

Hard cutoff and gauge theories

V. Branchina^{1,2,*} F. Contino^{3,4,†} R. Gandolfo^{1,2,‡} and A. Pernace^{1,2,§}

¹*Department of Physics, University of Catania, Via Santa Sofia 64, I-95123 Catania, Italy*

²*INFN-Catania, Via Santa Sofia 64, I-95123 Catania, Italy*

³*Scuola Superiore Meridionale, Largo San Marcellino 10, 80138 Napoli, Italy*

⁴*INFN-Napoli, Complesso Universitario di Monte S. Angelo, Via Cinthia Edificio 6, 80126 Napoli, Italy*



(Received 12 June 2026; accepted 13 July 2026; published 13 August 2026)

According to usual calculations, the use of a hard cutoff Λ in gauge theories leads to a violation of gauge invariance. This seems to generate a tension between gauge theories and the Wilsonian effective field theory paradigm, where Λ has the physical meaning of ultimate scale of the theory, the scale above which the latter has to be replaced by its UV completion. In the present work, considering the Euler-Heisenberg correction to the free Maxwell action, we present a way to introduce the Wilsonian hard UV cutoff Λ that preserves gauge invariance at the quantum level. For both scalar and fermionic QED, we recover the well-known Euler-Heisenberg result obtained within proper-time regularization, apart from terms that are generically cutoff suppressed. These terms, periodic in the inverse background field, might become relevant in regimes where the latter probes scales not much smaller than Λ . On the theoretical side, the methods developed in the present work represent a first step toward a new (closer in spirit to the Wegner-Houghton construction) realization of the Wilsonian renormalization group program in gauge theories.

DOI: [10.1103/sr3r-dqjd](https://doi.org/10.1103/sr3r-dqjd)

I. INTRODUCTION

As it is well known, a great progress in our understanding of renormalization in quantum field theory (QFT) comes from Wilson's lesson [1–6], according to which any QFT is an effective field theory (EFT), i.e. a theory defined with a built-in UV physical cutoff Λ , the scale above which it is no longer valid and has to be replaced by its UV completion. The corresponding Lagrangian $\mathcal{L}^{(\Lambda)}$ allows one to describe processes at energies $E \leq \Lambda$, and the “bare” parameters $g_i(\Lambda)$ in $\mathcal{L}^{(\Lambda)}$ keep track of the higher energy degrees of freedom ($E > \Lambda$) of the UV completion of the theory. A fundamental aspect of the Wilsonian paradigm consists in including in the effective action the quantum fluctuations through the progressive integration of modes in infinitesimal shells, starting from the UV scale Λ and moving toward the IR. At energy scales $\mu < \Lambda$, the parameters $g_i(\mu)$ result from the elimination of the modes

in the range $[\mu, \Lambda]$, and this gives rise to the renormalization group flow of the theory.

According to usual calculations, the use of a hard cutoff Λ in gauge theories leads to a violation of the Ward-Takahashi (Slavnov-Taylor) identities [7–10]. A simple example is given by the vacuum polarization tensor $\Pi_{\mu\nu}$ in QED. At one-loop order, a quadratically divergent longitudinal contribution arises, that can be traced back to a nongauge invariant term $\sim \Lambda^2 A_\mu A^\mu$ in the effective action. Faced with this problem, the common attitude is to abandon the idea of a hard cutoff and to resort to regularization methods that by construction preserve gauge invariance. Pauli-Villars, dimensional and zeta-function regularization, and proper time are among the most used ones. The physical meaning of these methods, however, is less transparent. Pauli and Villars themselves acknowledged that their method, based on the introduction of fictitious ghostlike particles with heavy masses M_i , had to be regarded as a “formalistic” “transitional stage of the theory,” awaiting a truly physical way to preserve gauge invariance at the quantum level [11]. Analogous considerations hold for dimensional and zeta-function regularization, where nongauge invariant terms are canceled due to the use of complex analysis techniques [12,13].

According to the above considerations, there seems to be a tension between the Wilsonian paradigm and gauge theories. In view of the deep physical understanding brought by Wilson's lesson, and of the central role played by gauge theories in the description of fundamental

*Contact author: branchina@ct.infn.it

†Contact author: f.contino@ssmeridionale.it

‡Contact author: riccardo.gandolfo@ct.infn.it

§Contact author: arcangelo.pernace@ct.infn.it

interactions, it seems of great importance to find a way to reconcile these two fundamental pillars of modern physics. Some attempts in this direction are in [14–47].

In the present work, we investigate the possibility of introducing a Wilsonian *hard* UV physical cutoff Λ that preserves gauge invariance. To this end, it is sufficient to consider the simple case of QED. Moreover, for the purposes of the present work, we can limit ourselves to the Euler-Heisenberg correction δS_{EH} [48–54] to the free Maxwell action $S_{\text{em}} = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}$ due to the quantum fluctuations of charged scalar and fermion fields. The inclusion of the electromagnetic field quantum fluctuations according to the methods discussed in the present work, as well as the case of non-Abelian gauge theories, will be considered elsewhere [55]. Following the suggestion coming from Wilson’s lesson, here we calculate the fluctuation determinant that appears in δS_{EH} as the product over the eigenvalues λ_n of the fluctuation operator, and implement the Wilsonian UV physical cutoff Λ as a hard cut over the spectrum: $\lambda_n \leq \Lambda^2$. We will consider the case of a constant magnetic field background, in which case the spectrum is given by the Landau levels. As we will see, our result coincides with the well-known Schwinger’s result [50] (obtained with proper-time regularization) apart from the appearance of field-dependent oscillatory terms. These terms are generically suppressed by inverse powers of the UV cutoff Λ , but may become relevant when the background field probes scales not much smaller than Λ .

Oscillatory contributions originating from the discreteness of Landau levels and from the presence of a sharp physical boundary are well known in condensed matter physics. The paradigmatic example is provided by de Haas-van Alphen oscillations, where the magnetization displays a periodic dependence on the inverse magnetic field as Landau levels cross the Fermi surface [56,57]. Related quantum oscillatory phenomena also occur in Dirac and Weyl materials, in moiré graphene systems, and in Hofstadter-type spectra realized in graphene-based superlattices and cold-atom optical lattices [58–63]. Of course, the physical setting is different from the one considered here: in those systems the sharp boundary is typically provided by a Fermi surface, a band edge or a lattice miniband. In the present work, instead, the sharp boundary is the Wilsonian spectral cutoff. To the best of our knowledge, this is the first time that oscillatory terms of this kind are discussed in the context of fundamental gauge theories.

The rest of the paper is organized as follows. In Sec. II, we recall the definition of the Euler-Heisenberg correction $\delta S_{\text{EH}}^\phi$ to the Maxwell action in the case of scalar QED and set up the tools for our analysis, paying attention to the scale M that takes care of the dimensions of the fluctuation operator. In Sec. II A, we solve the eigenvalue problem for the (Euclidean) fluctuation operator specifying to the case of a constant magnetic background $B = \text{const}$, taking

$E = 0$, and calculate the corresponding Euler-Heisenberg correction $\delta S_{\text{EH}}^\phi$. In Sec. III, we perform the same calculation for the case of fermionic QED. In Sec. IV, we further discuss the issue of the scale M used in the calculation of the fluctuation determinant, considering also the relation between our calculation (spectral hard cutoff) and proper-time regularization. Section V is for conclusions and outlooks.

II. EULER-HEISENBERG ACTION AND SCALAR QED

In this section we perform the calculation of the Euler-Heisenberg action introducing the Wilsonian UV physical cutoff Λ . Before moving to that, it is worthwhile to discuss an important issue related to the path integral formulation of QFTs. To illustrate the point, we consider the simple theory of a single component real scalar field (Euclidean signature),

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \quad (2.1)$$

For a constant background Φ , the one-loop correction to the classical potential is usually obtained directly from the configuration space path integral $\int \mathcal{D}\phi e^{-S[\phi]}$ and is written as

$$\delta V_{\text{1l}}(\Phi) = \text{Tr} \log \frac{-\partial_\mu \partial^\mu + m^2 + \frac{\lambda}{2} \Phi^2}{M^2}, \quad (2.2)$$

where M is a mass scale that is typically introduced to take care of the dimensions inside the logarithm. Here we stress an important point. Though after calculation of the trace in (2.2) the scale M appears in $\delta V_{\text{1l}}(\Phi)$ only in unimportant field-independent terms (vacuum energy), and there is then no need to specify this scale, there might be situations where this is not the case. As we will see, this is what happens in the case of the Euler-Heisenberg Lagrangian. These issues, and their connection with proper-time regularization, will be further discussed in Sec. IV.

To find which scale takes care of the dimensions in (2.2), it is useful to refer to the original phase space path integral of the theory (2.1). Let us consider the corresponding Hamiltonian density $\mathcal{H}(\pi, \phi) = \frac{1}{2} \pi^2 + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$ (quadratic in the conjugate momenta π). The vacuum persistence amplitude is

$$\begin{aligned} & \int \left(\prod_x \frac{a^3 d\pi_x}{2\pi} \right) \left(\prod_x d\phi_x \right) \\ & \times \exp \left(\sum_x a^4 \left[i\pi_x \partial_0 \phi - \frac{1}{2} \pi_x^2 - \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right] \right), \end{aligned} \quad (2.3)$$

where $\prod_x$ indicates the product over the points of a 4D grid of lattice spacing a , and $\partial_0\phi$ and $\nabla\phi$ indicate the discretized version of the (Euclidean) time derivative and of the gradient, respectively. The integration over the conjugate momenta π_x leads to

$$\int \prod_x d\left(\frac{a\phi_x}{\sqrt{2\pi}}\right) e^{-\sum_x a^4 \mathcal{L}(\phi_x, \partial_\mu\phi)}, \quad (2.4)$$

where $\mathcal{L}$ is the (discretized version of the) Lagrangian density (2.1), and $a\phi_x$ are the dimensionless integration variables. The resulting configuration space path integral measure $\prod_x d(a\phi_x/\sqrt{2\pi})$ is dimensionless, and the scale M that takes care of the dimensions [see (2.2)] turns out to be $M = \Lambda \equiv \frac{\sqrt{2\pi}}{a}$, that is nothing but the Wilsonian UV physical cutoff of the theory (not an arbitrary mass scale). This observation will be useful for our following analysis.

