

Explicit connections between Krylov and Nielsen complexity

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We establish a direct correspondence between Krylov and Nielsen complexity by choosing the Krylov basis to be part of the elementary gate set of Nielsen geometry and selecting a Nielsen complexity metric compatible with the Krylov metric. Up to normalization, the Krylov complexity of a Hermitian operator then equals the length squared of a straight-line trajectory on the manifold of unitaries that connects the identity operator with a precursor operator. The corresponding length provides an upper bound on Nielsen complexity that saturates whenever the straight line is a minimal geodesic. While for general systems we can only establish saturation in the limit of small precursors, we provide evidence that in broad classes of models and for suitable initial operators there is a precise correspondence between Krylov complexity and (the square of) Nielsen complexity for a finite range of precursors.

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I. INTRODUCTION

Quantum complexity has become a central concept at the intersection of quantum information, condensed-matter theory, and high-energy physics. One of the most influential approaches is *Nielsen complexity* [1–3], which reformulates circuit complexity as a geometric problem. In this framework, a circuit corresponds to a curve on the unitary group manifold, and the complexity is given by the length of the minimal geodesic connecting the chosen unitary operator with the identity operator [4–14]. A complementary measure, *Krylov complexity* [15], quantifies operator growth under Heisenberg evolution. Starting from an initial operator, one constructs an orthonormal Krylov basis via the Lanczos algorithm, thereby mapping the dynamics to a one-dimensional hopping problem along the Krylov chain. Krylov complexity directly measures the spread of the initial operator in Krylov space and provides a quantitative probe of operator growth, scrambling, and chaos [16–27]. For comprehensive reviews, see [28–30].

Despite their different origins, Nielsen complexity and Krylov complexity both have deep ties to holography. Nielsen’s geometric framework has motivated holographic

proposals linking circuit complexity to geometric quantities that capture the growth of the black hole interior [31–47]. Meanwhile, in models of two-dimensional gravity, Krylov state complexity provides a concrete realization of the proposed correspondence between complexity and size of black hole interiors [48–56]. Although the possible relation between Nielsen complexity and Krylov complexity has attracted significant recent interest [57–60], a direct and general correspondence remains elusive. Reference [59] showed that Nielsen complexity and Krylov state complexity can be expressed as distinct functions of a common q -matrix, thereby revealing a formal similarity between their mathematical structures. However, this relation is neither an equality nor an inequality. In the present paper, we show that the Krylov complexity of an operator is explicitly related to the Nielsen complexity of an associated precursor operator via an inequality that saturates in broad classes of models.

II. MAPPING KRYLOV TO NIELSEN COMPLEXITY

Krylov complexity characterizes the dynamical growth of a *Hermitian* operator under time evolution. We begin with a normalized initial operator $\mathcal{O}(t=0) \equiv \mathcal{O}_0$, defined with respect to a chosen inner product, such as the infinite-temperature Wightman inner product,

$$(\mathcal{O}|\mathcal{O}') := \frac{1}{\text{Tr}[1]} \text{Tr}[\mathcal{O}^\dagger \mathcal{O}']. \quad (1)$$

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The Heisenberg evolution of the operator under a time-independent Hamiltonian H , $\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$, is naturally formulated in operator space using the Liouvillian superoperator $\mathcal{L} \equiv [H, \cdot]$. Successive applications of $\mathcal{L}$ to the initial operator generate the Krylov subspace $\mathcal{K} = \text{span}\{\mathcal{O}_0, i\mathcal{L}\mathcal{O}_0, (i\mathcal{L})^2\mathcal{O}_0, \dots\}$. An orthonormal basis $\{|\mathcal{O}_n\rangle\}$ for this subspace can be constructed recursively via the Lanczos algorithm, i.e., by applying the Gram-Schmidt procedure to the Krylov sequence $\mathcal{K}$. Using this orthonormal and Hermitian basis, the time-evolved operator can be expanded as

$$|\mathcal{O}(t)\rangle \equiv e^{i\mathcal{L}t}|\mathcal{O}(0)\rangle = \sum_n \varphi_n(t)|\mathcal{O}_n\rangle, \quad (2)$$

