

Celestial decomposition of Wigner's particles

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We provide a detailed decomposition of Wigner's particles, defined as unitary irreducible representations of the Poincaré group, in terms of unitary representations of its Lorentz subgroup. As pointed out before us, this decomposition only involves Lorentz representations belonging to the principal continuous series, and further underpins the connection between scattering amplitudes and conformal correlation functions discussed in the context of celestial holography. We provide very explicit formulas for the decomposition of particles of arbitrary mass and (half-)integer spin and for any spacetime dimension. We emphasize that this decomposition is unique and comes equipped with a specific inner product on the corresponding Hilbert space. We also confirm that unitary translations mix Lorentz representations within the principal continuous series only, as required. Implications to the celestial holography program are indicated.

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I. INTRODUCTION

Unitary representations of symmetry groups play a fundamental role in quantum mechanics and quantum field theory (QFT). Observers related to one another by a continuous symmetry should measure the same transition probabilities, which implies that the latter are represented on the Hilbert space of physical states by unitary operators [1,2]. The physical Hilbert space is the carrier space of a unitary representation of the symmetry group. Unitary irreducible representations (UIRs) can then be used as the building blocks that can potentially appear within a specific theory.¹ Most notably this methodology led Wigner to propose a classification of stable particles in terms of UIRs of the inhomogeneous Lorentz group $ISO(1, d+1)$ (also known as the Poincaré group), which still constitutes the basis of our understanding of relativistic particle scattering [3]. Another big success of this approach lies in the study of conformal field theories (CFT), whose Hilbert space carries unitary representations of the pseudo-orthogonal group $SO(2, d)$. In that context group theory is so powerful that any given theory can in principle be solved nonperturbatively [4].

In recent years substantial effort has been put in revisiting the theory of particle scattering through the lens of the Lorentz group $SO(1, d+1) \subset ISO(1, d+1)$, with the hope of exploiting powerful methods developed in the context of conformal field theory. This program, which goes under the name of *celestial holography*, is reviewed for example in [5,6]. It should be emphasized that if one aims at describing a quantum theory where Poincaré symmetries are realized unitarily on the physical Hilbert space, then one necessarily needs to consider unitary representations of its Lorentz subgroup. One should therefore be very careful in trying to import methods from conformal field theory, since analytic continuation from $SO(2, d)$ to $SO(1, d+1)$ turns unitary representations into nonunitary ones (called reflection-positive instead). At this time, it is still unknown which of the many results from standard conformal field theory can be safely transposed to theories with unitary $SO(1, d+1)$ symmetry. Existing work on the subject points to even greater tractability in some aspects [7].

In this paper we intend to work out the decomposition of Wigner particles, defined as UIRs of the Poincaré group $ISO(1, d+1)$, into UIRs of its Lorentz subgroup. While this problem has been studied in the past [8–22], our aim is to present the solution in the language of *elementary representations*, which are carried by basis states $|\Delta, \vec{x}\rangle_\sigma$ labeled by a fixed scaling dimension Δ , a continuous parameter $\vec{x} \in \mathbb{R}^d$, and a finite set of spin indices σ .² For $d = 2$ which is most relevant to phenomenology, this has

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¹Beware of reducible but indecomposable representations, however.

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²Elementary representations cover both unitary and reflection-positive representations, depending on the value of Δ .

first been performed in [12] for massless particles of arbitrary integer spin and in [18] for massive scalar particles. The resulting decompositions have then independently reappeared in the context of celestial holography in a slightly disguised form relating plane waves to *conformal primary wave functions* [23–26], with extensions to spinning massive wave functions [27–29]. While the central decomposition formulas do appear in this second body of work, a complete group theoretic analysis was not provided. In particular, the physical Hilbert space was not explicitly described and questions of unitarity left aside. In this paper we close this gap, providing at the same time the generalization to arbitrary dimension d , mass, and (half-)integer spin.

The decomposition of Wigner’s UIRs into elementary representations is essentially a problem in harmonic analysis. Indeed, a generic one-particle state can be characterized by a square integrable momentum wave function $\psi(\hat{p})$, with $\hat{p}$ a point on the (forward) null cone $\text{LC}_{d+1}^+ \subset \mathbb{M}^{d+2}$ in the massless case, or a point on the negatively curved hyperboloid $\mathbb{H}^{d+1} \subset \mathbb{M}^{d+2}$ in the massive case. Here $\mathbb{M}^{d+2}$ denotes the $(d+2)$ -dimensional Minkowski spacetime. Both LC_{d+1}^+ and $\mathbb{H}^{d+1}$ are homogeneous spaces with respect to $\text{SO}(1, d+1)$; i.e., they can be realized as the quotient $\text{SO}(1, d+1)/H$ for some closed subgroup $H \in \text{SO}(1, d+1)$, and the decomposition we are looking for amounts to a decomposition of a generic square integrable function $\psi(\hat{p})$ over a complete set of harmonic functions carrying irreducible representations of $\text{SO}(1, d+1)$. While this philosophy underlies our approach, the description we provide will be rather practical, and we refer the interested reader to [30–32] for a detailed discussion of harmonic analysis in relation to group theory.

The paper is organized as follows. In Sec. II we briefly review the elementary representations of the Lorentz group $\text{SO}(1, d+1)$ [31,33–36]. They can be constructed using the method of induced representations, inducing from a representation $[\Delta, \rho]$ of $\mathbb{R} \times \text{SO}(d)$, with Δ a scaling dimension and ρ an irrep of $\text{SO}(d)$ acting on the spin degrees of freedom. We pay particular attention to the unitary representations of the *principal continuous series* where $\Delta \in \frac{d}{2} + i\mathbb{R}$, since these are the only ones which will appear in the aforementioned harmonic decomposition. In Sec. III we describe the decomposition of a generic massless particle UIR. As is well-known from the works of de Boer–Solodukhin [23] and Pasterski-Shao [25], a Mellin transform exchanges the particle energy ω for the scaling dimension Δ of the elementary representation, integrated over the continuous principal series. The representation of the little group $\text{SO}(d)$ from which Wigner’s UIRs are induced is then readily identified with the representation ρ of the corresponding elementary representation, such that the same decomposition applies irrespective of the particle spin. We explicitly discuss the compatibility of this

decomposition with the inner products that the one-particle Hilbert space and the Hilbert spaces of the principal continuous series come equipped with. We show how Wigner’s UIRs are turned into Lorentz UIRs, working infinitesimally in the main text, and with finite transformations in Appendix A. Finally, we show that unitary translations $U(a) = e^{-ia^\mu \bar{P}_\mu}$ mix the Lorentz UIRs within the principal continuous series, and we give the explicit formula describing this mixing. This contrasts with the widespread folklore that translations require to consider representations off the principal continuous series, which are nonunitary. In Sec. IV we turn to the decomposition of massive particle UIRs. In the spinless case, the decomposition given by Pasterski-Shao-Strominger [26] is used to easily reach analogous conclusions. Adding spin introduces substantial technical complications, however. The origin for this complication is the fact that massive particles carry a representation of the little group $\text{SO}(d+1)$, which needs to be decomposed over the representations ρ of $\text{SO}(d)$ labeling the elementary representations. Fortunately, for integer spin all the required technology has already been developed in a different context in [37,38]. In particular a resolution of the identity on the space of square integrable spinning wave functions was provided in [38], which has been used in [27] to discuss massive conformal primary wave functions for $d=2$. For half-integer spin, the corresponding resolution of identity has been provided by one of us in [28]. Using these formulas, we are able to describe in detail the decomposition of massive Wigner UIRs into elementary representations of the principal continuous series, for arbitrary dimension d and (half-)integer spin.

II. UNITARY REPRESENTATIONS OF THE LORENTZ GROUP

Unitary representations of the Lorentz group $\text{SO}(1, d+1)$ have been classified in [31,33–36]. For the purposes of this work the relevant unitary representations are those of the so-called *principal continuous series*. We will start by describing them in a language most suitable to the present construction and closely related to that of conformal field theory, mostly following the physicist-friendly review [39].

While the Lorentz algebra $\mathfrak{so}(1, d+1)$ is usually presented in terms of the abstract generators $J_{\mu\nu}$ as

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}), \quad (2.1)$$

with $\eta_{\mu\nu} = \text{diag}(-, +, \dots, +)$, it is very convenient to distinguish the indices $\mu=0$ and $\mu=d+1$ and define the generators (P_i, K_i, D) through

$$J_{0i} = \frac{K_i + P_i}{2}, \quad J_{(d+1)i} = \frac{K_i - P_i}{2}, \quad J_{0(d+1)} = -D, \quad (2.2)$$

such that the algebra (2.1) takes the form

$$\begin{aligned}
[D, P_i] &= iP_i, & [D, K_i] &= -iK_i, & [K_i, P_j] &= -2i(\delta_{ij}D - J_{ij}), \\
[J_{ij}, P_k] &= -i(\delta_{ik}P_j - \delta_{jk}P_i), & [J_{ij}, K_k] &= -i(\delta_{ik}K_j - \delta_{jk}K_i), \\
[J_{ij}, J_{kl}] &= -i(\delta_{ik}J_{jl} + \delta_{jl}J_{ik} - \delta_{il}J_{jk} - \delta_{jk}J_{il}).
\end{aligned} \tag{2.3}$$

Indeed the Bruhat decomposition of the Lorentz group is then given in terms of the subgroups

$$\begin{aligned}
N &= \{e^{ib^i K_i}\}, & A &= \{e^{i\lambda D}\}, \\
M &= \{e^{i\omega^{ij} J_{ij}}\}, & \tilde{N} &= \{e^{ix^i P_i}\}.
\end{aligned} \tag{2.4}$$

The basis generators $\langle J_{ij}, P_i, K_i, D \rangle$ with $i = 1, \dots, d$ are naturally associated with the conformal transformations of $\mathbb{R}^d$ with Cartesian coordinates x^i , which can be viewed as the homogeneous space

$$\mathbb{R}^d \cong \tilde{N} \cong \frac{\text{SO}(1, d+1)}{\text{MAN}}. \tag{2.5}$$

In this basis the quadratic Casimir operator is then given by

$$C_2^{\text{so}(1, d+1)} = \frac{1}{2} J_{\mu\nu} J^{\mu\nu} = iD(d+iD) - P_i K^i + \frac{1}{2} J_{ij} J^{ij}. \tag{2.6}$$

Conformal field representations can then be constructed using the method of induced representations, starting from a representation of the isotropy subgroup MAN spanned by $\langle J_{ij}, K_i, D \rangle$ [40]. The inducing representation is characterized by

$$D|\Delta\rangle_\sigma = i\Delta|\Delta\rangle_\sigma, \quad J_{ij}|\Delta\rangle_\sigma = (\Sigma_{ij})_\sigma^{\sigma'} |\Delta\rangle_{\sigma'}, \quad K_i|\Delta\rangle_\sigma = 0, \tag{2.7}$$

where $\Delta \in \mathbb{C}$ and $(\Sigma_{ij})_\sigma^{\sigma'}$ are Hermitian spin matrices furnishing a unitary representation of $\text{SO}(d)$. The induced representation is then obtained by “translating” this state,

$$|\Delta, \vec{x}\rangle_\sigma \equiv e^{ix^i P_i} |\Delta\rangle_\sigma. \tag{2.8}$$

Using the commutation relations of the generators (2.2) together with the defining relation (2.7), one can easily work out the action of the generators on the state (2.8), resulting in

$$\begin{aligned}
P_i |\Delta, \vec{x}\rangle &= -i\partial_i |\Delta, \vec{x}\rangle, \\
J_{ij} |\Delta, \vec{x}\rangle &= -i(x_i \partial_j - x_j \partial_i + i\Sigma_{ij}) |\Delta, \vec{x}\rangle, \\
D |\Delta, \vec{x}\rangle &= i(\Delta + x^i \partial_i) |\Delta, \vec{x}\rangle, \\
K_i |\Delta, \vec{x}\rangle &= i(2x_i \Delta + 2x_i x^j \partial_j - x^2 \partial_i + 2ix^j \Sigma_{ij}) |\Delta, \vec{x}\rangle,
\end{aligned} \tag{2.9}$$

where spin indices have been suppressed for convenience. This provides an *elementary representation* of the Lorentz

group, with value of the quadratic Casimir given by

$$C_2^{\text{so}(1, d+1)} = \Delta(\Delta - d) + C_2^{\text{so}(d)}, \tag{2.10}$$

where $C_2^{\text{so}(d)}$ is the Casimir value associated with the defining $\text{SO}(d)$ representation. For a totally symmetric traceless spin- ℓ representation, we have $C_2^{\text{so}(d)} = \ell(\ell + d - 2)$. We denote by $[\Delta, \ell]$ the corresponding representation. The unitarity of the induced representation (2.9) depends on a choice of invariant inner product and on the value of the scaling dimension Δ , to be discussed momentarily. Importantly, any admissible irreducible representation can be found within such an elementary representation [41–43]. Of primary interest to us are the unitary representations of the *principal continuous series* that exist for any scaling dimension

$$\Delta = \frac{d}{2} + i\nu, \quad \nu \in \mathbb{R}, \tag{2.11}$$

with Casimir value

$$\begin{aligned}
C_2^{\text{so}(1, d+1)} &= -(\nu^2 + d^2/4) + C_2^{\text{so}(d)} \\
&= -(\nu^2 + d^2/4) + \ell(\ell + d - 2),
\end{aligned} \tag{2.12}$$

where in the second equality we restricted to a symmetric spin- ℓ representation of $\text{SO}(d)$. These constitute irreducible unitary representations. Note that ν and $-\nu$ are associated with the same Casimir value, and there exists an isomorphism between these representations known as “shadow transform.”

The above discussion was phrased in terms of infinitesimal Lorentz transformations. To obtain the finite form of the transformations, we recall that the Lorentz group $\text{SO}(1, d+1)$ is naturally realized as the group of conformal isometries of $\mathbb{R}^d$, whose action we abstractly denote $\vec{x} \mapsto \vec{x}' = \Lambda \vec{x}$ for any $\vec{x} \in \mathbb{R}^d$. Explicit expressions can be found for example in [44]. Such conformal isometries satisfy

$$\frac{\partial x'^i}{\partial x^j} = \Omega(x) R^i_j(x), \quad \Omega(x) = \left| \frac{\partial \vec{x}'}{\partial \vec{x}} \right|^{\frac{1}{d}}, \tag{2.13}$$

with $|\frac{\partial \vec{x}'}{\partial \vec{x}}|$ the determinant of the transformation, and $R^i_j(x) \in \text{SO}(d)$ an orthogonal matrix with unit determinant. For any element $\Lambda \in \text{SO}(1, d+1)$, the group action (2.9) exponentiates to

$$U(\Lambda)|\Delta, \vec{x}\rangle_\sigma = \Omega(x)^\Delta D(R(x)^{-1})_{\sigma'} |\Delta, \vec{x}'\rangle_{\sigma'}, \quad (2.14)$$

where $D(R(x))$ is the spin- ℓ representation of $R(x) \in \text{SO}(d)$.

