0} \left(A^{(k)} + \sum_{m=2}^k \tilde{d}_{m-1}^{BS} \tilde{A}_{\nu=m}^{(k)} \right). \quad (35)$$

As in the GLS sum rule, the HQ contribution for BSR was calculated in [19], leading to the following replacement for $\tilde{d}_1^{BS} = d_1^{BS}$:

$$d_1^{BS} \rightarrow d_1^{BS} - \sum_{i=c,b,t} C_1(\xi_i), \quad \tilde{d}_1 \rightarrow \tilde{d}_1^{BS} - \sum_{i=c,b,t} C_1(\xi_i), \quad (36)$$

where ξ_i and $C_1(\xi)$ are given in Eqs. (17) and (20), respectively.

It is important to emphasize that the BSR perturbative structure is very similar to that of the GLS sum rule. Both sum rules are governed by very similar twist-2 parts. Their approximate equivalence has been discussed in the literature (see Ref. [10]). Moreover, there are speculations about the similarity between the twist-four parts (see Ref. [11]).³

Therefore, in agreement with discussions in [11], we can (with some precision) treat BSR and GLS sum rule as having the same dependence with only a different overall factor

$$C_{GLS}^{p+n}(Q^2) \approx \frac{18}{g_A} \Gamma_1^{p-n}(Q^2) \quad (37)$$

and, thus, for analytic QCD we have similarly

$$C_{A,GLS}^{p+n}(Q^2) \approx \frac{18}{g_A} \Gamma_{A,1}^{p-n}(Q^2). \quad (38)$$

³The similarity of perturbative parts for the structure functions $F_3(x, Q^2)$ and $g_1(x, Q^2)$ themselves has been explored in Ref. [23].

IV. RESULTS

By analogy with Refs. [7,24–28], where the BSR case was considered, we perform a fit of experimental data for the GLS sum rule within the framework of standard and analytic QCD with the usual and massive forms of the twist-four terms [see Eqs. (2), (3), (22), and (23), respectively].⁴

As in the BSR case (see Refs. [26,27]), for conventional PT in $\overline{MS}$ -scheme,⁵ in Figs. 1 and 2, we see a discrepancy between the QCD predictions and the experimental data, regardless of the form of the twist-four terms. Moreover, we see that the discrepancy increases with increasing PT order. This is the same as in the BSR case (see, e.g., Ref. [27]), and the reason is the same. With increasing PT order, the Landau pole of the strong coupling constant shifts toward higher Q^2 values (see Fig. 8 in the Appendix). Thus, the results for the GLS sum rule shift toward negative values, since the PT corrections have a negative sign.

Since the number of experimental points for the GLS sum rule is very small, we have added to Figs. 1 and 2 (but not used in the fits) the experimental BSR data [33] rescaled by the factor $18/g_A$. We see that the experimental data for the GLS sum rule lie slightly above these rescaled experimental BSR points. This observation is consistent with the results of [34], where the possibility was discussed that the experimental data for the GLS sum rule should be

⁴A fairly similar analysis of the GLS sum rule with the usual form of the twist-four term can be found in [29].

⁵Hereafter we will use the usual PT in $\overline{MS}$ -scheme with the renormalization scale $\mu_R = Q$. We do not consider resummations such as the principle of maximal conformality (PMC) [30,31], which leads to strong modifications of PT at low Q^2 values. Some discussions about resummations and applications for GLS sum rule based on the paper [32], will be done in the Conclusions.

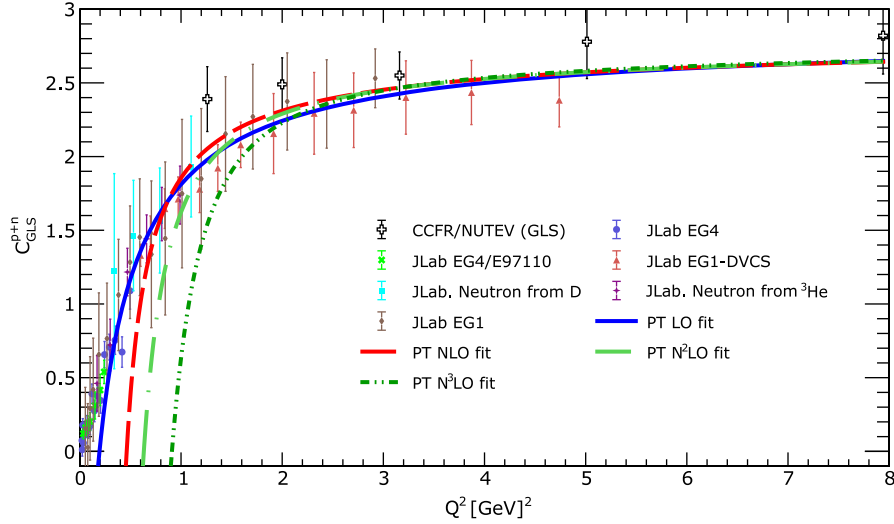


FIG. 2. As in Fig. 1 but for the massive twist-four case (3).

smaller than those published in [35] due to some unconsidered corrections.

In the case of analytic QCD, we have good agreement between the QCD predictions and the experimental data (see Tables I and II and Figs. 3 and 4).