Let us move now to the (Euclidean) action of scalar QED:

$$\begin{aligned} S[\phi^*, \phi, A_\mu] &= S_{\text{em}}[A_\mu] + S_0[\phi^*, \phi] + S_{\text{int}}[\phi^*, \phi, A_\mu] \\ &\equiv \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \int d^4x [\partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi] \\ &\quad + \int d^4x [j_\mu A^\mu + e^2 A_\mu A^\mu \phi^* \phi], \end{aligned} \quad (2.5)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $j_\mu = ie(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi)$. The Euler-Heisenberg correction $\delta S_{\text{EH}}^\phi$ to the free Maxwell action S_{em} is obtained integrating over the scalar fields [see (2.4)]

$$\delta S_{\text{EH}}^\phi = -\log \int \mathcal{D}(\phi^*/\Lambda) \mathcal{D}(\phi/\Lambda) e^{-(S_0[\phi^*, \phi] + S_{\text{int}}[\phi^*, \phi, A_\mu])}. \quad (2.6)$$

Performing the Gaussian integrations over ϕ and ϕ^* we get

$$\delta S_{\text{EH}}^\phi = \log \det \left(\frac{-D_\mu D^\mu + m^2}{\Lambda^2} \right) = \text{Tr} \log \left(\frac{-D_\mu D^\mu + m^2}{\Lambda^2} \right), \quad (2.7)$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative.

As anticipated in the Introduction, we now identify the Wilsonian UV physical cutoff Λ with a hard cut over the eigenvalues λ_n^ϕ of the operator $-D_\mu D^\mu + m^2$ that are included in the calculation of the trace in (2.7), i.e. we retain only the eigenvalues such that $\lambda_n^\phi \leq \Lambda^2$. From a physical point of view, we are considering quantum fluctuations up to the maximal energy scale Λ (down to the minimal distance a). Indicating with D_n^ϕ the degeneracy of λ_n^ϕ , we have

$$\delta S_{\text{EH}}^\phi = \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi \log \frac{\lambda_n^\phi}{\Lambda^2}. \quad (2.8)$$

In the next section, we specialize to the case of a constant magnetic (electric) field background. After solving the corresponding eigenvalue problem for the operator $-D_\mu D^\mu + m^2$, we calculate $\delta S_{\text{EH}}^\phi$ in (2.8) for this background. Since the eigenvalues are gauge invariant quantities, to solve the eigenvalue problem we will make a specific (and convenient) choice for the background gauge field A_μ .

A. Eigenvalue problem and calculation of $\delta S_{\text{EH}}^\phi$

In the present section, we consider the case of a constant magnetic field background $B \neq 0$ (vanishing electric field, $E = 0$). As it is clear from the explicit expression of the eigenvalues and of the corresponding degeneracies of $-D_\mu D^\mu + m^2$ [see (2.11) and (2.15)], in Euclidean signature the result for $\delta S_{\text{EH}}^\phi$ in the case of a constant electric field is obtained through the replacement $B \rightarrow E$. Differences appear after analytic continuation to Lorentzian signature.

Taking now the background gauge field A_μ to be

$$A_0 = A_2 = A_3 = 0, \quad A_1 = -Bx_2, \quad (2.9)$$

the eigenvalue problem for the operator $-D_\mu D^\mu + m^2$ reads as

$$(-\partial_0^2 - \partial_2^2 - \partial_3^2 - (\partial_1 - ieBx_2)^2 + m^2)\phi(x) = \lambda\phi(x). \quad (2.10)$$

This coincides with the eigenvalue problem for a free (nonrelativistic) particle of mass $1/2$ in the (x_0, x_3) plane, and with the Landau-level problem with frequency $2eB$ in the (x_1, x_2) plane. The eigenvalues are

$$\begin{aligned} \lambda_{n,k_0,k_3}^\phi &= k_0^2 + k_3^2 + 2eB \left(n + \frac{1}{2} \right) + m^2, \quad n = 0, 1, \dots; \\ k_i &= \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}, \end{aligned} \quad (2.11)$$

where L is the size of the quantization box (in the following we treat k_i as continuum variables as usual). From now on we use the notation $k^2 \equiv k_0^2 + k_3^2$. The eigenfunctions corresponding to the eigenvalues (2.11) are

$$\begin{aligned} \phi_{n,k_0,k_3;p_1} &= \left(\frac{2^n n!}{\sqrt{\pi}} \right)^{-\frac{1}{2}} \sqrt{eB} H_n \left(\sqrt{eB} \left(x_2 - \frac{p_1}{eB} \right) \right) \\ &\quad \times e^{-\frac{eB}{2} \left(x_2 - \frac{p_1}{eB} \right)^2} e^{ip_1 x_1} e^{i(k_0 x_0 + k_3 x_3)}, \end{aligned} \quad (2.12)$$

where $H_n(x)$ are Hermite functions and $p_1 \equiv \frac{2\pi n_1}{L}$, with n_1 integer. It is important to stress that n_1 is bounded. Actually,

$\frac{p_1}{eB}$ is the center of the Gaussian that appears in $\phi_{n,k_0,k_3;p_1}$, and lies in the range $[-L/2, L/2]$ of the coordinate x_2 , namely

$$p_1^{\min} \equiv -\frac{eB}{2}L \leq p_1 \leq \frac{eB}{2}L \equiv p_1^{\max}. \quad (2.13)$$

Therefore, the number of linearly independent eigenfunctions associated to the eigenvalue λ_{n,k_0,k_3}^ϕ is

$$\frac{L}{2\pi}(p_1^{\max} - p_1^{\min}) = \frac{eB}{2\pi}L^2, \quad (2.14)$$

i.e. each eigenvalue λ_{n,k_0,k_3}^ϕ has degeneracy

$$D_{n,k_0,k_3}^\phi = \frac{eB}{2\pi}L^2. \quad (2.15)$$

Having found the eigenvalues λ_{n,k_0,k_3}^ϕ and the corresponding degeneracies D_{n,k_0,k_3}^ϕ , we can now calculate the Euler-Heisenberg correction $\delta S_{\text{EH}}^\phi$ to the free Maxwell action. As said above, we implement the physical cutoff Λ as a hard cut on the spectrum, $\lambda_{n,k_0,k_3}^\phi \leq \Lambda^2$ [see (2.8)]. For the corresponding Lagrangian $\delta \mathcal{L}_{\text{EH}}^\phi = \delta S_{\text{EH}}^\phi / V$ (with $V = L^4$), we have

$$\delta \mathcal{L}_{\text{EH}}^\phi = \frac{eB}{2\pi} \sum_{n=0}^N \int^{(K_n)} \frac{d^2k}{(2\pi)^2} \log \left(\frac{k^2 + eB(2n+1) + m^2}{\Lambda^2} \right), \quad (2.16)$$

where $N \equiv \lfloor \frac{\Lambda^2 - m^2}{2eB} - \frac{1}{2} \rfloor$ (with $\lfloor x \rfloor$ the floor of x), $K_n \equiv \sqrt{\Lambda^2 - m^2 - eB(2n+1)}$, and $\int^{(K_n)}$ indicates that the integration is over the circle of radius K_n .

Performing the integration over k and the sum over n we get

$$\begin{aligned} \delta \mathcal{L}_{\text{EH}}^\phi = & -\frac{eB}{8\pi^2}(N+1) \left(\Lambda^2 - m^2 \log \frac{\Lambda^2}{2eB} \right) \\ & + \frac{e^2 B^2}{4\pi^2} \left[\frac{N+1}{2} (N - \log(2\pi)) \log \frac{\Lambda^2}{2eB} \right. \\ & \left. + \psi^{(-2)} \left(\frac{m^2}{2eB} + \frac{1}{2} \right) - \psi^{(-2)} \left(\frac{m^2}{2eB} + \frac{3}{2} + N \right) \right], \end{aligned} \quad (2.17)$$

where $\psi^{(-n)}(z)$ (with n positive integer) is the polygamma function of negative order $-n$ defined as [64] [$\Gamma(z)$ is the Euler gamma function]

$$\psi^{(-n)}(z) = \frac{1}{(n-2)!} \int_0^z dt (z-t)^{n-2} \log \Gamma(t) \quad \text{for } \text{Re}(z) > 0. \quad (2.18)$$

Defining

$$\xi \equiv \left(\frac{\Lambda^2 - m^2}{2eB} - \frac{1}{2} \right) - \left\lfloor \frac{\Lambda^2 - m^2}{2eB} - \frac{1}{2} \right\rfloor, \quad (2.19)$$

N can be written as

$$N = \frac{\Lambda^2 - m^2}{2eB} - \frac{1}{2} - \xi. \quad (2.20)$$

We now expand $\delta \mathcal{L}_{\text{EH}}^\phi$ for $m^2, eB \ll \Lambda^2$. We get [note that according to (2.19), $0 \leq \xi < 1$ independently of the specific values of Λ , eB and m^2]

$$\begin{aligned} \delta \mathcal{L}_{\text{EH}}^\phi = & -\frac{\Lambda^4}{64\pi^2} + \frac{m^2 \Lambda^2}{16\pi^2} - \frac{m^4}{32\pi^2} \left(\log \frac{\Lambda^2}{m^2} + \frac{3}{2} \right) + \frac{m^4}{32\pi^2} \left(\frac{3}{2} - \log \frac{m^2}{2eB} \right) - \frac{m^2 eB}{16\pi^2} \log(2\pi) \\ & + \frac{e^2 B^2}{96\pi^2} \left[\log \frac{\Lambda^2}{m^2} - 6 \log(2\pi A^4) \right] + \frac{e^2 B^2}{96\pi^2} \left[\log \frac{m^2}{2eB} + 24\psi^{(-2)} \left(\frac{m^2}{2eB} + \frac{1}{2} \right) \right] \\ & + \frac{e^2 B^2}{24\pi^2} \left[\frac{eB}{\Lambda^2} \xi(1-\xi)(1-2\xi) + \frac{e^2 B^2}{\Lambda^4} \left(\xi^2(1-\xi)^2 - \frac{1}{30} \right) + \mathcal{O} \left(\frac{e^3 B^3}{\Lambda^6} \right) \right], \end{aligned} \quad (2.21)$$

where A is the Glaisher-Kinkelin constant. The above equation is our result for the Euler-Heisenberg correction $\delta S_{\text{EH}}^\phi$ due to a scalar field [see (2.5)]. It is one of the central results of the present work. Before delving into a detailed discussion of the different terms in (2.21), we now make some comments.

For the first time the calculation of δS_{EH} has been performed resorting to a *hard* cutoff Λ , which has a precise physical meaning. Within the Wilsonian EFT paradigm, Λ is the ultimate scale above which the theory has to be replaced by its UV completion. Moreover, it is important to stress that, since Λ is implemented as a cut over the

eigenvalues of the fluctuation operator (that are gauge invariant), our result (2.21) is gauge invariant.

