

ϕ^3 theory at six loops

 Oliver Schnetz^{*}
II. Institut für Theoretische Physik, Luruper Chaussee 149, 22761 Hamburg, Germany


(Received 22 May 2025; accepted 24 June 2025; published 31 July 2025)

We present the renormalization functions of dimensionally regularized ϕ^3 theory in six dimensions up to loop order six in the minimal subtraction scheme.

 DOI: [10.1103/sp5q-km2c](https://doi.org/10.1103/sp5q-km2c)

I. INTRODUCTION

The calculation of the renormalization functions β , γ , and γ_m for a given quantum field theory (QFT) is a classical problem in particle physics [1]. The beta function determines the running of the coupling while γ and γ_m are the anomalous dimensions of the field and the mass. Knowledge of the renormalization functions also provides approximations for the critical exponents of phase transitions in certain universality classes of statistical models [2]. So, the calculation of β , γ , and γ_m to highest possible loop orders has impact beyond QFT.

The classical method to calculate these renormalization functions is based on momentum space. In many decades of research, very refined techniques were developed, most notably the reduction to master integrals [3,4] after infrared R^* reduction [5,6]. An overview over the classical methods is in [7] in the context of ϕ^4 theory. Outside ϕ^4 theory, the beta function is typically known to five loops [8–19].

A decade ago, the theory of graphical functions was developed by the author to calculate Feynman periods in four dimensions (residues of scalar Feynman integrals with a single logarithmic divergence) [20]. Graphical functions are single-valued functions on the complex plane with singularities at 0 and 1 that are defined by position space three-point integrals [21]. Many graphical functions can be expressed in the function space of generalized single-valued hyperlogarithms [22].

Using the theory of graphical functions, it was possible to calculate a significant number of Feynman periods. With these results, the author discovered a connection between QFT and motivic Galois theory, a deep and rich

mathematical structure which generalizes Galois theory to higher dimensions in the context of algebraic integrals (see, e.g., [23,24]). The discovery led to the formulation of the coaction conjectures in [25]. Motivic Galois theory (also named “the coaction principle” or “the cosmic Galois group”) is now a classic field in modern QFT (see, e.g., [26,27] and the references therein).

In collaboration with Michael Borinsky, the theory of graphical functions was generalized to even dimensions ≥ 4 [28]. It became possible to address ϕ^3 theory, which has a spin zero boson with a cubic self-interaction in six dimensions. The graphical function method provided a complete list of Feynman periods up to six loops and partial results up to nine loops [29,30]. All results are consistent with the coaction conjectures.

With graphical functions, dimensionally regularized ϕ^3 theory could readily be renormalized up to five loops [11]. The result was confirmed by a classical calculation [16]. The quest for six loops started right after the five loop result was obtained. A successful calculation of the field anomalous dimension (being significantly easier than the beta function), was finished in 2022 and published together with the seven loop beta function in ϕ^4 theory [31]. Note that the result $[\zeta(5, 3) = \sum_{k_1 > k_2 \geq 1} \frac{1}{k_1^5 k_2^3}]$

$$\begin{aligned} \gamma_6^{\phi^3} = & -\frac{1567}{72}\zeta(9) - \zeta(3)^3 - \frac{21}{10}\zeta(5, 3) - \frac{2}{3}\zeta(5)\zeta(3) \\ & + \frac{209}{324000}\pi^8 - \frac{25967}{2304}\zeta(7) - \frac{13}{14400}\pi^4\zeta(3) \\ & - \frac{11333}{10368}\zeta(3)^2 + \frac{25637}{7838208}\pi^6 + \frac{708913}{77760}\zeta(5) \\ & + \frac{1378253}{223948800}\pi^4 + \frac{574643}{46656}\zeta(3) + \frac{29506113557}{9674588160} \\ = & -0.331182447708\dots \end{aligned} \quad (1)$$

has a somewhat unexpected negative sign.

The calculation of the six loop beta function needed one more year. The result

^{*}Contact author: schnetz@mi.uni-erlangen.de

$$\begin{aligned}
\beta_6^{\phi^3} &= \frac{245045}{144}\zeta(9) + 37\zeta(3)^3 + \frac{3357}{40}\zeta(5,3) - \frac{11}{3}\zeta(5)\zeta(3) \\
&\quad - \frac{81733}{2016000}\pi^8 - \frac{456443}{1152}\zeta(7) + \frac{99}{800}\pi^4\zeta(3) \\
&\quad - \frac{2425}{384}\zeta(3)^2 + \frac{176425}{2612736}\pi^6 - \frac{24878747}{34560}\zeta(5) \\
&\quad + \frac{42654751}{74649600}\pi^4 - \frac{85523425}{186624}\zeta(3) - \frac{173655397121}{3224862720} \\
&= -241.455497609497...
\end{aligned} \tag{2}$$

was proposed in a talk at RadCor 2023 as a preliminary result [32]. Now, the result has undergone significant further testing (without corrections). Including $O(N)$ symmetry factors, it is contained in the Maple package HyperlogProcedures [30]. The article is supplemented by a human readable file with the renormalization functions in ϕ^3 theory and in ϕ^4 theory [33].