We will consider a Hilbert space admitting an action of $\text{SO}(1, d+1)$ consisting of the direct integral over the principal continuous series (2.11) with weight $\rho(\nu)$, and fixed $\text{SO}(d)$ spin for simplicity. The generalization to the direct sum of Hilbert spaces with different $\text{SO}(d)$ spins is trivial. This Hilbert space can be described as follows. Given a complex wave function $\psi_\sigma(\nu, \vec{x})$, we associate the state

$$|\psi\rangle = \sum_\sigma \int_{-\infty}^{\infty} d\nu \rho(\nu) \int d^d \vec{x} \psi_\sigma(\nu, \vec{x}) |\nu, \vec{x}\rangle_\sigma, \quad (2.15)$$

where the sum runs over all spin components, together with the inner product

$$\langle \phi | \psi \rangle = \sum_\sigma \int_{-\infty}^{\infty} d\nu \rho(\nu) \int d^d \vec{x} \phi_\sigma(\nu, \vec{x})^* \psi_\sigma(\nu, \vec{x}). \quad (2.16)$$

Up to this point the weight $\rho(\nu)$ is arbitrary, although when considering the decomposition of Poincaré representations it will take on specific forms. The Hilbert space furnishing the carrier space for the unitary representation is the set of states $|\psi\rangle$ with normalizable wave functions $\psi_\sigma(\nu, \vec{x})$, i.e., with $\langle \psi | \psi \rangle < \infty$.³ As physicists we are happy to write an inner product for the states $|\nu, \vec{x}\rangle_\sigma$ themselves even though they do not strictly belong to the Hilbert space, namely

$${}_{\sigma'} \langle \nu', \vec{y} | \nu, \vec{x} \rangle_\sigma = \frac{\delta(\nu - \nu') \delta(\vec{x} - \vec{y})}{\rho(\nu)} \delta_{\sigma\sigma'}, \quad (2.17)$$

which should therefore be understood in the sense of distributions. Note that if instead of a direct integral over the principal series, we wanted to consider a direct sum over a discrete set $\mathbf{S}$ of scaling parameters ν , i.e.,

$$|\psi\rangle = \sum_\sigma \sum_{\nu \in \mathbf{S}} \rho_\nu \int d^d \vec{x} \psi_{\sigma, \nu}(\vec{x}) |\vec{x}\rangle_{\sigma, \nu}, \quad (2.18)$$

the Dirac delta $\delta(\nu - \nu')$ in (2.17) would be replaced by a Kronecker delta $\delta_{\nu\nu'}$. Also note that the square integrable wave functions $\psi_\sigma(\nu, \vec{x}) \in L^2(\mathbb{R}^d)$ for any $\nu \in \mathbb{R}^+$ naturally carry the conjugate representation $[d - \Delta, \ell] = [\frac{d}{2} - i\nu, \ell]$. Indeed, using integration by parts, one can show that the same transformation for $|\psi\rangle_\sigma$ as given in (2.15) is obtained either by transforming $|\nu, \vec{x}\rangle_\sigma$ according to (2.9) or $\psi(\nu, \vec{x})_\sigma$ according to the conjugate representation. Since the conjugate representations $[\frac{d}{2} + i\nu, \ell]$ and $[\frac{d}{2} - i\nu, \ell]$ are

isomorphic, it is equivalent to work with the states $|\nu, \vec{x}\rangle_\sigma$ or with the wave functions $\psi_\sigma(\nu, \vec{x})$.

Finally, we can naturally associate an operator $O_{\Delta, \sigma}(\vec{x})$ with the state $|\Delta, \vec{x}\rangle_\sigma$ through the definition

$$|\Delta, \vec{x}\rangle_\sigma \equiv O_{\Delta, \sigma}(\vec{x})|0\rangle, \quad (2.19)$$

with $|0\rangle$ the “vacuum” state corresponding to the trivial representation. The inner product (2.17) can then be interpreted as a two-point correlation

$$\langle 0 | O_{\frac{d}{2} + i\nu', \sigma'}(\vec{y})^\dagger O_{\frac{d}{2} + i\nu, \sigma}(\vec{x}) | 0 \rangle = \frac{\delta(\nu - \nu') \delta(\vec{x} - \vec{y})}{\rho(\nu)} \delta_{\sigma\sigma'}. \quad (2.20)$$

By contrast, for the representations of the *complementary series* that exist for any $\Delta \in (0, d)$ at zero spin and for any $\Delta \in (1, d-1)$ at nonzero spin, the inner product and corresponding two-point functions take the form, up to normalization,

$${}_{\sigma'} \langle \Delta', \vec{y} | \Delta, \vec{x} \rangle_\sigma = \langle 0 | O_{\Delta', \sigma'}(\vec{y})^\dagger O_{\Delta, \sigma}(\vec{x}) | 0 \rangle = \frac{\delta(\Delta - \Delta')}{|\vec{x} - \vec{y}|^{2\Delta}} \delta_{\sigma\sigma'}. \quad (2.21)$$

Again the Dirac delta is to be replaced by a Kronecker delta for a discrete spectrum of scaling dimensions. Even though the two-point functions (2.20), contrary to (2.21), may appear unfamiliar from the perspective of standard conformal field theory which treats of unitary representations of $\text{SO}(2, d)$, they are completely appropriate to the principal continuous series representations of $\text{SO}(1, d+1)$.

III. DECOMPOSITION OF MASSLESS PARTICLES

In this section we study how massless particle states, defined as UIRs of the Poincaré group $\text{ISO}(1, d+1)$, decompose into UIRs of its Lorentz subgroup. The resulting Mellin decomposition of the momentum wave functions is well-known since the works of de Boer–Solodukhin [23] and Pasterski–Shao [25], and has appeared even earlier in the mathematics literature [12]. See also [45]. Here we revisit this decomposition, putting particular emphasis on the group theoretic aspects.

We recall that massless wave functions are defined on the forward null cone LC_{d+1}^+ , which is identified with the homogeneous space

$$\text{LC}_{d+1}^+ \cong \tilde{\text{NA}} \cong \frac{\text{SO}(1, d+1)}{\text{ISO}(d)}. \quad (3.1)$$

Following Wigner’s method of induced representation, massless particle states are defined starting from the reference null momentum

³Wave functions which coincide everywhere except on a set of measure zero are identified and correspond to the same state.

$$k^\mu = \omega_0(1, \vec{0}, 1), \quad (3.2)$$

expressed in terms of an arbitrary reference energy scale ω_0 , together with a UIR of the massless little group $\text{ISO}(d) \subset \text{SO}(1, d+1)$ generated by $\langle J_{ij}, K_i \rangle$. We can characterize this inducing representation by⁴

$$J_{ij}|k\rangle_\sigma = (\Sigma_{ij})_\sigma^{\sigma'}|k\rangle_{\sigma'}, \quad K_i|k\rangle_\sigma = 0, \quad \tilde{P}_\mu|k\rangle_\sigma = k_\mu|k\rangle_\sigma, \quad (3.3)$$

where Σ_{ij} is again a Hermitian matrix furnishing a unitary representation of $\text{SO}(d)$. Note that the operator $\tilde{P}_\mu$ above is the standard momentum generator, not to be confused with the boost generators P_i defined in (2.2). A generic momentum state is then obtained by applying the remaining boost generators,

$$|p\rangle_\sigma \equiv e^{ix^i P_i} e^{i \ln(\omega/\omega_0) D} |k\rangle_\sigma. \quad (3.4)$$

The last equation is just one specific way to implement Wigner's induced representation, resulting in the specific momentum parametrization [46,47]

$$p^\mu = \omega q^\mu(\vec{x}), \quad q^\mu(\vec{x}) \equiv (1 + x^2, 2\vec{x}, 1 - x^2). \quad (3.5)$$

The action of the Lorentz generators on the generic state (3.4) can be worked out using the algebra relations together with (3.3) and (3.4), yielding

$$\begin{aligned} P_i|p\rangle &= -i\partial_i|p\rangle, \\ J_{ij}|p\rangle &= -i(x_i\partial_j - x_j\partial_i + i\Sigma_{ij})|p\rangle, \\ D|p\rangle &= i(-\omega\partial_\omega + x^i\partial_i)|p\rangle, \\ K_i|p\rangle &= i(-2x_i\omega\partial_\omega + 2x_ix^j\partial_j - x^2\partial_i + 2ix^j\Sigma_{ij})|p\rangle, \end{aligned} \quad (3.6)$$

where spin indices have again been suppressed for convenience. The astute reader will have noticed that this is essentially the same as (2.9) up to the replacement $\Delta \leftrightarrow -\omega\partial_\omega$. Making this step precise will be the key to decomposing a massless particle UIR into Lorentz UIRs.

A generic state in the corresponding one-particle Hilbert space takes the form

$$|\psi\rangle = \sum_\sigma \int [d^{d+1}p(\omega, \vec{x})] \psi_\sigma(\omega, \vec{x}) |p(\omega, \vec{x})\rangle_\sigma, \quad (3.7)$$

with $\psi_\sigma(\omega, \vec{x})$ any complex wave function normalizable with respect to the inner product

$$\langle\phi|\psi\rangle = \sum_\sigma \int [d^{d+1}p(\omega, \vec{x})] \phi_\sigma(\omega, \vec{x})^* \psi_\sigma(\omega, \vec{x}), \quad (3.8)$$

and with $[d^{d+1}p(\omega, \vec{x})]$ the Lorentz-invariant measure on the null cone LC_{d+1}^+ ,

$$[d^{d+1}p(\omega, \vec{x})] = \omega^{d-1} d\omega d^d\vec{x}. \quad (3.9)$$

The complex wave functions $\psi_\sigma(\omega, \vec{x})$ again naturally carry the conjugate unitary representation, isomorphic to the original representation. Thus, the Hilbert space is isomorphic to the space of square integrable complex wave functions $L^2(\text{LC}_{d+1}^+, [d^{d+1}p])$. We note that in the limit of vanishing frequency ω , square integrability of the wave functions requires

$$\psi_\sigma(\omega, \vec{x}) = o(\omega^{-\frac{d}{2}}), \quad (\omega \rightarrow 0). \quad (3.10)$$

Additionally, even though the states $|p(\omega, \vec{x})\rangle_\sigma$ do not strictly belong to this Hilbert space, one can assign them the distributional inner product

$${}_\sigma\langle p_1|p_2\rangle_\sigma = (\omega_1)^{1-d} \delta(\omega_1 - \omega_2) \delta(\vec{x}_1 - \vec{x}_2) \delta_{\sigma\sigma'}, \quad (3.11)$$

consistently with (3.7) and (3.8).

Thus, we want to decompose the massless wave functions $\psi_\sigma(\omega, \vec{x})$ over a set of wave functions $\psi_\sigma(\Delta^*, \vec{x})$ carrying elementary representations of $\text{SO}(1, d+1)$ as described in Sec. II. This is easily done by making use of the Mellin transform

$$\psi_\sigma(\Delta^*, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\Delta^*-1} \psi_\sigma(\omega, \vec{x}). \quad (3.12)$$

Indeed, this implements the replacement $-\omega\partial_\omega \leftrightarrow \Delta^*$ mentioned after (3.6) such that $\psi_\sigma(\Delta^*, \vec{x})$ transforms by an elementary representation of the Lorentz group. While the present discussion is at the level of infinitesimal symmetry transformations, we give the finite version in Appendix A. For a wave function $\psi_\sigma(\omega, \vec{x})$ behaving as $O(\omega^{-\alpha})$ in the limit $\omega \rightarrow 0$, with $\alpha < \frac{d}{2}$ as imposed by (3.10), the Mellin transform (3.12) is holomorphic on the complex half-plane $\text{Re}(\Delta^*) > \alpha$. In particular, it is holomorphic in the variable Δ^* for $\text{Re}(\Delta^*) \geq \frac{d}{2}$. The inverse Mellin transform then provides the sought-for decomposition of the massless wave function into elementary representations,

$$\psi_\sigma(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi i}} \int_{c-i\infty}^{c+i\infty} d\Delta^* \omega^{-\Delta^*} \psi_\sigma(\Delta^*, \vec{x}), \quad (3.13)$$

with c chosen such that no singularity lies to the right of the integration path. With the appropriate choice $c = d/2$ this can be rewritten as

⁴We do not consider continuous/infinite-spin representations, where K_i would have nontrivial action.

$$\psi_\sigma(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\nu \omega^{-\frac{d}{2} + i\nu} \psi_\sigma(\nu, \vec{x}), \quad (3.14)$$

with $\psi_\sigma(\nu, \vec{x})$ transforming by the elementary representation $[\Delta^*, \ell]$ with scaling dimension $\Delta^* = \frac{d}{2} - i\nu$ and thus belonging to the continuous principal series. Even though the states $|p(\omega, \vec{x})\rangle_\sigma$ and $|\nu, \vec{x}\rangle_\sigma$ are not normalizable, a similar formal decomposition applies,

$$|p(\omega, \vec{x})\rangle_\sigma = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\nu \omega^{-\frac{d}{2} - i\nu} |\nu, \vec{x}\rangle_\sigma, \quad (3.15)$$

with inverse

$$|\nu, \vec{x}\rangle_\sigma = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\frac{d}{2} + i\nu - 1} |p(\omega, \vec{x})\rangle_\sigma. \quad (3.16)$$

Using (3.12) and (3.15), the generic massless particle state (3.7) can then be written

$$|\psi\rangle = \sum_\sigma \int_{-\infty}^{\infty} d\nu \int d^d \vec{x} \psi_\sigma(\nu, \vec{x}) |\nu, \vec{x}\rangle_\sigma, \quad (3.17)$$

where we recall that $|\nu, \vec{x}\rangle_\sigma$ has scaling dimension $\Delta = \frac{d}{2} + i\nu$ while $\psi_\sigma(\vec{x}, \nu)$ has conjugate scaling dimension $\Delta^* = \frac{d}{2} - i\nu$. This takes the form of a state (2.15) belonging to the carrier space of a reducible unitary Lorentz representation, with weight multiplicity $\rho(\nu) = 1$.

We should check the consistency of the Wigner inner product (3.8) with the Lorentz inner product (2.16).

Starting from (3.8) and plugging in the wave function decomposition (3.14), we get

$$\begin{aligned} \langle \phi | \psi \rangle &= \frac{1}{2\pi} \sum_\sigma \int d^d \vec{x} \int_{-\infty}^{\infty} d\nu d\nu' \phi_\sigma(\nu', \vec{x})^* \psi_\sigma(\nu, \vec{x}) \\ &\quad \times \int_0^\infty d\omega \omega^{i(\nu' - \nu) - 1} \\ &= \sum_\sigma \int_{-\infty}^{\infty} d\nu \int d^d \vec{x} \phi_\sigma(\nu, \vec{x})^* \psi_\sigma(\nu, \vec{x}), \end{aligned} \quad (3.18)$$

where we used the distributional identity

$$\int_0^\infty d\omega \omega^{i(\nu' - \nu) - 1} = 2\pi \delta(\nu' - \nu). \quad (3.19)$$

The final expression in (3.18) agrees with the inner product (2.16) that a Hilbert space consisting of the direct integral of Lorentz UIRs with weight $\rho(\nu) = 1$ comes equipped with.

Equation (3.15) provides the decomposition of a massless particle state of definite momentum into states carrying irreducible representations of the Lorentz group. Conversely, it shows how states of the unitary principal series can be assembled such as to form a massless representation of the full Poincaré group. From that perspective, unitary translations generated by $U(a) = e^{-ia^\mu \tilde{P}_\mu}$ connect distinct component Lorentz UIRs.⁵ We can explicitly work out their action using (3.16),

$$\begin{aligned} U(a)|\nu, \vec{x}\rangle_\sigma &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\frac{d}{2} + i\nu - 1} e^{-ia^\mu \tilde{P}_\mu} |p(\omega, \vec{x})\rangle_\sigma \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\frac{d}{2} + i\nu - 1} e^{-ia^\mu q_\mu(\vec{x})} |p(\omega, \vec{x})\rangle_\sigma \\ &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d\nu' \frac{\Gamma(\epsilon + i(\nu - \nu'))}{(ia^\mu q_\mu(\vec{x}) + \epsilon)^{i(\nu - \nu')}} \int_0^\infty d\omega \omega^{\frac{d}{2} + i\nu' - 1} |p(\omega, \vec{x})\rangle_\sigma \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu' \frac{\Gamma(\epsilon + i(\nu - \nu'))}{(ia^\mu q_\mu(\vec{x}) + \epsilon)^{i(\nu - \nu')}} |\nu', \vec{x}\rangle_\sigma \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu' e^{-\frac{\pi}{2}(\nu - \nu')} \frac{\Gamma(\epsilon + i(\nu - \nu'))}{(-a^\mu q_\mu(\vec{x}) + i\epsilon)^{i(\nu - \nu')}} |\nu', \vec{x}\rangle_\sigma, \end{aligned} \quad (3.20)$$

where we used the Mellin representation of the exponential function

$$e^{-i\omega x - \epsilon} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \Gamma(\epsilon - i\nu) (i\omega x + \epsilon)^{i\nu}, \quad \omega x \in \mathbb{R}, \quad \epsilon > 0. \quad (3.21)$$

Hence, we see that translations map a state of a given scaling dimension to states of all scaling dimensions,

though still restricted to the unitary principal series. Of course, this had to be the case; otherwise, the statement that the principal series provides a decomposition of the full massless Poincaré representation could not have been correct. We should contrast this result with the widespread

⁵Note that $\tilde{P}_\mu$ itself is unbounded and generically does not produce normalizable states.

statement that translations take states out of the principal series, which follows from unwarranted application of the unbounded operator $\tilde{P}_\mu$. Indeed, one naively finds, for instance,

$$\tilde{P}^0|\Delta, \vec{0}\rangle = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^\Delta |p(\omega, \vec{x})\rangle = |\Delta + 1, \vec{0}\rangle, \quad (3.22)$$

such that if one starts with $\Delta \in \frac{d}{2} + i\mathbb{R}$, then $\Delta + 1$ indeed lies off the unitary principal series. Generation of such non-normalizable states is avoided by working with the unitary operators $U(a) = e^{-ia^\mu \tilde{P}_\mu}$.