Summarizing the situation with the perturbative curves, one should note their unsatisfactory description of the experimental data, together with their unphysical behavior in the infrared region. The situation changes drastically once analytic QCD is applied. Tables I and II contain the results of comparing the experimental data with the theoretical expressions (22) and (23) taking into account the twist-four term in Table I and twist-four plus twist-six terms in Table II.

It is now instructive to compare these results with those obtained long ago: μ_4^{CCFR} [35], μ_4^{BK} [36] and μ_4^{GG} [37]

$$\begin{aligned}\mu_4^{\text{CCFR}} &= -0.15 \pm 0.15 \text{ GeV}^2, \\ \mu_4^{\text{BK}} &= -0.293 \pm 0.142 \text{ GeV}^2, \\ \mu_4^{\text{GG}} &= -0.471 \pm 0.036 \text{ GeV}^2\end{aligned}\quad (39)$$

TABLE I. The values of the fit parameters for the massive twist-four case shown in (23) [for the massless one shown in (22)].

	$M^2 [\text{GeV}^2]$	$\hat{\mu}_{A,4} [\mu_{A,4} \text{ GeV}^2]$	$\chi^2/(\text{d.o.f.})$
LO	0.448 ± 0.041	-0.212 ± 0.009 [-0.065 ± 0.002]	0.75 [1.63]
NLO	0.377 ± 0.040	-0.209 ± 0.011 [-0.057 ± 0.001]	0.80 [1.39]
N ² LO	0.358 ± 0.039	-0.213 ± 0.012 [-0.056 ± 0.001]	0.82 [1.25]
N ³ LO	0.354 ± 0.038	-0.215 ± 0.012 [-0.055 ± 0.002]	0.82 [1.25]

and with the more recent result (see Ref. [32]) within PMC approach [30,31]

$$\mu_4^{\text{PMC}} = -0.061 + 0.031 - 0.039 \text{ GeV}^2. \quad (40)$$

First of all, we will show that our results in Tables I and II, which are obtained within the framework of analytical QCD, can be compared with the results obtained in the conventional PT (39) and in the PMC approach (40). For simplicity, we do this at LO, where in the case of analytical QCD we have

$$D_A^{(1)}(Q^2) = \frac{4}{\beta_0} \left[a_s^{(1)} - \frac{\Lambda_0^2}{Q^2 - \Lambda_0^2} \right] \approx D^{(1)}(Q^2) - \frac{4\Lambda_0^2}{\beta_0 Q^2}, \quad (41)$$

where $\Lambda_0 = 142 \text{ MeV}$ is the LO QCD parameter [18] for the case of three active quarks (see also Refs. [38–40]). Thus, in the case of analytical QCD, there is an additional power-law contribution of $12\Lambda_0/(\beta_0 Q^2)$ to the twist-two component coming from the twist-four term. So, the twist-four parts in the analytic QCD case and in the usual PT are related as follows

TABLE II. The values of the fit parameters for the case with massless twist-four and twist-six terms shown in (22).

	$\mu_{A,4} \text{ GeV}^2$	$\mu_{MA,6} \text{ GeV}^4$	$\chi^2/(\text{d.o.f.})$
LO	-0.063 ± 0.002	0.0060 ± 0.0001	1.64
NLO	-0.056 ± 0.002	0.0054 ± 0.0001	1.24
N ² LO	-0.055 ± 0.001	0.0052 ± 0.0001	1.18
N ³ LO	-0.055 ± 0.001	0.0052 ± 0.0001	1.17

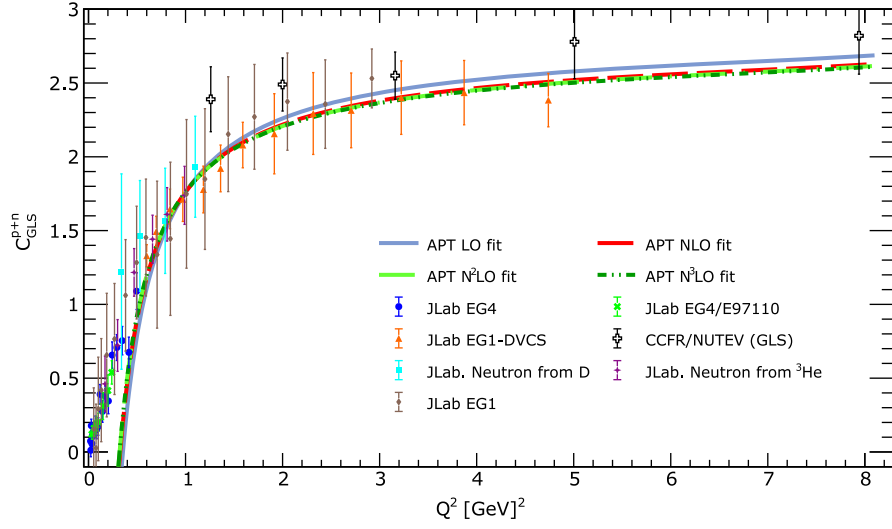


FIG. 3. As in Fig. 1 but for the APT result (22).

$$\mu_{A,4} = \mu_4 - \frac{12\Lambda_0^2}{\beta_0} \approx \mu_4 - 0.027 \text{ GeV}^2, \quad (42)$$

where the symbol $\approx$ shows here the case $f = 3$.

