

Anomaly of 4D Weyl fermions with discrete symmetries

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We derive explicit anomaly-index formulas for four-dimensional Weyl fermions charged under the finite symmetries $\text{Spin} \times \mathbb{Z}_n$ and $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F$. The strategy is to start from the standard perturbative anomaly indices for $\text{Spin} \times \text{U}(1)$ and $\text{Spin} \times_{\mathbb{Z}_2^F} \text{U}(1) = \text{Spin}^c$, and then restrict the continuous $\text{U}(1)$ symmetry to a finite cyclic subgroup. On the level of invertible field theories this gives natural homomorphisms, $\text{TP}_5(\text{Spin} \times \text{U}(1)) \longrightarrow \text{TP}_5(\text{Spin} \times \mathbb{Z}_n)$, $\text{TP}_5(\text{Spin}^c) \longrightarrow \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F)$. We compute these maps explicitly by evaluating reduced η invariants on geometric representatives of the finite anomaly groups. For $\text{Spin} \times \mathbb{Z}_n$, the relevant backgrounds are the five-dimensional lens-space bundle $X(n; 1, 1)$ and the product $L(n; 1) \times \text{K3}$. For $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F$, the relevant backgrounds are $L(m; 1, 1, 1)$ and, depending on the parity of m , either $L(m; 1) \times \text{Enriques}$ or $L(m; 1) \times \text{K3}$. The output is a pair of integer-valued anomaly indices for each finite symmetry. These indices are normalized in the cyclic factors of the finite anomaly group, so they can be used directly in anomaly-cancellation checks for fermions with discrete gauge or global symmetries. They also provide the input for symmetry-extension applications to topologically ordered quantum dark matter: in Standard Model examples, the indices identify the discrete $\mathbf{B} + \mathbf{L}$ mixed gauge-gravitational anomaly left by missing sterile right-handed neutrinos and the finite K -gauge topological sector that can match it.

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I. INTRODUCTION

A. Motivation

Anomalies constrain which symmetries can be consistently imposed in a quantum field theory. For four-dimensional Weyl fermions with continuous gauge symmetry, the perturbative anomaly is described by the familiar six-dimensional index polynomial. Discrete symmetries require a slightly different treatment. Although a discrete gauge field has no ordinary de Rham field strength, a fermion coupled to a finite background can still have a global anomaly. Such an anomaly is naturally measured by a five-dimensional invertible field theory, or equivalently by a bordism invariant.

This paper derives explicit anomaly indices for four-dimensional Weyl fermions with the finite symmetries

$$\text{Spin} \times \mathbb{Z}_n, \quad \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F. \quad (1)$$

The first case describes an ordinary spin theory with an independent cyclic charge symmetry. The second case is

the finite analog of a Spin^c symmetry: the order-2 element in $\mathbb{Z}_{2m}$ is identified with fermion parity. Our main goal is to express the resulting global anomalies in concrete integer coordinates of the finite anomaly groups, so that the formulas can be used directly in anomaly-cancellation tests.

One concrete application, developed in Refs. [1–3], is to use these finite anomaly coordinates as the input for symmetry extension. If a set of Weyl fermions has a nonzero 't Hooft anomaly for a finite symmetry G , but this anomaly pulls back to zero under an extension $1 \rightarrow K \rightarrow G_{\text{Tot}} \rightarrow G \rightarrow 1$, then a K -gauge topological field theory can carry the same G anomaly in the infrared. In the topologically ordered dark matter application of Ref. [2], the general symmetry-extension statements are organized prime by prime. For $G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F$ and $m = 2^p \cdot 3^r \cdot s$ with $2 \nmid s$ and $3 \nmid s$, the two-primary anomaly is canceled for 2^p (or 2^{p+1}) charge-one Weyl fermions, using a minimal $K = \mathbb{Z}_4$ (or $K = \mathbb{Z}_2^F$) extension. The three-primary anomaly can be canceled for 3^r charge-one Weyl fermions by a minimal $K = \mathbb{Z}_3$ extension, while an s -primary component is canceled by a trivial $K = \mathbb{Z}_1$ extension for s charge-one Weyl fermions. In the Standard Model application, G is a discrete baryon-plus-or-minus-lepton symmetry combined with spacetime spin structure, and the fermions whose anomaly is to be matched are the sterile right-handed neutrinos absent from the low-energy spectrum.

The later work [3] applies this idea to a generalized Standard Model with N_c colors and N_f families, but the

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anomaly-trivializing kernel is first a cyclic group $K = \mathbb{Z}_N$ determined by the finite anomaly and by N_f , not by the color number N_c . For the discrete $(\mathbf{B} + \mathbf{L})$ anomaly one finds minimal extension orders $N = 1, 3, 4, 12$ according to the two- and three-primary parts of N_f . Only after imposing the additional color-center matching condition $N = N_c$ can this $\mathbb{Z}_N$ gauge group be identified with the center of $SU(N_c)$. In the odd color-matched branch, the odd-color, fermionic-baryon condition and minimality then select $N = N_c = N_f = 3$. Thus the formulas derived below are not only a classification of finite-symmetry anomalies; they are the arithmetic data needed to test whether missing chiral fermions can be replaced by a symmetry-preserving topologically ordered sector, and when that sector can also be interpreted as a color-center topological quantum field theory (TQFT).

B. Previous work

Discrete gauge anomalies have been studied from several complementary viewpoints. Early work by Ibáñez and Ross [4], Preskill *et al.* [5], and Banks and Dine [6] emphasized anomaly cancellation conditions for finite symmetries and their relation to underlying continuous gauge symmetries. Ibáñez's later analysis [7] gave a more detailed account of the discrete anomaly constraints used in particle physics model building.

More recently, Hsieh [8] revisited discrete gauge anomalies using bordism and η invariants, while García-Etxebarria and Montero [9] developed the Dai–Freed viewpoint on global anomalies in particle physics. The mathematical tools underlying these computations, including η invariants of lens spaces and related fixed-point formulas, go back to work of Gilkey and collaborators [10–15]. The general classification of invertible field theories by bordism is due to Freed and Hopkins [16].

There are also important connections with anomaly matching, non-Abelian discrete symmetries, and fermionic symmetry-protected topological phases. Examples include discrete anomaly matching [17], non-Abelian discrete anomalies [18], and finite-group or fermionic symmetry-protected topological phases [19–25]. The anomaly constraints for theories with $\text{Spin} \times \mathbb{Z}_n$ and $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ symmetry were further studied in [24,25]. Motivated by

applications to topological quantum dark matter and the Standard Model family puzzle, related uses of these anomaly constraints appear in [1–3,26,27].

C. Contribution of this paper

The contribution of this paper is to give a direct bridge from the continuous $U(1)$ anomaly polynomial to finite-symmetry global anomaly indices. The bridge is implemented by the restriction homomorphisms

$$\begin{aligned} \text{TP}_5(\text{Spin} \times U(1)) &\longrightarrow \text{TP}_5(\text{Spin} \times \mathbb{Z}_n), \\ \text{TP}_5(\text{Spin}^c) &\longrightarrow \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}). \end{aligned} \quad (2)$$

We compute these maps explicitly by evaluating η invariants on geometric generators of the finite anomaly groups.

For $\text{Spin} \times \mathbb{Z}_n$, the relevant geometric representatives are the lens-space bundle $X(n; 1, 1)$, defined in (62), and the product $L(n; 1) \times K3$, where $K3$ is the $K3$ surface. The first representative detects the pure discrete gauge anomaly, and the second detects the mixed gauge-gravitational anomaly after a change of basis. This recovers and refines the formulas of [8], using the fixed-point formulas of [10,11,13].

For $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$, the relevant representatives are $L(m; 1, 1, 1)$ and either $L(m; 1) \times E$ for even m , where E denotes the Enriques surface, or $L(m; 1) \times K3$ for odd m . This gives the finite Spin^c -type anomaly indices and clarifies the normalization of the second cyclic summand relative to [8,9,12,15].

Here and below, $\text{TP}_r(G)$ denotes the group of deformation classes of r -dimensional invertible topological field theories with symmetry G , in the sense of Freed–Hopkins [16]. A useful way to interpret this notation is the following: an anomaly of a d -dimensional quantum field theory with symmetry G is represented by a $(d+1)$ -dimensional invertible field theory with the same symmetry. Thus deformation classes of anomalies of d -dimensional theories with symmetry G are classified by $\text{TP}_{d+1}(G)$. Since this paper studies four-dimensional Weyl fermions, the relevant anomaly groups are five-dimensional groups $\text{TP}_5(G)$.

The group $\text{TP}_r(G)$ is controlled by the G -bordism groups Ω_r^G and Ω_{r+1}^G . In this paper Ω_r^G means

$$\Omega_r^G = \{\text{closed } r\text{-manifolds } M \text{ equipped with a } G \text{ structure}\} / \text{bordism}. \quad (3)$$

Freed and Hopkins show that there is a split short exact sequence:

$$0 \rightarrow \text{Ext}^1(\Omega_r^G, \mathbb{Z}) \rightarrow \text{TP}_r(G) \rightarrow \text{Hom}(\Omega_{r+1}^G, \mathbb{Z}) \rightarrow 0. \quad (4)$$

Thus the torsion information in $\text{TP}_r(G)$ is detected by Ω_r^G , while the free part is detected by Ω_{r+1}^G . In the

five-dimensional finite-symmetry cases considered below, the relevant groups are finite, and we identify the torsion anomaly group $\text{TP}_5(G)$ with the corresponding finite bordism group Ω_5^G after choosing the geometric generators used in Secs. IV and V.

The final formulas for the two finite symmetries are given in (102) and (141). A compact summary appears in Table I.

TABLE I. Summary of our results.

	Spin $\times \mathbb{Z}_n$	Spin $\times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$
5D invertible theory	$\exp(-2\pi i \eta([M^5], \rho_q))$	$\exp(-2\pi i \eta([M^5], \tilde{\rho}_q))$
Manifold generators and values	$X(n; 1, 1)$ $\exp(2\pi i \frac{(n^2+3n+2)q^3}{6n})$	$L(m; 1, 1)$ $\exp(2\pi i \frac{(2m^2+m+1)q^3-(m+3)q}{48m})$
	$-L(n; 1) \times \text{K3}$ $\exp(2\pi i \frac{2q}{n})$	$-L(m; 1) \times \text{E for even } m$ $\exp(2\pi i \frac{q}{2m})$ $-L(m; 1) \times \text{K3 for odd } m$ $\exp(2\pi i \frac{q}{m})$
New manifold generators	$[M_1^5] = [X(n; 1, 1)]$ $[M_2^5] = -[L(n; 1) \times \text{K3}] - 6[X(n; 1, 1)]$	$[M_3^5] = [L(m; 1, 1)]$ $[M_4^5] = -[L(m; 1) \times \text{E}] + 12[L(m; 1, 1)]$ for even m $[M_4^5] = -[L(m; 1) \times \text{K3}] + 24[L(m; 1, 1)]$ for odd m
Rewrite 5D invertible theory	(102)	(141)

D. Main results and conventions

For ease of reference, we state the two final anomaly-index formulas at the beginning. The derivations, including the construction of the manifold generators and the evaluation of the relevant η invariants, are given in the later sections. Throughout the paper, all η invariants are understood modulo $\mathbb{Z}$ when they appear in the phase $\exp(-2\pi i \eta)$. We write the anomaly groups additively when discussing bordism generators and multiplicatively when discussing the associated invertible field theories.

We also fix the notation for bordism classes before it appears in the theorem statements. If M is a closed manifold equipped with the relevant tangential structure and background gauge field, then $[M]$ denotes its bordism class. A minus sign in front of a manifold denotes orientation reversal: $-M$ is the same manifold with the reversed orientation and the induced background structure. Equivalently, in the bordism group, $-[M] = [-M]$. A plus sign between manifolds or between their bordism classes means disjoint union, which is the group operation in bordism. Thus, for example,

$$[M] + [N] = [M \sqcup N], \quad (5)$$

and $k[M]$ means the disjoint union of k copies of M , with the coefficient understood modulo the order of the relevant cyclic summand when the bordism group is finite. These conventions apply to the formulas such as $[M^5] = k_1[M_1^5] + k_2[M_2^5]$ below.

We now explain the geometric symbols M_i^5 which occur in the theorem statements. For the symmetry $\text{Spin} \times \mathbb{Z}_n$, the two chosen five-dimensional generators are

$$\begin{aligned}
[M_1^5] &:= [X(n; 1, 1)], \\
[M_2^5] &:= -[L(n; 1) \times \text{K3}] - 6[X(n; 1, 1)].
\end{aligned} \quad (6)$$

Here $X(n; 1, 1)$ is the lens-space bundle defined in (62), while $L(n; 1) \times \text{K3}$ is the product of the one-dimensional lens space with the K3 surface. The second formula is an

identity in the bordism group: it means the bordism class represented by the orientation reversal of $L(n; 1) \times \text{K3}$, together with six oppositely oriented copies of $X(n; 1, 1)$. The corresponding integer-valued anomaly indices in Theorem 1 are obtained from the $\mathbb{R}/\mathbb{Z}$ -valued η invariants of these generators by multiplying by the orders of the two cyclic factors:

$$\begin{aligned}
\nu'_1(n, q) &= a_n \eta(M_1^5, \rho_q), \\
\nu'_2(n, q) &= b_n \eta(M_2^5, \rho_q).
\end{aligned} \quad (7)$$

Thus M_1^5 is the background used to measure the first cyclic coordinate, while M_2^5 is the background used to measure the second cyclic coordinate. Equivalently, if a general background has class $[M^5] = k_1[M_1^5] + k_2[M_2^5]$, then its anomalous phase is obtained by the additivity of the η invariant from the two numbers $\nu'_1(n, q)$ and $\nu'_2(n, q)$ as in (102).