IV. DECOMPOSITION OF MASSIVE PARTICLES

Now we turn to decomposition of the unitary irreducible representations of the Poincaré group $\text{ISO}(1, d+1)$ that describe massive particle states. Let us start by recalling Wigner's construction of the massive UIRs of the Poincaré group [3]. Within an irreducible representation, the quadratic Casimir operator $\mathcal{C}_2^{\text{iso}(1, d+1)} = \tilde{P}^\mu \tilde{P}_\mu$ should be proportional to the identity, and we thus require

$$\tilde{P}^\mu \tilde{P}_\mu = -m^2. \quad (4.1)$$

Henceforth, one starts with a reference momentum k^μ satisfying the above constraint, which can be chosen to be the momentum in its rest frame for simplicity,

$$k^\mu = m\delta_0^\mu. \quad (4.2)$$

The little group leaving this reference momentum invariant is spanned by spatial rotations and translations that are generated by $\langle J_{ab}, \tilde{P}_\mu \rangle$, respectively, with $a, b = 1, \dots, d+1$ the purely spatial indices. The representation of the little group, which will be used to induce the representation of the full Poincaré group, is given by

$$J_{ab}|k\rangle_\sigma = (\Sigma_{ab})_\sigma^{\sigma'} |k\rangle_{\sigma'}, \quad \tilde{P}_\mu |k\rangle_\sigma = k_\mu |k\rangle_\sigma, \quad (4.3)$$

where $(\Sigma_{ab})_\sigma^{\sigma'}$ is a set of Hermitian matrices furnishing a unitary irreducible representation of $\text{SO}(d+1)$. The induced representation is then obtained by boosting $|k\rangle_\sigma$ from its rest frame to a generic momentum frame. More specifically, given the Lorentz transformation $L(\hat{p})^\mu_\nu$ which boosts k^μ into the generic momentum $p^\mu = m\hat{p}^\mu$ with $\hat{p}^2 = -1$,

$$p^\mu = L(\hat{p})^\mu_\nu k^\nu, \quad (4.4)$$

with components and inverse components explicitly given by

$$\begin{aligned} L(\hat{p})^0_0 &= \hat{p}^0, \\ L(\hat{p})^a_0 &= L(\hat{p})^0_a = \hat{p}^a, \\ L(\hat{p})^a_b &= \delta^a_b + (\hat{p}^0 + 1)^{-1} \hat{p}^a \hat{p}_b, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} L^{-1}(\hat{p})^0_0 &= -\hat{p}_0, \\ L^{-1}(\hat{p})^a_0 &= L^{-1}(\hat{p})^0_a = -\hat{p}_a, \\ L^{-1}(\hat{p})^a_b &= \delta^a_b + (\hat{p}^0 + 1)^{-1} \hat{p}^a \hat{p}_b, \end{aligned} \quad (4.6)$$

respectively, one defines the boosted state by

$$|p\rangle_\sigma \equiv U(L(\hat{p}))|k\rangle_\sigma. \quad (4.7)$$

Here $U(L(\hat{p}))$ denotes a unitary representation of the boost element $L(\hat{p})$, to be determined following the method of induced representations. For a generic Lorentz transformation of the basis states $|p\rangle_\sigma$, the latter yields [2]

$$U(\Lambda)|p\rangle_\sigma = D(W(\Lambda, \hat{p})^{-1})_\sigma^{\sigma'} |\Lambda p\rangle_{\sigma'}, \quad (4.8)$$

where $W(\Lambda, \hat{p}) \in \text{SO}(d+1)$ is a rotation leaving the reference momentum k^μ invariant, given by

$$W(\Lambda, \hat{p}) = L^{-1}(\Lambda \hat{p}) \Lambda L(\hat{p}), \quad (4.9)$$

and where $D(W(\Lambda, \hat{p}))$ is its unitary spin matrix representation.

In the Hilbert space carrying this representation, a generic state takes the form

$$|\psi\rangle = \sum_\sigma \int [d^{d+1}\hat{p}] \psi_\sigma(\hat{p}) |p\rangle_\sigma, \quad (4.10)$$

with $\psi_\sigma(\hat{p})$ any complex wave function normalizable with respect to the inner product

$$\langle \phi | \psi \rangle = \sum_\sigma \int [d^{d+1}\hat{p}] \phi_\sigma(\hat{p})^* \psi_\sigma(\hat{p}), \quad (4.11)$$

and with $[d^{d+1}\hat{p}]$ the Lorentz-invariant measure on the hyperboloid $\mathbb{H}^{d+1}$,

$$[d^{d+1}\hat{p}] = \frac{d^{d+1}\hat{\mathbf{p}}}{\sqrt{1 + \|\hat{\mathbf{p}}\|^2}}, \quad \hat{p}^\mu = \left(\sqrt{1 + \|\hat{\mathbf{p}}\|^2}, \hat{\mathbf{p}} \right). \quad (4.12)$$

The complex wave functions $\psi_\sigma(\hat{p})$ again naturally carry the conjugate unitary representation, isomorphic to original representation. Thus the Hilbert space is isomorphic to the space of square integrable complex wave functions $L^2(\mathbb{H}^{d+1}, [d^{d+1}\hat{p}])$. Even though the states $|p\rangle_\sigma$ do not strictly belong to this Hilbert space, one can assign to them

the distributional inner product

$${}_{\sigma'} \langle p_1 | p_2 \rangle_{\sigma} = \sqrt{1 + \|\hat{\mathbf{p}}_1\|^2} \delta(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2) \delta_{\sigma\sigma'} \equiv \delta(\hat{p}_1 - \hat{p}_2), \quad (4.13)$$

consistently with (4.10) and (4.11).

We would like to decompose the wave functions $\psi_{\sigma}(\hat{p})$ into components carrying irreducible representations of the Lorentz group $\text{SO}(1, d+1)$ spanned by $\langle J_{\mu\nu} \rangle$, which precisely generate the isometries of the momentum mass-shell $\mathbb{H}^{d+1}$. This amounts to a problem of harmonic decomposition of $L^2(\mathbb{H}^{d+1}, [d^{d+1}\hat{p}])$, wherein the momentum mass-shell is viewed as the homogeneous space

$$\mathbb{H}^{d+1} \cong \frac{\text{SO}(1, d+1)}{\text{SO}(d+1)}. \quad (4.14)$$

The analysis we present here will be rather practical, and we refer the interested reader to [30,31] for a detailed discussion of harmonic analysis in relation to group theory. We first note that the set of normalizable wave functions $\psi_{\sigma}(\hat{p})$ does not provide an irreducible carrier space since the action of the quadratic Casimir $C_2^{\mathfrak{so}(1,d+1)} = \frac{1}{2} J_{\mu\nu} J^{\mu\nu}$ on these wave functions is not proportional to the identity, but rather to the Laplace-Beltrami operator ∇^2 on $\mathbb{H}^{d+1}$,

$$\begin{aligned} C_2^{\mathfrak{so}(1,d+1)} \psi_{\sigma}(\hat{p}) &= (\nabla^2 + C_2^{\mathfrak{so}(d+1)}) \psi_{\sigma}(\hat{p}) \\ &= (\nabla^2 + s(s+d-1)) \psi_{\sigma}(\hat{p}), \end{aligned} \quad (4.15)$$

where in the second equality we restricted our attention to totally symmetric spin- s representations of $\text{SO}(d+1)$. Hence, the idea is to decompose $\psi_{\sigma}(\hat{p})$ over a set of wave functions $G_{\nu,\ell}^{(s)}(\hat{p}; \dots)$ with definite Casimir value (2.12) and carrying an irreducible Lorentz representation of the principal series, i.e., satisfying the Laplace-Beltrami equation

$$\begin{aligned} (\nabla^2 + s(s+d-1)) G_{\nu,\ell}^{(s)}(\hat{p}; \dots) \\ = (-\nu^2 + d^2/4 + \ell(\ell+d-2)) G_{\nu,\ell}^{(s)}(\hat{p}; \dots). \end{aligned} \quad (4.16)$$

Here the dots ... refer to additional labels for the complete set of such functions $G_{\nu,\ell}^{(s)}(\hat{p}; \dots)$, which will be given momentarily. These labels will be analogs of the quantum numbers (j, m) labeling spherical harmonics $Y_{jm}(\Omega)$ that provide a basis of scalar functions on the sphere. Here we decompose wave functions over $\mathbb{H}^{d+1}$, whose noncompact nature requires the use of continuous quantum numbers rather than discrete ones.

The first point to discuss is the appearance of two distinct spin parameters s and ℓ , the first one being associated with the little group $\text{SO}(d+1)$ of the massive particle, and the

second one being associated with the subgroup $M = \text{SO}(d)$ defined in (2.4) and characterizing the elementary representations of the Lorentz group. The decomposition of the particle representation into Lorentz UIRs will thus feature a decomposition of the tensor irrep $V_s^{\text{SO}(d+1)}$ into the tensor irreps $V_{\ell}^{\text{SO}(d)}$, simply given by

$$V_s^{\text{SO}(d+1)} = \bigoplus_{\ell=0}^s V_{\ell}^{\text{SO}(d)}, \quad (4.17)$$

with multiplicity one. For ease of exposition, we will first describe the spinless case $s = \ell = 0$ before treating generic integer spin s , which requires introducing some heavier technology. We will then analyze the half-integer spin case, which requires us to consider spinor representations of $\text{SO}(d+1)$ and $\text{SO}(d)$.

A. Zero spin

For $s = \ell = 0$, the set of regular solutions to the Laplace-Beltrami equation (4.16) is given by

$$G_{\nu}(\hat{p}; \vec{x}) = \frac{1}{(-\hat{p} \cdot q(\vec{x}))^{\Delta_{\nu}}}, \quad \Delta_{\nu} = \frac{d}{2} + i\nu, \quad (4.18)$$

with $\vec{x} \in \mathbb{R}^d$ a continuous label and $q^{\mu}(\vec{x})$ the null vector given in (3.5). The solutions (4.18) with $\nu \in \mathbb{R}^+$ and $\vec{x} \in \mathbb{R}^d$ form a basis of normalizable wave functions in $L^2(\mathbb{H}^{d+1}, [d^{d+1}\hat{p}])$. Within this functional space, a resolution of the identity is given by [38]

$$\int_0^{\infty} d\nu \mu(\nu) \int d^d \vec{x} G_{-\nu}(\hat{p}_1; \vec{x}) G_{\nu}(\hat{p}_2; \vec{x}) = \delta(\hat{p}_1 - \hat{p}_2), \quad (4.19)$$

with $\mu(\nu)$ the Plancherel measure

$$\mu(\nu) = \frac{\Gamma(\frac{d}{2} + i\nu) \Gamma(\frac{d}{2} - i\nu)}{4\pi^{d+1} \Gamma(i\nu) \Gamma(-i\nu)}. \quad (4.20)$$

This allows one to decompose a generic wave function $\psi(\hat{p})$ as

$$\psi(\hat{p}) = \int_0^{\infty} d\nu \mu(\nu) \int d^d \vec{x} G_{\nu}(\hat{p}; \vec{x}) \psi(\nu, \vec{x}), \quad (4.21)$$

with

$$\psi(\nu, \vec{x}) \equiv \int [d^{d+1}\hat{p}] G_{-\nu}(\hat{p}; \vec{x}) \psi(\hat{p}). \quad (4.22)$$

Even though they are not normalizable, a similar decomposition can be written in terms of basis states $|p\rangle$ and $|\nu, \vec{x}\rangle$, namely

$$|p\rangle = \int_0^\infty d\nu \mu(\nu) \int d^d \vec{x} G_{-\nu}(\hat{p}; \vec{x}) |\nu, \vec{x}\rangle, \quad (4.23)$$

with inverse

$$|\nu, \vec{x}\rangle = \int [d^{d+1} \hat{p}] G_\nu(\hat{p}; \vec{x}) |p\rangle. \quad (4.24)$$

Let us now explicitly show that the states (4.24) do indeed transform according to the Lorentz representations of the principal series described in Sec. II. Acting with a Lorentz transformation $U(\Lambda)$, we have

$$\begin{aligned} U(\Lambda) |\nu, \vec{x}\rangle &= \int [d^{d+1} \hat{p}] G_\nu(\hat{p}; \vec{x}) |\Lambda p\rangle \\ &= \int [d^{d+1} \hat{p}] G_\nu(\Lambda^{-1} \hat{p}; \vec{x}) |p\rangle, \end{aligned} \quad (4.25)$$

where we used the transformation of the momentum state $U(\Lambda) |p\rangle = |\Lambda p\rangle$, and the Lorentz invariance of the integral measure. At this point we recall that the Lorentz group $SO(1, d+1)$ is naturally realized as the group of conformal isometries of $\mathbb{R}^d$, which we abstractly denote $\vec{x} \mapsto \vec{x}' = \Lambda \vec{x}$ for any $\vec{x} \in \mathbb{R}^d$. Under this transformation the null vector (3.5) satisfies [25]

$$q^\mu(\vec{x}') = \Omega(x) \Lambda^\mu_\nu q^\nu(\vec{x}), \quad \Omega(x) = \left| \frac{\partial \vec{x}'}{\partial \vec{x}} \right|^{1/d}, \quad (4.26)$$

which allows us to write

$$G_\nu(\Lambda^{-1} \hat{p}; \vec{x}) = \frac{1}{(-\hat{p} \cdot \Lambda q(\vec{x}))^{\Delta_\nu}} = \left(\frac{\Omega(x)}{-\hat{p} \cdot q(\vec{x}')} \right)^{\Delta_\nu}. \quad (4.27)$$

Plugging this back into (4.25) yields

$$U(\Lambda) |\nu, \vec{x}\rangle = \Omega(x)^{\Delta_\nu} |\nu, \vec{x}'\rangle, \quad (4.28)$$

which is the transformation law (2.14) associated with a scalar representation of the principal series.

Using (4.21)–(4.24), the generic one-particle state (4.10) can be written

$$|\psi\rangle = \int_0^\infty d\nu \mu(\nu) \int d^d \vec{x} \psi(\nu, \vec{x}) |\nu, \vec{x}\rangle, \quad (4.29)$$

which is of the general form (2.15) with $\rho(\nu) = \Theta(\nu) \mu(\nu)$. As for massless particles, one can check that the Lorentz inner product (2.16) is recovered from the particle inner product (4.11) together with the decomposition (4.29). Indeed, we can write

$$\begin{aligned} \langle \phi | \psi \rangle &= \int [d^{d+1} \hat{p}] \phi(\hat{p})^* \psi(\hat{p}) \\ &= \int_0^\infty d\nu \mu(\nu) \int d^d \vec{x} \psi(\nu, \vec{x})^* \phi(\nu, \vec{x}), \end{aligned} \quad (4.30)$$

using the completeness relation (4.19), or the orthogonality relation

$$\int [d^{d+1} \hat{p}] G_{-\nu'}(\hat{p}; \vec{w}) G_\nu(\hat{p}; \vec{x}) = \frac{\delta(\nu - \nu') \delta(\vec{w} - \vec{x})}{\mu(\nu)}, \quad \nu, \nu' \in \mathbb{R}^+. \quad (4.31)$$

Finally, we wish to compute the action of translations $U(a) = e^{-ia^\mu \hat{p}_\mu}$ on the basis states $|\nu, \vec{x}\rangle$ such that the latter indeed assemble into the carrier space of the massive particle representation. This can be done using (4.23) and (4.24), yielding

$$\begin{aligned} U(a) |\nu, \vec{x}\rangle &= \int [d^{d+1} \hat{p}] G_\nu(\hat{p}; \vec{x}) e^{-ia^\mu \hat{p}_\mu} |p\rangle \\ &= \int_0^\infty d\nu' \mu(\nu') \int d^d \vec{y} A_3(a^\mu; \nu, \vec{x}; \nu', \vec{y}) |\nu', \vec{y}\rangle, \end{aligned} \quad (4.32)$$

with

$$A_3(a^\mu; \nu, \vec{x}; \nu', \vec{y}) \equiv \int [d^{d+1} \hat{p}] G_\nu(\hat{p}; \vec{x}) G_{-\nu'}(\hat{p}; \vec{y}) e^{-ia^\mu \hat{p}_\mu}. \quad (4.33)$$

As in the massless case, Eq. (4.32) shows that translations mix states of all scaling dimensions within the principal series. The coefficient function (4.33) is recognized as a “mixed” massive celestial three-point function, with one leg $a^\mu \in \mathbb{M}^{d+2}$ still in “position space.” In Appendix B we explain how to explicitly evaluate this three-point function.