In the literature, the calculation of δS_{EH} has been performed in many other ways [48–54], resorting to regularization schemes that range from Pauli-Villars regularization to zeta function regularization, proper-time method, etc. As it is the case for the Wilsonian UV physical cutoff Λ introduced above, these techniques preserve gauge invariance. However, contrary to the use of the spectral hard cutoff, which has the precise physical meaning stressed above, these regularization techniques do not have an immediate physical interpretation.

In this respect, it is worthwhile to mention that Pauli and Villars themselves described their regularization method as a “formalistic” “transitional stage of the theory,” awaiting a truly physical way to preserve gauge invariance at the quantum level [11]. Similarly, in his seminal paper [50] Schwinger refers to the proper-time method as a technique to preserve gauge and Lorentz invariance, without delving into its physical interpretation.¹

Let us go back to (2.21). The terms in the first line are just inessential vacuum energy contributions. We consider now those in the second and third line. As it can be easily verified, these terms (that we obtained resorting to the physical *hard* cutoff Λ) coincide with the result for $\delta S_{\text{EH}}^\phi$ that is obtained resorting to proper-time regularization (see for instance [51–53] where the result is written in Lorentzian signature. Here we are using Euclidean signature, but in the case of a magnetic field background the difference consists simply in an overall minus sign),

$$[\delta \mathcal{L}_{\text{EH}}^\phi]_{\text{pt}} = -\frac{1}{16\pi^2} \int_{1/\Lambda^2}^{+\infty} \frac{ds}{s^3} \left(\frac{eBs}{\sinh(eBs)} - 1 \right) e^{-sm^2}. \quad (2.22)$$

We want to consider now the case of a constant electric field background E . As said at the beginning of the present section, the (Euclidean) result for this case ($E \neq 0$, $B = 0$) is obtained by making in (2.21) the replacement $B \rightarrow E$. Considering this case, and performing the analytic continuation of Eq. (2.21) to the Lorentzian signature ($E \rightarrow -iE$ and $\delta \mathcal{L}_{\text{EH}}^\phi \rightarrow -\delta \mathcal{L}_{\text{EH}}^\phi$), it is well known that $\delta \mathcal{L}_{\text{EH}}^\phi$ develops an imaginary part related to the production of e^+e^- pairs. Sticking as before only to the second and

third line of this equation, since (as stressed above) in Euclidean signature these two lines coincide with (2.22), we have that (after analytic continuation) the imaginary part of our $\delta \mathcal{L}_{\text{EH}}^\phi$ coincides with the imaginary part of $[\delta \mathcal{L}_{\text{EH}}^\phi]_{\text{pt}}$ (see for instance [50–53]), apart from contributions from the fourth line of (2.21) that are generically cutoff suppressed.

Let us finally comment on the terms in this fourth line. Since they depend on ξ , which is periodic in $1/eB$ with period $2(\Lambda^2 - m^2)^{-1}$ [see (2.19)], they are periodic in $1/eB$ with the same period. Moreover, since ξ lies in the range $0 \leq \xi < 1$, for values of the magnetic field B such that $eB \ll \Lambda^2$, these terms are cutoff suppressed, and our result (2.21) for $\delta \mathcal{L}_{\text{EH}}^\phi$ coincides with the result $[\delta \mathcal{L}_{\text{EH}}^\phi]_{\text{pt}}$ obtained resorting to proper-time regularization.

However, we have to stress again that the UV physical cutoff Λ is not a technical device introduced to render finite ill-defined expressions involved in the calculation of quantum fluctuations. It has instead the deep physical meaning of being the ultimate scale of the theory, above which the latter has to be replaced by its UV completion. Therefore, going back to our result (2.21), we see that for strong magnetic fields, more precisely for values of eB not much smaller than Λ^2 , the terms in the fourth line of this equation (absent in usual calculations of $\delta \mathcal{L}_{\text{EH}}^\phi$, in particular in $[\delta \mathcal{L}_{\text{EH}}^\phi]_{\text{pt}}$) may become relevant and provide significant modifications to physical quantities. This point deserves further investigation both from a theoretical and phenomenological point of view (further comments are in Sec. V).

In the next section, we perform a similar analysis for the case of fermionic QED.

III. EULER-HEISENBERG ACTION AND FERMIONIC QED

Let us consider the (Euclidean) action for fermionic QED

$$\begin{aligned} S[A, \psi, \bar{\psi}] &= S_{\text{em}}[A_\mu] + S_0[\bar{\psi}, \psi] + S_{\text{int}}[\bar{\psi}, \psi, A_\mu] \\ &\equiv \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} + \int d^4x \bar{\psi} (\not{\partial} + m) \psi + ie \\ &\quad \times \int d^4x \bar{\psi} \gamma_\mu \psi A^\mu, \end{aligned} \quad (3.1)$$

where $\not{\partial} = \gamma^\mu \partial_\mu$. γ^μ are the (Euclidean) gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \quad (3.2)$$

and

¹Apart from these historical remarks, it is worthwhile to observe that from the expression of the functional determinant of a given operator $\mathcal{O}$ regularized in terms of a proper-time cutoff $1/\Lambda^2$ (λ_n eigenvalues of $\mathcal{O}$ and D_n corresponding degeneracies), $\log \det(\mathcal{O}) = -\sum_n^\infty D_n \int_{1/\Lambda^2}^\infty \frac{ds}{s} e^{-s\lambda_n} = -\sum_n^\infty D_n \int_0^\infty \frac{ds}{s} \Theta(s - 1/\Lambda^2) e^{-s\lambda_n}$ [50], it is seen that $1/\Lambda^2$ implements a smooth suppression of the eigenvalues of the operator $\mathcal{O}$ greater than Λ^2 . Similar considerations hold for smooth versions of the proper-time cutoff where $\Theta(s - 1/\Lambda^2)$ is replaced by a smooth function $\rho(s, \Lambda)$ [65,66].

$$\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}. \quad (3.3)$$

Similarly to the scalar QED case considered in the previous section, the Euler-Heisenberg action $\delta S_{\text{EH}}^\psi$ is ($\mathcal{D} = \gamma^\mu D_\mu$):

$$\delta S_{\text{EH}}^\psi = -\log \det \left(\frac{\mathcal{D} + m}{\Lambda} \right) = -\text{Tr} \log \left(\frac{\mathcal{D} + m}{\Lambda} \right). \quad (3.4)$$

Again we implement the Wilsonian UV physical cutoff Λ with a hard cut over the eigenvalues λ_n^ψ of the operator $\mathcal{D} + m$. With $\mathcal{D}$ being anti-Hermitian, its eigenvalues $\lambda_n^\psi - m$ are purely imaginary. For each of them, the spectrum of $\mathcal{D}$ also contains the complex conjugate eigenvalue $(\lambda_n^\psi)^* - m$. The spectral cut is then implemented on $|\lambda_n^\psi|^2$ as $|\lambda_n^\psi|^2 \leq \Lambda^2$. We have then (D_n^ψ is the degeneracy of λ_n^ψ)

$$\delta S_{\text{EH}}^\psi = - \sum_{|\lambda_n^\psi|^2 \leq \Lambda^2} D_n^\psi \log \frac{|\lambda_n^\psi|^2}{\Lambda^2}. \quad (3.5)$$

In the next section, we solve the eigenvalue problem for the operator $\mathcal{D} + m$ in the case of a constant magnetic background, and calculate the corresponding $\delta S_{\text{EH}}^\psi$ in (3.5).

A. Eigenvalue problem and calculation of $\delta S_{\text{EH}}^\psi$

Similarly to what we did for scalar QED, we now consider the case of a constant magnetic field background $B \neq 0$ (in Euclidean signature, the result for $\delta S_{\text{EH}}^\psi$ for a constant electric field background is obtained through the replacement $B \rightarrow E$).

We now perform the calculation taking for the gauge field A_μ :

$$A_0 = A_2 = A_3 = 0, \quad A_1 = -Bx_2. \quad (3.6)$$

The eigenvalue problem for the Dirac operator $\mathcal{D} + m$ then reads as

$$(\gamma^\mu \partial_\mu - ieB\gamma^1 x_2 + m)\psi(x) = \lambda\psi(x). \quad (3.7)$$

As for the case of scalar QED of Sec. II, this problem can be solved analytically. The details of the calculation are given in the Appendix. The eigenvalues are

$$\lambda_{n,k_0,k_3}^\psi = m \pm i\sqrt{k_0^2 + k_3^2 + 2eBn},$$

$$n = 0, 1, \dots; \quad k_i = \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}. \quad (3.8)$$

From now on we use the notation $k^2 \equiv k_0^2 + k_3^2$. To obtain the eigenfunctions corresponding to the eigenvalues (3.8), we treat separately the cases $n = 0$ and $n \neq 0$. Let us begin from the latter.

1. Case $n \neq 0$

Let us begin by noting that, for any given n, k_0, k_3 , we have two complex conjugate eigenvalues of $\mathcal{D} + m$, namely $\lambda_{n,k_0,k_3}^\psi = m + i\sqrt{k^2 + 2eBn}$ and $(\lambda_{n,k_0,k_3}^\psi)^*$ [see (3.8)]. As shown in the Appendix, the eigenfunctions associated to $(\lambda_{n,k_0,k_3}^\psi)^*$ are obtained by applying the matrix γ_5 to the eigenfunctions associated to λ_{n,k_0,k_3}^ψ . A set of linearly independent eigenfunctions corresponding to the eigenvalue λ_{n,k_0,k_3}^ψ is given by [see (A27) and (A28)]

$$(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1} = \begin{pmatrix} \phi_{n-1,k_0,k_3;p_1} \\ 0 \\ \frac{k_0 + ik_3}{\sqrt{k^2 + 2eBn}} \phi_{n-1,k_0,k_3;p_1} \\ \frac{-i\sqrt{2eBn}}{\sqrt{k^2 + 2eBn}} \phi_{n,k_0,k_3;p_1} \end{pmatrix} \quad (3.9)$$

$$(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \phi_{n,k_0,k_3;p_1} \\ \frac{-i\sqrt{2eBn}}{\sqrt{k^2 + 2eBn}} \phi_{n-1,k_0,k_3;p_1} \\ \frac{k_0 - ik_3}{\sqrt{k^2 + 2eBn}} \phi_{n,k_0,k_3;p_1} \end{pmatrix}. \quad (3.10)$$

We have already encountered the functions $\phi_{n,k_0,k_3;p_1}$ when studying the scalar case; see Eq. (2.12) and the comments below. In particular, p_1 is the degeneracy index and lies in the range $p_1^{\min} \equiv -\frac{eB}{2}L \leq p_1 \leq \frac{eB}{2}L \equiv p_1^{\max}$ [see (2.13) and the comments below (A17)]. Therefore, the number of linearly independent eigenfunctions $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ ($(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$) associated to the eigenvalue λ_{n,k_0,k_3}^ψ is

$$\frac{L}{2\pi}(p_1^{\max} - p_1^{\min}) = \frac{eB}{2\pi}L^2, \quad (3.11)$$

and the degeneracy of the eigenvalue λ_{n,k_0,k_3}^ψ is

$$D_{n,k_0,k_3}^\psi = \frac{eB}{\pi}L^2. \quad (3.12)$$

Finally, the eigenfunctions $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ and $(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$ associated to the eigenvalue $(\lambda_{n,k_0,k_3}^\psi)^*$ are (as anticipated) $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1} = -\gamma_5(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ and $(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1} = -\gamma_5(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$, and the eigenvalue $(\lambda_{n,k_0,k_3}^\psi)^*$ has the same degeneracy (3.12) as λ_{n,k_0,k_3}^ψ .