For the finite Spin^c -type symmetry $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$, the two chosen five-dimensional generators are

$$\begin{aligned}
[M_3^5] &:= [L(m; 1, 1)], \\
[M_4^5] &:= -[\text{E} \times L(m; 1)] + 12[L(m; 1, 1)], \quad m \text{ even}, \\
[M_4^5] &:= -[\text{K3} \times L(m; 1)] + 24[L(m; 1, 1)], \quad m \text{ odd}.
\end{aligned} \quad (8)$$

Here $L(m; 1, 1)$ is the five-dimensional lens space, E is the Enriques surface, and K3 is the K3 surface. Again, the formulas for M_i^5 are bordism identities using orientation reversal and disjoint union. The anomaly indices in Theorem 2 are related to the η invariants of these generators by

$$\begin{aligned}
\tilde{\nu}'_1(m, q) &= \tilde{a}_m \eta(M_3^5, \tilde{\rho}_q), \\
\tilde{\nu}'_2(m, q) &= \tilde{b}_m \eta(M_4^5, \tilde{\rho}_q).
\end{aligned} \quad (9)$$

Consequently, if $[M^5] = k_3[M_3^5] + k_4[M_4^5]$, the anomalous phase is determined by the two integer coordinates $\tilde{\nu}'_1(m, q)$ and $\tilde{\nu}'_2(m, q)$ as in (141).

Theorem 1 (Integer-valued index for $\text{Spin} \times \mathbb{Z}_n$). Let $q \in \mathbb{Z}_n$ be the discrete charge of a left-handed Weyl fermion. With a_n, b_n as in (85), the image of the continuous $\text{Spin} \times \text{U}(1)$ anomaly class under the homomorphism

$$\text{TP}_5(\text{Spin} \times \text{U}(1)) \longrightarrow \text{TP}_5(\text{Spin} \times \mathbb{Z}_n) \quad (10)$$

is represented by

$$(\nu'_1(n, q), \nu'_2(n, q)) = \left(\frac{a_n(n^2 + 3n + 2)q^3}{6n} \mod a_n, \frac{2b_n(q - q^3)}{n} \mod b_n \right). \quad (11)$$

Equivalently, for $[M^5] = k_1[M_1^5] + k_2[M_2^5]$, the anomaly phase is the one displayed in (102).

Theorem 2 (Integer-valued index for $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$). Let $q \in \mathbb{Z}_{2m}$ be odd. With $\tilde{a}_m, \tilde{b}_m$ as in (125), the image of the continuous Spin^c anomaly class under the homomorphism

$$\text{TP}_5(\text{Spin}^c) \longrightarrow \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}) \quad (12)$$

is represented by

$$(\tilde{\nu}'_1(m, q), \tilde{\nu}'_2(m, q)) = \left(\tilde{a}_m \frac{(2m^2 + m + 1)q^3 - (m + 3)q}{48m} \mod \tilde{a}_m, \tilde{b}_m \frac{(m + 1) \gcd(m + 1, 2)(q^3 - q)}{4m} \mod \tilde{b}_m \right). \quad (13)$$

Equivalently, for $[M^5] = k_3[M_3^5] + k_4[M_4^5]$, the anomaly phase is the one displayed in (141).

The phrase “integer-valued index” means that the displayed coordinates lie in the cyclic factors of the finite anomaly group. The corresponding $\mathbb{R}/\mathbb{Z}$ -valued η invariants are recovered by dividing by the relevant cyclic orders.

We use the following conventions throughout. The cyclic group $\mathbb{Z}_n$ is written multiplicatively when it acts on complex vector spaces, so its elements are roots of unity λ with $\lambda^n = 1$. The one-dimensional representation of charge q is denoted by ρ_q , meaning $\rho_q(\lambda) = \lambda^q$. In the Spin^c -type finite symmetry $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$, the corresponding odd-charge representation is denoted by $\tilde{\rho}_q$, with q odd. The symbols $a_n, b_n, \tilde{a}_m, \tilde{b}_m$ denote the orders of the cyclic factors of the finite anomaly groups; their prime-primary values are recalled in (85) and (125). The notations $L(n; a_1, \dots, a_k)$, $X(n; a_1, a_2)$, K3, and E are introduced before the corresponding η -invariant computations.

E. Roadmap

In Sec. II, we review the index theorems on continuous backgrounds, deriving the classical result that captures both perturbative and mixed gravitational anomalies on $\text{Spin} \times \text{U}(1)$ and Spin^c manifolds.

In Sec. III, we introduce the η invariant and explain its role in the anomaly of fermions.

Section IV focuses on the discrete subgroup $\mathbb{Z}_n \subset \text{U}(1)$ in the $\text{Spin} \times \text{U}(1)$ theory. We compute the corresponding anomaly index by tracing the image of the cobordism group $\text{TP}_5(\text{Spin} \times \text{U}(1))$ under the natural homomorphism into

$\text{TP}_5(\text{Spin} \times \mathbb{Z}_n)$ and evaluating η invariants on manifold generators.

In Sec. V, we turn to the discrete subgroup $\mathbb{Z}_{2m} \subset \text{U}(1)$ in the $\text{Spin}^c = \text{Spin} \times_{\mathbb{Z}_2^F} \text{U}(1)$ theory. There, we derive the anomaly index for $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ by following the map from $\text{TP}_5(\text{Spin}^c)$ into $\text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m})$ and evaluating η invariants on manifold generators.

Finally, Sec. VI places our results in the context of earlier work and explores their implications for discrete dark matter constructions and anomaly cancellation in discrete extensions of the Standard Model, providing a new perspective for the family puzzle (generation problem) in the Standard Model [1–3]. We also discuss possible generalizations beyond a single cyclic symmetry.

By providing explicit formulas that descend from the continuous to the discrete setting, this paper unveils the topological underpinnings of 4D discrete anomalies and supplies a unified, computable framework for studying anomaly cancellation in theories with discrete-charged fermions.

II. ANOMALY OF 4D WEYL FERMION WITH CONTINUOUS SYMMETRY

We first review the continuous anomaly indices from which the finite-symmetry indices will be obtained. There are two continuous symmetry types. In the ordinary spin case the spacetime-internal symmetry is $\text{Spin} \times \text{U}(1)$. In the Spin^c case it is $\text{Spin} \times_{\mathbb{Z}_2^F} \text{U}(1) = \text{Spin}^c$, where the

central fermion parity in Spin is identified with the order-2 element of U(1). By the Atiyah-Singer index theorem [28,29], the local anomaly of a four-dimensional Weyl fermion is encoded by a six-dimensional anomaly polynomial,

$$I_6 = [\hat{A}(TM)\text{ch}(\mathcal{E})]_6. \quad (14)$$

Here TM is the tangent bundle of the six-dimensional auxiliary manifold, $\hat{A}(TM)$ is the $\hat{A}$ class, $\mathcal{E}$ is the complex bundle associated to the gauge representation of the fermion, and $[\cdot\cdot\cdot]_6$ denotes the degree-six component. We follow the standard normalization reviewed, for example, in [1,26].

The characteristic classes used below are

$$\begin{aligned} \hat{A}(TM) &= 1 - \frac{p_1}{24} + \frac{7p_1^2 - 4p_2}{5760} + \dots, \\ \text{ch}(\mathcal{E}) &= \text{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) \\ &\quad + \frac{1}{6}(c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) + 3c_3(\mathcal{E})) + \dots. \end{aligned} \quad (15)$$

Here $p_i = p_i(TM)$ are Pontryagin classes of the tangent bundle and $c_i = c_i(\mathcal{E})$ are Chern classes of the gauge bundle. For a single left-handed Weyl fermion of U(1) charge q , $\mathcal{E}$ is a complex line bundle. Hence $c_i(\mathcal{E}) = 0$ for $i > 1$, and the anomaly polynomial becomes

$$\begin{aligned} \exp\left(2\pi i \int_{M^6} I_6\right) &= \exp\left(2\pi i \int_{M^6} [\hat{A}(TM)\text{ch}(\mathcal{E})]_6\right) \\ &= \exp\left(2\pi i \int_{M^6} \frac{q^3 c_1^3}{6} - \frac{q c_1 p_1}{24}\right). \end{aligned} \quad (16)$$

In differential-form notation, if A is a U(1) connection normalized by $c_1 = dA/(2\pi)$, then the anomaly polynomial is the exterior derivative of a five-dimensional Chern-Simons form, $I_6 = dI_5/(2\pi)$, with

$$I_5 = q^3 \frac{Ac_1^2}{6} - q \frac{Ap_1}{24}. \quad (17)$$

Thus, if $\partial M^6 = M^5$,

$$\exp\left(2\pi i \int_{M^6} I_6\right) = \exp\left(i \int_{M^5} I_5\right). \quad (18)$$

The integrality of the index on closed M^6 implies that this five-dimensional invertible theory is independent of the choice of filling. Equivalently, if M^6 and M'^6 are two fillings of the same M^5 , then gluing M^6 to $-M'^6$ gives a closed six-manifold, on which $\exp(2\pi i \int I_6) = 1$.

For a collection of Weyl fermions with charges q , the total invertible field theory is obtained by summing the individual contributions:

$$\exp\left(i \int_{M^5} I_5\right) = \exp\left(i \int_{M^5} \sum_q \left(q^3 \frac{Ac_1^2}{6} - q \frac{Ap_1}{24}\right)\right). \quad (19)$$

Remark 1 (Normalization of the continuous generators). The continuous formulas in this section fix the normalization used throughout the discrete computation. In particular, the generators I_5^A, I_5^B , and $\tilde{I}_5^C, \tilde{I}_5^D$ are chosen so that the coefficients of a charge- q Weyl fermion are integers: $(q^3 - q)/6 \in \mathbb{Z}$ in the $\text{Spin} \times \text{U}(1)$ case and $(q^3 - q)/24 \in \mathbb{Z}$ for odd q in the Spin^c case.

A. The $\text{Spin} \times \text{U}(1)$ case

In the $\text{Spin} \times \text{U}(1)$ case, the relevant group of five-dimensional invertible theories is

$$\text{TP}_5(\text{Spin} \times \text{U}(1)) = \Omega_6^{\text{Spin} \times \text{U}(1)} \cong \mathbb{Z}^2. \quad (20)$$

The two integer generators can be chosen using the charge assignments

$$q = (1), \quad q = (2, -1, -1), \quad (21)$$

as in [1,26]. Substituting these charge assignments into (19) gives the two generators,

$$I_5^A = \frac{Ac_1^2}{6} - \frac{Ap_1}{24}, \quad I_5^B = Ac_1^2. \quad (22)$$

A fermion of arbitrary integer charge q then contributes

$$I_5 = qI_5^A + \frac{q^3 - q}{6} I_5^B. \quad (23)$$

The coefficient $(q^3 - q)/6$ is an integer because it is the product of three consecutive integers divided by 6.

B. The $\text{Spin} \times_{\mathbb{Z}_2} \text{U}(1) = \text{Spin}^c$ case

For Spin^c symmetry, fermions carry odd U(1) charge. Geometrically, the U(1) factor is not independent of the spin group: the element $-1 \in \text{U}(1)$ is identified with fermion parity. Therefore the properly normalized integral class is the first Chern class of the quotient $\text{U}(1)/\mathbb{Z}_2^F$, which we denote by $2c_1$. Rewriting the anomaly polynomial in terms of this integral class gives

$$\tilde{I}_6 = q^3 \frac{(2c_1)^3}{48} - q \frac{(2c_1)p_1}{48}. \quad (24)$$

If $2c_1 = d\tilde{A}/(2\pi)$, then $\tilde{I}_6 = d\tilde{I}_5/(2\pi)$, with

$$\tilde{I}_5 = q^3 \frac{\tilde{A}(2c_1)^2}{48} - q \frac{\tilde{A}p_1}{48}. \quad (25)$$

Thus, for $\partial M^6 = M^5$,

$$\exp\left(2\pi i \int_{M^6} \tilde{I}_6\right) = \exp\left(i \int_{M^5} \tilde{I}_5\right). \quad (26)$$

As above, integrality of the index on closed six-manifolds makes the five-dimensional invertible theory well defined.