B. Index-free and ambient space formalism

The treatment of symmetric tensor representation $V_s^{\text{SO}(d+1)}$, and their decomposition (4.17) in terms of representations $V_\ell^{\text{SO}(d)}$, will be facilitated by the use of *index-free* and *ambient space* formalisms. These were heavily used by the authors of [37,38] for their description of spinning representations in the context of the AdS/CFT correspondence, who incidentally also provided formulas which will play a key role in the present analysis.

Up to this point we have not given an explicit parametrization of the representations $V_\ell^{\text{SO}(d)}$ appearing in the above discussion. For later use it is useful to introduce their polynomial realization which is the basis of the *index-free* formalism. While we restrict here to totally symmetric traceless spin- ℓ tensor representations, a more general

discussion can be found for example in [48]. Given the totally symmetric and traceless (with respect to δ_{ij}) tensor field $f_\sigma(\vec{x}) = f_{i_1 \dots i_\ell}(\vec{x})$ in $\mathbb{R}^d$, we can construct a homogeneous polynomial of degree ℓ by contracting it with an auxiliary “polarization” vector $\vec{z} \in \mathbb{R}^d$,

$$f(\vec{x}, \vec{z}) \equiv f_{i_1 \dots i_\ell}(\vec{x}) z^{i_1} \dots z^{i_\ell}, \quad (4.34)$$

which can be shown to satisfy

$$(\vec{z} \cdot \partial_{\vec{z}} - \ell) f(\vec{x}, \vec{z}) = 0, \quad (\partial_{\vec{z}} \cdot \partial_{\vec{z}}) f(\vec{x}, \vec{z}) = 0. \quad (4.35)$$

Conversely, a polynomial satisfying (4.35) is equivalent to a totally symmetric and traceless tensor of rank ℓ . Restricted to the unit sphere $S^{d-1} \subset \mathbb{R}^d$ where $\vec{z} \cdot \vec{z} = 1$, such polynomials coincide with spherical harmonics of degree ℓ . The action of $\text{SO}(d)$ on $f(\vec{x}, \vec{z})$ is now explicitly implemented as a rotation of the polarization vector $\vec{z}$, which infinitesimally reads

$$\Sigma_{ij} f(\vec{x}, \vec{z}) = -i \left(z_i \frac{\partial}{\partial z^j} - z_j \frac{\partial}{\partial z^i} \right) f(\vec{x}, \vec{z}). \quad (4.36)$$

Following [37] it is convenient to complexify the polarization vector $\vec{z}$ and consider the restriction of $f(\vec{x}, \vec{z})$ to the submanifold $\vec{z} \cdot \vec{z} = 0$. This effectively removes any potential trace term such that the second condition in (4.35) becomes unnecessary.

The tensor field $f_{i_1 \dots i_\ell}(\vec{x})$ in $\mathbb{R}^d$ can also be encoded by a tensor field $f_{\mu_1 \dots \mu_\ell}(q)$ defined on the null cone $\text{LC}_{d+1}^+ \subset \mathbb{M}^{d+2}$ located at $q^2 \equiv \eta_{\mu\nu} q^\mu q^\nu = 0$, which is totally symmetric, traceless with respect to $\eta_{\mu\nu}$, and transverse with respect to q^μ , through the pullback

$$f_{i_1 \dots i_\ell}(\vec{x}) = \frac{\partial q^{\mu_1}}{\partial x^{i_1}} \dots \frac{\partial q^{\mu_\ell}}{\partial x^{i_\ell}} f_{\mu_1 \dots \mu_\ell}(q), \quad (4.37)$$

together with the parametrization $q^\mu(\vec{x})$ given in (3.5). This constitutes the basis of the *ambient space* formalism. Note that the extension of $f_{\mu_1 \dots \mu_\ell}(q)$ off the null cone $q^2 = 0$ is inconsequential. To get rid of the indices, we again construct a homogeneous polynomial of degree ℓ by contracting with the complex auxiliary vector Z^μ ,

$$f(q, Z) \equiv f_{\mu_1 \dots \mu_\ell}(q) Z^{\mu_1} \dots Z^{\mu_\ell}. \quad (4.38)$$

The polynomial $f(\vec{x}, \vec{z})$ is then obtained via

$$f(\vec{x}, \vec{z}) = f(q, Z), \quad (4.39)$$

with q^μ given as in (3.5) and

$$Z^\mu = (\vec{x} \cdot \vec{z}, \vec{z}, -\vec{x} \cdot \vec{z}). \quad (4.40)$$

The polarization vector Z^μ satisfies $Z^2 = \vec{z} \cdot \vec{z}$ and $Z \cdot q = 0$. Conversely, it was shown in [37] that a homogeneous polynomial $f(q, Z)$ of degree ℓ in $\mathbb{M}^{d+2}$, when restricted to the submanifold $Z^2 = Z \cdot q = q^2 = 0$, unambiguously encodes a symmetric traceless tensor field $f_{i_1 \dots i_\ell}(\vec{x})$ in $\mathbb{R}^d$. Its explicit construction proceeds through (4.37) and

$$f_{\mu_1 \dots \mu_\ell}(q) = \frac{1}{\ell!(h-1)_\ell} (D_Z) \dots (D_Z)_{\mu_\ell} f(q, Z), \quad (4.41)$$

with $h \equiv d/2$, $(a)_\ell \equiv \Gamma(a + \ell)/\Gamma(a)$ the Pochhammer symbol, and

$$(D_Z)_\mu \equiv \left(h - 1 + Z \cdot \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z^\mu}. \quad (4.42)$$

We note that this derivative operator can be used to implement tensor contractions at the level of the encoding polynomials, such as

$$f_{i_1 \dots i_\ell}(\vec{x}) g^{i_1 \dots i_\ell}(\vec{x}) = \frac{1}{\ell!(h-1)_\ell} f(q, D_Z) g(q, Z). \quad (4.43)$$

A similar description exists for the totally symmetric traceless spin- s tensor representations of $\text{SO}(d+1)$ [38]. Given the symmetric and traceless tensor field $h_\sigma(\hat{p}) = h_{a_1 \dots a_s}(\hat{p})$ in $\mathbb{H}^{d+1}$, there is a symmetric traceless tensor field $H_{\mu_1 \dots \mu_s}(\hat{p})$ in $\mathbb{M}^{d+2}$ such that⁶

$$h_{a_1 \dots a_s}(\mathbf{x}) = E(\hat{p})_{a_1}^{\mu_1} \dots E(\hat{p})_{a_s}^{\mu_s} H_{\mu_1 \dots \mu_s}(\hat{p}), \quad (4.44)$$

with $E(\hat{p})_a^\mu$ an orthonormal frame field satisfying $E_a^\mu \eta_{\mu\nu} E_b^\nu = \delta_{ab}$ and $E(\hat{p})_a^\mu \hat{p}_\mu = 0$. The latter can be explicitly chosen as

$$E(\hat{p})_a^0 = \hat{p}_a, \quad E(\hat{p})_a^b = \delta_a^b + (\hat{p}^0 + 1)^{-1} \hat{p}^b \hat{p}_a, \quad (4.45)$$

with inverse $E^{-1}(\hat{p})_\mu^a$ given by

$$E^{-1}(\hat{p})_0^a = -\hat{p}^a, \quad E^{-1}(\hat{p})_b^a = \delta_b^a + (\hat{p}^0 + 1)^{-1} \hat{p}^a \hat{p}_b. \quad (4.46)$$

They provide a resolution of the identity through the formula

$$\delta_\nu^\mu = -\hat{p}^\mu \hat{p}_\nu + E(\hat{p})_\nu^a E^{-1}(\hat{p})_\mu^a. \quad (4.47)$$

Again we construct a homogeneous polynomial of degree s by contracting with the complex polarization vector W^μ ,

⁶Note that the formulas in [38] are written with coordinate indices so that $\partial \hat{p}^\mu / \partial x^a$ appears in place of E_a^μ , while (4.44) is written with orthonormal frame indices.

$$H(\hat{p}, W) \equiv H_{\mu_1 \dots \mu_s}(\hat{p}) W^{\mu_1} \dots W^{\mu_s}, \quad (4.48)$$

satisfying

$$(W \cdot \partial_W - s)H(\hat{p}, W) = 0, \partial_W \cdot \partial_W H(\hat{p}, W) = 0. \quad (4.49)$$

The action of $SO(d+1)$ on $H(\hat{p}, W)$ is now explicitly implemented as rotations of the polarization vector W^μ , and infinitesimally reads

$$\Sigma_{ab} H(\hat{p}, W) = -i E_a^\mu E_b^\nu \left(W_\mu \frac{\partial}{\partial W^\nu} - W_\nu \frac{\partial}{\partial W^\mu} \right) H(\hat{p}, W). \quad (4.50)$$

Conversely, a homogeneous polynomial $H(\hat{p}, W)$ of degree s in $\mathbb{M}^{d+2}$, when restricted to the submanifold $W^2 = W \cdot \hat{p} = \hat{p}^2 + 1 = 0$, unambiguously encodes a symmetric traceless tensor field $h_{a_1 \dots a_s}(\hat{p})$ in $\mathbb{H}^{d+1}$. Its explicit construction proceeds through (4.44) and

$$H_{\mu_1 \dots \mu_s}(\hat{p}) = \frac{1}{s! \left(\frac{d-1}{2}\right)_s} (K_W)_{\mu_1} \dots (K_W)_{\mu_s} H(\hat{p}, W), \quad (4.51)$$

with

$$(K_W)_\mu \equiv \left(\frac{d-1}{2} + W \cdot \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W^\mu}. \quad (4.52)$$

Finally, the covariant derivative acting tangentially to $\mathbb{H}^{d+1} \subset \mathbb{M}^{d+2}$ takes the form

$$\mu_{s,\ell}(\nu) = \frac{2^{s-\ell} (\ell+1)_{s-\ell} (\ell + \frac{d-1}{2})_{s-\ell}}{(s-\ell)! (2\ell+d-1)_{s-\ell} (\ell+h-i\nu)_{s-\ell} (\ell+h+i\nu)_{s-\ell}} \left| \frac{\ell+h+i\nu-1}{h+i\nu-1} \right|^2 \mu(\nu), \quad (4.56)$$

with $\mu(\nu)$ as given in (4.20). The kernel $G_{\Delta,\ell}^{(s)}(\hat{p}, W; q, Z)$ is a homogeneous polynomial of degree s in W and degree ℓ in Z , and thus encodes a bitensor $G_{\Delta,\ell}^{(s)}(\hat{p}; q)_{a_1 \dots a_s; i_1 \dots i_\ell}$ traceless and symmetric in both sets of indices. Its explicit expression is

$$G_{\nu,\ell}^{(s)}(\hat{p}, W; q, Z) = (W \cdot \nabla)^{s-\ell} \Pi_{\nu,\ell}(\hat{p}, W; q, Z), \quad (4.57)$$

given in terms of the spin- ℓ “boundary-bulk propagator”

$$\Pi_{\nu,\ell}(\hat{p}, W; q, Z) \equiv \frac{(-\hat{p} \cdot q)(W \cdot Z) + (\hat{p} \cdot Z)(q \cdot W)^\ell}{(-\hat{p} \cdot q)^{\Delta_\nu + \ell}}, \quad (4.58)$$

$$\Delta_\nu = h + i\nu.$$

$$\nabla_\mu = \frac{\partial}{\partial \hat{p}^\mu} + \hat{p}_\mu \left(\hat{p} \cdot \frac{\partial}{\partial \hat{p}} \right) + W_\mu \left(\hat{p} \cdot \frac{\partial}{\partial W} \right). \quad (4.53)$$

The divergence of the tensor $H_{\mu_1 \dots \mu_s}(\hat{p})$, in index-free notation, can then be computed as

$$(\nabla \cdot H)(\hat{p}, W) = \frac{1}{s \left(\frac{d-3}{2} + s\right)} \nabla \cdot KH(\hat{p}, W). \quad (4.54)$$

Armed with this technology we are ready to resume our study of massive spinning particles and their decomposition into Lorentz UIRs.

C. Integer spin

Adopting the ambient space formalism with index-free notation, the generalization of the completeness relation (4.19) to the space of spin- s normalizable wave functions has been given in [38],

$$\begin{aligned} & \sum_{\ell=0}^s \frac{1}{\ell! (h-1)_\ell} \int_0^\infty d\nu \mu_{s,\ell}(\nu) \\ & \times \int d^d \vec{x} G_{\nu,\ell}^{(s)}(\hat{p}_1, W_1; q, D_Z) G_{-\nu,\ell}^{(s)}(\hat{p}_2, W_2; q, Z) \\ & = (W_1 \cdot W_2)^s \delta(\hat{p}_1, \hat{p}_2), \end{aligned} \quad (4.55)$$

where the null vector $q^\mu = q^\mu(\vec{x})$ implicitly depends on the integration variable $\vec{x} \in \mathbb{R}^d$ through (3.5), and

With respect to the ambient coordinates $\hat{p}^\mu$, the latter satisfies a Laplace-Beltrami equation on $\mathbb{H}^{d+1}$ as well as a divergence-free condition,

$$[\nabla^2 - \Delta_\nu(\Delta_\nu - d) + \ell] \Pi_{\nu,\ell}(\hat{p}, W; q, Z) = 0, \quad (4.59)$$

$$\nabla \cdot K \Pi_{\nu,\ell}(\hat{p}, W; q, Z) = 0. \quad (4.60)$$

Using (4.59) and the commutation relation [38]

$$[\nabla^2, (W \cdot \nabla)^n] = -n(d-1+2W \cdot \partial_W - n)(W \cdot \nabla)^n, \quad (4.61)$$

we can easily show

$$\begin{aligned} \nabla^2 G_{\nu,\ell}^{(s)}(\hat{p}, W; q, Z) &= [\Delta_\nu(\Delta_\nu - d) - \ell - (s-\ell)(d-1+s+\ell)] G_{\nu,\ell}^{(s)}(\hat{p}, W; q, Z) \\ &= [\Delta_\nu(\Delta_\nu - d) - s(s+d-1) + \ell(\ell+d-2)] G_{\nu,\ell}^{(s)}(\hat{p}, W; q, Z). \end{aligned} \quad (4.62)$$

We observe that this is exactly Eq. (4.16) that wave functions carrying a Lorentz UIR $[\Delta_\nu, \ell]$ of the spinning principal series must satisfy. It should now become clear that the summation $\sum_{\ell=0}^s$ in (4.55) implements the decomposition (4.17).