2. Case $n = 0$

Let us consider now the eigenvalues $\lambda_{0,k_0,k_3}^\psi = m + ik$. A set of linearly independent eigenfunctions corresponding to λ_{0,k_0,k_3}^ψ and $(\lambda_{0,k_0,k_3}^\psi)^*$ is [see (A29) and (A30)]

$$(\psi_\lambda^{(2)})_{0,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \phi_{0,k_0,k_3;p_1} \\ 0 \\ \frac{k_0-ik_3}{k} \phi_{0,k_0,k_3;p_1} \end{pmatrix} \quad (3.13)$$

and

$$(\psi_{\lambda^*}^{(2)})_{0,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \phi_{0,k_0,k_3;p_1} \\ 0 \\ -\frac{k_0-ik_3}{k} \phi_{0,k_0,k_3;p_1} \end{pmatrix}, \quad (3.14)$$

respectively. The degeneracy of both λ_{0,k_0,k_3}^ψ and $(\lambda_{0,k_0,k_3}^\psi)^*$ is $D_{0,k_0,k_3}^\psi = \frac{eB}{2\pi} L^2$ [see the comments below (A30)].

The complete set of orthogonal eigenfunctions that spans the field ψ space is then [see Eqs. (3.9), (3.10), (3.13) and (3.14)]

$$\begin{aligned} &(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}, & (\psi_{\lambda^*}^{(1)})_{n,k_0,k_3;p_1}, & (\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}, \\ &(\psi_{\lambda^*}^{(2)})_{n,k_0,k_3;p_1}, & (\psi_\lambda^{(2)})_{0,k_0,k_3;p_1}, & (\psi_{\lambda^*}^{(2)})_{0,k_0,k_3;p_1}. \end{aligned} \quad (3.15)$$

Similarly, the $\bar{\psi}$ space is spanned by

$$\begin{aligned} &(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}^\dagger, & (\psi_{\lambda^*}^{(1)})_{n,k_0,k_3;p_1}^\dagger, & (\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}^\dagger, \\ &(\psi_{\lambda^*}^{(2)})_{n,k_0,k_3;p_1}^\dagger, & (\psi_\lambda^{(2)})_{0,k_0,k_3;p_1}^\dagger, & (\psi_{\lambda^*}^{(2)})_{0,k_0,k_3;p_1}^\dagger. \end{aligned} \quad (3.16)$$

For the Euler-Heisenberg Lagrangian $\delta\mathcal{L}_{\text{EH}}^\psi = \delta S_{\text{EH}}^\psi/V$ we then have [see (3.5)]

$$\begin{aligned} \delta\mathcal{L}_{\text{EH}}^\psi = & -\frac{eB}{\pi} \sum_{n=1}^N \int^{(K_n)} \frac{d^2k}{(2\pi)^2} \log\left(\frac{k^2 + 2eBn + m^2}{\Lambda^2}\right) - \frac{eB}{2\pi} \\ & \times \int^{(\Lambda)} \frac{d^2k}{(2\pi)^2} \log\left(\frac{k^2 + m^2}{\Lambda^2}\right), \end{aligned} \quad (3.17)$$

where $N \equiv \lfloor \frac{\Lambda^2 - m^2}{2eB} \rfloor$, $K_n \equiv \sqrt{\Lambda^2 - m^2 - 2eBn}$, and $\int^{(K_n)}$ ($\int^{(\Lambda)}$) indicates that the integration is over the circle of radius K_n (radius Λ). Performing the integration over k and the sum over n we get

$$\begin{aligned} \delta\mathcal{L}_{\text{EH}}^\psi = & \frac{eB}{8\pi^2} \left[(2N+1)\Lambda^2 - m^2 \left(2N \log \frac{\Lambda^2}{2eB} + \log \frac{\Lambda^2}{m^2} + 1 \right) \right] \\ & - \frac{e^2 B^2}{2\pi^2} \left[\frac{N}{2} \left((N+1) \log \frac{\Lambda^2}{2eB} + \log(2\pi) \right) + \psi^{(-2)}\left(\frac{m^2}{2eB}\right) - \psi^{(-2)}\left(\frac{m^2}{2eB} + 1 + N\right) \right]. \end{aligned} \quad (3.18)$$

Similarly to the previous section, we now define

$$\xi \equiv \left(\frac{\Lambda^2 - m^2}{2eB} \right) - \left\lfloor \frac{\Lambda^2 - m^2}{2eB} \right\rfloor, \quad (3.19)$$

so that N can be written as

$$N = \frac{\Lambda^2 - m^2}{2eB} - \xi. \quad (3.20)$$

Expanding $\delta\mathcal{L}_{\text{EH}}^\psi$ for $m^2, eB \ll \Lambda^2$ (noting that $0 \leq \xi < 1$), we find

$$\begin{aligned} \delta\mathcal{L}_{\text{EH}}^\psi = & \frac{\Lambda^4}{32\pi^2} - \frac{m^2\Lambda^2}{8\pi^2} + \frac{m^4}{16\pi^2} \left(\log \frac{\Lambda^2}{m^2} + \frac{3}{2} \right) + \frac{m^4}{16\pi^2} \left(\log \frac{m^2}{2eB} - \frac{3}{2} \right) + \frac{m^2 eB}{8\pi^2} \left(\log \frac{m^2}{2eB} + \log(2\pi) - 1 \right) \\ & + \frac{e^2 B^2}{24\pi^2} \left[\log \frac{\Lambda^2}{m^2} + 6 \log(2\pi A^2) \right] + \frac{e^2 B^2}{24\pi^2} \left[\log \frac{m^2}{2eB} - 12\psi^{(-2)}\left(\frac{m^2}{2eB} + 1\right) \right] \\ & - \frac{e^2 B^2}{12\pi^2} \left[\frac{eB}{\Lambda^2} \xi(1-\xi)(1-2\xi) + \frac{e^2 B^2}{\Lambda^4} \left(\xi^2(1-\xi)^2 - \frac{1}{30} \right) + \mathcal{O}\left(\frac{e^3 B^3}{\Lambda^6}\right) \right]. \end{aligned} \quad (3.21)$$

The above equation is our result for the Euler-Heisenberg Lagrangian $\delta\mathcal{L}_{\text{EH}}^\psi$, another central outcome of the present work. Comparing (3.21) to our previous result (2.21) for $\delta\mathcal{L}_{\text{EH}}^\phi$ of scalar QED, it is clear that the comments made below (2.21) hold true also for $\delta\mathcal{L}_{\text{EH}}^\psi$. Here, we simply stress that the second and third lines of (3.21) coincide (after analytic continuation to Lorentzian signature) with the result for the Euler-Heisenberg Lagrangian obtained in the pioneering work [50] by Schwinger. As for the scalar QED case, the terms in the first line are vacuum energy terms, while those in the fourth line are generically cutoff-suppressed terms, periodic in $1/eB$ [see the fourth line of (2.21) and comments below].

In the next section, we go back to the question raised at the beginning of Sec. II, namely the issue of the scale M that takes care of the dimensions of the fluctuation operator in loop calculations.

IV. DISCUSSION

Starting from the original phase space path integral formulation of a QFT, in Sec. II we argued that the scale M that takes care of the dimensions of the fluctuation operator in loop calculations is the Wilsonian UV physical

cutoff Λ ($M = \Lambda$). In the present section, we discuss the consequences of choosing $M \neq \Lambda$. We will start from the case of scalar QED showing that, if to take care of the dimensions of the operator $-D_\mu D^\mu + m^2$ in (2.7) a generic fixed scale M is used rather than the Wilsonian physical scale Λ [i.e. if in (2.7) we replace Λ^2 with M^2 inside the logarithm], unphysical terms appear in $\delta S_{\text{EH}}^\phi$. We will see that similar considerations hold true for $\delta S_{\text{EH}}^\psi$. Moreover, we will show that $M = \Lambda$ is crucial to frame the proper-time method within the Wilsonian paradigm.

Let us begin then with scalar QED. As before, Λ is identified as a cut over the eigenvalues λ_n^ϕ of $-D_\mu D^\mu + m^2$ ($\lambda_n^\phi \leq \Lambda^2$). If we take $M \neq \Lambda$, for $\delta S_{\text{EH}}^\phi$ we have (D_n^φ degeneracy of λ_n^ϕ)

$$\delta S_{\text{EH}}^\phi = \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi \log \frac{\lambda_n^\phi}{M^2}. \quad (4.1)$$

Specializing (4.1) to the case of a constant magnetic field background B , the Euler-Heisenberg Lagrangian $\delta\mathcal{L}_{\text{EH}}^\phi$ turns out to be [see (2.19) for the definition of $\xi(B)$]

$$\begin{aligned} \delta\mathcal{L}_{\text{EH}}^\phi = & -\frac{\Lambda^4}{64\pi^2} \left(1 - 2 \log \frac{\Lambda^2}{M^2}\right) + \frac{m^2 \Lambda^2}{16\pi^2} \left(1 - \log \frac{\Lambda^2}{M^2}\right) - \frac{m^4}{32\pi^2} \left(\log \frac{\Lambda^2}{m^2} + \frac{3}{2} - \log \frac{\Lambda^2}{M^2}\right) \\ & + \frac{m^4}{32\pi^2} \left(\frac{3}{2} - \log \frac{m^2}{2eB}\right) - \frac{m^2 eB}{16\pi^2} \log(2\pi) \\ & + \frac{e^2 B^2}{96\pi^2} \left[\log \frac{\Lambda^2}{m^2} - 3 \log \frac{\Lambda^2}{M^2} - 6 \log(2\pi A^4)\right] + \frac{e^2 B^2}{96\pi^2} \left[\log \frac{m^2}{2eB} + 24\psi^{(-2)}\left(\frac{m^2}{2eB} + \frac{1}{2}\right)\right] \\ & + \frac{e^2 B^2}{24\pi^2} \left[3\xi(1-\xi) \log \frac{\Lambda^2}{M^2} + \frac{eB}{\Lambda^2} \xi(1-\xi)(1-2\xi) + \frac{e^2 B^2}{\Lambda^4} \left(\xi^2(1-\xi)^2 - \frac{1}{30}\right) + \mathcal{O}\left(\frac{e^3 B^3}{\Lambda^6}\right)\right]. \end{aligned} \quad (4.2)$$