with the initial condition $\varphi_n(0) = \delta_{n0}$ [61]. Krylov complexity is thus defined as [15,22,62]

$$C_K(\mathcal{O}(t)) := \sum_{m,n} f_{mn} \varphi_m(t) \varphi_n(t), \quad (3)$$

where the positive matrix f_{nm} , referred to as the Krylov metric, sets the weights for operator growth. A standard choice is $f_{mn} = f(n)\delta_{mn}$ with $f(n) = n$ [63], yielding a natural measure of spreading along the Krylov chain. Another interesting choice, consistent with the conjecture that the growth rate of Nielsen complexity should be proportional to operator size [64,65], is the Krylov metric $f_{mn} = n^2\delta_{mn}$, as discussed in part A of the Supplemental Material [66].

In contrast, Nielsen complexity is defined for unitary transformations, which mimic sequences of elementary unitary gates. The central idea of Nielsen's geometric formalism is to interpret the construction of a unitary operator $U(s)$ as a smooth trajectory on the manifold of unitaries, with control Hamiltonian (Hermitian velocity)

$$H_c(s) = i\partial_s U U^{-1}, \quad H_c(s) \equiv Y^I(s) T_I. \quad (4)$$

This Hamiltonian is expanded in a basis of Hermitian generators $\{T_I\}$ representing the elementary gates, with control functions $Y^I(s)$ playing the role of tangent vectors along the trajectory. Taking the initial point as the identity, $U(0) = \mathbb{1}$, and the end point as the target unitary, $U(1) = U_T$, Nielsen complexity is then defined as the minimal length (also known as minimal “cost”) among all admissible paths,

$$C_N(U_T) = \min_{Y^I(s)} \int_0^1 \sqrt{\sum_{I,J} G_{IJ} Y^I(s) Y^J(s)} ds. \quad (5)$$

Here, the positive, right-invariant “complexity metric” (or “cost function”) G_{IJ} encodes the weights assigned to each gate, also known as “penalty factors.” As a result,

minimizing the cost is equivalent to solving the geodesic equation (or Euler-Arnold equation).

A natural bridge between these two notions of complexity can be established by focusing on a special class of unitaries known as *precursors* [67], defined as

$$U_T = e^{iHt} e^{-iz\mathcal{O}(0)} e^{-iHt} = e^{-iz\mathcal{O}(t)}, \quad (6)$$

where $\mathcal{O}(t)$ is normalized using (1) and the positive parameter z is the norm of the Hermitian generator $z\mathcal{O}(t)$ of U_T . Equation (6) links the time-evolved Hermitian operator $\mathcal{O}(t)$ that enters Krylov complexity to U_T in Nielsen's formalism. As we show below, the parameter z will play a central role in relating the two notions of complexity. It is now straightforward to connect the identity to the target unitary via the straight-line trajectory $U(s) = e^{-isH_{\text{con}}}$, which corresponds to a constant control Hamiltonian,

$$H_c(s) \rightarrow H_{\text{con}} \equiv z\mathcal{O}(t) = z \sum_n \varphi_n(t) \mathcal{O}_n. \quad (7)$$

The next step is to choose the basis of Hermitian generators $\{T_I\}$ used in Nielsen's construction to contain the Krylov basis $\{|\mathcal{O}_n\rangle\}$, i.e., $\{\mathcal{O}_n\} \subseteq \{T_I\}$. A simple correspondence is then realized by identifying

$$T_n \sim \mathcal{O}_n, \quad Y^n \sim z\varphi_n. \quad (8)$$

Because the path $U(s)$ is a straight line with constant controls, its cost can be easily read off once the complexity metric is specified:

$$\text{Cost}(\mathcal{O}(t)) = z \sqrt{\sum_{m,n} G_{mn} \varphi_m \varphi_n} \geq C_N(U_T), \quad (9)$$

which provides an upper bound on the Nielsen complexity of the precursor U_T . Choosing G_{IJ} such that $G_{mn} = f_{mn}$ for the directions corresponding to the Krylov basis, the two measures obey