As in the spinless case, the completeness relation (4.55) allows us to decompose a generic spin- s wave function $\psi(\hat{p}, W)$ as

$$\psi(\hat{p}, W) = \sum_{\ell=0}^s \frac{1}{\ell!(h-1)_\ell} \int_0^\infty d\nu \mu_{s,\ell}(\nu) \times \int d^d \vec{x} G_{\nu,\ell}^{(s)}(\hat{p}, W; q, D_Z) \psi^{(\ell)}(\nu, q, Z), \quad (4.63)$$

with

$$\psi^{(\ell)}(\nu, q, Z) = \frac{1}{s! \binom{d-1}{2}_s} \int [d^{d+1} \hat{p}] G_{-\nu,\ell}^{(s)} \times (\hat{p}, K_W; q, Z) \psi(\hat{p}, W). \quad (4.64)$$

While $\psi(\hat{p}, W)$ is a degree- s polynomial in the variable W , $\psi^{(\ell)}(\nu, q, Z)$ is degree- ℓ polynomial in the variable Z . Even though they are not normalizable, a similar decomposition can be written in terms of basis states $|p, W\rangle \equiv |p\rangle_{\mu_1 \dots \mu_s} W^{\mu_1} \dots W^{\mu_s}$ and $|\nu, q, Z\rangle \equiv |\nu, \vec{x}\rangle_{\mu_1 \dots \mu_\ell} Z^{\mu_1} \dots Z^{\mu_\ell}$, namely

$$|p, W\rangle = \sum_{\ell=0}^s \frac{1}{\ell!(h-1)_\ell} \int_0^\infty d\nu \mu_{s,\ell}(\nu) \times \int d^d \vec{x} G_{-\nu,\ell}^{(s)}(\hat{p}, W; q, D_Z) |\nu, q, Z\rangle, \quad (4.65)$$

with inverse

$$|\nu, q, Z\rangle = \frac{1}{s! \binom{d-1}{2}_s} \int [d^{d+1} \hat{p}] G_{\nu,\ell}^{(s)}(\hat{p}, K_W; q, Z) |p, W\rangle. \quad (4.66)$$

Written with tensor indices, these equations amount to

$$|p\rangle_{a_1 \dots a_s} = \sum_{\ell=0}^s \int_0^\infty d\nu \mu_{s,\ell}(\nu) \times \int d^d \vec{x} [G_{-\nu,\ell}^{(s)}(\hat{p}; \vec{x})]_{a_1 \dots a_s}^{i_1 \dots i_\ell} |\nu, \vec{x}\rangle_{i_1 \dots i_\ell} \quad (4.67)$$

and

$$|\nu, \vec{x}\rangle_{i_1 \dots i_\ell} = \int [d^{d+1} \hat{p}] [G_{\nu,\ell}^{(s)}(\hat{p}; \vec{x})]_{i_1 \dots i_\ell}^{a_1 \dots a_s} |p\rangle_{a_1 \dots a_s}. \quad (4.68)$$

A generic one-particle state (4.10) can now be decomposed, using (4.63)–(4.66) together with the completeness

relation (4.55),⁷

$$|\psi\rangle = \frac{1}{s! \binom{d-1}{2}_s} \int [d^{d+1} \hat{p}] \psi(\hat{p}, K_W) |p, W\rangle = \sum_{\ell=0}^s \frac{1}{\ell!(h-1)_\ell} \int_0^\infty d\nu \mu_{s,\ell}(\nu) \times \int d^d \vec{x} \psi^{(\ell)}(\nu, q, D_Z) |\nu, q, Z\rangle. \quad (4.69)$$

As before, one can check that the Lorentz inner product (2.16) is recovered from the particle inner product (4.11) together with the decomposition (4.69). Using the same set of identities, we can indeed write

$$\langle \phi | \psi \rangle = \frac{1}{s! \binom{d-1}{2}_s} \int [d^{d+1} \hat{p}] \phi(\hat{p}, K_W)^* \psi(\hat{p}, W) = \sum_{\ell=0}^s \frac{1}{\ell!(h-1)_\ell} \int_0^\infty d\nu \mu_{s,\ell}(\nu) \times \int d^d \vec{x} \phi^{(\ell)}(\nu, q, D_Z)^* \psi^{(\ell)}(\nu, q, Z), \quad (4.70)$$

as expected from (2.16).

Let us show that the states (4.66) indeed transform by the Lorentz representations of the spinning principal series described in Sec. II. Acting with a Lorentz transformation $U(\Lambda)$, we have

$$U(\Lambda) |\nu, q, Z\rangle = \frac{1}{s! \binom{d-1}{2}_s} \int [d^{d+1} \hat{p}] G_{\nu,\ell}^{(s)}(\hat{p}, K_W; q, Z) U(\Lambda) |p, W\rangle. \quad (4.71)$$

At this point we need to specify the transformation $U(\Lambda) |p, W\rangle$ corresponding to the standard transformation (4.8) of the massive particle states. We claim that it is simply given by

$$U(\Lambda) |p, W\rangle = |\Lambda p, \Lambda W\rangle, \quad (4.72)$$

which in components reads

$$U(\Lambda) |p\rangle_{\mu_1 \dots \mu_s} = \Lambda^{\nu_1}_{\mu_1} \dots \Lambda^{\nu_s}_{\mu_s} |\Lambda p\rangle_{\nu_1 \dots \nu_s}. \quad (4.73)$$

Let us prove that (4.73) induces the correct transformation of massive vector particles $|p\rangle_a$, which straightforwardly generalizes to higher integer spin. Using (4.44), we can write

⁷Alternatively, one should be able to use the orthogonality relation given in Eq. (227) of [38].

$$\begin{aligned}
U(\Lambda)|p\rangle_a &= E(\hat{p})_a^\mu U(\Lambda)|p\rangle_\mu = E(\hat{p})_a^\mu \Lambda^\nu_\mu |\Lambda p\rangle_\nu \\
&= E(\hat{p})_a^\mu \Lambda^\nu_\mu (-\Lambda \hat{p})_\nu (\Lambda \hat{p})^\sigma \\
&\quad + E^{-1}(\Lambda \hat{p})_b^\nu E(\Lambda \hat{p})_b^\sigma |\Lambda p\rangle_\sigma \\
&= E^{-1}(\Lambda \hat{p})_b^\nu \Lambda^\nu_\mu E(\hat{p})_a^\mu |\Lambda p\rangle_b \\
&\quad - E(\hat{p})_a^\mu \Lambda^\nu_\mu (\Lambda \hat{p})_\nu (\Lambda \hat{p})^\sigma |\Lambda p\rangle_\sigma, \tag{4.74}
\end{aligned}$$

where we inserted the resolution of the identity (4.47) in writing the second line. The last term actually vanishes on account of the transversality $\hat{p}^\mu |p\rangle_\mu = 0$ of the ambient vector $|p\rangle_\mu$. We are therefore left with

$$U(\Lambda)|p\rangle_a = D(\Lambda, \hat{p})^b_a |\Lambda p\rangle_b, \tag{4.75}$$

where we defined the matrix

$$\begin{aligned}
U(\Lambda)|\nu, q, Z\rangle &= \frac{1}{s! \left(\frac{d-1}{2}\right)_s} \int [d^{d+1} \hat{p}] G_{\nu, \ell}^{(s)}(\hat{p}, K_W; q, Z) |\Lambda p, \Lambda W\rangle \\
&= \frac{1}{s! \left(\frac{d-1}{2}\right)_s} \int [d^{d+1} \hat{p}] G_{\nu, \ell}^{(s)}(\Lambda^{-1} \hat{p}, \Lambda^{-1} K_W; q, Z) |p, W'\rangle, \tag{4.77}
\end{aligned}$$

where we used the Lorentz invariance of the integral measure, and we renamed $W' = \Lambda W$ as well as used the explicit expression (4.52) to derive

$$(K_W)_\mu = \Lambda^\nu_\mu (K_{W'})_\nu, \quad K_W = \Lambda^{-1} K_{W'}. \tag{4.78}$$

From (4.57) and the relation (4.26), we can show that

$$\begin{aligned}
G_{\nu, \ell}^{(s)}(\Lambda^{-1} \hat{p}, \Lambda^{-1} K_W; q(\vec{x}), Z) \\
&= G_{\nu, \ell}^{(s)}(\hat{p}, K_W; \Lambda q(\vec{x}), \Lambda Z) \\
&= \Omega(x)^{\Delta_\nu} G_{\nu, \ell}^{(s)}(\hat{p}, K_W; q(\vec{x}'), \Lambda Z), \tag{4.79}
\end{aligned}$$

with $\vec{x}' = \Lambda \vec{x}$, such that

$$U(\Lambda)|\nu, q(\vec{x}), Z\rangle = \Omega(x)^{\Delta_\nu} |\nu, q(\vec{x}'), \Lambda Z\rangle. \tag{4.80}$$

In components this equation reads

$$U(\Lambda)|\nu, \vec{x}\rangle_{\mu_1 \dots \mu_\ell} = \Omega(x)^{\Delta_\nu} \Lambda^{\nu_1}_{\mu_1} \dots \Lambda^{\nu_s}_{\mu_s} |\nu, \vec{x}'\rangle_{\nu_1 \dots \nu_\ell}. \tag{4.81}$$

Now we apply (4.37) in order to extract the transformation of the physical states. We have

$$\begin{aligned}
U(\Lambda)|\nu, \vec{x}\rangle_{i_1 \dots i_\ell} &= \frac{\partial q^{\mu_1}}{\partial x^{i_1}} \dots \frac{\partial q^{\mu_\ell}}{\partial x^{i_\ell}} U(\Lambda)|\nu, \vec{x}\rangle_{\mu_1 \dots \mu_\ell} \\
&= \Omega(x)^{\Delta_\nu} \Lambda^{\nu_1}_{\mu_1} \frac{\partial q^{\mu_1}}{\partial x^{i_1}} \dots \Lambda^{\nu_s}_{\mu_s} \frac{\partial q^{\mu_\ell}}{\partial x^{i_\ell}} |\nu, \vec{x}'\rangle_{\nu_1 \dots \nu_\ell}. \tag{4.82}
\end{aligned}$$

Differentiating (4.26), we find

$$D(\Lambda, \hat{p})^b_a \equiv E^{-1}(\Lambda \hat{p})_b^\nu \Lambda^\nu_\mu E(\hat{p})_a^\mu. \tag{4.76}$$

Equation (4.75) is exactly the transformation (4.8) of a massive Wigner UIR, with $D(\Lambda, \hat{p})^b_a$ the explicit representation of the Wigner rotation (4.9) leaving the reference momentum (4.2) invariant. One can check that $E(\hat{p})_a^\mu = L(\hat{p})^\mu_a$ where $L(\hat{p})^\mu_\nu$ is the boost matrix (4.5) relating the generic momentum p^μ to the reference vector k^μ . This demonstrates the validity of the transformation (4.72). In Appendix A, we show that (4.72) also induces the correct Wigner rotations in the case of massless particles.

Returning to the Lorentz transformation of the states $|\nu, q, Z\rangle$, we thus have

$$\Lambda^\mu_\nu \frac{\partial q^\nu(\vec{x})}{\partial x^i} = \Omega(x)^{-1} \frac{\partial x'^j}{\partial x^i} \frac{\partial q^\mu(\vec{x}')}{\partial x'^j} + \frac{\partial}{\partial x^i} (\Omega(x)^{-1}) q^\mu(\vec{x}'). \tag{4.83}$$

The second term does not contribute to (4.82) thanks to the transversality $q^\mu(\vec{x}')|\nu, \vec{x}'\rangle_{\mu_1 \dots \mu_\ell} = 0$ of the ambient tensor $|\nu, \vec{x}'\rangle_{\mu_1 \dots \mu_\ell} = |\nu, q(\vec{x}')\rangle_{\mu_1 \dots \mu_\ell}$. Hence,

$$\begin{aligned}
U(\Lambda)|\nu, \vec{x}\rangle_{i_1 \dots i_\ell} &= \Omega(x)^{\Delta_\nu} \frac{\partial x'^{j_1}}{\partial x^{i_1}} \frac{\partial q^{\mu_1}(\vec{x}')}{\partial x'^{j_1}} \dots \frac{\partial x'^{j_\ell}}{\partial x^{i_\ell}} \\
&\quad \times \frac{\partial q^{\mu_\ell}(\vec{x}')}{\partial x'^{j_\ell}} |\nu, \vec{x}'\rangle_{\mu_1 \dots \mu_\ell} \\
&= \Omega(x)^{\Delta_\nu} \frac{\partial x'^{j_1}}{\partial x^{i_1}} \dots \frac{\partial x'^{j_\ell}}{\partial x^{i_\ell}} |\nu, \vec{x}'\rangle_{j_1 \dots j_\ell} \\
&= \Omega(x)^{\Delta_\nu} R^{j_1}_{i_1}(x) \dots R^{j_\ell}_{i_\ell}(x) |\nu, \vec{x}'\rangle_{j_1 \dots j_\ell}, \tag{4.84}
\end{aligned}$$

using (2.13) in the last equality. As expected, we recover the transformation (2.14) of a Lorentz UIR $[\Delta_\nu, \ell]$ of the spinning principal series.

D. Half spin

Finally, we turn to the case of massive particles with half-integer spin. These are characterized by a spinor representation of the little group $\text{SO}(d+1)$, i.e., a unitary representation of $\text{Spin}(d+1)$. As for tensors, it is useful to embed these spinors of $\text{SO}(d+1)$ into Lorentz-covariant spinors of $\text{SO}(1, d+1)$. The canonical way of describing

Lorentz-covariant spinors is to introduce the Dirac matrices Γ_μ of dimension $2^{\lfloor \frac{d}{2} \rfloor + 1}$ that satisfy the anticommutation relations

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\eta_{\mu\nu} \mathbb{1}. \quad (4.85)$$

The matrices Γ_a are Hermitian, while Γ_0 is anti-Hermitian. The Lorentz generators in this representation are then defined as

$$\Sigma_{\mu\nu} = \frac{i}{4} [\Gamma_\mu, \Gamma_\nu]. \quad (4.86)$$

In this way we can define spinor wave functions $\Psi(\hat{p})$ which are Dirac spinors of $\text{SO}(1, d+1)$ of dimension $2^{\lfloor \frac{d}{2} \rfloor + 1}$; i.e., under Lorentz transformation they behave as

$$U(\Lambda)\Psi(\hat{p}) = S(\Lambda)^{-1}\Psi(\Lambda\hat{p}), \quad (4.87)$$

with $S(\Lambda)$ the Dirac spinor representation of $\text{SO}(1, d+1)$. However, these are not irreducible spinors with respect to the massive little group $\text{SO}(d+1)$ generated by $\langle J_{ab} \rangle$. Since the matrix Γ_0 commutes with all the rotation matrices Σ_{ab} and further satisfies $(\Gamma_0)^2 = -\mathbb{1}$, it can be used to define the projectors

$$\Pi_\pm(\hat{k}) \equiv \frac{1}{2}(\mathbb{1} \pm i\Gamma_0), \quad (4.88)$$

which project onto the subspaces corresponding to irreducible spinor representations of $\text{SO}(d+1)$, each one of dimension $2^{\lfloor \frac{d}{2} \rfloor}$. In particular one can diagonalize Γ_0 and group its eigenspinors according to their eigenvalues,

$$\Gamma_0 u_\sigma^+(\hat{k}) = -iu_\sigma^+(\hat{k}), \quad \Gamma_0 u_\sigma^-(\hat{k}) = iu_\sigma^-(\hat{k}), \quad (4.89)$$

with the index running from $\sigma = 1, \dots, \lfloor \frac{d}{2} \rfloor$. This obviously implies

$$\Pi_\pm(\hat{k})u_\sigma^\pm(\hat{k}) = u_\sigma^\pm(\hat{k}), \quad \Pi_\pm(\hat{k})u_\sigma^\mp(\hat{k}) = 0. \quad (4.90)$$

The above equations, valid at the reference momentum $\hat{k}$, can be generalized to an arbitrary momentum $\hat{p}$ by applying the spinor representation $S(L(\hat{p}))$ of the boost element $L(\hat{p})$, and introducing

$$u_\sigma^\pm(\hat{p}) = S(L(\hat{p}))u_\sigma^\pm(\hat{k}), \quad (4.91)$$

$$\Pi_\pm(\hat{p}) = S(L(\hat{p}))\Pi_\pm(\hat{k})S(L(\hat{p}))^{-1} = \frac{1}{2}(\mathbb{1} \pm i\hat{p}^\mu \Gamma_\mu). \quad (4.92)$$

Introducing the dual spinors $\bar{u}_\sigma^\pm(\hat{k})$ and $\bar{u}_\sigma^\pm(\hat{p})$ satisfying

$$\begin{aligned} \bar{u}_\sigma^\pm(\hat{p})u_\sigma^\pm(\hat{p}) &= \bar{u}_\sigma^\pm(\hat{k})u_\sigma^\pm(\hat{k}) = \delta_{\sigma\sigma'}, \\ \bar{u}_\sigma^\pm(\hat{p})u_\sigma^\mp(\hat{p}) &= \bar{u}_\sigma^\pm(\hat{k})u_\sigma^\mp(\hat{k}) = 0, \end{aligned} \quad (4.93)$$

as well as

$$\bar{u}_\sigma^\pm(\hat{p}) = \bar{u}_\sigma^\pm(\hat{k})S(L(\hat{p}))^{-1}, \quad (4.94)$$

we can give a resolution of the projectors $\Pi_\pm(\hat{p})$,

$$\Pi_\pm(\hat{p}) = u^{\pm\sigma}(\hat{p})\bar{u}_\sigma^\pm(\hat{p}), \quad (4.95)$$

where σ indices are raised and lowered with the Kronecker delta $\delta_{\sigma\sigma'}$. The dual spinors $\bar{u}_\sigma^\pm(\hat{p})$ can be used to project onto the irreducible $\text{SO}(d+1)$ spinors transforming according to Wigner's induced representations, defined as

$$\psi_\sigma^\pm(\hat{p}) \equiv \bar{u}_\sigma^\pm(\hat{p})\Psi(\hat{p}). \quad (4.96)$$

Indeed, given the Lorentz transformation (4.87) of the Dirac spinor $\Psi(\hat{p})$, for the wave functions (4.96), we have

$$\begin{aligned} U(\Lambda)\psi_\sigma^\pm(\hat{p}) &= \bar{u}_\sigma^\pm(\hat{p})S(\Lambda)^{-1}\Psi(\Lambda\hat{p}) \\ &= \bar{u}_\sigma^\pm(\hat{p})S(\Lambda)^{-1}(u^{+\rho}(\Lambda\hat{p})\bar{u}_\rho^+(\Lambda\hat{p}) + u^{-\rho}(\Lambda\hat{p})\bar{u}_\rho^-(\Lambda\hat{p}))\Psi(\Lambda\hat{p}) \\ &= \bar{u}_\sigma^\pm(\hat{p})S(\Lambda)^{-1}u^{\pm\rho}(\Lambda\hat{p})\bar{u}_\rho^\pm(\Lambda\hat{p})\Psi(\Lambda\hat{p}) \\ &= \bar{u}_\sigma^\pm(\hat{k})S(L(\hat{p}))^{-1}S(\Lambda)^{-1}S(L(\Lambda\hat{p}))u^{\pm\rho}(\hat{k})\psi_\rho^\pm(\Lambda\hat{p}) \\ &= \bar{u}_\sigma^\pm(\hat{k})S(L^{-1}(\Lambda\hat{p})\Lambda L(\hat{p}))^{-1}u^{\pm\rho}(\hat{k})\psi_\rho^\pm(\Lambda\hat{p}) \\ &= [D^\pm(W(\Lambda, \hat{p}))^{-1}]_{\sigma'}^{\sigma} \psi_{\sigma'}^\pm(\Lambda\hat{p}), \end{aligned} \quad (4.97)$$

where $W(\Lambda, \hat{p}) \in \text{SO}(d+1)$ is the little group element (4.9) and

$$[D^\pm(W(\Lambda, \hat{p}))^{-1}]_{\sigma'}^{\sigma} \equiv \bar{u}_\sigma^\pm(\hat{k})S(W(\Lambda, \hat{p}))^{-1}u^{\pm\sigma'}(\hat{k}) \quad (4.98)$$

are the components of one of its unitary irreducible spinor representations. To summarize, a Lorentz spinor $\Psi(\hat{p})$ behaving as (4.87) breaks into two irreducible spinor particle representations $\psi_\sigma^\pm(\hat{p})$. Of course, up to this

point this is just a recap of the well-known theory of Dirac.