In Eq. (42), the difference between $\mu_{A,4}$ and μ_4 less than uncertainties in (39) and (40) and, thus, we can compare the results shown Tables I and II with ones in (39) and (40). Considering the results for $\mu_{A,4}$ in Table I, we see good agreement between $\mu_{A,4}$ and μ_4^{PMC} . Indeed, in Table I, $\mu_{A,4}$ decreases slightly with increasing PT order from $\mu_{A,4} = -0.065 \pm 0.002 \text{ GeV}^2$ at LO to $\mu_{A,4} = -0.055 \pm 0.002 \text{ GeV}^2$ at N^3LO and all the results are completely consistent with μ_4^{PMC} . We also see agreement with μ_4^{CCFR} , obtained by the CCFR collaboration, but disagreement with the old theoretical predictions μ_4^{BK} and μ_4^{GG} . When $\mu_{A,6}$ is

added, the values of $\mu_{A,4}$ do not change in their uncertainties. The values of $\mu_{A,6}$ are very small and opposite in sign to $\mu_{A,4}$. Moreover, the values of χ^2 decrease beyond LO.

One should pay attention to the different sign of twist-six contribution with respect to twist-four one. This property is in full agreement with correct analytic properties provided by the sign factor in the representation (4). Moreover, the decrease of its magnitude with respect to twist-four one may be compatible with appearance in (4) of the higher moments of decreasing function. In turn, the decrease of spectral function $\rho(s)$ with s growth does not seem unnatural. These properties are also manifested in the massive model.

In the case of the massive twist-four term, the values of $M^2 \hat{\mu}_{A,4}$ decrease slightly with increasing PT order: $M^2 \hat{\mu}_{A,4} = -0.095 \pm 0.005 \text{ GeV}^2$ in LO and

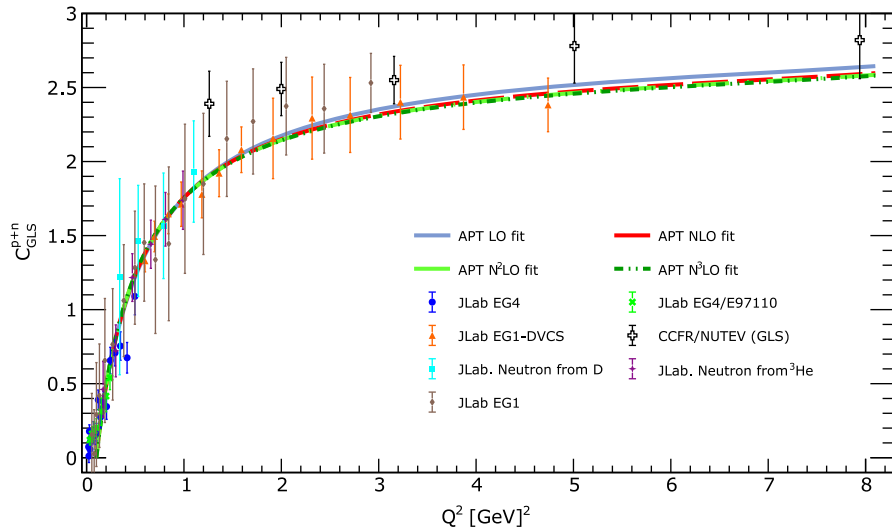


FIG. 4. As in Fig. 2 but for the APT result (23).

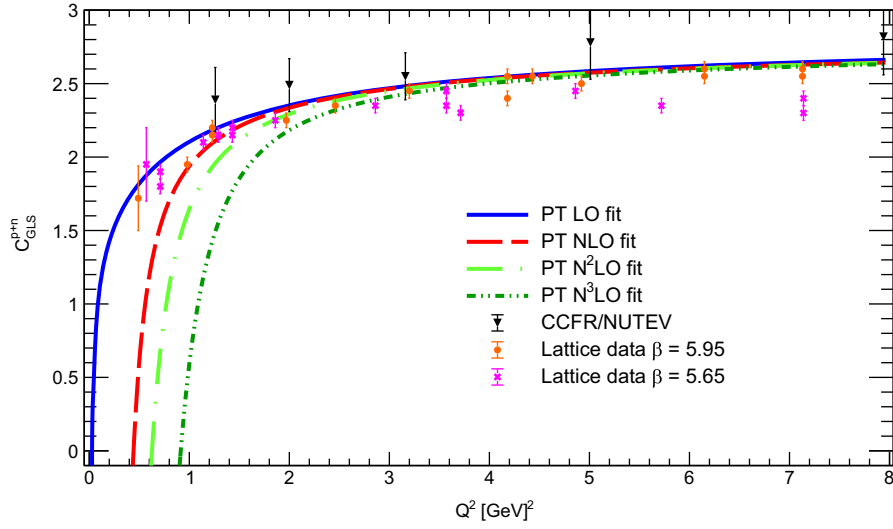


FIG. 5. As in Fig. 2 but with lattice data.

$M^2\hat{\mu}_{A,4} = -0.076 \pm 0.004 \text{ GeV}^2$ in $N^3\text{LO}$. In full accordance with the case of massless high twists shown above, we see good agreement between the results obtained for $M^2\hat{\mu}_{A,4}$ with μ_4^{PMC} and μ_4^{CCFR} and disagreement with the old theoretical predictions μ_4^{BK} and μ_4^{GG} . However, note that the value of $M^2\hat{\mu}_{A,4}$ is higher than the values of $\mu_{A,4}$ shown above. But the values of $M^2\hat{\mu}_{A,4}$ are divisible by $(Q^2 + M^2)$, which are significantly larger than Q^2 in the kinematic range. Note that the values of χ^2 are significantly lower in the case of the massive twist-four term.