The theory of graphical functions is explained in [11,31]. The calculation of β and γ_m has significant overlap with the calculation of γ from the six loop two-point function in [31]. To avoid copying material, we consider this article as an extension of [31] rather than an independent paper.

We still try to keep the article self contained on a superficial level. We focus on the ϕ^3 three-point function and explain the calculation of the beta function from a certain truncated limit in the next section. In Sec. III, we explain how to calculate this truncated three-point function. Finally, in Sec. IV, we give the full results for the beta function, the field anomalous dimension, and the mass anomalous dimension.

In this article, we do not calculate critical exponents for percolation or the Lee-Yang edge singularity [11].

First results for graphical functions in theories with particles of positive spin were developed in [34]. In particular, it is shown that a dimensional shift mechanism maps calculations in four dimensions to even dimensions ≥ 4 . The author plans to extend applications of graphical functions to gauge theories with the aim to calculate six loop renormalization functions in QED and QCD.

II. RENORMALIZATION

We work in dimension

$$D = 2\lambda + 2 = 6 - \epsilon \tag{3}$$

and use a straight forward approach to renormalization. For each Feynman graph, we calculate the ϵ expansion of the Feynman integral to the needed order and extract the Z factors Z_ϕ, Z_g, Z_m which scale the bare values of the field ϕ , the coupling g , and the mass m ; see, e.g., [1]. There exist much more refined techniques for calculating renormalization functions in momentum space (such as infrared rearrangement in the R^* operation). These techniques are to some extent built in features of the graphical functions method.

For the beta function, we consider the three-point function and restrict ourselves to one particle irreducible (1PI) graphs. We attach the three external legs to the external vertices z_0, z_1, z_2 and send one external vertex (say z_2) to infinity. This is strictly speaking not necessary because at low loop orders we can calculate the three-point integrals with graphical functions. However, the full structure of the three-point function is insignificant for renormalization. The limit $z_2 \rightarrow \infty$ has the same Z factors and is much easier to calculate. Taking this limit is very efficient to reach high loop orders.

Because the three-point function is 1PI, the limit $z_2 \rightarrow \infty$ has trivial scaling with a coefficient that corresponds to the truncated three-point function with deleted vertex z_2 [31]. The deletion of z_2 converts the three-point function into an effective two-point function. This two-point function is symmetric under $z_0 \leftrightarrow z_1$ and has trivial scaling behavior. Its coefficient is a scalar integral (depending on the regulator ϵ) which is a Feynman period in the sense of [35]. This Feynman period can be efficiently calculated with graphical functions using the freedom to choose new vertices z_0, z_1, z_2 in the graph. We obtain a reduction to a period graph G_{3pt} which is $G \setminus z_2$ with an extra edge closing the half-edges attached to z_0 and z_1 . The extra edge has weight

$$\nu_{3pt} = \frac{h_G \epsilon}{2\lambda} \tag{4}$$

while all other edges have weight 1; see Fig. 1. Here, h_G is the number of loops (independent cycles) in G . We obtain

$$\lim_{z_2 \rightarrow \infty} ||z_2||^{2\lambda} A_G(z_0, z_1, z_2) = A_{G \setminus z_2}(z_0, z_1) = \frac{P_{G_{3pt}} ||z_1 - z_0||^{-2+(h_G+1)\epsilon}}{\Gamma(\lambda)^2 \lambda \nu_{3pt} (\lambda \nu_{3pt} + 1) (\lambda - \lambda \nu_{3pt}) (\lambda - 1 - \lambda \nu_{3pt})}. \tag{5}$$

The denominator in (5) comes from integrating out the external edges attached to z_0 and z_1 in position space using

$$\bullet \xrightarrow{\nu} \bullet \xrightarrow{1} \bullet = \frac{1}{\Gamma(\lambda)(\lambda(1-\nu)+1)(\lambda\nu-1)} \bullet \xrightarrow{\nu-\frac{1}{\lambda}} \bullet, \tag{6}$$

where scaling weights of propagators are indicated above the edges. For the two-point function we obtain