$$C_N(e^{-iz\mathcal{O}(t)}) \leq z \sqrt{C_K(\mathcal{O}(t))}. \quad (10)$$

In fact, for targets close to the identity ($z \ll 1$) [68–70], the leading-order expansion yields [71]

$$C_N(e^{-iz\mathcal{O}(t)}) = z \sqrt{C_K(\mathcal{O}(t))} + O(z^3), \quad (11)$$

explicitly linking the two notions of complexity for “small” precursor operators in general systems. Geometrically, this reflects the fact that any Riemannian manifold is locally flat, so the minimal geodesic connecting two sufficiently nearby points is well approximated by the straight-line trajectory. Since the Krylov basis minimizes the cost function (3) among all bases with an increasing sequence

of penalty factors [22,26], it follows immediately from Eq. (11) that the Krylov basis also minimizes the Nielsen complexity of small precursors. This makes the inclusion (8) of the Krylov basis in the elementary gate set natural.

The remainder of this work addresses the question of whether the saturation of (10) extends to finite z , i.e., beyond the infinitesimal precursor limit. In other words, is the straight-line trajectory the shortest path from the identity operator to the target unitary U_T ? While for generic systems a trajectory generated by a constant control Hamiltonian H_{con} is not a geodesic, we will find that, for appropriate initial operators, the geodesic equation is satisfied for a large class of spin chains as well as for any physical fermionic system. For the inequality (10) to be saturated,

$$C_N(e^{-iz\mathcal{O}(t)}) = z\sqrt{C_K(\mathcal{O}(t))}, \quad (12)$$

it should moreover be checked that the geodesic is minimal.

To make the connection between the formalisms of Krylov and Nielsen complexities more explicit, we identify the inner product in both frameworks with Eq. (1) and choose the gate basis $\{T_I\}$ so that $(T_I, T_J) = \delta_{IJ}$. The geodesic equation can then be recast in the compact form [3]

$$\dot{H}_c + i\mathbf{G}^{-1}([H_c, \mathbf{G}(H_c)]) = 0, \quad (13)$$

where $\mathbf{G}(H_c) = \sum_{I,J} G_{IJ} Y^J T_I$ [72]. It follows immediately that a constant control Hamiltonian, $\dot{H}_c = 0$, constitutes a geodesic whenever $\mathbf{G}^{-1}([H_{\text{con}}, \mathbf{G}(H_{\text{con}})]) = 0$. We will now argue that this condition is satisfied for broad classes of models and suitable initial operators.

III. MODELS WITH STRAIGHT-LINE GEODESICS

We begin with Majorana fermions obeying the Clifford algebra

$$\{\psi_i, \psi_j\} = \delta_{ij}, \quad i, j = 1, \dots, N, \quad (14)$$

where $\{\cdot, \cdot\}$ denotes the anticommutator and N is even. The space of linear operators acting on the $2^{N/2}$ -dimensional Hilbert space, with operator basis $\{X_\alpha\} \equiv \{\psi_{i_1} \cdots \psi_{i_k}\}$ and $\alpha = 1, \dots, 2^N$, naturally decomposes into two orthogonal sectors. To see this, introduce the chirality operator

$$\psi^5 = (2i)^{N/2} \psi_1 \psi_2 \cdots \psi_N, \quad (15)$$

which satisfies $(\psi^5)^2 = \mathbf{1}$ and $\{\psi^5, \psi_i\} = 0$. It then follows that the trace of a Majorana string vanishes for odd k , namely,

$$\text{Tr}(\psi_{i_1} \cdots \psi_{i_k}) = 0 \quad (k \text{ odd}). \quad (16)$$

Accordingly, the operator space can be decomposed into two sectors, X_{odd} and X_{even} , with schematic algebraic structure

$$[X_{\text{odd}}, X_{\text{odd}}] \sim X_{\text{even}}, \quad [X_{\text{odd}}, X_{\text{even}}] \sim X_{\text{odd}}. \quad (17)$$