In the Figs. 3 and 4 we also show the experimental BSR data rescaled by the factor $18/g_A$. The results obtained using the massive twist-four term and using both the twist-four and twist-six terms are very similar. Therefore, in Figs. 3 and 4 we show the results obtained using the twist-four term.

With the massive form of the twist-4 term (see Fig. 4), the agreement with the rescaled BSR data extends down to the low- Q^2 region, remaining good for $Q^2 < 0.6 \text{ GeV}^2$. In contrast, the standard twist-four ansatz deteriorates at low Q^2 (see Fig. 3) and provides a reasonable description only for $Q^2 \geq 0.8 \text{ GeV}^2$. Of course, this correspondence occurs only for the rescaled BSR data, but not for the data according to the GLS sum rule, which are absent at such low values of Q^2 .

The available experimental information on the GLS sum rule still has large uncertainties, and the number of experimental points is very small. For this reason, it is useful to supplement the comparison with recent lattice QCD results from Ref. [6]. Note that the lattice “data” in [6] are quite different for different values of β . We will use the lattice data with a large β corresponding to the smallest value of the lattice parameter a .

In the case of standard PT in $\overline{\text{MS}}$ -scheme, the results obtained by fitting these lattice data are presented in Fig. 5. (Since the results in the cases of the usual and massive

twist-four terms are similar, we show only the results for the massive twist-four term). It can be seen that the LO results agree quite well with the lattice data, which cannot be said about the results for higher PT orders.⁶ Moreover, as already discussed for the case of the experimental data for the GLS sum rule, the discrepancy increases with increasing PT order. The reasons are the same as discussed above.

In the case of analytic QCD, we have good agreement with the lattice data. The results obtained by fitting these lattice data are presented in Table III and Fig. 6. The fitting parameters obtained with massless light quarks, with the addition of the charm contribution, and with the addition of the contribution of all heavy quarks are quite similar. Moreover, since there is no real difference between the results for the massless and massive twist-four terms, we present in Table III only the results for the massive twist-four term.

We see, that the values of $M^2\hat{\mu}_{A,4}$ decrease with increasing PT order: $M^2\hat{\mu}_{A,4} = -0.085 \pm 0.056 \text{ GeV}^2$ in LO and $M^2\hat{\mu}_{A,4} = -0.047 \pm 0.044 \text{ GeV}^2$ in $N^3\text{LO}$ in the case of massless quarks. When we added the contribution of heavy quarks, the situation did not change, but the values of $M^2\hat{\mu}_{A,4}$ increased: $M^2\hat{\mu}_{A,4} = -0.17 \pm 0.08 \text{ GeV}^2$ in LO and $M^2\hat{\mu}_{A,4} = -0.12 \pm 0.05 \text{ GeV}^2$ in $N^3\text{LO}$. We see a good agreement between these results and the results obtained above, as well as with μ_4^{PMC} and μ_4^{CCFR} and disagreement with μ_4^{BK} and μ_4^{GG} .

Note that we obtain rather large χ^2 values for the fits because the uncertainties in the lattice data are very small.

⁶Given the close analogy with the BSR case (see Ref. [41]), a detailed analysis of the low Q^2 range can lead to an incorrect description, even at LO. Unfortunately, in the case of the GLS sum rule, there is no information about the low Q^2 range, since the relevant experimental and lattice data are missing here.

TABLE III. The values of the twist-four parameters in (23) and (26) obtained from the lattice data.

	M^2 [GeV 2]	$10\hat{\mu}_{A,4}$	$\chi^2/(\text{d.o.f.})$
	(+ charm contributions)	(+ charm contributions)	(+ charm contributions)
	[+ HQ contributions]	[+ HQ contributions]	[+ HQ contributions]
LO	0.879 ± 0.338 (1.493 ± 0.391) [1.253 ± 0.432]	-0.97 ± 0.22 (-1.49 ± 0.43) [-1.33 ± 0.43]	4.84 (4.51) [4.79]
NLO	0.408 ± 0.277 (0.886 ± 0.400) [0.998 ± 0.337]	-1.09 ± 0.23 (-1.43 ± 0.33) [-1.29 ± 0.36]	3.50 (4.48) [4.15]
N 2 LO	0.400 ± 0.258 (0.785 ± 0.385) [0.878 ± 0.319]	-1.19 ± 0.75 (-1.49 ± 0.33) [-1.37 ± 0.42]	3.24 (4.04) [4.48]
N 3 LO	0.370 ± 0.247 (0.738 ± 0.373) [0.822 ± 0.306]	-1.26 ± 0.85 (-1.56 ± 0.55) [-1.43 ± 0.36]	3.12 (4.34) [4.02]

If we add experimental data to the considered data of the lattice, the values of χ^2 decrease, but the extracted parameters do not actually change when taking into account the corresponding uncertainties (see Tables III and IV).