For multiple Weyl fermions of odd charges q ,

$$\exp\left(i \int_{M^5} \tilde{I}_5\right) = \exp\left(i \int_{M^5} \sum_q \left(q^3 \frac{\tilde{A}(2c_1)^2}{48} - q \frac{\tilde{A}p_1}{48}\right)\right). \quad (27)$$

The two generators of $\text{TP}_5(\text{Spin}^c) = \Omega_6^{\text{Spin}^c} \cong \mathbb{Z}^2$ may be chosen from the charge assignments

$$q = (1), \quad q = (3, -1, -1, -1), \quad (28)$$

as in [1,26]. They are

$$\tilde{I}_5^C = \frac{\tilde{A}(2c_1)^2}{48} - \frac{\tilde{A}p_1}{48}, \quad \tilde{I}_5^D = \frac{1}{2} \tilde{A}(2c_1)^2. \quad (29)$$

A charge- q fermion with q odd contributes

$$\tilde{I}_5 = q\tilde{I}_5^C + \frac{q^3 - q}{24} \tilde{I}_5^D. \quad (30)$$

The coefficient $(q^3 - q)/24$ is integral for odd q , because $(q - 1)q(q + 1)$ is divisible by 3, and for odd q the product is also divisible by 8.

III. INTRODUCTION TO THE η INVARIANT

We now recall the spectral invariant that computes global anomalies. Let M^5 be a closed five-manifold equipped with the relevant tangential structure and finite background gauge field. If a Weyl fermion transforms in a representation ρ of the finite group, we write

$$\eta(M^5, \rho) \in \mathbb{R}/\mathbb{Z} \quad (31)$$

for the reduced η invariant of the five-dimensional Dirac operator coupled to ρ . With this convention, the corresponding five-dimensional invertible field theory has partition function

$$\exp(-2\pi i \eta(M^5, \rho)) \in \text{U}(1). \quad (32)$$

Only the class of $\eta(M^5, \rho)$ modulo $\mathbb{Z}$ enters this phase.

The Dai-Freed description [30] explains why this invariant is the correct global-anomaly phase. If a four-dimensional

fermion theory is placed on $M^4 = \partial M^5$, then its partition function is a product of the absolute value of the four-dimensional determinant and the five-dimensional phase,

$$Z_\psi = |\det(\mathcal{D}(M^4, \rho))| \exp(-2\pi i \eta(M^5, \rho)). \quad (33)$$

Here $\mathcal{D}(M^4, \rho)$ and $\mathcal{D}(M^5, \rho)$ are Dirac operators coupled to the same background data. The Grassmann Gaussian integral gives

$$\int [D\bar{\psi}][D\psi] \exp(-\bar{\psi} \mathcal{D}(M^4, \rho) \psi) = \det(\mathcal{D}(M^4, \rho)). \quad (34)$$

The determinant has a phase ambiguity; the reduced η invariant supplies a canonical five-dimensional refinement of that phase.

A useful physical interpretation is obtained from a domain-wall ratio of five-dimensional massive Dirac determinants [21,23,31,32]:

$$\begin{aligned} \exp(-2\pi i \eta(M^5, \rho)) &:= \lim_{m_0 \rightarrow \infty} \frac{\det(-i\mathcal{D}(M^5, \rho) - m_0)}{\det(-i\mathcal{D}(M^5, \rho) + m_0)} \\ &= \lim_{m_0 \rightarrow \infty} \prod_{\lambda_{M^5}} \frac{-i\lambda_{M^5} - m_0}{-i\lambda_{M^5} + m_0}, \end{aligned} \quad (35)$$

where λ_{M^5} runs over the eigenvalues of $\mathcal{D}(M^5, \rho)$. For a single eigenvalue, introduce $s(\lambda_{M^5}) \in (-1, 1]$ by

$$\begin{aligned} \exp(-\pi i s(\lambda_{M^5})) &:= \frac{-i\lambda_{M^5} - m_0}{-i\lambda_{M^5} + m_0} \\ &= \frac{-m_0^2 + \lambda_{M^5}^2 - 2m_0\lambda_{M^5}i}{m_0^2 + \lambda_{M^5}^2}. \end{aligned} \quad (36)$$

As $m_0 \rightarrow \infty$, $s(\lambda_{M^5})$ tends to $\text{sign}(\lambda_{M^5})$ for nonzero eigenvalues, while a zero eigenvalue contributes $s(0)=1$. Formally, this gives

$$\eta(M^5, \rho) = \frac{1}{2} \left(\sum_{\lambda_{M^5} \neq 0} \text{sign}(\lambda_{M^5}) + \dim \ker \mathcal{D}(M^5, \rho) \right). \quad (37)$$

The unregularized sign sum is not convergent. The precise definition uses the meromorphic continuation of

$$\eta(s) := \sum_{\lambda_{M^5} \neq 0} \text{sign}(\lambda_{M^5}) |\lambda_{M^5}|^{-s}, \quad (38)$$

which converges for $\Re(s)$ sufficiently large and is regular at $s = 0$ in our applications. The reduced η invariant is

$$\eta(M^5, \rho) = \frac{1}{2} (\eta(0) + \dim \ker \mathcal{D}(M^5, \rho)). \quad (39)$$

Thus $\eta(M^5, \rho)$ measures the spectral asymmetry of the five-dimensional Dirac operator, including zero modes. Its fractional part is the global anomaly phase of the four-dimensional fermion.

Remark 2 (Use of η invariants below). All later equalities involving η invariants are equalities in $\mathbb{R}/\mathbb{Z}$. By contrast, the primed quantities ν'_i and $\tilde{\nu}'_i$ are integer coordinates in finite cyclic groups. They are obtained from the $\mathbb{R}/\mathbb{Z}$ -valued η invariants by multiplying by the order of the relevant cyclic summand.

IV. FROM $\text{Spin} \times \text{U}(1)$ SYMMETRY TO $\text{Spin} \times \mathbb{Z}_n$ SYMMETRY

In this section, we restrict the continuous symmetry group $\text{Spin} \times \text{U}(1)$ to the finite subgroup $\text{Spin} \times \mathbb{Z}_n$. The goal is to express the anomaly of a four-dimensional left-handed Weyl fermion of discrete charge $q \in \mathbb{Z}_n$ as an element of the finite anomaly group $\text{TP}_5(\text{Spin} \times \mathbb{Z}_n)$. We first describe the geometric backgrounds and spin-structure conventions, then compute the relevant η invariants, and finally convert the resulting $\mathbb{R}/\mathbb{Z}$ -valued invariants into integer coordinates in the two cyclic summands of $\text{TP}_5(\text{Spin} \times \mathbb{Z}_n)$.

Throughout this section, λ denotes an element of $\mathbb{Z}_n$, written as an n -th root of unity. The charge- q representation is the one-dimensional character

$$\rho_q(\lambda) = \lambda^q, \quad (40)$$

so replacing q by $q + n$ gives the same representation. We use $RU(\mathbb{Z}_n)$ for the complex representation ring of $\mathbb{Z}_n$:

$$RU(\mathbb{Z}_n) = \bigoplus_{q=0}^{n-1} \rho_q \mathbb{Z}. \quad (41)$$

If $R = \sum_q k_q \rho_q$, then k_q is the number of left-handed Weyl fermions of discrete charge q .

A. Spin structures and geometric representatives

We first recall the definition of a spin structure in the form needed below. Let $E \rightarrow M$ be an oriented real vector bundle of rank r , equipped with a Euclidean metric. Denote by

$$P_{\text{SO}}(E) \longrightarrow M \quad (42)$$

the principal $\text{SO}(r)$ bundle of oriented orthonormal frames of E . Let

$$\lambda_{\text{Spin}}: \text{Spin}(r) \longrightarrow \text{SO}(r) \quad (43)$$

be the standard double cover. A spin structure on E is a principal $\text{Spin}(r)$ bundle,

$$P_{\text{Spin}}(E) \longrightarrow M, \quad (44)$$

together with a bundle map,

$$\theta: P_{\text{Spin}}(E) \longrightarrow P_{\text{SO}}(E), \quad (45)$$

covering the identity map on M and satisfying

$$\theta(p \cdot s) = \theta(p) \cdot \lambda_{\text{Spin}}(s), \quad p \in P_{\text{Spin}}(E), \quad s \in \text{Spin}(r). \quad (46)$$

Equivalently, a spin structure is a lift of the structure group of E from $\text{SO}(r)$ to $\text{Spin}(r)$; see Chap. II, Sec. 1 of Ref. [33]. Two spin structures are equivalent if the corresponding principal $\text{Spin}(r)$ bundles are isomorphic over $P_{\text{SO}}(E)$. A spin structure exists if and only if $w_2(E) = 0$, and, when it exists, the equivalence classes of spin structures form a torsor for $H^1(M; \mathbb{Z}_2)$; see Chap. II, Sec. 1 of Ref. [33]. For an oriented manifold M , a spin structure means a spin structure on TM .

We now introduce the lens spaces used in the computation. Let

$$L(n; a_1, a_2, \dots, a_k) := S^{2k-1} / (\rho_{a_1} \oplus \rho_{a_2} \oplus \dots \oplus \rho_{a_k}) \quad (47)$$

be the $(2k-1)$ -dimensional lens space. Here ρ_{a_i} is the one-dimensional complex representation of $\mathbb{Z}_n$ on which λ acts by multiplication by λ^{a_i} . We assume that the action is free, so each a_i is coprime to n . The quotient map $S^{2k-1} \rightarrow L(n; a_1, \dots, a_k)$ is a principal $\mathbb{Z}_n$ bundle, and the corresponding classifying map supplies the natural background $\mathbb{Z}_n$ gauge field. In particular,

$$\pi_1(L(n; a_1, \dots, a_k)) \cong \mathbb{Z}_n. \quad (48)$$

Let

$$V := \rho_{a_1} \oplus \dots \oplus \rho_{a_k}. \quad (49)$$

This is a complex k -dimensional representation of $\mathbb{Z}_n$. We write $V_{\mathbb{R}}$ for the same vector space regarded as a real $2k$ -dimensional representation, and we identify

$$S^{2k-1} = S(V_{\mathbb{R}}) \quad (50)$$

with the unit sphere in $V_{\mathbb{R}}$, using a $\mathbb{Z}_n$ -invariant Hermitian metric. The stable tangent-bundle formula used below is first an equivariant formula over the sphere:

$$TS(V_{\mathbb{R}}) \oplus \mathbf{1}_{\mathbb{R}} \cong S(V_{\mathbb{R}}) \times V_{\mathbb{R}}. \quad (51)$$

Here $S(V_{\mathbb{R}}) \times V_{\mathbb{R}}$ is the trivial real vector bundle over the sphere, but it carries the induced $\mathbb{Z}_n$ action: $g \in \mathbb{Z}_n$ sends (x, v) to (gx, gv) . At a point $x \in S(V_{\mathbb{R}})$,

$$T_x S(V_{\mathbb{R}}) = \{v \in V_{\mathbb{R}} \mid \langle v, x \rangle = 0\}, \quad (52)$$

and the additional trivial line is the radial normal line spanned by x . The isomorphism is given fiberwise by

$$\begin{aligned} T_x S(V_{\mathbb{R}}) \oplus \mathbb{R} &\longrightarrow V_{\mathbb{R}}, \\ (v, t) &\longmapsto v + tx. \end{aligned} \quad (53)$$

Since the $\mathbb{Z}_n$ action is unitary, it preserves the inner product and sends the radial vector x to the radial vector gx . Hence this stable isomorphism is $\mathbb{Z}_n$ equivariant. Passing to the quotient gives

$$TL(n; a_1, \dots, a_k) \oplus \mathbb{1}_{\mathbb{R}} \cong S^{2k-1} \times_{\mathbb{Z}_n} V_{\mathbb{R}}. \quad (54)$$

The notation $S^{2k-1} \times_{\mathbb{Z}_n} V_{\mathbb{R}}$ means the real vector bundle associated to the principal $\mathbb{Z}_n$ bundle $S^{2k-1} \rightarrow L(n; a_1, \dots, a_k)$ via the real representation $V_{\mathbb{R}}$.

Equivalently, if

$$E := S^{2k-1} \times_{\mathbb{Z}_n} V \quad (55)$$

is the associated complex vector bundle, then

$$E_{\mathbb{R}} \cong S^{2k-1} \times_{\mathbb{Z}_n} V_{\mathbb{R}}, \quad TL(n; a_1, \dots, a_k) \oplus \mathbb{1}_{\mathbb{R}} \cong E_{\mathbb{R}}. \quad (56)$$

Adding a trivial real line does not change w_2 , so

$$w_2(TL(n; a_1, \dots, a_k)) = w_2(E_{\mathbb{R}}). \quad (57)$$

For a complex vector bundle E , the underlying real bundle satisfies

$$w_2(E_{\mathbb{R}}) \equiv c_1(E) \pmod{2}; \quad (58)$$

see Sec. 14 of Ref. [34]. Let $u \in H^2(B\mathbb{Z}_n; \mathbb{Z}) \cong \mathbb{Z}_n$ be the standard generator and let $f: L(n; a_1, \dots, a_k) \rightarrow B\mathbb{Z}_n$ classify the principal $\mathbb{Z}_n$ bundle. Then

$$c_1(E) = (a_1 + \dots + a_k)f^*u. \quad (59)$$

Writing $\bar{u} = f^*(u \bmod 2)$, we obtain

$$w_2(TL(n; a_1, \dots, a_k)) = (a_1 + \dots + a_k)\bar{u}. \quad (60)$$

If n is odd, then $H^2(L(n; a_1, \dots, a_k); \mathbb{Z}_2) = 0$, so the lens space is spin and the spin structure is unique. If n is even, freeness implies that each a_i is odd, so $a_1 + \dots + a_k \equiv k \pmod{2}$. Thus $L(n; a_1, \dots, a_k)$ is spin precisely when k is even, except for the one-dimensional case $k = 1$, where $\bar{u} = 0$ and $L(n; a_1)$ is spin. In the spin case with even n , there are two inequivalent spin structures because

$$H^1(L(n; a_1, \dots, a_k); \mathbb{Z}_2) \cong \mathbb{Z}_2. \quad (61)$$

This agrees with the standard classification of spin structures on lens spaces; see Ref. [35].