Next, consider a general physical fermionic Hamiltonian. Due to the parity superselection rule [73], such a Hamiltonian necessarily belongs to the even sector, as the appearance of any term with an odd number of fermions would allow for faster than light communication across spatially separate regions. Therefore, if we choose an initial operator $\mathcal{O}_0$ belonging to the odd sector, its time-evolution is fully restricted to this sector due to Eq. (17). Adopting a block-diagonal ansatz for the complexity metric in these sectors,

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_{\text{odd}} & 0 \\ 0 & \mathbf{G}_{\text{even}} \end{pmatrix}, \quad (18)$$

we now examine whether the straight-line trajectory $H_c = H_{\text{con}} \in X_{\text{odd}}$ satisfies the geodesic equation (13). We observe that a simple solution arises if the penalty factor is homogeneous across Krylov subdirections: $\mathbf{G}(H_{\text{con}}) = \mathbf{G}_{\text{odd}}(H_{\text{con}}) \propto H_{\text{con}}$. For more general penalty factors, however, the commutator $[H_{\text{con}}, \mathbf{G}(H_{\text{con}})]$ does not vanish. Nevertheless, owing to the algebraic fact that $[H_{\text{con}}, \mathbf{G}(H_{\text{con}})] \in X_{\text{even}}$, substituting the constant H_{con} into the geodesic equation yields:

$$\mathbf{G}^{-1}([H_c, \mathbf{G}(H_c)]) \sim (G_{\text{even}})^{-1} G_{\text{odd}}. \quad (19)$$

Here, the subspace indices and the structure constants of the unitary group are suppressed for simplicity. One then finds that the constant solution $H_c = H_{\text{con}}$ asymptotically satisfies the geodesic equation in the limit $(G_{\text{even}})^{-1} G_{\text{odd}} \rightarrow 0$ (equivalently, $G_{\text{even}} \rightarrow \infty$ for fixed G_{odd}). This indicates that the straight-line trajectory within the Krylov subspace becomes a geodesic in the full operator space whenever the even sector is heavily penalized, i.e., when its contribution is effectively suppressed by a large penalty factor. With these penalty factors and an odd initial operator, the straight line therefore satisfies the geodesic equation in any physical fermionic Hamiltonian. We note that while we need to start with an odd seed operator $\mathcal{O}_0 \in X_{\text{odd}}$ for our construction to work, physical measurements of odd operators are themselves forbidden by superselection. However, the Krylov complexity of $\mathcal{O}_0$ is completely controlled by its autocorrelation function. This, in turn, depends solely on the even parity combination $\mathcal{O}_0(t)\mathcal{O}_0$ and is therefore a well-defined physical quantity.

Another class of systems that display the same algebraic structure is given by a subset of spin-1/2 chains. First, note that any spin chain model is equivalent to some fermionic model via the Jordan-Wigner transform. However, the superselection rule we discussed does not apply here, since

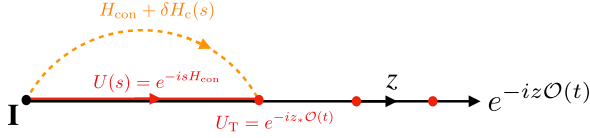


FIG. 1. Conjugate points (red dot) along the geodesic (black straight-line) are defined as the points where the perturbation satisfying the geodesic deviation equation vanishes.

the condition of locality is not required for these auxiliary (nonphysical) fermions. To arrive at the algebra (17), we instead start by considering a global spin-flip operator:

$$\mathcal{P}_x = \prod_{j=1}^N S_j^x. \quad (20)$$

We use the x direction for concreteness, but what follows can also be done for the other directions. With the usual spin algebra, it can be shown that

$$\mathcal{P}_x S_j^x = S_j^x \mathcal{P}_x, \quad \mathcal{P}_x S_j^y = -S_j^y \mathcal{P}_x, \quad \mathcal{P}_x S_j^z = -S_j^z \mathcal{P}_x. \quad (21)$$

With these relations, it is easy to see that given any Pauli string, the operator $\mathcal{P}_x$ counts the total number of spin- y and spin- z operators in the string, since it will commute with it when this number is even, and anticommute when it is odd. With this assignment, the operator space can once again be decomposed into an even and an odd sector, which follow the algebra of Eq. (17). As before, we need our Hamiltonian to lie in the even sector and the initial operator to lie in the odd sector. Some examples of common spin chain Hamiltonians belonging to this class are the longitudinal or the transverse field Ising chain, as well as the Heisenberg chain.