Table IV contains the results of fits of experimental data and lattice data using theoretical predictions (23), taking into account only the massive twist-four term. As in the case, considered above and shown in Table III, the values of $M^2\hat{\mu}_{A,4}$ decrease with increasing PT order: $M^2\hat{\mu}_{A,4} = -0.10 \pm 0.07 \text{ GeV}^2$ in LO and $M^2\hat{\mu}_{A,4} = -0.048 \pm 0.062 \text{ GeV}^2$ in N 3 LO in the case of massless quarks. When we added the charm contribution, the situation is similar, but the values of $M^2\hat{\mu}_{A,4}$ increased: $M^2\hat{\mu}_{A,4} = -0.18 \pm 0.08 \text{ GeV}^2$ in LO and $M^2\hat{\mu}_{A,4} = -0.13 \pm 0.06 \text{ GeV}^2$ in N 3 LO. We see a good agreement between

these results and the results obtained above, as well as with μ_4^{PMC} and μ_4^{CCFR} and disagreement with μ_4^{BK} and μ_4^{GG} .

In Fig. 7, we show the results for the GLS sum rule obtained in Table III, along with the BSR results scaled by a factor of $18/g_A$. The BSR results were obtained under the same conditions in Ref. [27]. As can be seen from Fig. 7, the ratio (38) is valid only for large values of Q^2 : $Q^2 \geq 5 \text{ GeV}^2$. For smaller values of Q^2 , $C_{A,\text{GLS}}^{p+n}(Q^2)$ and $(18/g_A)\Gamma_{A,1}^{p-n}(Q^2)$ become different. The difference arises due to the different parameter values in the twist-four terms in the cases of the GLS and Bjorken sum rules. Indeed, the values of $\hat{\mu}_{A,4}$ shown in Tables III and IV are significantly smaller than the corresponding values for the twist-four magnitudes obtained for the BSR case (see Table I in [27]) and rescaled by a factor of $(18/g_A)$. Thus, the ratio (38),

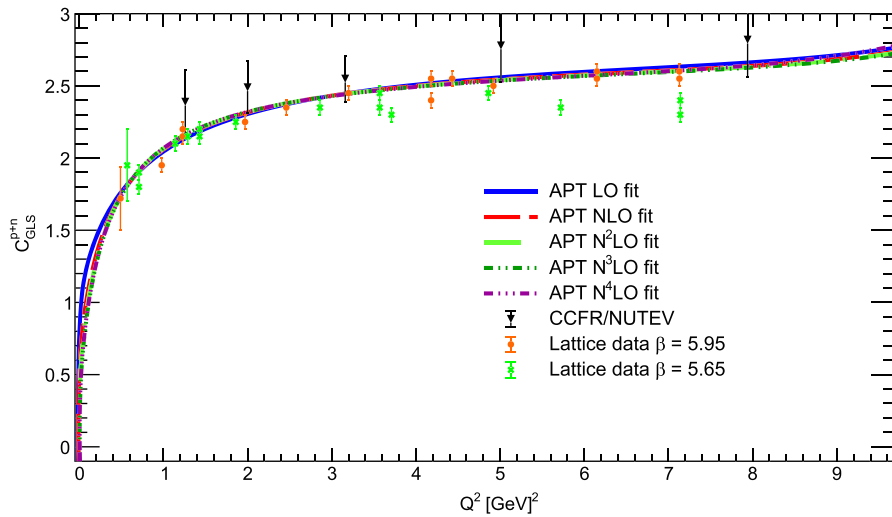


FIG. 6. As in Fig. 4 but with the lattice data.

TABLE IV. The values of the twist-four parameters in (23) and (26) obtained from experimental and lattice data.

	M^2 [GeV 2] for the massless case	$10\hat{\mu}_{A,4}$ for the massless case	$\chi^2/(\text{d.o.f.})$ the massless case
	(+ charm contributions)	(+ charm contributions)	(+ charm contributions)
	[+ HQ contributions]	[+ HQ contributions]	[+ HQ contributions]
LO	1.283 ± 0.819 (1.466 ± 0.592) [1.699 ± 0.595]	-0.78 ± 0.22 (-1.23 ± 0.23) [-1.21 ± 0.21]	3.18 (3.16) [3.34]
NLO	0.609 ± 0.556 (1.108 ± 0.463) [1.225 ± 0.512]	-0.94 ± 0.18 (-1.23 ± 0.30) [-1.20 ± 0.27]	2.58 (2.75) [2.94]
N 2 LO	0.477 ± 0.491 (1.006 ± 0.435) [1.143 ± 0.481]	-1.03 ± 0.75 (-1.27 ± 0.32) [-1.22 ± 0.29]	2.41 (2.43) [2.84]
N 3 LO	0.429 ± 0.457 (0.957 ± 0.415) [1.088 ± 0.458]	-1.11 ± 0.85 (-1.31 ± 0.33) [-1.25 ± 0.30]	2.33 (2.59) [2.79]