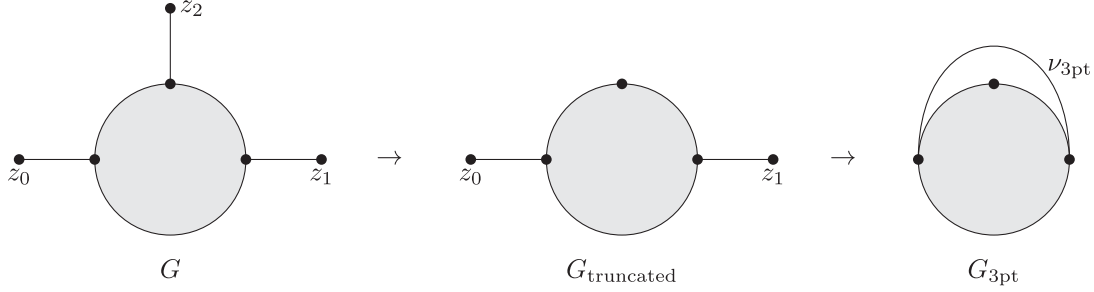


FIG. 1. The reduction of the three-point graph G to the period graph $G_{3\text{pt}}$ with glued edge of weight $\nu_{3\text{pt}} = h_G \epsilon / 2\lambda$.

$$A_G(z_0, z_1) = \frac{P_{G_{2\text{pt}}} \|z_1 - z_0\|^{-4+(h_G+1)\epsilon}}{\Gamma(\lambda)^2 \lambda \nu_{3\text{pt}} (\lambda \nu_{3\text{pt}} - 1) (\lambda - \lambda \nu_{3\text{pt}}) (\lambda + 1 - \lambda \nu_{3\text{pt}})}, \quad (7)$$

where $G_{2\text{pt}}$ is the corresponding two-point period which has a glued edge of weight

$$\nu_{2\text{pt}} = \frac{-2 + h_G \epsilon}{2\lambda}. \quad (8)$$

For the ℓ loop result, we need to calculate 1PI graphs of loop orders $k \leq \ell$ to order $\epsilon^{\ell-k-1}$. We determine the Laurent expansions of Z_ϕ, Z_g, Z_m such that the result is regular for $\epsilon \rightarrow 0$.

Because we work with leg-fixed Feynman graphs, we get $3!/2 = 3$ times the number of Feynman graph of a non-leg-fixed approach. By graph isomorphism, the exact number is a bit smaller. At loop order six in ϕ^3 theory, one obtains 37307 nonisomorphic graphs of Feynman periods which have to be evaluated to order ϵ^{-1} . In practice, the proliferation of graphs is not problematic because in most cases the calculations for permuted external vertices are either identical or very similar. Graphical functions are quite efficient, so that computation time is not an issue. The difficulty of the calculation is to find ways to calculate the few hard topologies; see Sec. III.

To extract critical exponents for both the Lee-Yang and percolation problems, it is necessary to include combinatorial symmetry factors. The calculation of the symmetry factors is explained in [11].

From the calculation of the anomalous field dimension γ , we know Z_ϕ to loop order six [31]. The Z factors Z_g and Z_m

are determined such that the truncated three-point function has no poles in ϵ (with different symmetry factors).

The results for the renormalization functions are available in HyperlogProcedures [30] which also includes ϵ expansions for all individual graphs. As a consistency test and to prepare future higher loop orders, in many cases the ϵ expansion is 1 order higher than what is necessary for the six loop results. Currently, however, the calculation of the seven loop renormalization functions in ϕ^3 theory is out of reach.

III. THE TRUNCATED THREE-POINT FUNCTION

The main tools for calculating the period $P_{G_{3\text{pt}}}$ are completion and local integration by parts (IBP) as explained in Sec. 8 of [31]. In particular, the $Y\bar{\nabla}\bar{Y}\bar{Y}$ identity depicted in Fig. 2 is of central importance. It is an extension of the uniqueness identity (the star-triangle identity) to noninteger dimensions [36–39]. Note that the two graphs with $\bar{Y}$ topology on the right-hand side are suppressed by one power in ϵ if $\nu_1 + \nu_2 = 2 + O(\epsilon)$.

Integration by parts is only used locally at fixed vertices. It is not possible to apply the Laporta algorithm [40,41] to graphs of loop order six (also see [42] and the references therein). A more systematic use of IBP equations was employed for the calculation of the residues of (primitive) three-point graphs without subdivergences [29].