Five-dimensional lens spaces are not spin for all n , so the first universal spin representative is instead the lens-space bundle

$$X(n; a_1, a_2) := S(H \otimes H \oplus \mathbb{1})/\tau(a_1, a_2). \quad (62)$$

Here H is the Hopf line bundle over S^2 , $S(H \otimes H \oplus \mathbb{1})$ is the unit sphere bundle of the rank-two complex bundle $H \otimes H \oplus \mathbb{1}$, and

$$\tau(a_1, a_2) = \rho_{a_1} \oplus \rho_{a_2} \quad (63)$$

is the fiberwise $\mathbb{Z}_n$ action with weights a_1 and a_2 . When a_1 and a_2 are coprime to n , the action is fixed-point free. By [13], $X(n; a_1, a_2)$ carries a natural spin structure and a natural $\mathbb{Z}_n$ background, with

$$\pi_1(X(n; a_1, a_2)) \cong \mathbb{Z}_n. \quad (64)$$

The lift of the $\mathbb{Z}_n$ action to the spin bundle is determined by a square root of the determinant character

$$\det(\rho_{a_1} \oplus \rho_{a_2}) = \rho_{a_1+a_2}. \quad (65)$$

Equivalently, one chooses $b \in \mathbb{Z}/n\mathbb{Z}$ such that

$$2b \equiv a_1 + a_2 \pmod{n}. \quad (66)$$

For odd n , this congruence has a unique solution. For even n , the numbers a_1 and a_2 are odd, so $a_1 + a_2$ is even and the congruence has exactly two solutions. These two solutions differ by the nontrivial class in $H^1(X(n; a_1, a_2); \mathbb{Z}_2)$. In the even case, the phrase “positive square root” means that we choose one of these two lifts once and for all.

The two five-dimensional geometric backgrounds used below are

$$X(n; 1, 1) \quad \text{and} \quad L(n; 1) \times \text{K3}. \quad (67)$$

Here $X(n; 1, 1) = S(H \otimes H \oplus \mathbb{1})/\tau(1, 1)$ is a lens-space bundle over S^2 with fiber $L(n; 1, 1)$, where

$$\tau(1, 1)(\lambda) = \text{diag}(\lambda, \lambda). \quad (68)$$

The second background uses the one-dimensional lens space $L(n; 1) = S^1/\rho_1$ and a K3 surface. The K3 surface is spin because for any complex manifold Y ,

$$w_2(TY_{\mathbb{R}}) \equiv c_1(TY) \pmod{2}, \quad (69)$$

and for $Y = \text{K3}$ the canonical bundle is trivial. Since $c_1(K_Y) = -c_1(TY)$, we have $c_1(\text{TK3}) = 0$; hence $w_2(\text{TK3}) = 0$. Moreover, K3 is simply connected, so its spin structure is unique. Therefore $L(n; 1) \times \text{K3}$ is a natural $\text{Spin} \times \mathbb{Z}_n$ background.

B. The two basic η invariants

The anomaly phase of a left-handed Weyl fermion of discrete charge q on a five-dimensional $\text{Spin} \times \mathbb{Z}_n$ background is

$$\exp(-2\pi i \eta(M^5, \rho_q)). \quad (70)$$

We first compute this phase on the two geometric backgrounds above.

For $X(n; 1, 1)$, the equivariant Lefschetz formula for the spin Dirac operator (see Theorem 4.7 in Ref. [13]) gives

$$\eta(X(n; 1, 1), \rho_q) = \frac{1}{n} \sum_{\substack{\lambda^n=1 \\ \lambda \neq 1}} \lambda^q \frac{\lambda(1+\lambda)}{(1-\lambda)^3}. \quad (71)$$

The factor λ^q is the character of the twisting representation ρ_q , while the rational function in λ is the local fixed-point contribution of the spin operator.

To evaluate this root-of-unity sum, compare it with the standard formula for the seven-dimensional lens space

$$\begin{aligned} L(n; 1, 1, 1, 1) &= S^7 / \tau(1, 1, 1, 1), \\ \tau(1, 1, 1, 1)(\lambda) &= \text{diag}(\lambda, \lambda, \lambda, \lambda). \end{aligned} \quad (72)$$

By Theorem 4.6 in Ref. [13],

$$\eta(L(n; 1, 1, 1, 1), R) = \frac{1}{n} \sum_{\substack{\lambda^n=1 \\ \lambda \neq 1}} \text{Tr}(R(\lambda)) \frac{\lambda^2}{(1-\lambda)^4}. \quad (73)$$

For the virtual representation $R = (\rho_1 - \rho_{-1})\rho_q$, one has

$$\begin{aligned} \text{Tr}(R(\lambda)) \frac{\lambda^2}{(1-\lambda)^4} &= (\lambda^{q+1} - \lambda^{q-1}) \frac{\lambda^2}{(1-\lambda)^4} \\ &= -\lambda^q \frac{\lambda(1+\lambda)}{(1-\lambda)^3}. \end{aligned} \quad (74)$$

Therefore

$$\eta(-X(n; 1, 1), \rho_q) = \eta(L(n; 1, 1, 1, 1), (\rho_1 - \rho_{-1})\rho_q), \quad (75)$$

where the minus sign denotes orientation reversal.

By [10], the seven-dimensional lens-space invariant can be written in terms of the $\hat{A}$ polynomials:

$$\eta(L(n; 1, 1, 1, 1), \rho_q) = -\frac{1}{n} \hat{A}_4\left(q + \frac{n}{2}; n, 1, 1, 1, 1\right) \mod \mathbb{Z}, \quad (76)$$

where

$$\hat{A}_k(t, \vec{x}) := \sum_{a+2b=k} \frac{t^a \hat{A}_b(\vec{x})}{a!}. \quad (77)$$

Applying this formula to $(\rho_1 - \rho_{-1})\rho_q$, or equivalently taking the corresponding finite difference, gives the cubic expression computed in [8]:

$$\eta(X(n; 1, 1), \rho_q) = \frac{(n^2 + 3n + 2)q^3}{6n} \mod \mathbb{Z}. \quad (78)$$

This is the pure discrete gauge contribution.

The second invariant is the product invariant on $L(n; 1) \times \text{K3}$. For an odd-dimensional spin manifold A and an even-dimensional spin manifold B , the product formula gives

$$\eta(A \times B, \rho) = \eta(A, \rho) \hat{A}(B); \quad (79)$$

see Refs. [9,11,12,15]. Since $\hat{A}(\text{K3}) = 2$, we obtain

$$\eta(L(n; 1) \times \text{K3}, \rho_q) = 2\eta(L(n; 1), \rho_q). \quad (80)$$

With the spin-structure convention fixed above, [9,12] gives

$$\begin{aligned} \eta(L(n; 1), \rho_q) &= \frac{1}{n} \sum_{\substack{\lambda \in \mathbb{Z}_n \\ \lambda \neq 1}} (\lambda^q - 1) \frac{\sqrt{\lambda}}{\lambda - 1} \\ &= \frac{1}{n} \sum_{\substack{\lambda \in \mathbb{Z}_n \\ \lambda \neq 1}} \sum_{r=1}^q \lambda^{\frac{2r-1}{2}} \\ &= -\frac{q}{n} \mod \mathbb{Z}. \end{aligned} \quad (81)$$

Therefore

$$\eta(-L(n; 1) \times \text{K3}, \rho_q) = \frac{2q}{n} \mod \mathbb{Z}. \quad (82)$$

This is the linear, or mixed discrete gauge-gravitational, contribution.

We denote these two $\mathbb{R}/\mathbb{Z}$ -valued invariants by

$$\nu_1(n, q) := \eta(X(n; 1, 1), \rho_q) = \frac{(n^2 + 3n + 2)q^3}{6n} \mod \mathbb{Z},$$

$$\nu_2(n, q) := \eta(-L(n; 1) \times \text{K3}, \rho_q) = \frac{2q}{n} \mod \mathbb{Z}. \quad (83)$$

C. Integer coordinates in $\text{TP}_5(\text{Spin} \times \mathbb{Z}_n)$

We now convert the $\mathbb{R}/\mathbb{Z}$ -valued invariants into integer coordinates in the finite anomaly group. There is a natural restriction homomorphism,

$$\text{TP}_5(\text{Spin} \times \text{U}(1)) = \mathbb{Z}^2 \longrightarrow \text{TP}_5(\text{Spin} \times \mathbb{Z}_n). \quad (84)$$

It sends the two continuous generators I_5^A and I_5^B to pairs (ι_1^A, ι_2^A) and (ι_1^B, ι_2^B) , respectively. By [8],

$$\begin{aligned} \text{TP}_5(\text{Spin} \times \mathbb{Z}_n) &= \Omega_5^{\text{Spin} \times \mathbb{Z}_n} = \mathbb{Z}_{a_n} \times \mathbb{Z}_{b_n} \\ &= \begin{cases} 0 & n = 2, \\ \mathbb{Z}_n \times \mathbb{Z}_{\frac{n}{4}} & n = 2^\nu, \nu > 1, \\ \mathbb{Z}_{3n} \times \mathbb{Z}_{\frac{n}{3}} & n = 3^\nu, \\ \mathbb{Z}_n \times \mathbb{Z}_n & n = p^\nu, p > 3 \text{ is prime.} \end{cases} \end{aligned} \quad (85)$$

Remark 3 (Prime-primary notation). The displayed formula records the prime-primary cases. For a general integer n , one decomposes $\mathbb{Z}_n$ into its primary components and combines the cyclic factors by the Chinese remainder theorem. Equivalently, if $n = \prod_p p^{\nu_p}$, then a_n and b_n are obtained by multiplying the corresponding prime-primary orders in (85). Thus the two-primary part contributes $(a_{2^\nu}, b_{2^\nu}) = (2^\nu, 2^{\nu-2})$ for $\nu > 1$, the three-primary part contributes $(3^{\nu+1}, 3^{\nu-1})$, and a prime $p > 3$ contributes (p^ν, p^ν) . The case $n = 2$ is trivial. The formulas below are written in terms of the resulting cyclic orders a_n and b_n .

Remark 4 (Normalization of ν_2). The second $\mathbb{R}/\mathbb{Z}$ -valued invariant is not yet the integer coordinate of the second cyclic summand. To obtain that coordinate, we are free to add an integral multiple of the first invariant ν_1 . Geometrically this is a change of basis among the two five-dimensional generators; algebraically it is the replacement of one cyclic generator by that generator plus a multiple of the other one. The choice made below is the one for which multiplication by b_n gives an integer-valued invariant in $\mathbb{Z}_{b_n}$.

We seek indices $\nu_1'(n, q) \in \mathbb{Z}_{a_n}$ and $\nu_2'(n, q) \in \mathbb{Z}_{b_n}$ such that

$$\begin{aligned} q\iota_1^A + \frac{q^3 - q}{6}\iota_1^B &= \nu_1'(n, q), \\ q\iota_2^A + \frac{q^3 - q}{6}\iota_2^B &= \nu_2'(n, q). \end{aligned} \quad (86)$$

Equivalently, (ν_1', ν_2') is the image of the continuous anomaly

$$qI_5^A + \frac{q^3 - q}{6}I_5^B \quad (87)$$

under the restriction map. The relations

$$\begin{aligned} \iota_1^A &= \nu_1'(n, 1), \\ \iota_1^B &= \nu_1'(n, 2) + 2\nu_1'(n, -1), \\ \iota_2^A &= \nu_2'(n, 1), \\ \iota_2^B &= \nu_2'(n, 2) + 2\nu_2'(n, -1) \end{aligned} \quad (88)$$

will be used to read off the images of I_5^A and I_5^B .

The first coordinate is obtained by multiplying the cubic invariant by the order of the first cyclic factor:

$$\begin{aligned} \nu_1'(n, q) &= a_n \nu_1(n, q) \\ &= \frac{a_n(n^2 + 3n + 2)q^3}{6n} \mod a_n. \end{aligned} \quad (89)$$

This is integer valued because

$$\begin{aligned} \iota_1^A &= \frac{a_n(n^2 + 3n + 2)}{6n} \in \mathbb{Z}, \\ \iota_1^B &= \frac{a_n(n^2 + 3n + 2)}{n} \in \mathbb{Z}, \end{aligned} \quad (90)$$

as follows from the prime-primary description of a_n in (85).