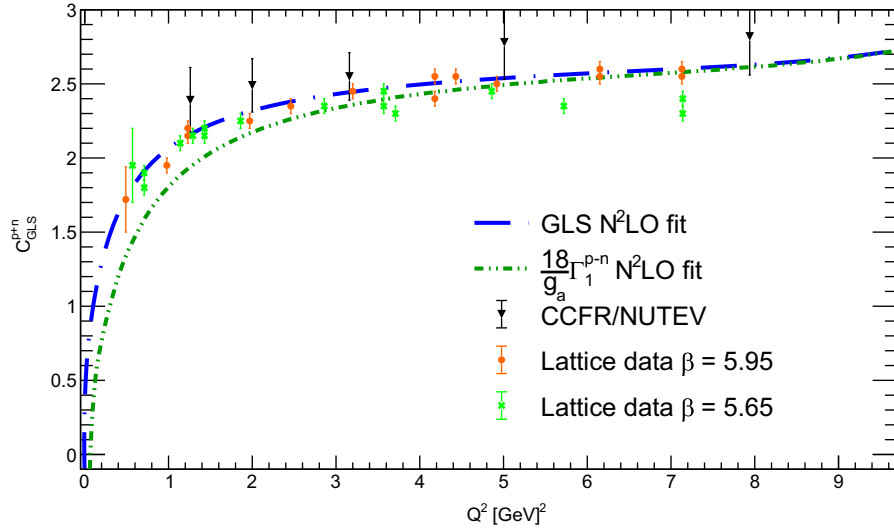


FIG. 7. As in Fig. 6 but with the rescaled BSR curve (38) (the green curve) added.

based on the assumption in [11], seems to be fulfilled only for rather large values of Q^2 , $Q^2 \geq 5 \text{ GeV}^2$.

V. CONCLUSIONS

We have investigated the Gross–Llewellyn Smith sum rule within the framework of conventional and analytic QCD. A comparison was performed between experimental data, recent lattice calculations, and perturbative QCD based on conventional and analytic versions of perturbation theory with different parametrizations for the twist-four contribution. We showed that conventional perturbation theory beyond LO in $\overline{MS}$ scheme with $\mu_R = Q$ fails to reproduce the data, while the analytic version, especially with the massive high-twist terms, demonstrates good agreement with experimental data and lattice calculations.

In all considered cases, we see good agreement between our results for $\mu_{A,4}$ and $\hat{\mu}_{A,4}$ and also μ_4^{PMC} obtained in [32] using the PMC approach [30] and its single scale version [31]. We also see agreement with μ_4^{CCFR} [35], obtained by CCFR collaboration, but disagreement with the old theoretical predictions μ_4^{BK} [36] and μ_4^{GG} [37]. Note that usually the values of χ^2 decrease when we add the twist-six term.

We also discussed the relation between the Gross–Llewellyn Smith sum rule and the Bjorken sum rule. We found a similarity between the Gross–Llewellyn Smith sum rule and the rescaled Bjorken sum rule, with the additional factor $18/g_A$, at rather large Q^2 values, $Q^2 \geq 5 \text{ GeV}^2$, where the corresponding twist-two parts give the main contributions. However, at smaller Q^2 values ($Q^2 \leq 2 \text{ GeV}^2$), there is a difference between them.

We have shown that this is due to the different values of the twist-four terms obtained here for the GLS sum rule (see Tables I–IV) and for the BSR case in Ref. [27].

Note that the disadvantage of traditional perturbation theory is related to the fact that the strong coupling constant has the Landau pole at $Q^2 \sim \Lambda^2$, and thus, at low values of Q^2 , perturbation theory becomes inapplicable. Indeed, at low values of Q^2 , the twist-two part of the GLS summation rules becomes very large (and negative). The situation changes dramatically when summations of perturbative series are taken into account. For example, in the PMC approach [30,31], which are now quite popular, the effective strong coupling scale for the GLS sum rule becomes $Q_* = 0.86$ GeV (see Ref. [32]), which greatly changes the situation because $Q_* \gg \Lambda$ and the Landau pole are not reached. In this case, the correspondence between experimental data and theoretical predictions are restored. Note that the summation in the case of the Bjorken sum rule performed in [42], also improves similar matching.

However, in the present research we mainly focus on the applicability of analytical QCD, and traditional perturbation theory plays an auxiliary role. Therefore, we have considered only the traditional perturbation theory in the $\overline{MS}$ scheme with $\mu_R = Q$.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [6].

APPENDIX: ANALYTIC COUPLING

Here we present a brief overview of QCD with analytic QCD couplings in the first three PT orders. More general results, including three other representations of analytic couplings for spacelike processes, can be found in [38] (see also Ref. [39]). Representations for analytic couplings for timelike processes are given in [40].

The strong coupling constant $\alpha_s(Q^2)$ satisfies the renormalization group equation

$$L \equiv \ln \frac{Q^2}{\Lambda^2} = \int^{\bar{a}_s(Q^2)} \frac{da}{\beta(a)}, \quad (\text{A1})$$

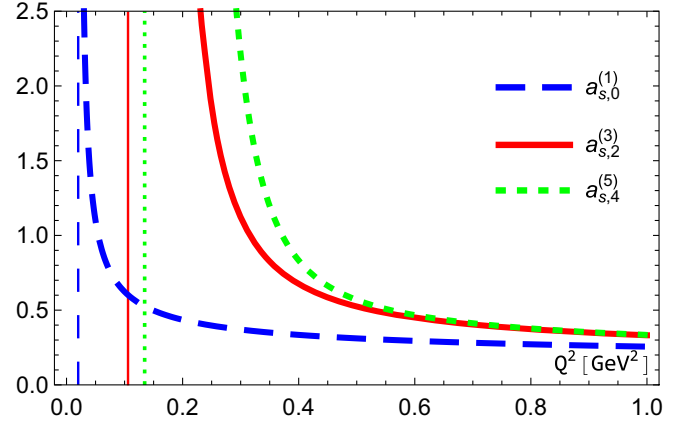


FIG. 8. The results for $a_{s,i}^{(i+1)}(Q^2)$ and $(\Lambda_i^{f=3})^2$ (vertical lines) with $i = 0, 2, 4$. Here and in subsequent figures, the $\Lambda_i^{f=3}$ values from [18] are used.

where $\bar{a}_s(Q^2)$, $a_s(Q^2)$ and $\beta(a)$ are defined in Eqs. (9) and (10) of the main text.