Truncation of the edge attached to z_2 gives a vertex of degree two in the graph $G_{3\text{pt}}$. This vertex can be eliminated

$$\begin{array}{c} \nu_1 \quad \nu_2 \\ \diagdown \quad \diagup \\ \text{---} 1 \text{---} \end{array} = \frac{1}{\Gamma(\lambda)(\lambda\nu_1 - 1)(\lambda\nu_2 - 1)} \begin{array}{c} \frac{1}{\lambda} \\ \diagdown \quad \diagup \\ \nu_1 - \frac{1}{\lambda} \quad \nu_2 - \frac{1}{\lambda} \\ \text{---} 1 \text{---} \end{array} - \left(\frac{\lambda\nu_\infty}{\lambda\nu_1 - 1} \begin{array}{c} \frac{1}{\lambda} \\ \diagdown \quad \diagup \\ \nu_1 - \frac{1}{\lambda} \quad \nu_2 \\ \text{---} 1 \text{---} \end{array} + \frac{\lambda\nu_\infty}{\lambda\nu_2 - 1} \begin{array}{c} \frac{1}{\lambda} \\ \diagdown \quad \diagup \\ \nu_1 \quad \nu_2 - \frac{1}{\lambda} \\ \text{---} 1 \text{---} \end{array} \right)$$

FIG. 2. The $Y\bar{\nabla}\bar{Y}\bar{Y}$ identity, where $\nu_\infty = 1 + \frac{2}{\lambda} - \nu_1 - \nu_2$. This identity is particularly useful for $\nu_1 = \nu_2 = 1$ in $6 - \epsilon$ dimensions.

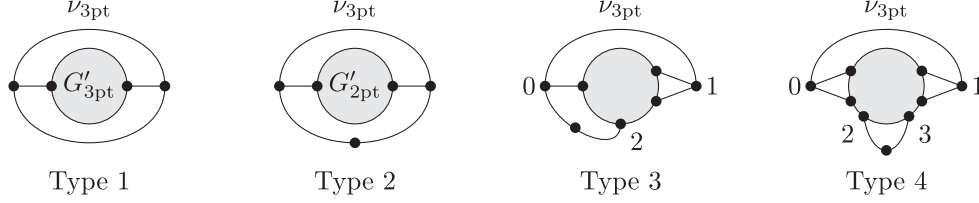


FIG. 3. The four types of the period graph G_{3pt} . In types 3 and 4, the two-chain connects to one or none of the vertices of the glued edge (and they are not of Type 1).

with (6),

$$\bullet \xrightarrow{1} \bullet \xrightarrow{1} \bullet = \frac{1}{\Gamma(\lambda)(\lambda-1)} \bullet \xrightarrow{1-\frac{1}{\lambda}} \bullet. \quad (9)$$

We classify the three-point graphs G_{3pt} into four types according to Fig. 3.

Types 1 and 2 are the cases when an edge or the two-chain at the truncated vertex z_2 are attached to the vertices of the glued edge. In these cases, one gets a reduction to lower loop order and higher degree in ϵ . The glued edge combines with the edge(s) in the lower arcs in Fig. 3 and we obtain from (6),

$$P_{G_{3pt}}^{\text{Type 1}} = \frac{P_{G'_{3pt}}}{\Gamma(\lambda)^2(1-\lambda\nu_{3pt})(2-\lambda\nu_{3pt})(\lambda-1+\lambda\nu_{3pt})(\lambda-2+\lambda\nu_{3pt})}$$

$$P_{G_{3pt}}^{\text{Type 2}} = \frac{P_{G'_{2pt}}}{\Gamma(\lambda)^3(\lambda-1)(2-\lambda\nu_{3pt})(3-\lambda\nu_{3pt})(\lambda-2+\lambda\nu_{3pt})(\lambda-3+\lambda\nu_{3pt})}. \quad (10)$$

On the right-hand sides, the graphs G'_{3pt} and G'_{2pt} are the internal reduced three-point and two-point period graphs. These graphs are shaded in Fig. 3 with an additional glued edge of weight (4) or (8), respectively, with loop order $h_G - 1$. Because $\lambda - 2 + \lambda\nu_{3pt} = (h_G - 1)\epsilon/2$, we need to calculate the reduced graphs to one extra order in ϵ . The result can be taken from the contribution of loop order $h_G - 1$ to the truncated three-point function or the two-point function because this contribution has to be calculated

to one order in ϵ higher than the contribution to loop order h_G ; see Sec. II.