The comparison between (4.2) and (2.21) shows that, apart from inessential additional vacuum energy terms [first line of (4.2)], the differences due to the use of a generic fixed scale M (rather than Λ) reside in the term $-3 \log \frac{\Lambda^2}{M^2}$ in the third line of (4.2) and in the term $3\xi(B)(1-\xi(B)) \log \frac{\Lambda^2}{M^2}$ in the fourth line. From a simple look to (4.2), we see that the former term has a dramatic impact on the beta function $\beta(e)$ for the running charge. In fact, for $M = \Lambda$ this term vanishes and the beta function $\beta(e)$ is solely due to the term $\log \frac{\Lambda^2}{m^2}$ (third line) that leads to $\beta(e) = \frac{e^3}{48\pi^2}$, the well-known one-loop result for scalar QED. On the contrary, for generic $M \neq \Lambda$, the additional term $-3 \log \frac{\Lambda^2}{M^2}$ modifies the beta function leading to $\beta(e) = -\frac{e^3}{36\pi^2}$, which not only has a different coefficient in front of e^3 , but even worse it is opposite in sign (Landau pole $\rightarrow$ asymptotic freedom). Needless to say, this is an

absurd result that further supports $M = \Lambda$ (see the discussion at the beginning of Sec. II). Let us also observe that the other additional term due to $M \neq \Lambda$ [first term in the fourth line of (4.2)] is a term periodic in $1/eB$, with period $2(\Lambda^2 - m^2)^{-1}$, that significantly changes the field dependence of $\delta\mathcal{L}_{\text{EH}}^\phi$, giving rise to spurious large amplitude oscillations.

We now repeat the same calculation with $M \neq \Lambda$ in the case of fermionic QED [see (3.5)],

$$\delta S_{\text{EH}}^\psi = - \sum_{|\lambda_n^\psi|^2 \leq \Lambda^2} D_n^\psi \log \frac{|\lambda_n^\psi|^2}{M^2}. \quad (4.3)$$

In the case of a constant magnetic field background B we get

$$\begin{aligned}
\delta\mathcal{L}_{\text{EH}}^\psi = & \frac{\Lambda^4}{32\pi^2} \left(1 - 2\log\frac{\Lambda^2}{M^2}\right) - \frac{m^2\Lambda^2}{8\pi^2} \left(1 - \log\frac{\Lambda^2}{M^2}\right) + \frac{m^4}{16\pi^2} \left(\log\frac{\Lambda^2}{m^2} + \frac{3}{2} - \log\frac{\Lambda^2}{M^2}\right) \\
& + \frac{m^4}{16\pi^2} \left(\log\frac{m^2}{2eB} - \frac{3}{2}\right) + \frac{m^2eB}{8\pi^2} \left(\log\frac{m^2}{2eB} + \log(2\pi) - 1\right) \\
& + \frac{e^2B^2}{24\pi^2} \left[\log\frac{\Lambda^2}{m^2} + 6\log(2\pi A^2)\right] + \frac{e^2B^2}{24\pi^2} \left[\log\frac{m^2}{2eB} - 12\psi^{(-2)}\left(\frac{m^2}{2eB} + 1\right)\right] \\
& - \frac{e^2B^2}{12\pi^2} \left[3\xi(1-\xi)\log\frac{\Lambda^2}{M^2} + \frac{eB}{\Lambda^2}\xi(1-\xi)(1-2\xi) + \frac{e^2B^2}{\Lambda^4}\left(\xi^2(1-\xi)^2 - \frac{1}{30}\right) + \mathcal{O}\left(\frac{e^3B^3}{\Lambda^6}\right)\right]. \quad (4.4)
\end{aligned}$$

Differently from what happens for scalar QED, in this case there are no additional terms of the kind $e^2B^2 \log \Lambda^2$ that modify the one-loop beta function $\beta(e)$. This is due to the difference between $\lambda_n^\phi = k^2 + eB(2n+1) + m^2$ in (4.1) (λ_n^ϕ eigenvalues of $-D_\mu D^\mu + m^2$) and $|\lambda_n^\psi|^2 = k^2 + 2eBn + m^2$ in (4.3) (λ_n^ψ and $(\lambda_n^\psi)^*$ eigenvalues of $\mathcal{D} + m$). The eigenvalue λ_n^ϕ has an extra eB term compared to $|\lambda_n^\psi|^2$ that, as it can be easily seen, is responsible for the term $-3\log \Lambda^2$ that appears in $\delta\mathcal{L}_{\text{EH}}^\phi$. However, as for the scalar QED case, the choice $M \neq \Lambda$ gives rise to a term periodic in $1/eB$ with period $2(\Lambda^2 - m^2)^{-1}$ [first term in the fourth line of (4.4)], that again sensibly changes the field dependence of $\delta\mathcal{L}_{\text{EH}}^\psi$ giving rise to spurious large amplitude oscillations.

Let us consider now the issue of the scale M to be used in the fluctuation determinant from a different perspective, comparing our calculation in terms of a hard cutoff Λ to its definition within the proper-time method. We show that, taking $M = \Lambda$, the proper-time definition is recovered from the expression written in terms of the Wilsonian UV hard cutoff Λ . Conversely, we will see that for a generic $M \neq \Lambda$ this connection cannot be established.

We focus on $\delta S_{\text{EH}}^\phi$ in (4.1) (the same analysis can be repeated for $\delta S_{\text{EH}}^\psi$) and write $\log \frac{\lambda_n^\phi}{M^2}$ using the identity $\log \frac{A}{B} = \int_0^\infty \frac{ds}{s} (e^{-sB} - e^{-sA})$,

$$\delta S_{\text{EH}}^\phi = \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi \log \frac{\lambda_n^\phi}{M^2} = \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi \int_0^\infty \frac{ds}{s} (e^{-sM^2} - e^{-s\lambda_n^\phi}). \quad (4.5)$$

The latter can be trivially rewritten as

$$\delta S_{\text{EH}}^\phi = \lim_{\Lambda_{\text{pt}} \rightarrow \infty} \int_{1/\Lambda_{\text{pt}}^2}^\infty \frac{ds}{s} \left(\sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi e^{-sM^2} - \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi e^{-s\lambda_n^\phi} \right). \quad (4.6)$$

Up to this point we have only used elementary identities. Now we want to establish the connection between the above calculation of $\delta S_{\text{EH}}^\phi$ performed using a Wilsonian UV hard cutoff Λ and the usual calculation performed resorting

to the proper-time method [50,52,53]. In this scheme one sums over the whole (infinite) spectrum of eigenvalues and regularizes UV divergences introducing a lower integration bound $1/\Lambda_{\text{pt}}^2$ in the proper-time integral (see footnote 1). To establish the aforementioned connection from (4.6), we have then to send in the right-hand side $\Lambda \rightarrow \infty$ while keeping Λ_{pt} finite, and see if we recover the proper-time definition. In other words, we have to perform the replacement

$$\begin{aligned}
& \lim_{\Lambda_{\text{pt}} \rightarrow \infty} \int_{1/\Lambda_{\text{pt}}^2}^\infty \frac{ds}{s} \left(\sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi e^{-sM^2} - \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi e^{-s\lambda_n^\phi} \right) \\
& \rightarrow \int_{1/\Lambda_{\text{pt}}^2}^\infty \frac{ds}{s} \lim_{\Lambda \rightarrow \infty} \left(\sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi e^{-sM^2} - \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi e^{-s\lambda_n^\phi} \right) \\
& = - \int_{1/\Lambda_{\text{pt}}^2}^\infty \frac{ds}{s} \sum_n D_n^\phi e^{-s\lambda_n^\phi} + \int_{1/\Lambda_{\text{pt}}^2}^\infty \frac{ds}{s} \lim_{\Lambda \rightarrow \infty} \left(e^{-sM^2} \sum_{\lambda_n^\phi \leq \Lambda^2} D_n^\phi \right). \quad (4.7)
\end{aligned}$$

The first term in the third line of the above equation is easily recognized as the usual *definition* of the functional determinant within the proper-time regularization. Therefore, in order for the third line of (4.7) to coincide with the proper-time definition of the fluctuation determinant, the second term in this line has to vanish. For a generic scale $M \neq \Lambda$, this term contains field dependent Λ -divergences that cannot be discarded tout-court. Instead, for $M = \Lambda$ such term vanishes thanks to the presence of the factor $e^{-s\Lambda^2}$, and the proper-time result is recovered.

V. CONCLUSIONS AND OUTLOOKS

In the present work we have addressed a question which, in our opinion, lies at the interface between two fundamental aspects of modern quantum field theory. On the one hand, according to Wilson's lesson, a QFT has to be regarded, in general, as an effective field theory endowed with a physical UV cutoff Λ , the scale above which the theory has to be replaced by its UV completion. On the other hand, in gauge theories the naive use of a hard cutoff

is known to lead to violations of gauge invariance, as happens for instance in the one-loop calculation of the vacuum polarization tensor in QED, where a quadratically divergent longitudinal term is generated once a hard cutoff is imposed on the loop momentum. This has led to the common view that, in gauge theories, hard cutoffs should be abandoned in favor of regularization schemes that preserve gauge invariance.

The aim of the present paper was to show that this conclusion is not unavoidable. We have considered the Euler-Heisenberg correction to the free Maxwell action as a clean and explicit setting where the issue can be studied in detail. The central point of our analysis is that the Wilsonian cutoff should be implemented as a hard spectral cutoff on the eigenvalues of the gauge-covariant fluctuation operator. In scalar QED this amounts to retaining only the eigenvalues λ of $-D^2 + m^2$ such that $\lambda \leq \Lambda^2$, while in fermionic QED the corresponding prescription is implemented on the squared modulus of the eigenvalues of $\not{D} + m$. Since the spectrum of these operators is gauge invariant, the resulting one-loop determinant is gauge invariant from the outset.