In the perturbative regime, when $Q^2 \gg \Lambda^2$, Eq. (A1) can be solved iteratively in the form of a $1/L$ -expansion,⁷ which can be written as a series of corrections to the well-known LO expression

$$a_{s,0}^{(1)}(Q^2) = \frac{1}{L_0},$$

$$a_{s,i}^{(i+1)}(Q^2) = a_{s,i}^{(1)}(Q^2) + \sum_{m=2}^i \delta_{s,i}^{(m)}(Q^2), \quad (i = 0, 1, 2, \dots), \quad (\text{A2})$$

where

$$L_i = \ln t_i, \quad t_i = \frac{1}{z_i} = \frac{Q^2}{\Lambda_i^2} \quad (\text{A3})$$

and

$$\delta_{s,i}^{(2)}(Q^2) = -\frac{b_1 \ln L_i}{L_i^2},$$

$$\delta_{s,i}^{(3)}(Q^2) = \frac{1}{L_i^3} [b_1^2 (\ln^2 L_i - \ln L_i - 1) + b_2], \quad (\text{A4})$$

i.e., $a_s^{(i+1)}(Q^2)$ ($i = 0, 1, 2$) contain poles and other singularities at $Q^2 = \Lambda_i^2$. The results for Λ_i can be found, for example, in Ref. [18].

Figure 8 shows that the strong couplings $a_{s,i}^{(i+1)}(Q^2)$ become singular at $Q^2 \sim \Lambda_i^2$, and these singularities shift to higher Q^2 values as the PT order increases.

⁷The $1/L$ -expansion provides a good approximation for the solution of Eq. (A1) at $Q^2 \geq 10$ GeV² (see, e.g., Ref. [43]).

We can raise the results (A2) to the power ν and then reconstruct $\tilde{a}_\nu^{(k)}(Q^2)$ (11) using the relations between $\tilde{a}_\nu$ and a_s^ν obtained in [16]. This procedure is described in more details in Appendix A of [38]. Here we present only the final results, which take the form⁸

$$\begin{aligned}\tilde{a}_{\nu,0}^{(1)}(Q^2) &= (a_{s,0}^{(1)}(Q^2))^\nu = \frac{1}{L_0^\nu}, \\ \tilde{a}_{\nu,i}^{(i+1)}(Q^2) &= \tilde{a}_{\nu,i}^{(1)}(Q^2) + \sum_{m=1}^i C_m^{\nu+m} \tilde{\delta}_{\nu,i}^{(m+1)}(Q^2), \\ \tilde{\delta}_{\nu,i}^{(m+1)}(Q^2) &= \hat{R}_m \frac{1}{L_i^{\nu+m}}, \quad C_m^{\nu+m} = \frac{\Gamma(\nu+m)}{m! \Gamma(\nu)},\end{aligned}\quad (\text{A5})$$

where

$$\begin{aligned}\hat{R}_1 &= b_1 \left[\hat{Z}_1(\nu) + \frac{d}{d\nu} \right], \\ \hat{R}_2 &= b_2 + b_1^2 \left[\frac{d^2}{(d\nu)^2} + 2\hat{Z}_1(\nu+1) \frac{d}{d\nu} + \hat{Z}_2(\nu+1) \right]\end{aligned}\quad (\text{A6})$$

and $\hat{Z}_j(\nu)$ ($j = 1, 2$) are combinations of Euler Ψ -functions and their derivatives.

The representation (A5) of the $\tilde{\delta}_{\nu,i}^{(m+1)}(Q^2)$ corrections as $\hat{R}_m$ -operators is very important and allows us to obtain in a similar way the higher-order results for the $1/L$ -expansion of analytic couplings [see Eqs. (A12) and (A13) below], avoiding the need to compute rather complicated integral expressions.

Let us now return to the problem of the IR singularity of the coupling constant. In [46], an effective approach was developed to eliminate the Landau pole, based on a dispersion relation that connects the new analytic coupling $A_{\text{MA}}(Q^2)$ with the spectral function $r_{\text{pt}}(s)$ derived from PT. At LO, this gives

$$\begin{aligned}A_{\text{MA}}^{(1)}(Q^2) &= \frac{1}{\pi} \int_0^{+\infty} \frac{ds}{(s+t)} r_{\text{pt}}^{(1)}(s), \\ r_{\text{pt}}^{(1)}(s) &= \text{Im } a_s^{(1)}(-s - i\epsilon).\end{aligned}\quad (\text{A7})$$

This approach is commonly referred to as the *minimal approach* (MA) (see, e.g., Ref. [47]) or *analytic perturbation theory* (APT) [46].

A further development of APT is the fractional APT (FAPT) [44], which extends the construction principles to PT series involving noninteger powers of the coupling. In quantum field theory, such series arise for quantities with nonzero anomalous dimensions.

Here we summarize the fundamental properties of MA couplings, as derived in [38] using the

⁸The expansion (A5) is similar to those used in Refs. [44,45] for expanding $(a_{s,i}^{(i+1)}(Q^2))^\nu$ in powers of $a_{s,i}^{(1)}(Q^2)$.