Types 3 and 4 are the cases when no edges exist that directly connect the vertices of the glued edge. Then, we distinguish between the case that the truncated two-chain is attached to a vertex of the glued edge (type 3) or not (type 4). Because the glued edge has a weight that vanishes for $\epsilon = 0$, one can efficiently use the local IBP identity at its vertices,

$$\begin{aligned} \bullet \xrightarrow{1} \bullet &= \frac{1}{\Gamma(\lambda)(\lambda\nu_1 - 1)(\lambda\nu_\infty - 1)} \bullet \xrightarrow{\nu_{3pt}} \bullet \\ &\quad - \frac{\lambda\nu_{3pt}}{\lambda\nu_1 - 1} \bullet \xrightarrow{\nu_{3pt} + \frac{1}{\lambda}} \bullet - \frac{\lambda\nu_{3pt}}{\lambda\nu_\infty - 1} \bullet \xrightarrow{\nu_{3pt} + \frac{1}{\lambda}} \bullet \end{aligned} \quad (11)$$

where $\nu_\infty = 1 + \frac{2}{\lambda} - \nu_1 - \nu_{3pt}$.

In both types, 3 and 4, there exists a vertex of the glued edge with $\nu_1 = 1$. In this setup, (11) is particularly useful because the two rightmost graphs are suppressed by one order in ϵ . We obtain

$$\begin{aligned} \bullet \xrightarrow{1} \bullet &= \frac{1}{\Gamma(\lambda)(\lambda-1)(1-\lambda\nu_{3pt})} \bullet \xrightarrow{1-\frac{1}{\lambda}} \bullet \\ &\quad - \frac{\lambda\nu_{3pt}}{\lambda-1} \bullet \xrightarrow{1-\frac{1}{\lambda}} \bullet - \frac{\lambda\nu_{3pt}}{1-\lambda\nu_{3pt}} \bullet \xrightarrow{1-\frac{1}{\lambda}} \bullet \end{aligned} \quad (12)$$

In Type 3, we have an identity with $\nu_1 = 1 - 1/\lambda$. In this case, the vertex is singular and produces terms with ϵ^{-1} . Also here, (11) can be useful although a simplification is not evident.

Another option is to use completion and thereafter local IBP at vertex ∞ as explained in Secs. 8.1 and 8.2 of [31]. Completion adds a vertex ∞ which connects to all other vertices such that the weighted degree of every vertex is D/λ . Local IBP gives identities for sums of graphs which differ by locally adding and subtracting triangles with edges of weights $1/\lambda$, $-1/\lambda$, $1/\lambda$. Local IBP at ∞ is formulated in terms of graphs G_{kl}^{ij} which are G_{3pt} with an

edge ij of weight $1/\lambda$ and an edge kl of weight $-1/\lambda$. In our setup, $ij = 01$ are the vertices of the glued edge. This edge has weight $\nu_{3pt} + 1/\lambda$ in G_{kl}^{01} for $k\ell \neq 01$ while $G_{01}^{01} = G_{3pt}$. In graphs of types 3 and 4, we label the vertices where the two-chain connects to the rest of the graph with 2 and 3 (see Fig. 3). We define $P_{G_{kk}^{01}} = 0$ for all k and use the abbreviations

$$\alpha = 2 + \epsilon, \quad \beta = 2 - h_G \epsilon, \quad \gamma = 2 - (h_G - 3)\epsilon. \quad (13)$$

For graphs of type 3 we get the identities

$$\begin{aligned} P_{G_{3pt}}^{\text{Type 3}} &= P_{G_{01}^{01}} - \frac{\alpha}{\beta} (P_{G_{02}^{01}} - P_{G_{21}^{01}}) + \frac{\beta + 2}{\beta} P_{G_{0i}^{01}} - \frac{\epsilon}{\beta} \sum_{k \geq 3} (P_{G_{0k}^{01}} - P_{G_{ik}^{01}}) \\ &= P_{G_{0i}^{01}} - \frac{\alpha}{\beta + 2} (P_{G_{12}^{01}} - P_{G_{2i}^{01}}) + \frac{\beta}{\beta + 2} P_{G_{0i}^{01}} - \frac{\epsilon}{\beta + 2} \sum_{k \geq 3} (P_{G_{1k}^{01}} - P_{G_{ik}^{01}}), \end{aligned} \quad (14)$$

with $i \geq 2$. For general i , the right-hand sides are not simpler than $P_{G_{3pt}}$. However, it may happen that there exists a particularly convenient value for i such that the right hand side can be calculated in a case where $P_{G_{3pt}}$ is not accessible by other methods.