The second coordinate must be chosen so that it is integer valued in $\mathbb{Z}_{b_n}$. We therefore allow a change of basis by adding an integral multiple of ν_1 to ν_2 :

$$\begin{aligned} \nu_2'(n, q) &= b_n(k\nu_1(n, q) + \nu_2(n, q)) \mod b_n \\ &= b_n \left(k \frac{(n^2 + 3n + 2)q^3}{6n} + \frac{2q}{n} \right) \mod b_n. \end{aligned} \quad (91)$$

The condition that the corresponding images of I_5^A and I_5^B be integral is

$$\begin{aligned} \iota_2^A &= b_n \left(k \frac{n^2 + 3n + 2}{6n} + \frac{2}{n} \right) \in \mathbb{Z}, \\ \iota_2^B &= \frac{b_n k(n^2 + 3n + 2)}{n} \in \mathbb{Z}. \end{aligned} \quad (92)$$

Using the possible values of b_n , the uniform choice $k = -6$ satisfies both integrality conditions and gives the basis of integer-valued coordinates used below. Hence

$$(\nu_1'(n, q), \nu_2'(n, q)) = \left(\frac{a_n(n^2 + 3n + 2)q^3}{6n} \mod a_n, \frac{2b_n(q - q^3)}{n} \mod b_n \right) \quad (93)$$

in $\text{TP}_5(\text{Spin} \times \mathbb{Z}_n) = \mathbb{Z}_{a_n} \times \mathbb{Z}_{b_n}$. Equivalently,

$$I_5^A \mapsto \left(\frac{a_n(n^2 + 3n + 2)}{6n} \mod a_n, 0 \mod b_n \right), \quad I_5^B \mapsto \left(\frac{2a_n}{n} \mod a_n, -\frac{12b_n}{n} \mod b_n \right). \quad (94)$$

D. Geometric generators and the anomaly phase

Set

$$\begin{aligned} [M_1^5] &:= [X(n; 1, 1)], \\ [M_2^5] &:= -[L(n; 1) \times K3] - 6[X(n; 1, 1)]. \end{aligned} \quad (95)$$

Then

$$\begin{aligned} \nu'_1(n, q) &= a_n \eta(M_1^5, \rho_q), \\ \nu'_2(n, q) &= b_n \eta(M_2^5, \rho_q). \end{aligned} \quad (96)$$

Indeed,

$$\begin{aligned} \eta(M_2^5, \rho_q) &= \frac{2q}{n} - 6 \frac{(n^2 + 3n + 2)q^3}{6n} \\ &= \frac{2(q - q^3)}{n} \pmod{\mathbb{Z}}, \end{aligned} \quad (97)$$

because $(n^2 + 3n)q^3/n = (n + 3)q^3 \in \mathbb{Z}$. The shift by $-6[X(n; 1, 1)]$ is precisely the change of basis that removes the cubic component from the mixed generator.

The first class detects the cubic, or pure discrete gauge, anomaly. The second class detects the independent mixed discrete gauge-gravitational anomaly after this change of basis.

The fact that M_1^5 and M_2^5 generate the two cyclic summands is checked by these normalized η invariants. For $q = 1$,

$$\nu'_1(n, 1) = \frac{a_n(n^2 + 3n + 2)}{6n} \quad (98)$$

is a unit modulo a_n in each nontrivial prime-primary case, and hence is a generator of $\mathbb{Z}_{a_n}$; moreover $\nu'_2(n, 1) = 0$. For $q = 2$,

$$\nu'_2(n, 2) = -\frac{12b_n}{n} \quad (99)$$

is a unit modulo b_n in each nontrivial second factor, and hence is a generator of $\mathbb{Z}_{b_n}$. Therefore $[M_1^5]$ has exact order a_n and $[M_2^5]$ has exact order b_n . Since

$$\Omega_5^{\text{Spin} \times \mathbb{Z}_n} \cong \mathbb{Z}_{a_n} \oplus \mathbb{Z}_{b_n}, \quad (100)$$

the two classes form a basis for the two cyclic summands.

Consequently, any five-dimensional $\text{Spin} \times \mathbb{Z}_n$ background has a unique coordinate expression,

$$[M^5] = k_1[M_1^5] + k_2[M_2^5], \quad k_1 \in \mathbb{Z}_{a_n}, \quad k_2 \in \mathbb{Z}_{b_n}. \quad (101)$$

Here the group law is induced by disjoint union, and the coefficients are taken modulo the orders of the two cyclic factors. The coefficients k_1 and k_2 are background-dependent

topological coordinates; they should not be interpreted as numbers of fermions. The fermion-dependent data are instead the indices $\nu'_1(n, q)$ and $\nu'_2(n, q)$.

With these conventions, the anomaly phase on a general background is

$$\begin{aligned} &\exp(-2\pi i \eta([M^5], \rho_q)) \\ &= \exp(-2\pi i k_1 \eta([M_1^5], \rho_q) - 2\pi i k_2 \eta([M_2^5], \rho_q)) \\ &= \exp\left(-\frac{2\pi i}{a_n} k_1 \nu'_1(n, q) - \frac{2\pi i}{b_n} k_2 \nu'_2(n, q)\right) \\ &= \exp\left(-\frac{2\pi i}{a_n} k_1 \frac{a_n(n^2 + 3n + 2)q^3}{6n} - \frac{2\pi i}{b_n} k_2 \frac{2b_n(q - q^3)}{n}\right). \end{aligned} \quad (102)$$

V. FROM $\text{Spin} \times_{\mathbb{Z}_2^F} \text{U}(1) = \text{Spin}^c$ SYMMETRY TO $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ SYMMETRY

We next restrict the $\text{U}(1)$ factor in the $\text{Spin}^c = \text{Spin} \times_{\mathbb{Z}_2^F} \text{U}(1)$ theory to its finite subgroup $\mathbb{Z}_{2m}$. The resulting finite symmetry is $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$. Unlike the $\text{Spin} \times \mathbb{Z}_n$ case, the finite cyclic symmetry is not independent of the spin structure: the order-2 central element in $\mathbb{Z}_{2m}$ is identified with fermion parity. Consequently, fermions carry odd $\mathbb{Z}_{2m}$ charge.

Throughout this section, ζ denotes a primitive $2m$ -th root of unity. For an odd integer q , the one-dimensional representation used to twist the Dirac operator is

$$\tilde{\rho}_q(\zeta) = \zeta^q, \quad q \in 2\mathbb{Z} + 1. \quad (103)$$

The condition that q be odd means that the central element $\zeta^m = -1$ acts as fermion parity. We write $RU^o(\mathbb{Z}_{2m})$ for the additive subgroup of $RU(\mathbb{Z}_{2m})$ generated by these odd-charge representations. Thus a collection of left-handed Weyl fermions of odd charges q is represented by an element $\sum_q k_q \tilde{\rho}_q \in RU^o(\mathbb{Z}_{2m})$.

A. Finite Spin^c -type structures and geometric representatives

We first recall the tangential structure that replaces an ordinary spin structure. A $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ structure on an oriented manifold M can be described as a spin structure on the stabilized real bundle

$$TM \oplus f^* \xi. \quad (104)$$

Here $f: M \rightarrow B\mathbb{Z}_m$ is a classifying map, and ξ is the underlying real two-plane bundle of the complex line bundle over $B\mathbb{Z}_m$ induced by the embedding $\mathbb{Z}_m \rightarrow \text{U}(1)$. The appearance of $B\mathbb{Z}_m$, rather than $B\mathbb{Z}_{2m}$, comes from the quotient by fermion parity: after identifying the order-2 subgroup of $\mathbb{Z}_{2m}$ with $\mathbb{Z}_2^F$, the remaining determinant data

are classified by a $\mathbb{Z}_m$ background. Thus M admits such a structure precisely when

$$w_2(TM) = w_2(f^*\xi) \quad (105)$$

for some classifying map f .

Let

$$\beta_m: H^1(-; \mathbb{Z}_m) \longrightarrow H^2(-; \mathbb{Z}) \quad (106)$$

be the Bockstein homomorphism associated with the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}_m \rightarrow 0$. Then $c_1(f^*\xi) = \beta_m[f]$. Similarly, the exact sequence $0 \rightarrow \mathbb{Z}_2 \xrightarrow{m} \mathbb{Z}_{2m} \rightarrow \mathbb{Z}_m \rightarrow 0$ gives a Bockstein homomorphism,

$$\beta_{m,2}: H^1(-; \mathbb{Z}_m) \longrightarrow H^2(-; \mathbb{Z}_2), \quad (107)$$

and $w_2(f^*\xi) = \beta_{m,2}[f]$. The corresponding long exact sequence in cohomology contains

$$\cdots \rightarrow H^1(-; \mathbb{Z}_m) \xrightarrow{\beta_{m,2}} H^2(-; \mathbb{Z}_2) \xrightarrow{m} H^2(-; \mathbb{Z}_{2m}) \rightarrow \cdots \quad (108)$$

Consequently, if $mw_2(TM) = 0$, then $w_2(TM)$ lies in the image of $\beta_{m,2}$, and M admits a $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ structure.

This criterion explains the geometric representatives used below. First, consider the Enriques surface E . Its first Chern class is a nonzero two-torsion class in $H^2(E; \mathbb{Z})$, and

$$w_2(TE) = c_1(TE) \pmod{2} \quad (109)$$

is nonzero in $H^2(E; \mathbb{Z}_2)$. Therefore E is not spin. For even m , however, $mw_2(TE) = 0$, so E admits a $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ structure. This is why the product $L(m; 1) \times E$ appears in the even- m calculation.

Lens spaces also carry natural $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ structures. For

$$L(m; a_1, \dots, a_k) = S^{2k-1}/(\rho_{a_1} \oplus \cdots \oplus \rho_{a_k}), \quad (110)$$

the quotient map is a principal $\mathbb{Z}_m$ bundle with

$$\pi_1(L(m; a_1, \dots, a_k)) \cong \mathbb{Z}_m. \quad (111)$$

The associated complex line bundle determines the real two-plane bundle $f^*\xi$, and the stable tangent-bundle calculation for lens spaces gives $w_2(TL) = w_2(f^*\xi)$ in the cases used here. Hence the lens spaces below are natural finite Spin^c -type backgrounds.

For even m , we use the two five-dimensional backgrounds,

$$L(m; 1, 1, 1) \quad \text{and} \quad L(m; 1) \times E. \quad (112)$$

Here $L(m; 1, 1, 1) = S^5/\tau(1, 1, 1)$, with $\tau(1, 1, 1)(\lambda) = \text{diag}(\lambda, \lambda, \lambda)$, and $L(m; 1)$ is the one-dimensional lens space. For odd m , the group $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ is equivalent to $\text{Spin} \times \mathbb{Z}_m$: the order-2 subgroup of $\mathbb{Z}_{2m}$ splits off and is identified with fermion parity, leaving a genuine $\mathbb{Z}_m$ background. In that case we use

$$L(m; 1, 1, 1) \quad \text{and} \quad L(m; 1) \times K3. \quad (113)$$

B. The two basic η invariants

The anomaly phase of a left-handed Weyl fermion of odd charge q on a five-dimensional $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ background is

$$\exp(-2\pi i \eta(M^5, \tilde{\rho}_q)). \quad (114)$$

We now compute this phase on the two types of geometric backgrounds introduced above.

The invariant of $L(m; 1, 1, 1)$ is computed from the Lefschetz fixed point formula of Theorem 4.6 in Ref. [13] and the $\hat{A}$ -polynomial formula, as in [8]. The formula is often written in terms of an ordinary $\mathbb{Z}_m$ representation. For an odd charge q , the corresponding $\mathbb{Z}_m$ character has charge $(q-1)/2$. Thus, if

$$R = \bigoplus_q k_q \rho_{\frac{q-1}{2}} \in RU(\mathbb{Z}_m) \quad (115)$$

corresponds to $\tilde{R} = \bigoplus_q k_q \tilde{\rho}_q \in RU^o(\mathbb{Z}_{2m})$, then

$$\eta(L(m; 1, 1, 1), R) = \frac{1}{m} \sum_{\substack{\lambda^m=1 \\ \lambda \neq 1}} \text{Tr}(R(\lambda)) \frac{\lambda^2}{(1-\lambda)^3}. \quad (116)$$

Comparing this expression with the seven-dimensional lens-space formula

$$\eta_{\text{Spin} \times \mathbb{Z}_m}(L(m; 1, 1, 1, 1), R) = \frac{1}{m} \sum_{\substack{\lambda^m=1 \\ \lambda \neq 1}} \text{Tr}(R(\lambda)) \frac{\lambda^2}{(1-\lambda)^4}, \quad (117)$$

one obtains

$$\eta(L(m; 1, 1, 1), \tilde{\rho}_q) = \eta_{\text{Spin} \times \mathbb{Z}_m}(L(m; 1, 1, 1, 1), (\rho_0 - \rho_1) \rho_{\frac{q-1}{2}}). \quad (118)$$

Using (76), Hsieh obtains [8]

$$\eta(L(m; 1, 1, 1), \tilde{\rho}_q) = \frac{(2m^2 + m + 1)q^3 - (m + 3)q}{48m} \pmod{\mathbb{Z}}. \quad (119)$$

This is the cubic finite Spin^c -type contribution.