Having shown that the straight line is a geodesic in these models, we now turn to the question of whether it realizes the shortest-length trajectory.

IV. MINIMAL GEODESICS

Nielsen complexity is defined as the minimal cost over all admissible paths, so it is essential to verify that the corresponding geodesic is indeed minimal, both locally and globally. A geodesic is locally length minimizing if and only if it contains no conjugate points along its trajectory [74], as illustrated in Fig. 1.

To investigate the presence of such conjugate points in the models discussed above, we parametrize small perturbations around the reference trajectory as

$$\delta H_c(s) = \delta H_o(s) + \delta H_e(s). \quad (22)$$

Solving the perturbed geodesic equation with arbitrary penalty factors in these models is generally difficult.

To gain intuition, we solve it for a special case, namely homogeneous penalties in the even and odd sectors. Furthermore, in part B of the Supplemental Material [66], we solve the perturbed geodesic equation for the Berger sphere with arbitrary penalties. In all these cases, the location of the first conjugate point is bounded from below by $\pi/2$, which can be taken to suggest that also in more general situations the first conjugate point will stay a finite distance away from $z = 0$.

So, we assume homogeneous penalties in the even and odd sectors,

$$\mathbf{G}(X_{\text{even}}) = G_{\text{even}} X_{\text{even}}, \quad \mathbf{G}(X_{\text{odd}}) = G_{\text{odd}} X_{\text{odd}}. \quad (23)$$

Under this simplification, the straight-line trajectory with a constant control Hamiltonian $H_{\text{con}} = z\mathcal{O}(t)$ can be shown to satisfy the geodesic equation (13), thereby representing a geodesic in the space of unitaries. The associated geodesic deviation (Jacobi) equation governing small perturbations δH_c then takes the form [75]

$$i \frac{d}{ds} \delta H_o = (\alpha - 1) [H_{\text{con}}, \delta H_e], \quad i \frac{d}{ds} \delta H_e = 0, \quad (24)$$

where we have introduced the ratio $\alpha \equiv G_{\text{even}}/G_{\text{odd}}$. After fixing the initial conditions, the corresponding solution of the geodesic deviation equation (i.e., the Jacobi fields) can be expressed as

$$\delta H_c(s) = \delta H_c(0) - i(\alpha - 1)s[H_{\text{con}}, \delta H_e(0)]. \quad (25)$$

Solutions for conjugate points in a closely related setup were analyzed in detail in Ref. [75]. The essential observation is that the first-order perturbation at the endpoint of the straight-line trajectory is given by

$$(U^{-1} \delta U)|_{s=1} = \int_0^1 d\tilde{s} e^{iH_{\text{con}}\tilde{s}} \delta H_c(\tilde{s}) e^{-iH_{\text{con}}\tilde{s}} \equiv \Delta[\delta H_c], \quad (26)$$

which can be interpreted as the action of a superoperator Δ on the perturbation δH_c . The conjugate points are therefore determined by the zeros of this operator, i.e., by the condition $(U^{-1} \delta U)|_{s=1} = 0$. To evaluate the integral in Eq. (26) explicitly, we expand in the eigenbasis of the Hermitian operator $\mathcal{O}(t)$, defined by $\mathcal{O}(t)|n\rangle = \omega_n|n\rangle$ [76]. Assuming the initial perturbations take the form

$$\delta H_c(0) = \sum_{m,n} S_{mn} |m\rangle \langle n|, \quad \delta H_e(0) = \sum_{m,n} M_{mn} |m\rangle \langle n|,$$