As in the case of massive particles with integer spin, the use of the Lorentz-covariant spinors $\Psi(\hat{p})$ will facilitate the decomposition into elementary representations of Lorentz. The resolution of the identity we need has been given by one of us in [28],

$$\int_0^\infty d\nu \mu_{\frac{1}{2}}(\nu) \int d^d \vec{x} \frac{\Pi_\pm(\hat{p}_1) q^\mu(\vec{x}) \Gamma_\mu \Pi_\pm(\hat{p}_2)}{(-\hat{p}_1 \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}} (-\hat{p}_2 \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}}} = \mp i \delta(\hat{p}_1 - \hat{p}_2) \Pi_\pm(\hat{p}_2), \quad (4.99)$$

with

$$\mu_{\frac{1}{2}}(\nu) = \frac{1}{2\pi^{d+1}} \frac{\Gamma(\frac{d+1}{2} + i\nu) \Gamma(\frac{d+1}{2} - i\nu)}{\Gamma(\frac{1}{2} + i\nu) \Gamma(\frac{1}{2} - i\nu)}. \quad (4.100)$$

The corresponding orthogonality relation in this case is obtained from Appendix C of [28],

$$\begin{aligned} & \pm 2i \int [d^{d+1} \hat{p}] \frac{(-1 \quad \gamma^i x_i) \Pi_\pm(\hat{p}) (\gamma^{i' x_{i'}})}{(-\hat{p} \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}} (-\hat{p} \cdot q(\vec{x}'))^{\Delta_{\nu'} + \frac{1}{2}}} \\ &= \frac{\delta(\nu - \nu') \delta^{(d)}(\vec{x} - \vec{x}')}{\mu_{\frac{1}{2}}(\nu)}, \end{aligned} \quad (4.101)$$

for $\nu, \nu' \in \mathbb{R}^+$. The presence of the projector $\Pi_\pm(\hat{p})$ indicates that the resolution of the identity (4.99) is within the subspace of Lorentz-covariant Dirac spinors that correspond to a single set of Wigner particles. Acting with (4.99) on a generic spinor wave function $\Psi(\hat{p}_2)$ and integrating over $\hat{p}_2$ thus yields

$$\begin{aligned} \Pi_\pm(\hat{p}) \Psi(\hat{p}) &= \pm i \Pi_\pm(\hat{p}) \int_0^\infty d\nu \mu_{\frac{1}{2}}(\nu) \\ &\quad \times \int d^d \vec{x} \frac{q^\mu(\vec{x}) \Gamma_\mu \Psi_\pm(\nu, \vec{x})}{(-\hat{p} \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}}}, \end{aligned} \quad (4.102)$$

with

$$\Psi_\pm(\nu, \vec{x}) = \int [d^{d+1} \hat{p}] \frac{\Pi_\pm(\hat{p}) \Psi(\hat{p})}{(-\hat{p} \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}}}. \quad (4.103)$$

This decomposition of the particle wave function $\Pi_\pm(\hat{p}) \Psi(\hat{p})$ is completely analogous to the spinless case described in (4.21) and (4.22) up to the shift $\Delta_\nu \mapsto \Delta_\nu + \frac{1}{2}$. The qualitatively new ingredient is the presence of the operator $q^\mu(\vec{x}) \Gamma_\mu$ in (4.102), whose role is to ensure that $q^\mu(\vec{x}) \Gamma_\mu \Psi_\pm(\nu, \vec{x})$ carries an elementary representation of the Lorentz group. To understand this point better, let us focus on the locus $\vec{x} = 0$ where the inducing representation (2.7) in the construction of elementary representation of Lorentz is defined. In particular, within an elementary

representation, the generator $D = -J_{0(d+1)}$ is proportional to the identity. This is not the case of the Dirac representation $\Sigma_{0(d+1)} = \frac{i}{4} [\Gamma_0, \Gamma_{d+1}]$, and we should therefore project onto the subspaces of spinors that have a common eigenvalue with respect to $\Sigma_{0(d+1)}$. This is achieved with the projectors

$$P_\pm(0) = \frac{1}{2} (1 \pm 2i \Sigma_{0(d+1)}), \quad (4.104)$$

which project onto subspaces that carry spinor representations of $\text{SO}(d)$ of dimension $2^{\frac{d}{2}}$, the correct dimension for Dirac spinors of $\text{SO}(d)$. To be very explicit, let us adopt the following representation of the gamma matrices Γ_μ :

$$\Gamma_i = \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_{d+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4.105)$$

in terms of the lower-dimensional gamma matrices γ_i (and we denote the $2^{\frac{d}{2}} \times 2^{\frac{d}{2}}$ unit matrix by 1). Within this representation, the rotations Σ_{ij} and boost matrix $\Sigma_{0(d+1)}$ take the explicit form

$$\Sigma_{ij} = \frac{i}{4} \begin{pmatrix} [\gamma_i, \gamma_j] & 0 \\ 0 & [\gamma_i, \gamma_j] \end{pmatrix}, \quad \Sigma_{0(d+1)} = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.106)$$

Thus, if we write a generic Dirac spinor of $\text{SO}(1, d+1)$ as

$$\Psi(0) = \begin{pmatrix} \psi_+(0) \\ \psi_-(0) \end{pmatrix}, \quad (4.107)$$

we see that $\psi_\pm(0)$ are Dirac spinors of $\text{SO}(d)$, and that the role of (4.104) is precisely to project onto the subspaces they span,

$$P_+(0) \Psi(0) = \begin{pmatrix} \psi_+(0) \\ 0 \end{pmatrix}, \quad P_-(0) \Psi(0) = \begin{pmatrix} 0 \\ \psi_-(0) \end{pmatrix}. \quad (4.108)$$

One can extract the Dirac spinors of $\text{SO}(d)$ by using the basis $v_\pm^\sigma(0)$ of Lorentz-covariant spinors, grouped into the following matrices:

$$v_+(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_-(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.109)$$

with duals

$$\bar{v}_+(0) = (1 \quad 0), \quad \bar{v}_-(0) = (0 \quad 1), \quad (4.110)$$

such that

$$\bar{v}_\pm(0)\Psi(0) = \psi_\pm(0). \quad (4.111)$$

To get the full structure (2.7) of the elementary representation at $\vec{x} = 0$, we should also check that the generators $K_i = J_{0i} + J_{(d+1)i}$ act trivially. Their representation on Lorentz-covariant spinors is given by

$$K_i^{(\Gamma)} \equiv \Sigma_{0i} + \Sigma_{(d+1)i} = i \begin{pmatrix} 0 & 0 \\ \gamma_i & 0 \end{pmatrix}, \quad (4.112)$$

such that its action on a generic spinor $\Psi(0)$ yields

$$K_i^{(\Gamma)}\Psi(0) = \begin{pmatrix} 0 \\ i\gamma_i\psi_+(0) \end{pmatrix}. \quad (4.113)$$

Since only $P_-(0)\Psi(0)$ is annihilated by $K_i^{(\Gamma)}$, it is the only component of $\Psi(0)$ which can be associated with an elementary representation of Lorentz. We can now “translate” the above spinors to a generic point $\vec{x}$ by application of the operators $e^{-ix^i P_i}$ in the Lorentz-covariant spinor representation. We have

$$P_i^{(\Gamma)} \equiv \Sigma_{0i} - \Sigma_{(d+1)i} = i \begin{pmatrix} 0 & \gamma_i \\ 0 & 0 \end{pmatrix}, \quad (4.114)$$

such that the latter is given by

$$e^{-ix^i P_i^{(\Gamma)}} = \mathbb{1} - ix^i P_i^{(\Gamma)} = \begin{pmatrix} 1 & x^i \gamma_i \\ 0 & 1 \end{pmatrix}. \quad (4.115)$$

We therefore have the translated basis spinors

$$\begin{aligned} v_+(\vec{x}) &\equiv e^{-ix^i P_i^{(\Gamma)}} v_+(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ v_-(\vec{x}) &\equiv e^{-ix^i P_i^{(\Gamma)}} v_-(0) = \begin{pmatrix} x^i \gamma_i \\ 1 \end{pmatrix}, \end{aligned} \quad (4.116)$$

with dual spinors

$$\bar{v}_+(\vec{x}) = (1 \quad -x^i \gamma_i), \quad \bar{v}_-(\vec{x}) = (0 \quad 1). \quad (4.117)$$

An explicit computation then shows that the quantity $q^\mu(\vec{x})\Gamma_\mu$ of interest can be simply written as

$$q^\mu(\vec{x})\Gamma_\mu = 2 \begin{pmatrix} x^i \gamma_i & -x^2 \\ 1 & -x^i \gamma_i \end{pmatrix} = 2v_-(\vec{x})\bar{v}_+(\vec{x}). \quad (4.118)$$

Applied on a generic Dirac spinor

$$\Psi(\vec{x}) = \begin{pmatrix} \psi_1(\vec{x}) \\ \psi_2(\vec{x}) \end{pmatrix}, \quad (4.119)$$

it thus yields

$$\begin{aligned} q^\mu(\vec{x})\Gamma_\mu\Psi(\vec{x}) &= 2v_-^\sigma(\vec{x})[\bar{v}_+(\vec{x})\Psi(\vec{x})]_\sigma \\ &= 2v_-^\sigma(\vec{x})[\psi_1(\vec{x}) - x^i \gamma_i \psi_2(\vec{x})]_\sigma, \end{aligned} \quad (4.120)$$

where we reinstated σ indices for clarity. On the one hand, the above quantity is a linear combination of the Dirac spinors $v_-^\sigma(\vec{x})$ that are annihilated by $K_i^{(\Gamma)}$ at $\vec{x} = 0$. On the other hand, it is proportional to the $\text{SO}(d)$ Dirac spinor wave function $\psi_1(\vec{x}) - x^i \gamma_i \psi_2(\vec{x})$ which, provided $\Psi(\vec{x})$ transforms like a Lorentz-covariant conformal spinor [see (4.123) below], is known to be the only combination of $\psi_1(\vec{x})$ and $\psi_2(\vec{x})$ which carries an elementary representation of the Lorentz group [49]. With this understanding at hand, we rewrite the decomposition (4.102) in the slightly more transparent way,

$$\begin{aligned} \Pi_\pm(\hat{p})\Psi(\hat{p}) &= \pm 2i\Pi_\pm(\hat{p}) \int_0^\infty d\nu \mu_{\frac{1}{2}}(\nu) \\ &\times \int d^d \vec{x} \frac{v_-^\sigma(\vec{x})[\bar{v}_+(\vec{x})\Psi_\pm(\nu, \vec{x})]_\sigma}{(-\hat{p} \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}}}. \end{aligned} \quad (4.121)$$

These equations express the decomposition of the Lorentz-covariant Dirac spinors $\Pi_\pm(\hat{p})\Psi(\hat{p})$ in terms of $\bar{v}_+(\vec{x})\Psi_\pm(\nu, \vec{x})$ which are Dirac spinors of $\text{SO}(d)$. Let us now complete the proof that these transform according to elementary representations of the principal continuous series. Starting from

$$U(\Lambda)\Pi_\pm(\hat{p})\Psi(\hat{p}) = S(\Lambda)^{-1}\Pi_\pm(\Lambda\hat{p})\Psi(\Lambda\hat{p}), \quad (4.122)$$

we have

$$\begin{aligned} U(\Lambda)\Psi_\pm(\nu, \vec{x}) &= S(\Lambda)^{-1} \int [d^{d+1}\hat{p}] \frac{\Pi_\pm(\Lambda\hat{p})\Psi(\Lambda\hat{p})}{(-\hat{p} \cdot q(\vec{x}))^{\Delta_\nu + \frac{1}{2}}} \\ &= S(\Lambda)^{-1} \int [d^{d+1}\hat{p}] \frac{\Pi_\pm(\hat{p})\Psi(\hat{p})}{(-\hat{p} \cdot \Lambda q(\vec{x}))^{\Delta_\nu + \frac{1}{2}}} \\ &= \Omega(x)^{\Delta_\nu + \frac{1}{2}} S(\Lambda)^{-1} \Psi_\pm(\nu, \vec{x}). \end{aligned} \quad (4.123)$$

This is the “conformal Lorentz-covariant spinor transformation” mentioned above. Calling the result obtained by Isono [49], we thus have

$$\begin{aligned} U(\Lambda)[\bar{v}_+(\vec{x})\Psi_\pm(\nu, \vec{x})] \\ = \Omega(x)^{\Delta_\nu} \tilde{S}(R(\vec{x}))^{-1}[\bar{v}_+(\vec{x}')\Psi_\pm(\nu, \vec{x}')], \end{aligned} \quad (4.124)$$

with $\tilde{S}(R(x))$ the Dirac spinor representation of $R(x) \in \text{SO}(d)$ given in (2.13), which is to say that $\bar{v}_+(\vec{x})\Psi_\pm(\nu, \vec{x})$ transforms according to the unitary representation $[\Delta_\nu^*, \tilde{S}]$ of the principal continuous series.⁸ As in the previous sections, the decomposition formula (4.121) is also easily adapted to the basis states instead of the wave functions.

$$\Pi_\pm(\hat{p})\Psi(\hat{p}, W) = \pm 2i\Pi_\pm(\hat{p}) \sum_{\ell=0}^s \frac{1}{\ell!(h-1)_\ell} \int_0^\infty d\nu \mu_{s+\frac{1}{2}, \ell}(\nu) \int d^d \vec{x} G_{\nu, \ell+\frac{1}{2}}^{(s)}(\hat{p}, W; q, D_Z) v_-^\sigma(\vec{x}) [\bar{v}_+(\vec{x})\Psi_\pm^{(\ell)}(\nu, \vec{x}, Z)]_\sigma, \quad (4.125)$$

with

$$\Psi_\pm^{(\ell)}(\nu, q, Z) = \frac{1}{s! \binom{d-1}{2}_s} \int [d^{d+1} \hat{p}] G_{-\nu, \ell+\frac{1}{2}}^{(s)}(\hat{p}, K_W; q, Z) \Pi_\pm(\hat{p})\Psi(\hat{p}, W), \quad (4.126)$$

with $\mu_{s+\frac{1}{2}, \ell}(\nu)$ a measure left to determine. This provides the decomposition of a Lorentz-covariant tensor-spinor $\Pi_\pm(\hat{p})\Psi(\hat{p}, W)$ describing spin- $(s+\frac{1}{2})$ particles, into elementary representations $\Phi_\pm^{(\ell)}(\nu, \vec{x}, Z) \equiv \bar{v}_+(\vec{x})\Psi_\pm^{(\ell)}(\nu, \vec{x}, Z)$ carrying tensor-spinor representations of $\text{SO}(d)$. While we are confident that (4.125) and (4.126) provide the correct decomposition, at this point we have not been able to determine the measure $\mu_{s+\frac{1}{2}, \ell}(\nu)$. We hope to resolve this point in the near future, perhaps adapting the determination of $\mu_{s, \ell}(\nu)$ given in Appendix D of [38]. Note that without further conditions, the tensor-spinors appearing here are reducible. To start with, working with an irreducible set of spin- $(s+\frac{1}{2})$ wave functions $\Pi_\pm(\hat{p})\Psi(\hat{p}, W)$ instructs us to supplement the condition $\Gamma^\mu \Psi(\hat{p})_{\mu \dots \mu_s} = 0$. Second, we need to decompose the conformal tensor-spinor $\Phi_\pm^{(\ell)}(\nu, \vec{x}, Z)$ over its γ -trace and γ -traceless components [50]. Defining

$$\mathbf{P}_i^j = \alpha_\ell \gamma_i \gamma^j, \quad \bar{\mathbf{P}}_i^j = \delta_i^j - \alpha_\ell \gamma_i \gamma^j, \quad \alpha_\ell = \frac{\ell}{2(h-1+\ell)}, \quad (4.127)$$

the γ -traceless component is explicitly given by

$$(\Pi^{(\ell+\frac{1}{2})}\Phi_\pm^{(\ell)}(\nu, \vec{x}))_{i_1 \dots i_\ell} \equiv \frac{1}{\ell!} \bar{\mathbf{P}}_{\{i_1}^j \Phi_{i_2 \dots i_\ell\}j}, \quad (4.128)$$

since it can be shown to satisfy

$$\gamma^{i_1} (\Pi^{(\ell+\frac{1}{2})}\Phi_\pm^{(\ell)}(\nu, \vec{x}))_{i_1 \dots i_\ell} = 0. \quad (4.129)$$

⁸Note that the Dirac spinor representation of $\text{SO}(d)$ can be further reduced to two Weyl spinors when d is odd.