For both scalar and fermionic QED we have carried out the calculation explicitly in a constant magnetic (electric) background. In the scalar case, the spectrum reduces to the standard Landau-level spectrum of the operator $-D^2 + m^2$, with the physical cutoff Λ selecting a finite number of levels and, for each of them, a finite domain in the remaining continuous momenta. In the fermionic case, the corresponding eigenvalue problem for the Euclidean Dirac operator was solved by taking into account the pairing of complex-conjugate eigenvalues. In both cases, the Euler-Heisenberg Lagrangian was obtained by performing the finite spectral sum and the momentum integrals before expanding in the regime $m^2, eB \ll \Lambda^2$.

The result of this procedure is twofold. First, the leading terms reproduce the usual Euler-Heisenberg effective Lagrangian obtained with proper-time regularization, up to vacuum-energy contributions and after the standard analytic continuation to Lorentzian signature. In particular, the logarithmic terms responsible for the one-loop running of the electric charge are correctly recovered. Second, the hard spectral cutoff produces terms which are absent in the proper-time result. These terms depend on the fractional part of the maximal number of Landau levels included below the cutoff and are therefore periodic functions of $1/eB$. When $eB \ll \Lambda^2$ they are suppressed by powers of eB/Λ^2 , and the usual (proper-time) result is recovered (further comments on these terms are at the end of the section).

Another important point discussed in the present paper concerns the scale which makes the argument of the logarithm of the fluctuation determinant dimensionless. Starting from the phase space path integral, we argued that this scale is not an arbitrary fixed scale M , but it is precisely

the Wilsonian UV physical cutoff Λ . This identification is not a matter of convention. We have shown that, if one keeps $M \neq \Lambda$, spurious field-dependent contributions appear in the Euler-Heisenberg action. In scalar QED these terms would even modify the coefficient and the sign of the one-loop beta function. Moreover, in scalar as well as in fermionic QED, spurious large-amplitude oscillatory terms are generated. Conversely, for $M = \Lambda$, all these unphysical contributions are absent. We have also shown that $M = \Lambda$ allows one to establish a connection between our calculation of the fluctuation determinant with the Wilsonian hard cutoff Λ and its definition within the framework of proper-time regularization.

The results of the present work provide a concrete example in which a hard cutoff, endowed with a direct physical meaning, can be introduced in a gauge theory without spoiling gauge invariance at the quantum level. In this sense, they represent a first step toward a formulation of the Wilsonian renormalization group in gauge theories closer in spirit to the Wegner-Houghton construction [4], where the integration of modes is performed with a sharp cutoff. In fact, we suggest that in gauge theories the notion of “mode” should be tied to the spectrum of gauge-covariant operators rather than to the Fourier momentum of the free theory.

Several directions deserve further investigation. First, it would be important to clarify the relation between the hard spectral cutoff used here and lattice gauge theory. In the lattice formulation, gauge invariance is implemented at the discretized level through link variables rather than through the gauge potentials themselves. In perturbation theory this leads, for example, to additional contributions such as seagull diagrams, which cancel the quadratically divergent gauge-noninvariant terms produced by a naive momentum cutoff in the vacuum polarization tensor. Here, gauge-noninvariant terms are absent due to the use of a hard cutoff over the eigenvalues.

A closely related problem is the comparison between the continuum spectral determinant studied in this work and the determinant obtained from lattice QED in a constant magnetic background, where the relevant spectral problem is connected with the Hofstadter spectrum [58,59]. Such a comparison would allow one to understand in detail how the Landau level structure emerges from a microscopic gauge-invariant lattice regularization, and how the periodic terms found here are represented in the lattice theory.

Another natural development is the construction of an exact or approximate Wegner-Houghton-type renormalization group equation [4] for QED, and more generally for gauge theories. The calculations and results of the present work suggest that such a formulation is not excluded by gauge invariance, provided the cutoff is imposed spectrally.

Finally, a comment on the terms periodic in $1/eB$ found in both scalar QED [fourth line of (2.21)] and fermionic QED [fourth line of (3.21)]. These terms deserve further

study both from a theoretical and a phenomenological point of view. They originate from the discreteness of the spectrum below a sharp physical cutoff. In this respect, they are reminiscent of the oscillatory contributions that appear in phenomena such as de Haas-van Alphen oscillations [56,57]. Whether in gauge theories analogous effects can have observable consequences in situations where the background field probes scales not much smaller than the UV physical cutoff Λ is an open question.

ACKNOWLEDGMENTS

We would like to thank D. Zappalà for useful discussions. The work of V. B., F. C., R. G. and A. P. is carried out within the INFN project QGSKY.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: EIGENVALUE PROBLEM FOR $\not{D} + m$: CONSTANT MAGNETIC BACKGROUND

In this Appendix, we solve the eigenvalue problem for the (Euclidean) Dirac operator $\not{D} + m$ for a constant magnetic field background B . For the gauge field A_μ we take

$$A_0 = A_2 = A_3 = 0, \quad A_1 = -Bx_2, \quad (\text{A1})$$

so that the eigenvalue problem reads as

$$(\gamma^\mu \partial_\mu - ieB\gamma^1 x_2 + m)\psi(x) = \lambda\psi(x). \quad (\text{A2})$$

We now write ψ as

$$\psi(x) = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}, \quad (\text{A3})$$

where $\varphi(x)$ and $\chi(x)$ are two-component spinors,

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad \chi(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix}. \quad (\text{A4})$$

Inserting (3.2) and (A3) in (A2) we get

$$C_1\chi(x) \equiv (\partial_0 - i\sigma^j \partial_j - eBx_2\sigma^1)\chi(x) = i\tilde{\lambda}\varphi(x) \quad (\text{A5})$$

$$C_2\varphi(x) \equiv (\partial_0 + i\sigma^j \partial_j + eBx_2\sigma^1)\varphi(x) = i\tilde{\lambda}\chi(x), \quad (\text{A6})$$

where $\tilde{\lambda}$ is defined as $\tilde{\lambda} \equiv -i(\lambda - m)$. With $i\tilde{\lambda}$ being the eigenvalues of the anti-Hermitian operator $\not{D}$, $\tilde{\lambda}$ are real numbers. The system (A5)–(A6) can be solved in two

equivalent ways, either using first (A6), writing $\chi = (i\tilde{\lambda})^{-1}C_2\varphi$ and inserting this latter relation in (A5), or using first (A5), writing $\varphi = (i\tilde{\lambda})^{-1}C_1\chi$ and inserting this relation in (A6).

Let us insert (A6) in (A5), i.e. we write $\chi = (i\tilde{\lambda})^{-1}C_2\varphi$. After some trivial algebra we get

$$\left[(-\partial_0^2 - \partial_3^2) + (-\partial_1^2 - \partial_2^2 + 2ieBx_2\partial_1 + e^2B^2x_2^2) + eB\sigma^3\right]\varphi(x) = \tilde{\lambda}^2\varphi(x), \quad (\text{A7})$$

that corresponds to the system

$$\begin{aligned} &(-\partial_0^2 - \partial_3^2 - \partial_1^2 - \partial_2^2 + 2ieBx_2\partial_1 + e^2B^2x_2^2)\varphi_1 \\ &= (\tilde{\lambda}^2 - eB)\varphi_1 \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} &(-\partial_0^2 - \partial_3^2 - \partial_1^2 - \partial_2^2 + 2ieBx_2\partial_1 + e^2B^2x_2^2)\varphi_2 \\ &= (\tilde{\lambda}^2 + eB)\varphi_2. \end{aligned} \quad (\text{A9})$$

In the left-hand side of both (A8) and (A9) we recognize the eigenvalue problem for a free (nonrelativistic) particle of mass $1/2$ in the (x_0, x_3) plane and the Landau-level problem with frequency $2eB$ in the (x_1, x_2) plane. For $(\tilde{\lambda}^2 - eB)$ and $(\tilde{\lambda}^2 + eB)$ we then have

$$\begin{aligned} &(\tilde{\lambda}^\psi)^2 \pm eB = k_0^2 + k_3^2 + 2eB\left(n + \frac{1}{2}\right), \\ &n = 0, 1, \dots, \quad k_i = \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}. \end{aligned} \quad (\text{A10})$$

Therefore, for $\tilde{\lambda}$ we have

$$\begin{aligned} &\tilde{\lambda}_{n,k_0,k_3}^\psi = \pm \sqrt{k_0^2 + k_3^2 + 2eBn}, \quad n = 0, 1, \dots, \\ &k_i = \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}, \end{aligned} \quad (\text{A11})$$

and the eigenvalues λ are [see the comments below (A6)]

$$\begin{aligned} &\lambda_{n,k_0,k_3}^\psi = m + i\sqrt{k_0^2 + k_3^2 + 2eBn}, \quad n = 0, 1, \dots, \\ &k_i = \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}, \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} &(\lambda_{n,k_0,k_3}^\psi)^* = m - i\sqrt{k_0^2 + k_3^2 + 2eBn}, \quad n = 0, 1, \dots, \\ &k_i = \frac{2\pi n_i}{L}, \quad n_i \in \mathbb{Z}. \end{aligned} \quad (\text{A13})$$

From now on we will use the notation $k^2 \equiv k_0^2 + k_3^2$. To obtain now the eigenfunctions corresponding to the eigenvalues (3.8), we treat separately the case $n = 0$ and the case $n \neq 0$. Let us begin from the latter.