For graphs of type 4 we get two duality formulas for $i = 1 - j = 0, 1$,

$$\begin{aligned} P_{G_{3pt}}^{\text{Type 4}} &= \frac{\alpha^2}{\beta\gamma} P_{G_{23}^{01}} - \frac{3\alpha\epsilon}{\beta\gamma} (P_{G_{12}^{01}} + P_{G_{13}^{01}}) \\ &\quad - \epsilon \sum_{k \geq 4} \left(\frac{1}{\beta} P_{G_{ik}^{01}} + \frac{1}{\gamma} P_{G_{jk}^{01}} \right) - \frac{\epsilon^2}{\beta\gamma} \sum_{4 \leq k < \ell} P_{G_{kl}^{01}} \end{aligned} \quad (15)$$

in which, up to terms of order ϵ , the weights of the edges 01 and 23 are swapped.

There exists a plethora of similar formulas that can be useful for various topologies. Some of these formulas were implemented in HyperlogProcedures [30] to complete the calculation of the six loop beta function in ϕ^3 theory.

IV. RESULTS

In this section we summarize the six loop results for the renormalization functions of ϕ^3 theory in the minimal subtraction scheme. The results were calculated using HyperlogProcedures [30] over several weeks on a server with 36 cores and 780 GB RAM. We confirm the five loop results for the Lee-Yang edge singularity in [11] and obtain for the beta function β^{ϕ^3} , the anomalous field dimension γ^{ϕ^3} , and the mass anomalous dimension $\gamma_m^{\phi^3}$,

$$\begin{aligned} \beta^{\phi^3}(g) &= \left(\frac{245045}{144} \zeta(9) + 37\zeta(3)^3 + \frac{3357}{40} \zeta(5, 3) - \frac{11}{3} \zeta(5)\zeta(3) - \frac{81733}{2016000} \pi^8 - \frac{456443}{1152} \zeta(7) + \frac{99}{800} \pi^4 \zeta(3) \right. \\ &\quad \left. - \frac{2425}{384} \zeta(3)^2 + \frac{176425}{2612736} \pi^6 - \frac{24878747}{34560} \zeta(5) + \frac{42654751}{74649600} \pi^4 - \frac{85523425}{186624} \zeta(3) - \frac{173655397121}{3224862720} \right) g^{13} \\ &\quad + \left(-\frac{5495}{64} \zeta(7) + \frac{99}{16} \zeta(3)^2 + \frac{5}{2016} \pi^6 + \frac{151795}{3456} \zeta(5) - \frac{46519}{829440} \pi^4 + \frac{366647}{6912} \zeta(3) + \frac{102052031}{6718464} \right) g^{11} \\ &\quad + \left(\frac{5}{3} \zeta(5) + \frac{1}{192} \pi^4 - \frac{4891}{864} \zeta(3) - \frac{3404365}{746496} \right) g^9 + \left(\frac{5}{8} \zeta(3) + \frac{33085}{20736} \right) g^7 - \frac{125}{144} g^5 + \frac{3}{4} g^3 \\ &\approx -241.45550g^{13} + 43.78256g^{11} - 9.12961g^9 + 2.34682g^7 - 0.86806g^5 + 0.75g^3, \end{aligned} \quad (16)$$

$$\begin{aligned}
\gamma^{\phi^3}(g) = & \left(-\frac{1567}{72}\zeta(9) - \zeta(3)^3 - \frac{21}{10}\zeta(5,3) - \frac{2}{3}\zeta(5)\zeta(3) + \frac{209}{324000}\pi^8 - \frac{25967}{2304}\zeta(7) - \frac{13}{14400}\pi^4\zeta(3) \right. \\
& - \frac{11333}{10368}\zeta(3)^2 + \frac{25637}{7838208}\pi^6 + \frac{708913}{77760}\zeta(5) + \frac{1378253}{223948800}\pi^4 + \frac{574643}{46656}\zeta(3) + \frac{29506113557}{9674588160} \Big) g^{12} \\
& + \left(\frac{147}{64}\zeta(7) - \frac{25}{144}\zeta(3)^2 - \frac{25}{54432}\pi^6 + \frac{4471}{10368}\zeta(5) - \frac{5651}{2488320}\pi^4 - \frac{56693}{62208}\zeta(3) - \frac{16492987}{20155392} \right) g^{10} \\
& + \left(-\frac{5}{18}\zeta(5) + \frac{7}{8640}\pi^4 + \frac{35}{864}\zeta(3) + \frac{53449}{248832} \right) g^8 + \left(\frac{1}{24}\zeta(3) - \frac{5195}{62208} \right) g^6 + \frac{13}{432}g^4 - \frac{1}{12}g^2 \\
& \approx -0.33118g^{12} - 0.06420g^{10} + 0.05438g^8 - 0.03342g^6 + 0.03009g^4 - 0.08333g^2,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\gamma_m^{\phi^3}(g) = & \left(-\frac{248179}{144}\zeta(9) - 38\zeta(3)^3 - \frac{3441}{40}\zeta(5,3) + 3\zeta(5)\zeta(3) + \frac{747301}{18144000}\pi^8 + \frac{886919}{2304}\zeta(7) - \frac{359}{2880}\pi^4\zeta(3) \right. \\
& + \frac{27071}{5184}\zeta(3)^2 - \frac{251819}{3919104}\pi^6 + \frac{45348875}{62208}\zeta(5) - \frac{316465}{559872}\pi^4 + \frac{29273999}{62208}\zeta(3) + \frac{13761807623}{241864704} \Big) g^{12} \\
& + \left(\frac{2821}{32}\zeta(7) - \frac{229}{36}\zeta(3)^2 - \frac{5}{1701}\pi^6 - \frac{225457}{5184}\zeta(5) + \frac{66953}{1244160}\pi^4 - \frac{839129}{15552}\zeta(3) - \frac{40331135}{2519424} \right) g^{10} \\
& + \left(-\frac{35}{18}\zeta(5) - \frac{19}{4320}\pi^4 + \frac{821}{144}\zeta(3) + \frac{445589}{93312} \right) g^8 + \left(-\frac{7}{12}\zeta(3) - \frac{52225}{31104} \right) g^6 + \frac{97}{108}g^4 - \frac{5}{6}g^2 \\
& \approx 241.12432g^{12} - 43.84676g^{10} + 9.18399g^8 - 2.38024g^6 + 0.89815g^4 - 0.83333g^2.
\end{aligned} \tag{18}$$