The second invariant is a product invariant. The one-dimensional lens-space calculation gives [9,12,15]

$$\begin{aligned}\eta(L(m; 1), \tilde{\rho}_q) &= \frac{1}{2m} \sum_{\substack{\lambda \in \mathbb{Z}_{2m} \\ \lambda \neq 1}} (\lambda^q - 1) \frac{\lambda}{\lambda - 1} \\ &= \frac{1}{2m} \sum_{\substack{\lambda \in \mathbb{Z}_{2m} \\ \lambda \neq 1}} \sum_{r=1}^q \lambda^r \\ &= -\frac{q}{2m} \pmod{\mathbb{Z}}.\end{aligned}\quad (120)$$

Using the product formula (79), and using $\hat{A}(E) = 1$, we find for even m that

$$\eta(-L(m; 1) \times E, \tilde{\rho}_q) = \frac{q}{2m} \pmod{\mathbb{Z}}. \quad (121)$$

For odd m , we use K3 instead. Since $\hat{A}(K3) = 2$,

$$\eta(-L(m; 1) \times K3, \tilde{\rho}_q) = \frac{q}{m} \pmod{\mathbb{Z}}. \quad (122)$$

It is convenient to write these two product cases in the uniform form:

$$\begin{aligned}\tilde{\nu}_1(m, q) &:= \eta(L(m; 1, 1, 1), \tilde{\rho}_q) \\ &= \frac{(2m^2 + m + 1)q^3 - (m + 3)q}{48m} \pmod{\mathbb{Z}}, \\ \tilde{\nu}_2(m, q) &:= \frac{\gcd(m+1, 2)q}{2m} \pmod{\mathbb{Z}}.\end{aligned}\quad (123)$$

For even m , $\tilde{\nu}_2$ is represented by $-L(m; 1) \times E$; for odd m , it is represented by $-L(m; 1) \times K3$.

C. Integer coordinates in $\text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m})$

We now convert the $\mathbb{R}/\mathbb{Z}$ -valued invariants into integer coordinates in the finite anomaly group. There is a natural restriction homomorphism,

$$\text{TP}_5(\text{Spin}^c) = \mathbb{Z}^2 \longrightarrow \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}), \quad (124)$$

sending the continuous generators $\tilde{I}_5^C$ and $\tilde{I}_5^D$ to (ι_1^C, ι_2^C) and (ι_1^D, ι_2^D) , respectively. By [8],

$$\begin{aligned}\text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}) &= \Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}} = \mathbb{Z}_{\tilde{a}_m} \times \mathbb{Z}_{\tilde{b}_m} \\ &= \begin{cases} 0 & m = 1, \\ \mathbb{Z}_{8m} \times \mathbb{Z}_{\frac{m}{2}} & m = 2^\nu > 1, \\ \mathbb{Z}_{3m} \times \mathbb{Z}_{\frac{m}{3}} & m = 3^\nu, \\ \mathbb{Z}_m \times \mathbb{Z}_m & m = p^\nu, p > 3 \text{ is prime.} \end{cases}\end{aligned}\quad (125)$$

Remark 5 (Prime-primary notation). As in the $\text{Spin} \times_{\mathbb{Z}_n}$ case, the displayed group structure records

the prime-primary cases. For a general m , the orders $\tilde{a}_m$ and $\tilde{b}_m$ are obtained by multiplying the prime-primary orders. The two-primary contribution is $(8m, m/2)$ for $m = 2^\nu > 1$, the three-primary contribution is $(3m, m/3)$, and a prime $p > 3$ contributes (m, m) on the p -primary component. The case $m = 1$ is trivial.

Remark 6 (Normalization of $\tilde{\nu}_2$). The second $\mathbb{R}/\mathbb{Z}$ -valued invariant is not yet the integer coordinate of the second cyclic summand. To obtain that coordinate, we are free to add an integral multiple of the first invariant $\tilde{\nu}_1$. Geometrically this is a change of basis among the two five-dimensional generators; algebraically it is the replacement of one cyclic generator by that generator plus a multiple of the other one. The choice made below is the one for which multiplication by $\tilde{b}_m$ gives an integer-valued invariant in $\mathbb{Z}_{\tilde{b}_m}$.

We seek indices $\tilde{\nu}'_1(m, q) \in \mathbb{Z}_{\tilde{a}_m}$ and $\tilde{\nu}'_2(m, q) \in \mathbb{Z}_{\tilde{b}_m}$ satisfying

$$\begin{aligned}q\iota_1^C + \frac{q^3 - q}{24}\iota_1^D &= \tilde{\nu}'_1(m, q), \\ q\iota_2^C + \frac{q^3 - q}{24}\iota_2^D &= \tilde{\nu}'_2(m, q).\end{aligned}\quad (126)$$

Equivalently, $(\tilde{\nu}'_1, \tilde{\nu}'_2)$ is the image of

$$q\tilde{I}_5^C + \frac{q^3 - q}{24}\tilde{I}_5^D \quad (127)$$

under the restriction map; see Eq. (29). The useful identities are

$$\begin{aligned}\iota_1^C &= \tilde{\nu}'_1(m, 1), \\ \iota_1^D &= \tilde{\nu}'_1(m, 3) + 3\tilde{\nu}'_1(m, -1), \\ \iota_2^C &= \tilde{\nu}'_2(m, 1), \\ \iota_2^D &= \tilde{\nu}'_2(m, 3) + 3\tilde{\nu}'_2(m, -1).\end{aligned}\quad (128)$$

The pair $(\exp(-2\pi i \tilde{\nu}_1), \exp(-2\pi i \tilde{\nu}_2))$ gives $\mathbb{R}/\mathbb{Z}$ -valued generators [8].¹

The first normalized invariant is obtained by multiplying the first η invariant by the order of the first cyclic factor:

$$\begin{aligned}\tilde{\nu}'_1(m, q) &= \tilde{a}_m \tilde{\nu}_1(m, q) \\ &= \frac{\tilde{a}_m((2m^2 + m + 1)q^3 - (m + 3)q)}{48m} \pmod{\tilde{a}_m}.\end{aligned}\quad (129)$$

It is integer valued because

¹The $\tilde{\nu}_2$ in [8] is our $\tilde{\nu}_2$ multiplied by $\frac{m+1}{\gcd(m+1, 2)}$. Since this number is coprime to $\tilde{b}_m$, the multiplication does not change the cyclic generator.

$$\begin{aligned} \iota_1^C &= \frac{\tilde{a}_m(m^2 - 1)}{24m} \in \mathbb{Z}, \\ \iota_1^D &= \frac{\tilde{a}_m(m + 1)}{2m} \in \mathbb{Z}, \end{aligned} \quad (130)$$

as follows from the prime-primary cases in (125).

The second coordinate must be chosen so that it is integer valued in $\mathbb{Z}_{\tilde{b}_m}$. We therefore allow a change of basis by adding an integral multiple of $\tilde{\nu}_1$ to $\tilde{\nu}_2$:

$$\begin{aligned} \tilde{\nu}'_2(m, q) &= \tilde{b}_m(k\tilde{\nu}_1(m, q) + \tilde{\nu}_2(m, q)) \\ &= \tilde{b}_m \left(\frac{k((2m^2 + m + 1)q^3 - (m + 3)q)}{48m} \right. \\ &\quad \left. + \frac{\gcd(m + 1, 2)q}{2m} \right) \mod \tilde{b}_m. \end{aligned} \quad (131)$$

The integrality conditions are

$$\begin{aligned} \iota_2^C &= \tilde{b}_m \left(\frac{k(m^2 - 1)}{24m} + \frac{\gcd(m + 1, 2)}{2m} \right) \in \mathbb{Z}, \\ \iota_2^D &= \tilde{b}_m \frac{k(m + 1)}{2m} \in \mathbb{Z}. \end{aligned} \quad (132)$$

Using the possible values of $\tilde{b}_m$, a uniform way to satisfy the second condition is to write $k = 12l$. The first condition is then solved by choosing $l = \gcd(m + 1, 2)$; equivalently, $k = 12$ for even m and $k = 24$ for odd m . This gives the basis of integer-valued coordinates used below. Therefore

$$\begin{aligned} &(\tilde{\nu}'_1(m, q), \tilde{\nu}'_2(m, q)) \\ &= \left(\frac{\tilde{a}_m((2m^2 + m + 1)q^3 - (m + 3)q)}{48m} \mod \tilde{a}_m, \frac{\tilde{b}_m(m + 1) \gcd(m + 1, 2)(q^3 - q)}{4m} \mod \tilde{b}_m \right) \end{aligned} \quad (133)$$

in $\text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}) = \mathbb{Z}_{\tilde{a}_m} \times \mathbb{Z}_{\tilde{b}_m}$. Equivalently,

$$\begin{aligned} \tilde{I}_5^C &\mapsto \left(\frac{\tilde{a}_m(m^2 - 1)}{24m} \mod \tilde{a}_m, 0 \mod \tilde{b}_m \right), \\ \tilde{I}_5^D &\mapsto \left(\frac{\tilde{a}_m(m + 1)}{2m} \mod \tilde{a}_m, \frac{6\tilde{b}_m(m + 1) \gcd(m + 1, 2)}{m} \mod \tilde{b}_m \right). \end{aligned} \quad (134)$$

D. Geometric generators and the anomaly phase

Define the geometric generators

$$\begin{aligned} [M_3^5] &:= [L(m; 1, 1, 1)], \\ [M_4^5] &:= -[E \times L(m; 1)] + 12[L(m; 1, 1, 1)], \quad m \text{ even}, \\ [M_4^5] &:= -[K3 \times L(m; 1)] + 24[L(m; 1, 1, 1)], \quad m \text{ odd}. \end{aligned} \quad (135)$$

Then

$$\begin{aligned} \tilde{\nu}'_1(m, q) &= \tilde{a}_m \eta(M_3^5, \tilde{\rho}_q), \\ \tilde{\nu}'_2(m, q) &= \tilde{b}_m \eta(M_4^5, \tilde{\rho}_q). \end{aligned} \quad (136)$$

Indeed, the coefficient 12 for even m and 24 for odd m is exactly the change of basis described above. It removes the part of the second $\mathbb{R}/\mathbb{Z}$ -valued invariant which is not integral after multiplication by $\tilde{b}_m$.

The two classes M_3^5 and M_4^5 generate the two cyclic summands. For $q = 1$,

$$\tilde{\nu}'_1(m, 1) = \frac{\tilde{a}_m(m^2 - 1)}{24m} \quad (137)$$

is a unit modulo $\tilde{a}_m$ in each nontrivial prime-primary case, and hence is a generator of $\mathbb{Z}_{\tilde{a}_m}$; moreover $\tilde{\nu}'_2(m, 1) = 0$. For $q = 3$,

$$\tilde{\nu}'_2(m, 3) = \frac{6\tilde{b}_m(m + 1) \gcd(m + 1, 2)}{m} \quad (138)$$

is a unit modulo $\tilde{b}_m$ in each nontrivial second factor, and hence is a generator of $\mathbb{Z}_{\tilde{b}_m}$. Hence $[M_3^5]$ has exact order $\tilde{a}_m$, $[M_4^5]$ has exact order $\tilde{b}_m$, and the two classes form a basis for

$$\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}} \cong \mathbb{Z}_{\tilde{a}_m} \oplus \mathbb{Z}_{\tilde{b}_m}. \quad (139)$$

Consequently, any five-dimensional $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$ background has a unique coordinate expression,

$$[M^5] = k_3[M_3^5] + k_4[M_4^5], \quad k_3 \in \mathbb{Z}_{\tilde{a}_m}, \quad k_4 \in \mathbb{Z}_{\tilde{b}_m}. \quad (140)$$

As before, k_3 and k_4 are background-dependent topological coordinates, not numbers of fermions. The fermion-dependent data are the anomaly indices $\tilde{\nu}'_1(m, q)$ and $\tilde{\nu}'_2(m, q)$.

The anomalous phase on a general background is therefore

$$\begin{aligned} & \exp(-2\pi i \eta([M^5], \tilde{\rho}_q)) \\ &= \exp(-2\pi i k_3 \eta([M^5_3], \tilde{\rho}_q) - 2\pi i k_4 \eta([M^5_4], \tilde{\rho}_q)) \\ &= \exp\left(-\frac{2\pi i}{\tilde{a}_m} k_3 \tilde{\nu}'_1(m, q) - \frac{2\pi i}{\tilde{b}_m} k_4 \tilde{\nu}'_2(m, q)\right) \\ &= \exp\left(-\frac{2\pi i}{\tilde{a}_m} k_3 \frac{\tilde{a}_m((2m^2 + m + 1)q^3 - (m + 3)q)}{48m} \right. \\ & \quad \left. - \frac{2\pi i}{\tilde{b}_m} k_4 \frac{\tilde{b}_m(m + 1) \gcd(m + 1, 2)(q^3 - q)}{4m}\right). \end{aligned} \quad (141)$$

VI. DISCUSSIONS AND APPLICATIONS

In this section, we compare our results with the results in [8,9,11,12,15] and discuss the applications of our results in topological quantum dark matter and the family puzzle (generation problem) in the Standard Model [1–3] and possible generalizations beyond a single cyclic symmetry.