1. Case $n \neq 0$

For any given n , we have two complex conjugate eigenvalues of $\mathcal{D} + m$, namely $\lambda_{n,k_0,k_3}^\psi = m + i\tilde{\lambda}_{n,k_0,k_3}^\psi \equiv m + i\sqrt{k^2 + 2eBn}$ and its complex conjugate [see (A12) and (A13)]. We will see that the eigenfunctions associated to $\tilde{\lambda}_{n,k_0,k_3}^\psi$ are obtained by applying the matrix γ_5 to the eigenfunctions associated to λ_{n,k_0,k_3}^ψ (this is due to the fact that γ_5 and γ^μ anticommute). Let us write $\chi = (i\tilde{\lambda})^{-1}C_2\varphi$, take $\tilde{\lambda} = \tilde{\lambda}_{n,k_0,k_3}^\psi \equiv \sqrt{k^2 + 2eBn}$ and go back to the system

$$(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1} = \begin{pmatrix} \varphi_1(x) \\ 0 \\ (i\tilde{\lambda}_{n,k_0,k_3}^\psi)^{-1}C_2 \begin{pmatrix} \varphi_1(x) \\ 0 \end{pmatrix} \end{pmatrix},$$

Resorting again to the well-known solution of the Landau-level problem with frequency $2eB$, we see that [see (2.12) and comments therein]

$$\varphi_1(x) = \phi_{n-1,k_0,k_3;p_1} \quad (\text{A16})$$

$$\varphi_2(x) = \phi_{n,k_0,k_3;p_1}. \quad (\text{A17})$$

Since

$$p_1^{\min} \equiv -\frac{eB}{2}L \leq p_1 \leq \frac{eB}{2}L \equiv p_1^{\max}, \quad (\text{A18})$$

the number of linearly independent eigenfunctions of type $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ (of type $(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$) associated to the eigenvalue $\lambda_{n,k_0,k_3}^\psi = m + i\sqrt{k^2 + 2eBn}$ is

$$\frac{L}{2\pi}(p_1^{\max} - p_1^{\min}) = \frac{eB}{2\pi}L^2. \quad (\text{A19})$$

The eigenvalue λ_{n,k_0,k_3}^ψ has then degeneracy

$$D_{n,k_0,k_3}^\psi = \frac{eB}{\pi}L^2. \quad (\text{A20})$$

The final expression of $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ and $(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$ is obtained once we calculate $C_2\varphi^{(1)}$ and $C_2\varphi^{(2)}$ in (A15). To this end, we now write explicitly the operator C_2 :

$$C_2 = \partial_0 + i\sigma_j\partial_j + eBx_2\sigma^1 \\ = \begin{pmatrix} \partial_0 + i\partial_3 & \partial_2 + i\partial_1 + eBx_2 \\ -\partial_2 + i\partial_1 + eBx_2 & \partial_0 - i\partial_3 \end{pmatrix}. \quad (\text{A21})$$

(A8)–(A9). To obtain the linearly independent eigenfunctions corresponding to λ_{n,k_0,k_3}^ψ we search for solutions of the form

$$\varphi^{(1)}(x) = \begin{pmatrix} \varphi_1(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \varphi^{(2)}(x) = \begin{pmatrix} 0 \\ \varphi_2(x) \end{pmatrix}. \quad (\text{A14})$$

This gives rise to the eigenfunctions of $\mathcal{D} + m$ (up to a normalization constant),

$$(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \varphi_2(x) \\ (i\tilde{\lambda}_{n,k_0,k_3}^\psi)^{-1}C_2 \begin{pmatrix} 0 \\ \varphi_2(x) \end{pmatrix} \end{pmatrix}. \quad (\text{A15})$$

Since C_2 acts on $\varphi^{(1)}$ and $\varphi^{(2)}$, which depend on x_0, x_1 and x_3 only through the plane waves $e^{ik_0x_0}, e^{ip_1x_1}$ and $e^{ik_3x_3}$, we can replace ∂_0 with ik_0 , ∂_1 with ip_1 and ∂_3 with ik_3 , thus getting

$$C_2 = \begin{pmatrix} ik_0 - k_3 & \partial_2 - p_1 + eBx_2 \\ -\partial_2 - p_1 + eBx_2 & ik_0 + k_3 \end{pmatrix} \\ \equiv \begin{pmatrix} ik_0 - k_3 & \tilde{\partial}_2 + eB\tilde{x}_2 \\ -\tilde{\partial}_2 + eB\tilde{x}_2 & ik_0 + k_3 \end{pmatrix}, \quad (\text{A22})$$

where we have defined $\tilde{x}_2 \equiv x_2 - \frac{p_1}{eB}$ and $\tilde{\partial}_2 \equiv \frac{\partial}{\partial \tilde{x}_2}$. We can now write C_2 in terms of the creation and annihilation operators,

$$b^\dagger = -\frac{1}{\sqrt{2eB}}(\tilde{\partial}_2 - eB\tilde{x}_2), \quad b = \frac{1}{\sqrt{2eB}}(\tilde{\partial}_2 + eB\tilde{x}_2) \quad (\text{A23})$$

as

$$C_2 = \begin{pmatrix} ik_0 - k_3 & \sqrt{2eB}b \\ \sqrt{2eB}b^\dagger & ik_0 + k_3 \end{pmatrix}. \quad (\text{A24})$$

We then get

$$C_2\varphi^{(1)} = \begin{pmatrix} (ik_0 - k_3)\phi_{n-1,k_0,k_3;p_1} \\ \sqrt{2eBn}\phi_{n,k_0,k_3;p_1} \end{pmatrix} \quad (\text{A25})$$

$$C_2\varphi^{(2)} = \begin{pmatrix} \sqrt{2eBn}\phi_{n-1,k_0,k_3;p_1} \\ (ik_0 + k_3)\phi_{n,k_0,k_3;p_1} \end{pmatrix}, \quad (\text{A26})$$

and from (A15) we have

$$(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1} = \begin{pmatrix} \phi_{n-1,k_0,k_3;p_1} \\ 0 \\ \frac{k_0+ik_3}{\sqrt{k^2+2eBn}} \phi_{n-1,k_0,k_3;p_1} \\ \frac{-i\sqrt{2eBn}}{\sqrt{k^2+2eBn}} \phi_{n,k_0,k_3;p_1} \end{pmatrix} \quad (\text{A27})$$

$$(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \phi_{n,k_0,k_3;p_1} \\ \frac{-i\sqrt{2eBn}}{\sqrt{k^2+2eBn}} \phi_{n-1,k_0,k_3;p_1} \\ \frac{k_0-ik_3}{\sqrt{k^2+2eBn}} \phi_{n,k_0,k_3;p_1} \end{pmatrix}. \quad (\text{A28})$$

Finally, the eigenfunctions $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ and $(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$ associated to the complex conjugate eigenvalue $(\lambda_{n,k_0,k_3}^\psi)^*$ [see (A13)] are obtained from (A27) and (A28) through the replacement $\tilde{\lambda}_{n,k_0,k_3}^\psi = \sqrt{k^2+2eBn} \rightarrow -\tilde{\lambda}_{n,k_0,k_3}^\psi$. Using (3.3), we see that (as anticipated) $(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1} = -\gamma_5(\psi_\lambda^{(1)})_{n,k_0,k_3;p_1}$ and $(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1} = -\gamma_5(\psi_\lambda^{(2)})_{n,k_0,k_3;p_1}$. Therefore, the eigenvalue $(\lambda_{n,k_0,k_3}^\psi)^*$ has the same degeneracy D_{n,k_0,k_3}^ψ [see (A20)] as λ_{n,k_0,k_3}^ψ .

2. Case $n=0$

Let us consider now the case $\tilde{\lambda}_{0,k_0,k_3}^\psi = \sqrt{k_0^2+k_3^2} = k$ (that corresponds to the eigenvalues $\lambda_{0,k_0,k_3}^\psi = m+ik$ and $\lambda_{0,k_0,k_3}^* = m-ik$ of $\mathcal{D}+m$). From (A8) and (A9) we see that solutions of the kind $\varphi^{(1)}$ are not allowed (since they are not normalizable), and that only solutions of the kind $\varphi^{(2)}$ are acceptable. The linearly independent eigenfunctions corresponding to λ_{0,k_0,k_3}^ψ and $(\lambda_{0,k_0,k_3}^\psi)^*$ are

$$(\psi_\lambda^{(2)})_{0,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \phi_{0,k_0,k_3;p_1} \\ 0 \\ \frac{k_0-ik_3}{k} \phi_{0,k_0,k_3;p_1} \end{pmatrix} \quad (\text{A29})$$

and

$$(\psi_{\lambda^*}^{(2)})_{0,k_0,k_3;p_1} = \begin{pmatrix} 0 \\ \phi_{0,k_0,k_3;p_1} \\ 0 \\ \frac{ik_3-k_0}{k} \phi_{0,k_0,k_3;p_1} \end{pmatrix}, \quad (\text{A30})$$

respectively. The eigenvalues λ_{0,k_0,k_3}^ψ and $(\lambda_{0,k_0,k_3}^\psi)^*$ have then both degeneracy $D_{0,k_0,k_3}^\psi = \frac{eB}{2\pi} L^2$ [see the comments below (A17)].

To summarize, we have found the complete set of orthogonal eigenfunctions [see (A27), (A28), (A29), (A30)],

$$\begin{pmatrix} \psi_\lambda^{(1)} \end{pmatrix}_{n,k_0,k_3;p_1}, \quad \begin{pmatrix} \psi_{\lambda^*}^{(1)} \end{pmatrix}_{n,k_0,k_3;p_1}, \quad \begin{pmatrix} \psi_\lambda^{(2)} \end{pmatrix}_{n,k_0,k_3;p_1}, \\ \begin{pmatrix} \psi_{\lambda^*}^{(2)} \end{pmatrix}_{n,k_0,k_3;p_1}, \quad \begin{pmatrix} \psi_\lambda^{(2)} \end{pmatrix}_{0,k_0,k_3;p_1}, \quad \begin{pmatrix} \psi_{\lambda^*}^{(2)} \end{pmatrix}_{0,k_0,k_3;p_1}, \quad (\text{A31})$$

that spans the space of the field ψ . Similarly, the space of $\bar{\psi}$ is spanned by the set

$$\begin{pmatrix} \psi_\lambda^{(1)} \end{pmatrix}_{n,k_0,k_3;p_1}^\dagger, \quad \begin{pmatrix} \psi_{\lambda^*}^{(1)} \end{pmatrix}_{n,k_0,k_3;p_1}^\dagger, \quad \begin{pmatrix} \psi_\lambda^{(2)} \end{pmatrix}_{n,k_0,k_3;p_1}^\dagger, \\ \begin{pmatrix} \psi_{\lambda^*}^{(2)} \end{pmatrix}_{n,k_0,k_3;p_1}^\dagger, \quad \begin{pmatrix} \psi_\lambda^{\text{iny}(2)} \end{pmatrix}_{0,k_0,k_3;p_1}^\dagger, \quad \begin{pmatrix} \psi_{\lambda^*}^{(2)} \end{pmatrix}_{0,k_0,k_3;p_1}^\dagger. \quad (\text{A32})$$

-
- [1] K. G. Wilson, Renormalization group and critical phenomena. 1. Renormalization group and the Kadanoff scaling picture, *Phys. Rev. B* **4**, 3174 (1971).
[2] K. G. Wilson, Renormalization group and critical phenomena. 2. Phase space cell analysis of critical behavior, *Phys. Rev. B* **4**, 3184 (1971).
[3] K. G. Wilson and M. E. Fisher, Critical exponents in 3.99 dimensions, *Phys. Rev. Lett.* **28**, 240 (1972).
[4] F. J. Wegner and A. Houghton, Renormalization group equation for critical phenomena, *Phys. Rev. A* **8**, 401 (1973).