E. Half-integer spin

While we treated massive spin- $\frac{1}{2}$ particles in the previous subsection, one can in principle generalize to arbitrary half-integer spin $s+\frac{1}{2}$. All one needs to do is to consider the Lorentz-covariant Dirac spinor $\Pi_\pm(\hat{p})\Psi(\hat{p}, W)$ whose components are homogeneous polynomials of degree s in W^μ . One obtains

Similarly we define the γ -trace component

$$(\Pi^{(\ell-\frac{1}{2})}\Phi_\pm^{(\ell)}(\nu, \vec{x}))_{i_1 \dots i_\ell} \equiv \frac{1}{\ell!} \mathbf{P}_{\{i_1}^j \Phi_{i_2 \dots i_\ell\}j}, \quad (4.130)$$

and we can write the decomposition

$$\begin{aligned} \Phi_\pm^{(\ell)}(\nu, \vec{x})_{i_1 \dots i_\ell} &= (\Pi^{(\ell+\frac{1}{2})}\Phi_\pm^{(\ell)}(\nu, \vec{x}))_{i_1 \dots i_\ell} \\ &\quad + (\Pi^{(\ell-\frac{1}{2})}\Phi_\pm^{(\ell)}(\nu, \vec{x}))_{i_1 \dots i_\ell}. \end{aligned} \quad (4.131)$$

The wave functions $\Pi^{(\ell \pm \frac{1}{2})}\Phi_\pm^{(\ell)}(\nu, \vec{x})$ carry the two irreducible spin- $(\ell \pm \frac{1}{2})$ representations of $\text{SO}(d)$ [50].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: MASSLESS WIGNER ROTATIONS

In this appendix we use the ambient space formalism to derive the finite form of the transformation of the spin- ℓ states $|\nu, \vec{x}\rangle_\sigma = |\nu, \vec{x}\rangle_{i_1 \dots i_\ell}$ given in (3.16) that appear in the decomposition of massless particle states. Given the particle states $|p\rangle_{i_1 \dots i_\ell}$ with $p = \omega q$ as given in (3.5), we introduce the ambient tensor

$$|p, Z\rangle = |p\rangle_{\mu_1 \dots \mu_\ell} Z^{\mu_1} \dots Z^{\mu_\ell}, \quad (\text{A1})$$

following the general discussion around (4.38).

We first give the explicit expression of the boost matrix $L(q)^\mu{}_\nu$ relating a generic null momentum $q^\mu(\vec{x})$ to the reference momentum

$$k^\mu = (1, \vec{0}, 1). \quad (\text{A2})$$

Introducing a second null vector $n^\mu = (1, \vec{0}, -1)$ satisfying $n \cdot q(\vec{x}) = -2$, we can write its components explicitly as

$$\begin{aligned} L(q)^\mu{}_0 &= \frac{1}{2}(q^\mu + n^\mu), \\ L(q)^\mu{}_i &= \frac{1}{2} \frac{\partial q^\mu(\vec{x})}{\partial x^i}, \\ L(q)^\mu{}_{d+1} &= \frac{1}{2}(q^\mu - n^\mu), \end{aligned} \quad (\text{A3})$$

and for its inverse we have

$$\begin{aligned} L^{-1}(q)^0{}_\mu &= -\frac{1}{2}(q_\mu + n_\mu), \\ L^{-1}(q)^i{}_\mu &= \frac{1}{2} \delta^{ij} \frac{\partial q_\mu}{\partial x^j}, \\ L^{-1}(q)^{d+1}{}_\mu &= \frac{1}{2}(q_\mu - n_\mu). \end{aligned} \quad (\text{A4})$$

This immediately yields a resolution of the identity in the form

$$\delta^\mu{}_\nu = L(q)^\mu{}_\alpha L^{-1}(q)^\alpha{}_\nu = -\frac{1}{2}(q^\mu n_\nu + n^\mu q_\nu) + \frac{1}{4} \frac{\partial q^\mu}{\partial x^i} \delta^{ij} \frac{\partial q_\nu}{\partial x^j}. \quad (\text{A5})$$

With this boost matrix at hand, we can show that the transformation

$$\begin{aligned} U(\Lambda)|p, Z\rangle &= |\Lambda p, \Lambda Z\rangle, \\ U(\Lambda)|p\rangle_{\mu_1 \dots \mu_\ell} &= \Lambda^{\nu_1}{}_{\mu_1} \dots \Lambda^{\nu_\ell}{}_{\mu_\ell} |\Lambda p\rangle_{\nu_1 \dots \nu_\ell}, \end{aligned} \quad (\text{A6})$$

is the ambient version of Wigner's transformation (4.8). Let us prove it for a vector representation, also known as a photon, for simplicity of notations but without loss of generality. Thus, starting from

$$U(\Lambda)|p\rangle_\mu = \Lambda^\nu{}_\mu |\Lambda p\rangle_\nu, \quad (\text{A7})$$

for the physical states we get

$$\begin{aligned} U(\Lambda)|p\rangle_i &= \frac{\partial q^\mu}{\partial x^i} U(\Lambda)|p\rangle_\mu = \frac{\partial q^\mu}{\partial x^i} \Lambda^\nu{}_\mu |\Lambda p\rangle_\nu \\ &= \frac{\partial q^\mu}{\partial x^i} \Lambda^\nu{}_\mu \left(-\frac{1}{2} (\Lambda q^\sigma n_\nu + n^\sigma \Lambda q_\nu) \right. \\ &\quad \left. + \frac{1}{4} \frac{\partial \Lambda q^\sigma}{\partial x^j} \delta^{jk} \frac{\partial \Lambda q_\nu}{\partial x^k} \right) |\Lambda p\rangle_\sigma \\ &= \frac{1}{4} \frac{\partial q^\mu}{\partial x^i} \Lambda^\nu{}_\mu \delta^{jk} \frac{\partial \Lambda q_\nu}{\partial x^k} |\Lambda p\rangle_j \\ &= L^{-1}(\Lambda q)^j{}_\nu \Lambda^\nu{}_\mu L(q)^\mu{}_i |\Lambda p\rangle_j, \end{aligned} \quad (\text{A8})$$

where we inserted the resolution of the identity (A5), and where we used that $\Lambda q^\sigma |\Lambda p\rangle_\sigma = 0$ and $q_\mu \partial_i q^\mu = 0$. This is indeed of the form

$$U(\Lambda)|p\rangle_i = D(W(q, \Lambda))^j{}_i |\Lambda p\rangle_j, \quad (\text{A9})$$

with

$$D(W(q, \Lambda))^j{}_i = L^{-1}(\Lambda q)^j{}_\nu \Lambda^\nu{}_\mu L(q)^\mu{}_i, \quad (\text{A10})$$

the explicit representation of the Wigner rotation (4.9). This demonstrates the validity of the transformation (A6).

We can then use this to derive the transformation of the states (3.16) under finite Lorentz symmetries. Using again (4.26), we have

$$\begin{aligned} U(\Lambda)|\nu, q(\vec{x}), Z\rangle &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\Delta_\nu - 1} U(\Lambda)|\omega q(\vec{x}), Z\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\Delta_\nu - 1} |\omega \Lambda q(\vec{x}), \Lambda Z\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \omega^{\Delta_\nu - 1} |\omega' q(\vec{x}'), \Lambda Z\rangle \\ &= \frac{\Omega(x)^{\Delta_\nu}}{\sqrt{2\pi}} \int_0^\infty d\omega' \omega'^{\Delta_\nu - 1} |\omega' q(\vec{x}'), \Lambda Z\rangle \\ &= \Omega(x)^{\Delta_\nu} |\nu, q(\vec{x}'), \Lambda Z\rangle, \end{aligned} \quad (\text{A11})$$

where we performed the variable change

$$\omega' = \Omega(x)^{-1} \omega. \quad (\text{A12})$$

As shown after (4.80), for the physical components this implies

$$U(\Lambda)|\nu, \vec{x}\rangle_{i_1 \dots i_\ell} = \Omega(x)^{\Delta_\nu} R^{j_1}{}_{i_1}(x) \dots R^{j_\ell}{}_{i_\ell}(x) |\nu, \vec{x}'\rangle_{j_1 \dots j_\ell}, \quad (\text{A13})$$

as appropriate to a spinning representation of the principal series.

APPENDIX B: THE MIXED THREE-POINT AMPLITUDE

Here we study the mixed three-point amplitude (4.33) appearing in the decomposition of the translated massive states. We start by computing the integral

$$\mathcal{A}_{\Delta_1, \Delta_2, m}(q_1, q_2; a) = \int [d^{d+1}\hat{p}] (-\hat{p} \cdot q_1)^{\Delta_1} (-\hat{p} \cdot q_2)^{\Delta_2} e^{i m a^\mu \hat{p}_\mu} (a^\mu \in \mathbb{R}^{1, d+1}), \quad (\text{B1})$$

which coincides with (4.33) upon setting $\Delta_1 = d/2 + i\nu$ and $\Delta_2 = d/2 - i\nu'$. To achieve this goal, let us start with the convergent integral

$$\mathcal{A}_{\Delta_1, \Delta_2, m}^{\text{conv}}(q_1, q_2; a) = \int [d^{d+1}\hat{p}] (-\hat{p} \cdot q_1)^{\Delta_1} (-\hat{p} \cdot q_2)^{\Delta_2} e^{m a \cdot \hat{p}} (a^2 < 0, a^0 > 0), \quad (\text{B2})$$

where we assumed a to be timelike and future-directed to ensure $a \cdot \hat{p} < 0$, which guarantees the convergence of the integral. We will eventually send $m \rightarrow im$ to retrieve the solution of (B1) inside the light cone. The solution will then be analytically extended to the other region of $a^\mu \in \mathbb{R}^{1, d+1}$ by inserting an appropriate $i\epsilon$ -prescription.

Let us begin our computation by using the Mellin-Barnes representation of the exponential function,

$$e^{-x} = \int_{c-i\infty}^{c+i\infty} \frac{du}{2\pi i} \Gamma(u) x^{-u}, \quad c > 0, \quad \Re(x) > 0, \quad (\text{B3})$$

to rewrite

$$\mathcal{A}_{\Delta_1, \Delta_2, m}^{\text{conv}}(q_1, q_2; a) = \int_{-i\infty}^{+i\infty} \frac{du}{2\pi i} \Gamma(u) \int [d^{d+1}\hat{p}] (-\hat{p} \cdot q_1)^{\Delta_1} (-\hat{p} \cdot q_2)^{\Delta_2} (-m \hat{p} \cdot a)^{-u}, \quad (\text{B4})$$

where the integration contour is chosen to lie to the right of the imaginary axis ($c \rightarrow 0^+$). Let us focus on the resolution of

$$\mathcal{I}_{\Delta_0, \Delta_1, \Delta_2}(a, q_1, q_2) = \int [d^{d+1}\hat{p}] (-m \hat{p} \cdot a)^{-\Delta_0} (-\hat{p} \cdot q_1)^{\Delta_1} (-\hat{p} \cdot q_2)^{\Delta_2}, \quad (\text{B5})$$

where we set $u \equiv \Delta_0$. Using the Feynman-Schwinger parametrization, we get

$$\mathcal{I}_{\Delta_0, \Delta_1, \Delta_2}(a, q_1, q_2) = \prod_{k=0}^2 \int_0^\infty \frac{dt_k}{t_k} \frac{t_k^{\Delta_k}}{\Gamma(\Delta_k)} \int [d^{d+1}\hat{p}] e^{\hat{p} \cdot T}, \quad (\text{B6})$$

with $T = t_0 X + t_1 q_1 + t_2 q_2$ and $X = ma$. We can solve the integral in $\hat{p}$ in the rest frame of T . Since T is timelike and future-directed, in its rest frame $T^\mu = |T|(1, \vec{0}, 0)^\mu$, with $|T| = \sqrt{-T^2}$. Then, parametrizing $\hat{p}^\mu$ with

$$\hat{p}^\mu = \left(\frac{1 + y^2 + |\vec{z}|^2}{2y}, \frac{\vec{z}}{y}, \frac{1 - y^2 + |\vec{z}|^2}{2y} \right)^\mu, \quad y > 0, \quad \vec{z} \in \mathbb{R}^d, \quad (\text{B7})$$

yields

$$\begin{aligned} \int [d^{d+1}\hat{p}] e^{\hat{p} \cdot T} &= \int_0^\infty \frac{dy}{y^{d+1}} \int d^d z e^{\frac{|T|}{2y}(1+y^2+|\vec{z}|^2)} = \left(\frac{2\pi}{|T|} \right)^{d/2} \int_0^\infty \frac{dy}{y^{\frac{d+2}{2}}} e^{-\frac{|T|}{2y}(1+y^2)} \\ &\stackrel{y \rightarrow y/2|T|}{=} (4\pi)^{d/2} \int_0^\infty \frac{dy}{y^{\frac{d+2}{2}}} e^{\frac{T^2}{y} - \frac{y}{4}}. \end{aligned} \quad (\text{B8})$$

After sending $t_k \rightarrow \sqrt{y} t_k$, we can compute the integral in y ,

$$\int_0^\infty \frac{dy}{y^{\frac{d+2}{2}}} y^{\frac{\Delta_0 + \Delta_1 + \Delta_2}{2}} e^{-y/4} = 2^{\Sigma\Delta - d} \Gamma\left(\frac{\Sigma\Delta - d}{2}\right), \quad \Sigma\Delta \equiv \Delta_0 + \Delta_1 + \Delta_2, \quad (\text{B9})$$

so that (B5) reduces to

$$\mathcal{I}_{\Delta_0, \Delta_1, \Delta_2}(a, q_1, q_2) = \pi^{d/2} 2^{\Sigma \Delta} \Gamma\left(\frac{\Sigma \Delta - d}{2}\right) \prod_{k=0}^2 \int_0^\infty \frac{dt_k}{t_k} \frac{t_k^{\Delta_k}}{\Gamma(\Delta_k)} e^{T^2}. \quad (\text{B10})$$