Our results are summarized in Table I.

For $\text{Spin} \times \mathbb{Z}_n$, the generator $X(n; 1, 1)$ gives the cubic invariant ν_1 , in agreement with the formulas of [8]. The product $-L(n; 1) \times \text{K3}$ gives the linear mixed gauge-gravitational invariant, using the product formula for η invariants as in [9,11,12,15]. The shifted generator

$$M_2^5 = -L(n; 1) \times \text{K3} - 6X(n; 1, 1) \quad (142)$$

is introduced so that the two independent cyclic coordinates in $\mathbb{Z}_{a_n} \oplus \mathbb{Z}_{b_n}$ are separated: M_1^5 detects the pure discrete gauge anomaly, while M_2^5 detects the independent mixed gauge-gravitational anomaly.

For $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}$, the generator $L(m; 1, 1, 1)$ gives $\tilde{\nu}_1$, again matching the lens-space formulas in [8]. The second $\mathbb{R}/\mathbb{Z}$ -valued invariant is represented by $-L(m; 1) \times \text{E}$ for even m , and by $-L(m; 1) \times \text{K3}$ for odd m , using [9,11,12,15]. Our normalization differs from the normalization of $\tilde{\nu}_2$ in [8] by multiplication by a unit in the relevant cyclic group. This gives a minimal integer coordinate for the second cyclic summand.

We now spell out the applications of the anomaly indices in Refs. [1–3].

A. Application I: Symmetry extension as anomaly matching

An anomaly that is nonzero for G can sometimes be trivialized after a symmetry extension [36]. Let

$$1 \longrightarrow K \longrightarrow G_{\text{Tot}} \xrightarrow{r} G \longrightarrow 1 \quad (143)$$

be an extension of the symmetry group. The map $r: G_{\text{Tot}} \rightarrow G$ induces a pullback map on anomaly groups,

$$r^*: \text{TP}_5(G) \longrightarrow \text{TP}_5(G_{\text{Tot}}). \quad (144)$$

If an anomaly class $\alpha \in \text{TP}_5(G)$ satisfies

$$r^* \alpha = 0 \quad \text{in} \quad \text{TP}_5(G_{\text{Tot}}), \quad (145)$$

then α lies in the kernel of the pullback. The physical interpretation used in Refs. [1,2] is that the anomalous G -symmetric Weyl-fermion sector can be replaced, at low energies, by a G -symmetric topologically ordered sector obtained by dynamically gauging the finite kernel K . The resulting K -gauge TQFT has no ordinary local particle operators replacing the fermion field itself; instead it contains extended topological excitations, but it carries exactly the same G 't Hooft anomaly as the original fermions. The anomaly becomes trivial only after it is viewed as an anomaly of the enlarged symmetry G_{Tot} .

Reference [2] makes this application precise in terms of the prime-primary decomposition of the finite anomaly group. For charge- $q = 1$ Weyl fermions, it gives the following useful symmetry-extension rules. For the finite Spin^c -type symmetry

$$G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2^{k+1}}^F, \quad (146)$$

the anomaly of 2^k charge-one Weyl fermions is trivialized by the minimal $K = \mathbb{Z}_4$ extension,

$$1 \longrightarrow \mathbb{Z}_4 \longrightarrow \text{Spin} \times \mathbb{Z}_{2^{k+2}} \longrightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2^{k+1}}^F \longrightarrow 1. \quad (147)$$

The smaller fermion-parity extension by $K = \mathbb{Z}_2^F$ does not trivialize this 2^k -fermion anomaly; instead, a $K = \mathbb{Z}_2^F$ extension trivializes the anomaly of 2^{k+1} charge-one Weyl fermions through

$$1 \longrightarrow \mathbb{Z}_2^F \longrightarrow \text{Spin} \times \mathbb{Z}_{2^{k+1}} \longrightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2^{k+1}}^F \longrightarrow 1. \quad (148)$$

For the ordinary spin symmetry $G = \text{Spin} \times \mathbb{Z}_{2^{k+1}}$, the anomaly of 2^k charge-one Weyl fermions is trivialized by

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin} \times \mathbb{Z}_{2^{k+2}} \longrightarrow \text{Spin} \times \mathbb{Z}_{2^{k+1}} \longrightarrow 1. \quad (149)$$

Thus the two-primary statement is not simply that one may add an arbitrary $\mathbb{Z}_2$ gauge theory: the number of Weyl fermions, the fermion-parity quotient, and the choice between $K = \mathbb{Z}_4$ and $K = \mathbb{Z}_2$ all matter.

Reference [2] also records the odd-primary rules. For $G = \text{Spin} \times \mathbb{Z}_{3^r s}$ with $2 \nmid s$ and $3 \nmid s$, the three-primary anomaly of charge-one Weyl fermions is trivialized for 3^r fermions by a minimal $K = \mathbb{Z}_3$ extension, while an s -primary component is trivialized for s charge-one Weyl fermions by a trivial $K = \mathbb{Z}_1$ extension. The three-primary

mechanisms apply to the group-cohomology subclass of the finite anomaly, hence admit bosonic analogs. In contrast, the two-primary examples above are intrinsically fermionic because the role of fermion parity in the spacetime-internal symmetry is essential.

The practical role of the present paper is to give the anomaly class α in explicit integer coordinates. For a spectrum of Weyl fermions with charges q_i , one simply adds the coordinates

$$\left(\sum_i \nu'_1(n, q_i), \sum_i \nu'_2(n, q_i) \right) \in \mathbb{Z}_{a_n} \oplus \mathbb{Z}_{b_n} \quad (150)$$

for $\text{Spin} \times \mathbb{Z}_n$, or

$$\left(\sum_i \tilde{\nu}'_1(m, q_i), \sum_i \tilde{\nu}'_2(m, q_i) \right) \in \mathbb{Z}_{\tilde{a}_m} \oplus \mathbb{Z}_{\tilde{b}_m} \quad (151)$$

for $\text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F$. Vanishing from both components is the anomaly-cancellation condition for gauging the finite symmetry. A nonzero pair is the anomaly that must either be matched by infrared degrees of freedom or be trivialized by a suitable symmetry extension. This is why the normalization of the two cyclic coordinates is important: the proposed K -gauge topological sector must match these finite-group coordinates exactly.

B. Application II: Topologically ordered quantum dark matter

References [1,2] apply this mechanism to Weyl fermions with the fermionic discrete symmetries studied here,

$$G = \text{Spin} \times \mathbb{Z}_n \quad \text{or} \quad G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F. \quad (152)$$

The symmetry-extension problem is to find the finite kernel K and the total symmetry G_{Tot} for which the nonperturbative global anomaly computed from the Atiyah-Patodi-Singer η invariant becomes trivial after pullback. When this happens, the anomalous Weyl fermion can be replaced, within the same anomaly class, by a four-dimensional anomalous G -symmetric discrete K -gauge TQFT. The general results of Ref. [2], summarized above, determine which kernel is appropriate: for $G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m}^F$ and $m = 2^p \cdot 3^r \cdot s$ with $2 \nmid s$ and $3 \nmid s$, $K = \mathbb{Z}_4$ (or $K = \mathbb{Z}_2^F$) is minimal for the two-primary branch when the number of charge-one Weyl fermions is 2^p (or 2^{p+1}), $K = \mathbb{Z}_3$ is minimal for the three-primary branch when the number of charge-one Weyl fermions is 3^r , and $K = \mathbb{Z}_1$ is trivial for the s -primary branch when the number of charge-one Weyl fermions is s .

In the Standard Model application, one considers the theory with 15 Weyl fermions per family, omitting the 16th fermion, the sterile right-handed neutrino ν_R . The omission of ν_R leaves a nonperturbative mixed gauge-gravitational

global anomaly involving baryon and lepton number. Electroweak $\text{SU}(2)$ instantons preserve $\text{U}(1)_{\mathbf{B}-\mathbf{L}}$ but break $\text{U}(1)_{\mathbf{B}+\mathbf{L}}$ to the fermionic discrete subgroup $\mathbb{Z}_{2N_f, \mathbf{B}+\mathbf{L}}^F$. The symmetry relevant to the family-puzzle construction is therefore

$$G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2N_f, \mathbf{B}+\mathbf{L}}^F. \quad (153)$$

For $N_f = 3$, this becomes

$$G = \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{6, \mathbf{B}+\mathbf{L}}^F, \quad (154)$$

and the anomaly of the three missing sterile neutrinos lies in the three-primary branch of the finite anomaly classification. Accordingly, Ref. [2] identifies a symmetry-extension TQFT with kernel $K = \mathbb{Z}_3$ that can reproduce the anomaly contribution of the three missing ν_R fields in the infrared. The omission of ν_R also produces a closely related contribution to the $(\mathbf{B} - \mathbf{L})$ anomaly, with the opposite lepton-number sign; however, $\text{U}(1)_{\mathbf{B}-\mathbf{L}}$ is not broken to a finite subgroup by $\text{SU}(2)$ instantons and is therefore not the discrete symmetry used in the family-puzzle construction considered below.

The proposal in Refs. [1–3] is that the missing anomaly can be carried instead by a symmetry-preserving K -gauge fermionic TQFT, where the minimal finite gauge group is determined by the relevant torsion branch of the finite anomaly and is not assumed at the outset to be the color center. This is the sense in which the topological sector is “topologically ordered quantum dark matter”: it is a dark sector constrained not by an ordinary perturbative gauge anomaly alone, but by the same finite nonperturbative anomaly coordinates computed in this paper, and it does not require an Anderson-Higgs symmetry-breaking mechanism.

C. Application III: Anomaly-trivializing extension and color-center matching

Reference [3] sharpens the preceding application by separating which parts of a $\mathbb{Z}_n$ anomaly can be killed by finite symmetry extension. The $H^5(\mathbb{Z}_n, \text{U}(1)) \simeq \mathbb{Z}_n$ group-cohomology component, represented by the 5D cocycle $A_{\mathbb{Z}_n}(\beta_{(n,n)} A_{\mathbb{Z}_n})(\beta_{(n,n)} A_{\mathbb{Z}_n})$, can be trivialized by

$$1 \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_{n^2} \longrightarrow \mathbb{Z}_n \longrightarrow 1, \quad (155)$$

whereas a generic beyond-group-cohomology term involving the Pontryagin class, written there schematically as $A_{\mathbb{Z}_n} p_1$, cannot be trivialized by a finite group extension except in the special cases $n = 2$ and $n = 3$.

For $n = 3$, one has

$$\text{TP}_5(\text{Spin} \times \mathbb{Z}_3) \cong \mathbb{Z}_9, \quad a_3 = 9, \quad b_3 = 1, \quad (156)$$

so the group-cohomology and Pontryagin-sensitive contributions are combined into a single nonsplit $\mathbb{Z}_9$ invariant rather than appearing as two independent cyclic coordinates. Indeed, the coordinates derived in this paper reduce to

$$(\nu'_1(3, q), \nu'_2(3, q)) = (q^3 \bmod 9, 0). \quad (157)$$

Since the mod-three Pontryagin contribution $A_{\mathbb{Z}_3} p_1$ vanishes, the anomaly of three charge-one Weyl fermions is the order-3 element $3 \in \mathbb{Z}_9$. It belongs to the group-cohomology subgroup and is trivialized by

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow \mathbb{Z}_9 \longrightarrow \mathbb{Z}_3 \longrightarrow 1. \quad (158)$$

For the generalized Standard Model with N_c colors and N_f families, Ref. [3] first determines the minimal cyclic gauge group $K = \mathbb{Z}_N$ of the anomalous TQFT that can replace the N_f missing sterile right-handed neutrinos. The anomaly-trivializing extension is

$$1 \longrightarrow \mathbb{Z}_N \longrightarrow \mathbb{Z}_{NN_f} \longrightarrow \mathbb{Z}_{N_f} \longrightarrow 1, \quad (159)$$

or, including spacetime fermion parity,

$$1 \longrightarrow \mathbb{Z}_N \longrightarrow \text{Spin} \times \mathbb{Z}_{NN_f} \longrightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2N_f, \mathbf{B}+\mathbf{L}}^F \longrightarrow 1. \quad (160)$$

Here N is determined by the anomaly and by N_f ; it is not assumed *a priori* to be the color number N_c . The minimal extension order is

$$\begin{aligned} N = 1, & \quad N_f \geq 1, & 2 \nmid N_f, & \quad 3 \nmid N_f, \\ N = 3, & \quad N_f \geq 3, & 2 \nmid N_f, & \quad 3 \mid N_f, \\ N = 4, & \quad N_f \geq 2, & 2 \mid N_f, & \quad 3 \nmid N_f, \\ N = 12, & \quad N_f \geq 6, & 2 \mid N_f, & \quad 3 \mid N_f. \end{aligned} \quad (161)$$

Thus the finite anomaly of N_f charge-one fermions determines an anomaly-trivializing $\mathbb{Z}_N$ gauge sector, not directly a color-center sector. For the four-dimensional Standard Model case with $n = N_f = 3$, the anomaly is represented by a five-dimensional invertible theory, and the construction gives a $\text{Spin} \times \mathbb{Z}_3$ -symmetric four-dimensional $\mathbb{Z}_3$ -gauge TQFT that can reproduce the anomaly contribution of the three missing sterile right-handed neutrinos while preserving the discrete $\mathbb{Z}_{6, \mathbf{B}+\mathbf{L}}^F$ symmetry.