- [5] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, *Phys. Rep.* **12**, 75 (1974).
[6] K. G. Wilson, The renormalization group: Critical phenomena and the Kondo problem, *Rev. Mod. Phys.* **47**, 773 (1975).
[7] J. C. Ward, An identity in quantum electrodynamics, *Phys. Rev.* **78**, 182 (1950).
[8] Y. Takahashi, On the generalized Ward identity, *Nuovo Cimento* **6**, 371 (1957).
[9] J. C. Taylor, Ward identities and charge renormalization of the Yang-Mills field, *Nucl. Phys.* **B33**, 436 (1971).

- [10] A. A. Slavnov, Ward identities in gauge theories, *Theor. Math. Phys.* **10**, 99 (1972).
- [11] W. Pauli and F. Villars, On the invariant regularization in relativistic quantum theory, *Rev. Mod. Phys.* **21**, 434 (1949).
- [12] C. Branchina, V. Branchina, F. Contino, and N. Darvishi, Dimensional regularization, Wilsonian RG, and the naturalness and hierarchy problem, *Phys. Rev. D* **106**, 065007 (2022).
- [13] C. Branchina, V. Branchina, and F. Contino, Physical tuning and naturalness, *Phys. Rev. D* **107**, 096012 (2023).
- [14] M. Bonini, M. D’Attanasio, and G. Marchesini, Ward identities and Wilson renormalization group for QED, *Nucl. Phys.* **B418**, 81 (1994).
- [15] M. Bonini, M. D’Attanasio, and G. Marchesini, Renormalization group flow for SU(2) Yang-Mills theory and gauge invariance, *Nucl. Phys.* **B421**, 429 (1994).
- [16] M. Bonini, M. D’Attanasio, and G. Marchesini, BRS symmetry for Yang-Mills theory with exact renormalization group, *Nucl. Phys.* **B437**, 163 (1995).
- [17] M. Bonini, M. D’Attanasio, and G. Marchesini, BRS symmetry from renormalization group flow, *Phys. Lett. B* **346**, 87 (1995).
- [18] M. D’Attanasio and T. R. Morris, Gauge invariance, the quantum action principle, and the renormalization group, *Phys. Lett. B* **378**, 213 (1996).
- [19] M. Bonini, G. Marchesini, and M. Simionato, Beta function and infrared renormalons in the exact Wilson renormalization group in Yang-Mills theory, *Nucl. Phys.* **B483**, 475 (1997).
- [20] M. Simionato, Gauge consistent Wilson renormalization group: Abelian case, *Int. J. Mod. Phys. A* **15**, 2121 (2000).
- [21] M. Simionato, Gauge consistent Wilson renormalization group. 2: NonAbelian case, *Int. J. Mod. Phys. A* **15**, 2153 (2000).
- [22] T. R. Morris, A gauge invariant exact renormalization group. 1, *Nucl. Phys.* **B573**, 97 (2000).
- [23] M. Simionato, On the consistency of the exact renormalization group approach applied to gauge theories in algebraic noncovariant gauges, *Int. J. Mod. Phys. A* **15**, 4811 (2000).
- [24] T. R. Morris, A gauge invariant exact renormalization group. 2, *J. High Energy Phys.* **12** (2000) 012.
- [25] H. Gies and C. Wetterich, Renormalization flow of bound states, *Phys. Rev. D* **65**, 065001 (2002).
- [26] H. Gies, Running coupling in Yang-Mills theory: A flow equation study, *Phys. Rev. D* **66**, 025006 (2002).
- [27] S. Arnone, A. Gatti, and T. R. Morris, A proposal for a manifestly gauge invariant and universal calculus in Yang-Mills theory, *Phys. Rev. D* **67**, 085003 (2003).
- [28] H. Gies and J. Jaeckel, Renormalization flow of QED, *Phys. Rev. Lett.* **93**, 110405 (2004).
- [29] H. Gies and J. Jaeckel, Chiral phase structure of QCD with many flavors, *Eur. Phys. J. C* **46**, 433 (2006).
- [30] S. Arnone, T. R. Morris, and O. J. Rosten, Manifestly gauge invariant QED, *J. High Energy Phys.* **10** (2005) 115.
- [31] J. M. Pawłowski, Aspects of the functional renormalisation group, *Ann. Phys. (N.Y.)* **322**, 2831 (2007).
- [32] T. R. Morris and O. J. Rosten, Manifestly gauge invariant QCD, *J. Phys. A* **39**, 11657 (2006).
- [33] S. Arnone, T. R. Morris, and O. J. Rosten, A generalised manifestly gauge invariant exact renormalisation group for SU(N) Yang-Mills, *Eur. Phys. J. C* **50**, 467 (2007).
- [34] B. J. Schaefer, J. M. Pawłowski, and J. Wambach, The phase structure of the Polyakov–Quark-Meson model, *Phys. Rev. D* **76**, 074023 (2007).
- [35] C. S. Fischer, A. Maas, and J. M. Pawłowski, On the infrared behavior of Landau gauge Yang-Mills theory, *Ann. Phys. (Amsterdam)* **324**, 2408 (2009).
- [36] J. Braun, H. Gies, and J. M. Pawłowski, Quark confinement from color confinement, *Phys. Lett. B* **684**, 262 (2010).
- [37] H. Gies, Introduction to the functional RG and applications to gauge theories, *Lect. Notes Phys.* **852**, 287 (2012).
- [38] W. j. Fu, J. M. Pawłowski, and F. Rennecke, QCD phase structure at finite temperature and density, *Phys. Rev. D* **101**, 054032 (2020).
- [39] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawłowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, *Phys. Rep.* **910**, 1 (2021).
- [40] C. Branchina, V. Branchina, F. Contino, and A. Pernace, Does the cosmological constant really indicate the existence of a dark dimension?, *Int. J. Geom. Methods Mod. Phys.* **22**, 2450305 (2025).
- [41] C. Branchina, V. Branchina, F. Contino, and A. Pernace, Dark dimension and the effective field theory limit, *Int. J. Geom. Methods Mod. Phys.* **22**, 2450303 (2025).
- [42] C. Branchina, V. Branchina, F. Contino, and A. Pernace, Path integral measure and the cosmological constant, *Phys. Rev. D* **111**, 105018 (2025).
- [43] C. Branchina, V. Branchina, F. Contino, and A. Pernace, Path integral measure and RG equations for gravity, *Phys. Rev. D* **111**, 125021 (2025).
- [44] C. Branchina, V. Branchina, F. Contino, R. Gandolfo, and A. Pernace, Diffeomorphism invariance of the effective gravitational action, *Phys. Rev. D* **112**, 045002 (2025).
- [45] C. Branchina, V. Branchina, F. Contino, R. Gandolfo, and A. Pernace, Gravity and the Higgs boson mass, [arXiv:2507.13832](https://arxiv.org/abs/2507.13832).
- [46] G. Giacometti, D. Rizzo, and D. Zappala, Universal content of the proper time flow in scalar and Yang-Mills theories, *Phys. Rev. D* **113**, 045020 (2026).
- [47] G. Giacometti, K. Kowalska, D. Rizzo, E. M. Sessolo, and D. Zappala, Quantum gravity contributions to the gauge and Yukawa couplings in proper time flow, *Phys. Rev. D* **114**, 016020 (2026).
- [48] W. Heisenberg and H. Euler, Consequences of Dirac’s theory of positrons, *Z. Phys.* **98**, 714 (1936).
- [49] V. Weisskopf, The electrodynamics of the vacuum based on the quantum theory of the electron, *Kong. Dan. Vid. Sel. Mat. Fys. Med.* **14N6**, 1 (1936).
- [50] J. S. Schwinger, On gauge invariance and vacuum polarization, *Phys. Rev.* **82**, 664 (1951).
- [51] W. Dittrich, One loop effective potentials in QED, *J. Phys. A* **9**, 1171 (1976).
- [52] G. V. Dunne, Heisenberg-Euler effective Lagrangians: Basics and extensions, [arXiv:hep-th/0406216](https://arxiv.org/abs/hep-th/0406216).
- [53] G. V. Dunne, The Heisenberg-Euler effective action: 75 years on, *Int. J. Mod. Phys. A* **27**, 1260004 (2012).

- [54] H. Gies and F. Karbstein, An addendum to the Heisenberg-Euler effective action beyond one loop, *J. High Energy Phys.* **03** (2017) 108.
- [55] V. Branchina, F. Contino, R. Gandolfo, and A. Pernace (to be published).
- [56] W. J. de Haas and P. M. van Alphen, The dependence of the susceptibility of diamagnetic metals upon the field, *Proc. Acad. Sci. Amsterdam* **33**, 1106 (1930).
- [57] D. Shoenberg, *Magnetic Oscillations in Metals* (Cambridge University Press, Cambridge, England, 1984).
- [58] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields, *Phys. Rev. B* **14**, 2239 (1976).
- [59] S. Janecek, E. Krotscheck, and Y. Zhang, Two-dimensional Bloch electrons in perpendicular magnetic fields: An exact calculation of the Hofstadter butterfly spectrum, *Phys. Rev. B* **87**, 235429 (2013).
- [60] C. R. Dean, L. Wang, P. Maher, C. Forsythe, F. Ghahari, Y. Gao, J. Katoch, M. Ishigami, P. Moon, M. Koshino, T. Taniguchi, K. Watanabe, K. L. Shepard, J. Hone, and P. Kim, Hofstadter's butterfly and the fractal quantum Hall effect in Moiré superlattices, *Nature (London)* **497**, 598 (2013).
- [61] B. Hunt, J. D. Sanchez-Yamagishi, A. F. Young, K. Watanabe, T. Taniguchi, P. Moon, M. Koshino, P. Jarillo-Herrero, and R. C. Ashoori, Massive Dirac Fermions and hofstadter butterfly in a van der Waals heterostructure, *Science* **340**, 1427 (2013).
- [62] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Realization of the hofstadter Hamiltonian with ultracold atoms in optical lattices, *Phys. Rev. Lett.* **111**, 185301 (2013).
- [63] M. Bocarsly, M. Uzan, I. Roy, S. Grover, J. Xiao, Z. Dong, M. Labendik, A. Uri, M. E. Huber, Y. Myasoedov, K. Watanabe, T. Taniguchi, B. Yan, L. S. Levitov, and E. Zeldov, De Haas-van Alphen spectroscopy and magnetic breakdown in Moiré graphene, *Science* **383**, 42 (2024).
- [64] V. S. Adamchik, Polygamma functions of negative order, *J. Comput. Appl. Math.* **100**, 191 (1998).
- [65] M. Oleszczuk, A symmetry preserving cutoff regularization, *Z. Phys. C* **64**, 533 (1994).
- [66] S. B. Liao, Connection between momentum cutoff and operator cutoff regularizations, *Phys. Rev. D* **53**, 2020 (1996).