Plugging the expression $T^2 = t_0^2 X^2 + 2t_0 t_1 X \cdot q_1 + 2t_0 t_2 X \cdot q_1 + 2t_1 t_2 q_1 \cdot q_2$ in the last equation and using again the Mellin-Barnes representation of the exponential functions, we obtain

$$\begin{aligned} \mathcal{I}_{\Delta_0, \Delta_1, \Delta_2}(a, q_1, q_2) &= \pi^{d/2} 2^{\Sigma \Delta} \Gamma\left(\frac{\Sigma \Delta - d}{2}\right) \prod_{k=0}^2 \int_0^\infty \frac{dt_k}{t_k} \frac{t_k^{\Delta_k}}{\Gamma(\Delta_k)} \int_{c_{00}-i\infty}^{c_{00}+i\infty} \frac{ds_{00}}{2\pi i} \Gamma(s_{00}) (-t_0^2 X^2)^{-s_{00}} \\ &\times \int_{c_{01}-i\infty}^{c_{01}+i\infty} \frac{ds_{01}}{2\pi i} \Gamma(s_{01}) (-2t_0 t_1 X \cdot q_1)^{-s_{01}} \int_{c_{02}-i\infty}^{c_{02}+i\infty} \frac{ds_{02}}{2\pi i} \Gamma(s_{02}) (-2t_0 t_2 X \cdot q_2)^{-s_{02}} \\ &\times \int_{c_{12}-i\infty}^{c_{12}+i\infty} \frac{ds_{12}}{2\pi i} \Gamma(s_{12}) (-2t_1 t_2 q_2 \cdot q_1)^{-s_{12}}. \end{aligned} \quad (\text{B11})$$

We also have

$$\int_0^\infty \frac{dt}{t} t^\alpha = 2\pi \delta(\alpha), \quad (\Re(\alpha) > 0). \quad (\text{B12})$$

Therefore, each integral in t_k gives a different delta function, which allows us to straightforwardly evaluate three out of the four remaining Mellin-Barnes integrals. Setting $s_{01} = \Delta_1 - s_{12}$, $s_{02} = \Delta_2 - s_{12}$, and $s_{00} = s_{12} + \frac{1}{2}(u - \Delta_1 - \Delta_2)$ yields

$$\begin{aligned} \mathcal{A}_{\Delta_1, \Delta_2, m}^{\text{conv}}(q_1, q_2; a) &= \frac{\pi^{d/2} (-a^2)^{\frac{\Delta_1 + \Delta_2}{2}}}{\Gamma(\Delta_1) \Gamma(\Delta_2) (-a \cdot q_1)^{\Delta_1} (-a \cdot q_2)^{\Delta_2}} \int_{-i\infty}^{+i\infty} \frac{du}{2\pi i} \Gamma\left(\frac{u + \Delta_1 + \Delta_2 - d}{2}\right) \\ &\times \left(\frac{m\sqrt{-a^2}}{2}\right)^{-u} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{ds_{12}}{2\pi i} \Gamma(s_{12}) \Gamma(\Delta_1 - s_{12}) \Gamma(\Delta_2 - s_{12}) \Gamma\left(s_{12} + \frac{u - \Delta_1 - \Delta_2}{2}\right) z^{-s_{12}}, \end{aligned} \quad (\text{B13})$$

with

$$z = \frac{1}{2} \frac{(-a^2)(-q_1 \cdot q_2)}{(-a \cdot q_1)(-a \cdot q_2)}, \quad (\text{B14})$$

which is positive for a^μ timelike and future-directed, as assumed in our analysis. Solving first the integral in u yields

$$\mathcal{A}_{\Delta_1, \Delta_2, m}^{\text{conv}}(q_1, q_2; a) = \frac{4(2\pi)^{d/2} (-ma \cdot q_1)^{\Delta_2 - \frac{d}{2}} (-ma \cdot q_2)^{\Delta_1 - \frac{d}{2}}}{\Gamma(\Delta_1) \Gamma(\Delta_2) (-q_1 \cdot q_2)^{\Delta_1 + \Delta_2 - \frac{d}{2}}} \mathcal{M}_{\Delta_1, \Delta_2}^{(m)}(q_1, q_2; a), \quad (\text{B15})$$

with

$$\begin{aligned} \mathcal{M}_{\Delta_1, \Delta_2}^{(m)}(q_1, q_2; a) &= \int_{-i\infty}^{+i\infty} \frac{ds_{12}}{2\pi i} \Gamma\left(\frac{d}{2} - \Delta_1 + s_{12}\right) \Gamma\left(\frac{d}{2} - \Delta_2 + s_{12}\right) \Gamma\left(\Delta_1 + \Delta_2 - \frac{d}{2} - s_{12}\right) \\ &\times \left(\frac{(-q_1 \cdot q_2)}{(-ma \cdot q_1)(-ma \cdot q_2)}\right)^{s_{12}} \left(m\sqrt{-a^2}\right)^{s_{12}} K_{s_{12}}\left(m\sqrt{-a^2}\right). \end{aligned} \quad (\text{B16})$$

This last integral representation is particularly suitable for studying the distribution of poles. Alternatively, to obtain a closed-form expression for (B2), it is more convenient to first perform the integral in s_{12} . Integrals of this form have been extensively studied in the appendix of [51]. Specifically, our integral can be recast as follows:

$$\begin{aligned} & \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{ds_{12}}{2\pi i} \Gamma(s_{12}) \Gamma(\Delta_1 - s_{12}) \Gamma(\Delta_2 - s_{12}) \Gamma\left(s_{12} + \frac{1}{2}(u - \Delta_1 - \Delta_2)\right) z^{-s_{12}} \\ &= \zeta^d \frac{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\frac{u+\Delta_1-\Delta_2}{2}) \Gamma(\frac{u-\Delta_1+\Delta_2}{2})}{\Gamma(\frac{u+\Delta_1+\Delta_2}{2})} {}_2F_1\left(\Delta_1, \Delta_2, \frac{u+\Delta_1+\Delta_2}{2}; 1 - \zeta^2\right), \end{aligned} \quad (\text{B17})$$

where $\zeta = \sqrt{z}$, for $z < 1$, and $\zeta = 1/\sqrt{z}$, for $z > 1$, with z given by (B14). For $z = 1$, Eq. (B17) reduces to Barnes' first lemma, ensuring the analyticity of the solution for any $z > 0$. This yields

$$\mathcal{A}_{\Delta_1, \Delta_2, m}^{\text{conv}}(q_1, q_2; a) = \frac{\pi^{d/2} (-a^2)^{\frac{\Delta_1+\Delta_2}{2}}}{(-2a \cdot q_1)^{\Delta_1} (-2a \cdot q_2)^{\Delta_2}} \mathcal{B}_{\Delta_1, \Delta_2, m}(q_1, q_2; a), \quad (\text{B18})$$

where

$$\mathcal{B}_{\Delta_1, \Delta_2, m}(q_1, q_2; a) = \zeta^d \int_{-i\infty}^{+i\infty} \frac{du}{2\pi i} \left(\frac{m\sqrt{-a^2}}{2} \right)^{-u} \frac{\Gamma(\frac{u+\Delta_1+\Delta_2-d}{2}) \Gamma(\frac{u+\Delta_1-\Delta_2}{2}) \Gamma(\frac{u-\Delta_1+\Delta_2}{2})}{\Gamma(\frac{u+\Delta_1+\Delta_2}{2})} {}_2F_1\left(\Delta_1, \Delta_2, \frac{u+\Delta_1+\Delta_2}{2}; 1 - \zeta^2\right). \quad (\text{B19})$$

This last integral can be explicitly solved by rewriting

$${}_2F_1\left(\Delta_1, \Delta_2; \frac{u+\Delta_1+\Delta_2}{2}, 1 - \zeta^2\right) = (\zeta^2)^{\frac{u-\Delta_1-\Delta_2}{2}} {}_2F_1\left(\frac{u-\Delta_1+\Delta_2}{2}, \frac{u+\Delta_1-\Delta_2}{2}; \frac{u+\Delta_1+\Delta_2}{2}, 1 - \zeta^2\right), \quad (\text{B20})$$

and then using the *mixed integral theorem* on page 143 of [52]. This theorem provides a solution in terms of a linear combination of Appell-like series, ensuring the convergence of (B2). Although the expression is quite cumbersome, the convergence conditions of the mixed integral theorem still hold true when the solution is analytically continued to $m \rightarrow e^{i(\frac{\pi}{2}-\epsilon)} m$. At the final point of this analytic continuation, we recover the closed-form expression of (B1) for a^μ timelike and future-directed. This last result can then be

analytically extended to the whole Minkowski spacetime using suitable transformations, as in [53], where the authors split Minkowski into four regions—two inside and two outside the light cone—and used analytical transformations to connect points across them. In this procedure, all analytic continuations, which ultimately correspond to rotations in the complex plane in a suitable set of coordinates, are uniquely fixed by the requirement of not crossing any branch cut of (B1).

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- [1] E. Wigner, *Gruppentheorie und ihre anwendung auf die quantenmechanik der atomspektren* (Verlag, Braunschweig, 1931).
 - [2] S. Weinberg, *The Quantum Theory of Fields. Vol. 1: Foundations* (Cambridge University Press, Cambridge, England, 2005).
 - [3] E. P. Wigner, On unitary representations of the inhomogeneous Lorentz group, *Ann. Math.* **40**, 149 (1939).
 - [4] D. Poland, S. Rychkov, and A. Vichi, The conformal bootstrap: Theory, numerical techniques, and applications, *Rev. Mod. Phys.* **91**, 015002 (2019).
 - [5] S. Pasterski, M. Pate, and A.-M. Raclariu, Celestial holography, in *2022 Snowmass Summer Study* (2021), [arXiv:2111.11392](#).
 - [6] T. McLoughlin, A. Puhm, and A.-M. Raclariu, The SAGEX review on scattering amplitudes, Chapter 11: Soft theorems and celestial amplitudes, *J. Phys. A* **55**, 443012 (2022).
 - [7] A. Gadde, In search of conformal theories, [arXiv:1702.07362](#).
 - [8] H.-D. Joos, Zur darstellungstheorie der inhomogenen lorentzgruppe als grundlage quantenmechanischer kinematik, *Protein Sci.* **10**, 65 (2006).
 - [9] V. Cantoni, On the structure of the representations of the poincaré group with zero mass and discrete helicity, *Rend. Circ. Mat. Palermo* **24**, 35 (1975).
 - [10] A. Chakrabarti, M. Levy-Nahas, and R. Seneor, “Lorentz Basis” of the Poincaré Group, *J. Math. Phys. (N.Y.)* **9**, 1274 (1968).
 - [11] A. Chakrabarti, Lorentz basis of the poincaré group. II, *J. Math. Phys. (N.Y.)* **12**, 1274 (1968).
 - [12] N. Mukunda, Zero-mass representations of the Poincaré group in an $O(3, 1)$ basis, *J. Math. Phys. (N.Y.)* **9**, 532 (1968).

- [13] I. S. Shapiro, Expansion of the scattering amplitude in relativistic spherical functions, *J. Exp. Theor. Phys.* **16**, 1219 (1963).
- [14] G. B. Smith, Matrix element expansion of a spin wavefunction, *J. Math. Phys. (N.Y.)* **19**, 581 (1978).
- [15] W. Rühl, The convolution of fourier transforms and its application to the decomposition of the momentum operator on the homogeneous Lorentz group, *Il Nuovo Cimento A* (1971–1996) **63**, 1131 (1969).
- [16] A. W. Weidemann, Quantum fields in a “Lorentz basis”, *Il Nuovo Cimento A* (1965–1970) **57**, 221 (1980).
- [17] J. S. Lomont and H. E. Moses, The representations of the inhomogeneous Lorentz group in terms of an angular momentum basis, *J. Math. Phys. (N.Y.)* **5**, 294 (1964).
- [18] W. MacDowell and R. Roskies, Reduction of the Poincaré group with respect to the Lorentz group, *J. Math. Phys. (N.Y.)* **13**, 1585 (1972).
- [19] K.-C. Chou and L. G. Zastavenko, Integral transformations of the i. s. Shapiro type for particles of zero mass, in *Selected Papers of K C Chou* (World Scientific, Singapore, 2009), pp. 77–80.
- [20] K.-C. Chou and L. G. Zastavenko, The shapiro integral transformation, in *Selected Papers of K C Chou* (World Scientific, Singapore, 2009), pp. 33–38.
- [21] J. Zmuidzinas, Unitary representations of the Lorentz group on 4-vector manifolds, *J. Math. Phys. (N.Y.)* **7**, 764 (1966).
- [22] A. D. Steiger, Poincaré-irreducible tensor operators for positive-mass one-particle states. I, *J. Math. Phys. (N.Y.)* **12**, 1497 (1971).
- [23] J. de Boer and S. N. Solodukhin, A holographic reduction of Minkowski space-time, *Nucl. Phys.* **B665**, 545 (2003).
- [24] C. Cheung, A. de la Fuente, and R. Sundrum, 4D scattering amplitudes and asymptotic symmetries from 2D CFT, *J. High Energy Phys.* **01** (2017) 112.
- [25] S. Pasterski and S.-H. Shao, Conformal basis for flat space amplitudes, *Phys. Rev. D* **96**, 065022 (2017).
- [26] S. Pasterski, S.-H. Shao, and A. Strominger, Flat space amplitudes and conformal symmetry of the celestial sphere, *Phys. Rev. D* **96**, 065026 (2017).
- [27] Y. T. A. Law and M. Zlotnikov, Massive spinning bosons on the celestial sphere, *J. High Energy Phys.* **06** (2020) 079.
- [28] L. Iacobacci and W. Mück, Conformal primary basis for Dirac spinors, *Phys. Rev. D* **102**, 106025 (2020).
- [29] S. A. Narayanan, Massive celestial fermions, *J. High Energy Phys.* **12** (2020) 074.
- [30] A. O. Barut and R. Raczka, *Theory Of Group Representations And Applications* (World Scientific, Singapore, 1986).
- [31] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov, *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory* (Springer-Verlag, Berlin, 1977), Vol. 63.
- [32] C.-M. Chang, R. Liu, and W.-J. Ma, Split representation in celestial holography, [arXiv:2311.08736](https://arxiv.org/abs/2311.08736).
- [33] P. A. M. Dirac, Unitary representations of the Lorentz group, *Proc. R. Soc. A* **183**, 284 (1945).
- [34] H. Harish-Chandra, Infinite irreducible representations of the Lorentz group, *Proc. R. Soc. A* **189**, 372 (1947).
- [35] V. Bargmann, Irreducible unitary representations of the Lorentz group, *Ann. Math.* **48**, 568 (1947).
- [36] I. M. Gel’fand and M. A. Naimark, Unitary representations of the Lorentz group, *Izv. Ross. Akad. Nauk Ser. Mat.* **11**, 411 (1947).
- [37] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, Spinning conformal correlators, *J. High Energy Phys.* **11** (2011) 071.
- [38] M. S. Costa, V. Gonçalves, and J. a. Penedones, Spinning AdS propagators, *J. High Energy Phys.* **09** (2014) 064.
- [39] Z. Sun, A note on the representations of $SO(1, d + 1)$, *Rev. Math. Phys.* **37**, 2430007 (2025).
- [40] G. Mack and A. Salam, Finite component field representations of the conformal group, *Ann. Phys. (N.Y.)* **53**, 174 (1969).
- [41] R. P. Langlands, On the classification of irreducible representations of real algebraic groups, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Vol. 31 of Math. Surveys Monogr. (American Mathematical Society, Providence, 1989), pp. 101–170.
- [42] A. W. Knap and G. Zuckerman, Classification theorems for representations of semisimple lie groups, in *Non-Commutative Harmonic Analysis* (Springer, Berlin, Heidelberg, 1977), pp. 138–159.
- [43] A. W. Knap and G. J. Zuckerman, Classification of irreducible tempered representations of semisimple groups, *Ann. Math.* **116**, 389 (1982).
- [44] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).
- [45] S. Banerjee, Null infinity and unitary representation of the Poincaré group, *J. High Energy Phys.* **01** (2019) 205.
- [46] K. Nguyen and P. West, Carrollian conformal fields and flat holography, *Universe* **9**, 385 (2023).
- [47] K. Nguyen, Carrollian conformal correlators and massless scattering amplitudes, *J. High Energy Phys.* **01** (2024) 076.
- [48] X. Bekaert and N. Boulanger, The unitary representations of the poincaré group in any spacetime dimension, *SciPost Phys. Lect. Notes* **30**, 1 (2021).
- [49] H. Isono, On conformal correlators and blocks with spinors in general dimensions, *Phys. Rev. D* **96**, 065011 (2017).
- [50] J. Bures, F. Sommen, V. Soucek, and P. V. Lancker, Symmetric analogues of Rarita–Schwinger equations, *Ann. Glob. Anal. Geom.* **21**, 215 (2002).
- [51] C. Sleight, A mellin space approach to cosmological correlators, *J. High Energy Phys.* **01** (2020) 090.
- [52] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, England, 2008).
- [53] L. Iacobacci, C. Sleight, and M. Taronna, From celestial correlators to AdS, and back, *J. High Energy Phys.* **06** (2023) 053.