The color-center interpretation is an additional matching condition. If the TQFT gauge group is to coincide with the center of the color gauge group, then one imposes

$$\mathbb{Z}_N = \mathbb{Z}_{N_c} = Z(\text{SU}(N_c)). \quad (162)$$

In the faithful quark/baryon notation this color-matched symmetry extension is

$$\begin{aligned} 1 &\longrightarrow \mathbb{Z}_{N_c} \longrightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2N_c N_f, \mathbf{Q}+N_c \mathbf{L}}^F \\ &\longrightarrow \text{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2N_f, \mathbf{B}+\mathbf{L}}^F \longrightarrow 1. \end{aligned} \quad (163)$$

The clean Standard-Model-like identification occurs in the odd color-matched branch, where N_f and N_c are odd and $N = N_c$; in this branch the baryons are fermions and the $\mathbb{Z}_N$ gauge TQFT can be interpreted as the color center. For even N_f , the minimal anomaly-trivializing extension contains a two-primary part, $\mathbb{Z}_4$ or $\mathbb{Z}_{12}$. One may choose an even $N_c = N$ to match finite orders, but this is an even-color branch rather than the baryon-fermion branch. Therefore, only after imposing the extra odd color-matching condition does the anomaly-trivializing extension coincide with the Standard-Model-like color symmetry extension. In this color-matched situation the unique minimal nontrivial solution is

$$N = N_c = N_f = 3. \quad (164)$$

Thus the anomaly-index formulas derived in this paper supply the arithmetic input for the conclusion: the anomaly first selects the TQFT gauge group $K = \mathbb{Z}_N$, and the familiar three-color interpretation follows only after the additional color-center matching $N = N_c$ is imposed.

D. Extensions to more general finite groups

We conclude with some comments on possible generalizations beyond a single cyclic symmetry. Let

$$A = \prod_{a=1}^r \mathbb{Z}_{n_a} \quad (165)$$

be a finite Abelian group, and let $T = \text{U}(1)^r$. The coordinatewise embedding $\iota: A \hookrightarrow T$ induces the restriction map

$$\iota^*: \text{TP}_5(\text{Spin} \times T) \longrightarrow \text{TP}_5(\text{Spin} \times A). \quad (166)$$

Every finite-dimensional complex representation of A decomposes into one-dimensional characters, and every such character extends to a character of T . Consequently, there is no representation-theoretic obstruction to applying the continuous-to-discrete construction to Weyl fermions with a finite Abelian symmetry.

More explicitly, let $q_{i,a} \in \mathbb{Z}$ be integer lifts of the discrete charges of the i th Weyl fermion under the factors $\mathbb{Z}_{n_a}$. Writing c_1^a for the first Chern class of the a th $\text{U}(1)$ factor, the continuous six-dimensional anomaly polynomial is

$$I_6 = \sum_i \left[\frac{1}{6} \left(\sum_{a=1}^r q_{i,a} c_1^a \right)^3 - \frac{p_1}{24} \left(\sum_{a=1}^r q_{i,a} c_1^a \right) \right]. \quad (167)$$

The terms involving more than one index a encode mixed gauge anomalies between different cyclic factors, while the

terms linear in c_1^a give the mixed gauge-gravitational anomalies. Restricting this anomaly class along ι therefore gives the anomaly of an arbitrary collection of Weyl fermions transforming under A .

This observation concerns anomalies generated by Weyl fermions. It does not imply that the restriction map ι^* is surjective onto the full group $\text{TP}_5(\text{Spin} \times A)$ of invertible field theories. The bordism group $\Omega_5^{\text{Spin} \times A}$ can contain additional mixed torsion classes that are not generated by free-fermion representations. Its complete computation may involve nontrivial differentials and extension problems in the Atiyah-Hirzebruch or Adams spectral sequence and generally requires more geometric representatives than the two lens-space-type generators used for a single cyclic group. Detecting all such classes requires a direct bordism calculation and the evaluation of η invariants on a sufficiently large set of generators.

The fermionic central product can be treated similarly. Suppose that A contains a distinguished central element z of order 2, and choose the embedding $\iota: A \hookrightarrow T$ so that $\iota(z)$ is the order-2 element of T identified with fermion parity. One then obtains

$$\iota^*: \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} T) \longrightarrow \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} A). \quad (168)$$

The relevant fermion representations are those on which z acts as -1 .

The same general principle applies to a non-Abelian finite group A . Every finite group admits a faithful finite-dimensional unitary representation and hence an embedding,

$$\iota: A \hookrightarrow H, \quad (169)$$

into a compact Lie group H , for example $H = \text{U}(N)$. Although the group inclusion runs from A to H , restriction of anomaly theories is contravariant and gives

$$\iota^*: \text{TP}_5(\text{Spin} \times H) \longrightarrow \text{TP}_5(\text{Spin} \times A). \quad (170)$$

If A has a distinguished central involution z and $\iota(z) = z_H$ is a central involution of H , the corresponding map is

$$\iota^*: \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} H) \longrightarrow \text{TP}_5(\text{Spin} \times_{\mathbb{Z}_2^F} A). \quad (171)$$

Unlike the Abelian case, a representation of A need not extend to a representation of a fixed continuous group H . For any specified finite set of representations one may choose a sufficiently large continuous parent, but there is generally no canonical choice of H . Moreover, the restriction from a chosen H need not detect every element of $\Omega_5^{\text{Spin} \times A}$; finite non-Abelian groups can possess intrinsically discrete anomaly classes that are not obtained from that continuous parent. A complete classification therefore again requires a direct bordism and η -invariant analysis.

Finally, restriction of the background symmetry requires only the inclusion $A \hookrightarrow H$ and does not assume a physical symmetry-breaking mechanism. Realizing $H \rightarrow A$ dynamically additionally requires a Higgs representation and potential whose unbroken stabilizer is A , which is a model-dependent question. These Abelian and non-Abelian generalizations provide natural directions for future work.

E. Conclusion

The computations above give a direct bridge from perturbative $\text{U}(1)$ anomaly data to finite-symmetry global anomaly data. The final answers are expressed in integer coordinates of the finite anomaly groups, while the derivation keeps track of the underlying η invariants, spin or Spin^c -type structures, and geometric manifold generators. This makes the formulas suitable both for comparison with bordism classifications and for practical anomaly-cancellation tests in model building.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [1] Juven Wang, Topological quantum dark matter via global anomaly cancellation, [arXiv:2502.21319](#).
 - [2] Zheyang Wan and Juven Wang, Anomalous $(3+1)\text{D}$ fermionic topological quantum field theories via symmetry extension, [arXiv:2512.25038](#).

- [3] Zheyang Wan, Juven Wang, and Shing-Tung Yau, Fermion families and Pontryagin class: Topological field theory via colour symmetry extension, [arXiv:2605.26202](#).
- [4] Luis E. Ibanez and Graham G. Ross, Discrete gauge symmetry anomalies, *Phys. Lett. B* **260**, 291 (1991).

- [5] John Preskill, Sandip P. Trivedi, Frank Wilczek, and Mark B. Wise, Cosmology and broken discrete symmetry, *Nucl. Phys.* **B363**, 207 (1991).
- [6] Tom Banks and Michael Dine, Note on discrete gauge anomalies, *Phys. Rev. D* **45**, 1424 (1992).
- [7] Luis E. Ibanez, More about discrete gauge anomalies, *Nucl. Phys.* **B398**, 301 (1993).
- [8] Chang-Tse Hsieh, Discrete gauge anomalies revisited, [arXiv:1808.02881](#).
- [9] Iñaki García-Etxebarria and Miguel Montero, Dai-Freed anomalies in particle physics, *J. High Energy Phys.* **08** (2019) 003.
- [10] P. B. Gilkey, The eta invariant and the K -theory of odd-dimensional spherical space forms, *Inventiones Math.* **76**, 421 (1984).
- [11] Anthony Bahri and Peter Gilkey, The eta invariant, Pin^c bordism, and equivariant Spin^c bordism for cyclic 2-groups, *Pac. J. Math.* **128**, 1 (1987).
- [12] Peter B. Gilkey and Boris Botvinnik, The eta invariant and the equivariant spin bordism of spherical space form 2 groups, in *New developments in differential geometry (Debrecen, 1994)*, Math. Appl. (Kluwer Academic Publication, Dordrecht, 1996), Vol. 350, pp. 213–223.
- [13] Boris Botvinnik, Peter Gilkey, and Stephan Stolz, The Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology, *J. Diff. Geom.* **45**, 374 (1997).
- [14] Egidio Barrera-Yanez and Peter B. Gilkey, The eta invariant and the connective K -theory of the classifying space for cyclic 2 groups, *Ann. Glob. Anal. Geom.* **17**, 289 (1999).
- [15] Peter B. Gilkey, *The Geometry of Spherical Space Form Groups*, 2nd ed., Series in Pure Mathematics (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018), Vol. 28, pp. xiii+493.
- [16] Daniel S. Freed and Michael J. Hopkins, Reflection positivity and invertible topological phases, *Geom. Topol.* **25**, 1165 (2021).
- [17] Csaba Csaki and Hitoshi Murayama, Discrete anomaly matching, *Nucl. Phys.* **B515**, 114 (1998).
- [18] Takeshi Araki, Tatsuo Kobayashi, Jisuke Kubo, Saul Ramos-Sanchez, Michael Ratz, and Patrick K. S. Vaudrevange, (Non-)Abelian discrete anomalies, *Nucl. Phys.* **B805**, 124 (2008).
- [19] Juven C. Wang, Zheng-Cheng Gu, and Xiao-Gang Wen, Field theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology and beyond, *Phys. Rev. Lett.* **114**, 031601 (2015).
- [20] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang, Fermionic symmetry protected topological phases and cobordisms, *J. High Energy Phys.* **12** (2015) 052.
- [21] Kazuya Yonekura, Dai-Freed theorem and topological phases of matter, *J. High Energy Phys.* **09** (2016) 022.
- [22] Meng Guo, Kantaro Ohmori, Pavel Putrov, Zheyang Wan, and Juven Wang, Fermionic finite-group Gauge theories and interacting symmetric/crystalline orders via cobordisms, *Commun. Math. Phys.* **376**, 1073 (2020).
- [23] Edward Witten and Kazuya Yonekura, Anomaly Inflow and the η -invariant, in *The Shoucheng Zhang Memorial Workshop* (World Scientific, 2021).
- [24] Clay Córdova and Kantaro Ohmori, Anomaly constraints on gapped phases with discrete chiral symmetry, *Phys. Rev. D* **102**, 025011 (2020).
- [25] Meng Cheng, Juven Wang, and Xinping Yang, $(3+1)$ D boundary topological order of $(4+1)$ D fermionic SPT state, *Phys. Rev. B* **113**, 125105 (2026).
- [26] Pavel Putrov and Juven Wang, Categorical symmetry of the standard model from gravitational anomaly, *Phys. Rev. D* **110**, 125028 (2024).
- [27] Juven Wang, Topological leptogenesis, *Phys. Rev. Res.* **8**, L022024 (2026).
- [28] Luis Alvarez-Gaume and Edward Witten, Gravitational anomalies, *Nucl. Phys.* **B234**, 269 (1984).
- [29] Luis Alvarez-Gaume and Paul H. Ginsparg, The structure of gauge and gravitational anomalies, *Ann. Phys. (N.Y.)* **161**, 423 (1985); **171**, 233(E) (1986).
- [30] Xianzhe Dai and Daniel S. Freed, η -invariants and determinant lines, *J. Math. Phys. (N.Y.)* **35**, 5155 (1994).
- [31] Edward Witten, Fermion path integrals and topological phases, *Rev. Mod. Phys.* **88**, 035001 (2016).
- [32] Edward Witten, The “parity” anomaly on an unorientable manifold, *Phys. Rev. B* **94**, 195150 (2016).
- [33] H. Blaine Lawson and Marie-Louise Michelsohn, *Spin Geometry*, Princeton Mathematical Series (Princeton University Press, Princeton, NJ, 1989), Vol. 38.
- [34] John W. Milnor and James D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies (Princeton University Press, Princeton, NJ, 1974), Vol. 76.
- [35] A. Franc, Spin structures and Killing spinors on lens spaces, *J. Geom. Phys.* **4**, 277 (1987).
- [36] Juven Wang, Xiao-Gang Wen, and Edward Witten, Symmetric gapped interfaces of SPT and set states: Systematic constructions, *Phys. Rev. X* **8** (2018).