the integration in Eq. (26) yields

$$\frac{(e^{iz\omega_{mn}} - 1)S_{mn} + (\alpha - 1)(e^{iz\omega_{mn}}(1 - iz\omega_{mn}) - 1)M_{mn}}{iz\omega_{mn}},$$

where $\omega_{mn} = \omega_m - \omega_n$. The solution for the geodesic perturbation $\delta H_c(s)$ in Eq. (25) is determined by an arbitrary initial perturbation spanning the operator space, i.e., $\delta H_c(0) = \sum_\alpha c_\alpha X_\alpha$. The existence of a conjugate point, $\Delta[\delta H_c] = 0$, corresponds to the vanishing of the determinant:

$$0 = \det \Delta_{\alpha\beta} \equiv \det (\text{Tr}(X_\alpha \Delta[X_\beta])). \quad (27)$$

For concreteness, we perform the numerics that follow in the SYK model [77,78]. (We obtained similar results for spin chains.) The model consists of N Majorana fermions with all-to-all random interactions [79]. Its Hamiltonian is

$$H = i^{q/2} \sum_{1 \leq i_1 < \dots < i_q \leq N} J_{i_1 \dots i_q} \psi_{i_1} \dots \psi_{i_q}, \quad (28)$$

with q taken to be even and where the couplings $J_{i_1 \dots i_q}$ are drawn from a Gaussian ensemble with vanishing mean and variance

$$\langle J_{i_1 i_2 \dots i_q}^2 \rangle = \frac{2^{q-1} \mathcal{J}^2 (q-1)!}{q N^{q-1}}. \quad (29)$$

Here $\mathcal{J}$ sets the characteristic energy scale of the model. In what follows we set the parameters to $N = 8$, $q = 4$ and $\mathcal{J} = 1$, and start with the initial operator $\mathcal{O}(0) = \sqrt{2}\psi_1$. Numerical evaluation of the determinant in Eq. (27) for this setup is shown in Fig. 2. Although obtaining a closed-form expression for the exact locations of the conjugate points is challenging, it is straightforward to observe that, in the special case $\alpha = 1$, corresponding to the bi-invariant complexity metric, the determinant vanishes at $z = \frac{2\pi k}{\omega_{\max}}$ for integer k . The first conjugate point therefore occurs at $z_* = \frac{2\pi}{\omega_{\max}}$, where $\omega_{\max}$ denotes the maximum spectral gap. Choosing $\mathcal{O}(t) = \sqrt{2}\psi_1(t)$ gives $\omega_{\max} = 2$, and thus $z_* = \pi$. Intuitively, reducing the penalty assigned to even-sector operators (i.e., decreasing α) brings the conjugate points closer, whereas increasing α pushes them further away. This trend is confirmed by our numerical results. More importantly, the location of the first conjugate point is bounded below:

$$(\text{first conjugate point :}) \quad z_*(\alpha) \geq \frac{\pi}{2}, \quad (30)$$

with the lower bound saturated at $\alpha = 0$. In part B of the Supplemental Material [66], we analytically verify that this bound also holds for the Berger sphere with arbitrary penalties. We thus find that, in all models that we investigated, the straight-line trajectory (7) remains a locally minimal geodesic for any precursor operator with $z \leq \frac{\pi}{2}$. This motivates us to expect that, for the models discussed previously, with penalty schedules of interest, the straight-line