The six loop results can be expressed in terms of transcendentals in ϕ^4 theory [43,44].

The author did not attempt to obtain improved values for critical exponents for percolation or the Lee-Yang edge singularity.

ACKNOWLEDGMENTS

The author thanks Sven-Olaf Moch for support. During most of the work on this project the author was financed by the DFG Grants No. SCHN 1240/2 and No. SCHN 1240/3.

DATA AVAILABILITY

The data that support the findings of this article are openly available [30,33].

-
- [1] J. C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
 - [2] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, International Series of Monographs on Physics 113, (Clarendon Press, Oxford, 2002).
 - [3] K. G. Chetyrkin and F. V. Tkachov, Integration by parts: The algorithm to calculate β -functions in 4 loops, *Nucl. Phys. B* **192**, 159 (1981).
 - [4] F. V. Tkachov, A theorem on analytical calculability of 4-loop renormalization group functions, *Phys. Lett.* **100B**, 65 (1981).
 - [5] K. G. Chetyrkin and V. A. Smirnov, R^* -operation corrected, *Phys. Lett.* **144B**, 419 (1984).
 - [6] K. G. Chetyrkin and F. V. Tkachov, Infrared R -operation and ultraviolet counterterms in the MS-scheme, *Phys. Lett.* **114B**, 340 (1982).
 - [7] M. V. Kompaniets and E. Panzer, Minimally subtracted six loop renormalization of $O(n)$ -symmetric ϕ^4 theory and critical exponents, *Phys. Rev. D* **96**, 036016 (2017).
 - [8] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Five-loop fermion anomalous dimension for a general gauge group from four-loop massless propagators, *J. High Energy Phys.* **04** (2017) 119.
 - [9] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Five-loop running of the QCD coupling constant, *Phys. Rev. Lett.* **118**, 082002 (2017).