$1/L$ -expansion. As already discussed above, for the standard coupling, this expansion is only valid for large Q^2 , i.e., $Q^2 \gg \Lambda^2$. However, as demonstrated in [38], the situation is entirely different for analytic couplings: the $1/L$ -expansion is applicable for all values of the argument. This is because nonleading corrections in the expansion vanish not only as $Q^2 \rightarrow \infty$ but also as $Q^2 \rightarrow 0$,⁹ resulting only in small, nonzero corrections in the region $Q^2 \sim \Lambda^2$.

In the LO case, the spectral integral (A7) can be evaluated analytically. The MA coupling $A_{\text{MA},\nu,0}^{(1)}$ has the following form [44]

$$\begin{aligned}A_{\text{MA},\nu,0}^{(1)}(Q^2) &= \left(a_{\nu,0}^{(1)}(Q^2) \right)^\nu - \frac{\text{Li}_{1-\nu}(z_0)}{\Gamma(\nu)} \\ &= \frac{1}{L_0^\nu} - \frac{\text{Li}_{1-\nu}(z_0)}{\Gamma(\nu)} \equiv \frac{1}{L_0^\nu} - \Delta_{\nu,0}^{(1)},\end{aligned}\quad (\text{A8})$$

where

$$\text{Li}_\nu(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^\nu} \quad (\text{A9})$$

is the polylogarithm.

For $\nu = 1$, we recover the famous Shirkov-Solovtsov result [46]

$$A_{\text{MA},0}^{(1)}(Q^2) \equiv A_{\text{MA},\nu=1,0}^{(1)}(Q^2) = \frac{1}{L_0} - \frac{z_0}{1-z_0}. \quad (\text{A10})$$

Note that the result (A10) can be derived directly from the integral form (A7), as was done in Ref. [46].

Beyond LO, the spectral integral becomes considerably more complicated, even for numerical methods [48]. To avoid the need for direct numerical or analytical computations, and following Eq. (A8) for the LO analytic coupling, we consider the derivatives of the MA coupling, defined as

$$\tilde{A}_{\text{MA},n+1}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n A_{\text{MA}}(Q^2)}{(dL)^n}. \quad (\text{A11})$$

By analogy with the ordinary coupling, and using the results (A5), we obtain for the MA analytic coupling $\tilde{A}_{\text{MA},\nu,i}^{(i+1)}$ the following expression:

$$\tilde{A}_{\text{MA},\nu,i}^{(i+1)}(Q^2) = \tilde{A}_{\text{MA},\nu,i}^{(1)}(Q^2) + \sum_{m=1}^i C_m^{\nu+m} \tilde{\delta}_{\nu,i}^{(m+1)}(Q^2), \quad (\text{A12})$$

where $\tilde{A}_{\text{MA},\nu,i}^{(1)}$ is given in Eq. (A8) and

⁹This observation was previously made in [46].

$$\tilde{\delta}_{A,\nu,i}^{(m+1)}(Q^2) = \tilde{\delta}_{\nu,i}^{(m+1)}(Q^2) - \hat{R}_m \left(\frac{\text{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right), \quad (\text{A13})$$

with $\tilde{\delta}_{\nu,i}^{(m+1)}(Q^2)$ and $\hat{R}_m$ given in Eqs. (A5) and (A6), respectively.

The relations (A8) reflect the fact that the MA procedure (A8) commutes with the operation $d/(d\nu)$. Thus, to obtain (A8), we assume that the form (A5) for the usual coupling a_s at higher orders applies exactly in the same way to the MA coupling.

After some calculations, we obtain the following expressions without explicit operators:

$$\tilde{\Delta}_{\nu,i}^{(i+1)} = \tilde{\Delta}_{\nu,i}^{(1)} + \sum_{m=1}^i C_m^{\nu+m} \bar{R}_m(z_i) \left(\frac{\text{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right), \quad (\text{A14})$$

where

$$\begin{aligned} \bar{R}_1(z) &= b_1 [\bar{\gamma}_E + \text{M}_{-\nu,1}(z)], \\ \bar{R}_2(z) &= b_2 + b_1^2 [\text{M}_{-\nu-1,2}(z) + 2\bar{\gamma}_E \text{M}_{-\nu-1,1}(z) + \bar{\gamma}_E^2 - \zeta_2], \end{aligned} \quad (\text{A15})$$

with the Euler constant γ_E and

$$\begin{aligned} \bar{\gamma}_E &= \gamma_E - 1, \\ \text{Li}_{\nu,k}(z) &= (-1)^k \frac{d^k}{(d\nu)^k} \text{Li}_{\nu}(z) = \sum_{m=1}^{\infty} \frac{z^m \ln^k m}{m^{\nu}}, \\ \text{M}_{\nu,k}(z) &= \frac{\text{Li}_{\nu,k}(z)}{\text{Li}_{\nu}(z)}. \end{aligned} \quad (\text{A16})$$

We see that the $\Psi(\nu)$ -function and its derivatives have completely canceled out. Moreover, the differences between $A_{\text{MA},\nu=1,i}^{(i+1)}(Q^2)$ are rather small and have nonzero values only around the position $Q^2 = \Lambda_i^2$ (see Figs. 3 and 4 in Ref. [38]).

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