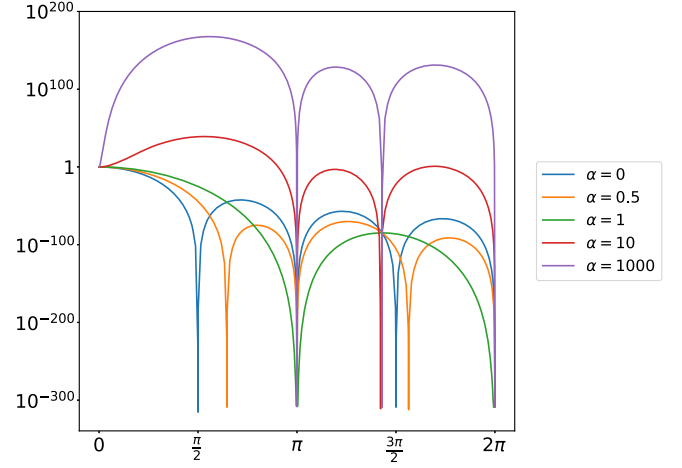


FIG. 2. Absolute value of the determinant in Eq. (27) for the SYK model with $N = 8$, $q = 4$, $\mathcal{J} = 1$ and initial operator $\mathcal{O}(0) = \sqrt{2}\psi_1$, as a function of the precursor size z for various penalty ratios $\alpha \equiv G_{\text{even}}/G_{\text{odd}}$. The same result would be found for other even Hamiltonians and odd initial operators, as long as the latter has eigenvalues ± 1 .

trajectory (7) will be a locally minimal geodesic for a finite range of z .

Our conjugate-point analysis was simplified by the specific choice $\mathcal{O}_0 = \sqrt{2}\psi_1$, for which $\mathcal{O}(t)^2 = 1$, so the spectrum of $\mathcal{O}(t)$ is fixed to ± 1 . For a more general odd initial operator, the conjugate-point locations, and hence the precise range of finite- z saturation, may depend on the spectrum of $\mathcal{O}(t)$. We do expect a finite saturation interval in many such cases, even though its end point need not coincide with the one found for $\mathcal{O}_0 = \sqrt{2}\psi_1$. In contrast, for even initial operators, we generally do not expect our inequality between Nielsen and Krylov complexity to saturate for finite z .

To complete our construction, we now discuss the global minimality of geodesics. Since the unitary group $\text{SU}(N)$ is compact, geodesic loops can occur: beyond a certain distance, two distinct geodesics may connect the same end points with equal length. In the SYK model, starting from the Hermitian operator $\mathcal{O}(0) = \sqrt{2}\psi_1$, whose eigenvalues are fixed at ± 1 , it is straightforward to see that the geodesic forms a closed loop at $z = 2\pi$, since

$$U_{\text{T}}|_{z=2\pi} = 1 = U(s=0). \quad (31)$$

Combining this with the local minimality condition, we conclude that the locally minimizing geodesic remains globally minimizing up to the midpoint $z = \pi$. In this regime, we establish an explicit correspondence between the Krylov complexity $C_K(\mathcal{O}(t))$ and the Nielsen complexity $C_N(e^{-iz\mathcal{O}(t)})$ by showing that the straight-line trajectory $U(s) = e^{-isz\mathcal{O}}$ is the globally minimizing geodesic underlying both measures of complexity.

We comment on the relation of our results with Ref. [60], which demonstrated that Krylov complexity does not satisfy the triangle inequality and therefore cannot be interpreted as a distance measure on the space of operators. First, we have shown that Krylov complexity is naturally interpreted as the square of a distance, rather than a distance itself—a possibility that was also considered in Ref. [60]. Second, in our prescription the distance does not involve the target operator itself, but rather a precursor associated with this operator. Third, and crucially, $z\sqrt{C_K(\mathcal{O}(t))}$ is mapped to a distance between the identity and the precursor of $\mathcal{O}(t)$. Since all distances in the Nielsen geometry are measured relative to the identity, our correspondence does not give rise to an obvious triangle construction that would require Krylov complexity to satisfy the triangle inequality.

V. SUMMARY

In this work, we have established a direct correspondence between Krylov complexity and Nielsen complexity by embedding the Krylov basis and weight metric into the elementary gate set and complexity metric of Nielsen’s geometric approach. We showed that Krylov complexity is mapped to the square of the length of a straight-line trajectory on the manifold of unitaries, which generally serves as an upper bound on Nielsen complexity. The bound generally saturates in the limit of small precursors ($z \rightarrow 0$). We presented evidence that, for suitable initial operators, it also saturates for a finite range of z in any physical fermionic model, as well as in a large class of spin chain models. Our results thus establish a concrete geometric bridge between two distinct yet complementary measures of quantum complexity.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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