- [10] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, and J. Rittinger, Vector correlator in massless QCD at order $\mathcal{O}(\alpha_s^4)$ and the QED β -function at five Loop, *J. High Energy Phys.* **07** (2012) 017.
- [11] M. Borinsky, J. A. Gracey, M. V. Kompaniets, and O. Schnetz, Five loop renormalization of ϕ^3 theory with applications to the Lee-Yang edge singularity and percolation theory, *Phys. Rev. D* **103**, 116024 (2021).
- [12] K. G. Chetyrkin, G. Falcioni, F. Herzog, and J. A. M. Vermaseren, Five-loop renormalisation of QCD in covariant gauges, *J. High Energy Phys.* **10** (2017) 179.
- [13] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, The five-loop beta function of Yang-Mills theory with fermions, *J. High Energy Phys.* **02** (2017) 090.
- [14] F. Herzog, S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, Five-loop contributions to low- N non-singlet anomalous dimensions in QCD, *Phys. Lett. B* **790**, 436 (2019).
- [15] A. L. Kataev and S. A. Larin, Analytical five-loop expressions for the renormalization group QED β -function in different renormalization schemes, *JETP Lett.* **96**, 61 (2012).
- [16] M. V. Kompaniets and A. Pikelner, Critical exponents from five-loop scalar theory renormalization near six-dimensions, *Phys. Lett. B* **817**, 136331 (2021).
- [17] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, Five-loop quark mass and field anomalous dimensions for a general gauge group, *J. High Energy Phys.* **01** (2017) 081.
- [18] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, Complete renormalization of QCD at five loops, *J. High Energy Phys.* **03** (2017) 020.
- [19] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, The five-loop Beta function for a general gauge group and anomalous dimensions beyond Feynman gauge, *J. High Energy Phys.* **10** (2017) 166.
- [20] O. Schnetz, Graphical functions and single-valued multiple polylogarithms, *Commun. Number Theory Phys.* **8**, 589 (2014).
- [21] M. Golz, E. Panzer, and O. Schnetz, Graphical functions in parametric space, *Lett. Math. Phys.* **107**, 1177 (2017).
- [22] O. Schnetz, Generalized single-valued hyperlogarithms, *arXiv:2111.11246*.
- [23] F. C. S. Brown, Feynman amplitudes, coaction principle, and cosmic Galois group, *Commun. Num. Theor. Phys.* **11**, 453 (2017).
- [24] F. C. S. Brown, Notes on motivic periods, *Commun. Num. Theor. Phys.* **11**, 557 (2017).
- [25] E. Panzer and O. Schnetz, The Galois coaction on ϕ^4 periods, *Commun. Num. Theor. Phys.* **11**, 657 (2017).
- [26] S. Caron-Huot, L. J. Dixon, F. Dulat, M. von Hippel, A. J. McLeod, and G. Papathanasiou, The cosmic Galois group and extended Steinmann relations for planar $N = 4$ SYM amplitudes, *J. High Energy Phys.* **09** (2019) 061.
- [27] O. Schnetz, The Galois coaction on the electron anomalous magnetic moment, *Commun. Num. Theor. Phys.* **12**, 335 (2018).
- [28] M. Borinsky and O. Schnetz, Graphical functions in even dimensions, *Commun. Num. Theor. Phys.* **16**, 515 (2022).
- [29] M. Borinsky and O. Schnetz, Recursive computation of Feynman periods, *J. High Energy Phys.* **08** (2022) 291.
- [30] O. Schnetz, HyperlogProcedures, Version 0.8, Maple package available on the homepage of the author at <https://www.physik.uni-hamburg.de/th2/ag-moch/personen/schnetz-oliver.html> (2025).
- [31] O. Schnetz, ϕ^4 theory at seven loops, *Phys. Rev. D* **107**, 036002 (2023).
- [32] O. Schnetz, Loop calculations with graphical functions, in *talk presented at RadCor*, Crieff, UK (2023).
- [33] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/sp5q-km2c> for renormalization functions in ϕ^3 theory and in ϕ^4 theory.
- [34] O. Schnetz, Graphical functions with spin, *J. High Energy Phys.* **06** (2025) 053.
- [35] O. Schnetz, Numbers and functions in quantum field theory, *Phys. Rev. D* **97**, 085018 (2018).
- [36] M. d'Eramo, L. Peliti, and G. Parisi, Theoretical predictions for critical exponents at the λ -point of Bose liquids, *Lett. Nuovo Cimento* **2**, 878 (1971).
- [37] S. Jeffries and K. A. Yeats, A degree preserving delta wye transformation with applications to 6-regular graphs and Feynman periods, *Ann. Inst. Henri Poincaré D* **11**, 641 (2024).
- [38] D. I. Kazakov, The method of uniqueness, a new powerful technique for multiloop calculations, *Phys. Lett.* **133B**, 406 (1983).
- [39] G. Parisi, On self-consistency conditions in conformal covariant field theory, *Lett. Nuovo Cimento* **4**, 777 (1972).
- [40] S. Laporta, High-precision calculation of multiloop Feynman integrals by difference equations, *Int. J. Mod. Phys. A* **15**, 5087 (2000).
- [41] S. Laporta and E. Remiddi, The analytical value of the electron ($g - 2$) at order α^3 in QED, *Phys. Lett. B* **379**, 283 (1996).
- [42] S. Weinzierl, *Feynman Integrals* (Springer Nature, Cham, Switzerland, 2022).
- [43] D. J. Broadhurst and D. Kreimer, Knots and numbers in ϕ^4 theory to 7 loops and beyond, *Int. J. Mod. Phys. C* **06**, 519 (1995).
- [44] O. Schnetz, Quantum periods: A census of ϕ^4 transcendentals, *Commun. Num. Theor. Phys.* **4**, 1 (2010).