

Gauge and parametrization dependence of quantum Einstein gravity within the proper time flow

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Proper time functional flow equations have garnered significant attention in recent years, as they are particularly suitable in analyzing nonperturbative contexts. By resorting to this flow, we investigate the regulator and gauge dependence in quantum Einstein gravity within the asymptotic safety framework, considering various regularization schemes. Our findings indicate that some details of the regulator have minor influence on the critical properties of the theory. In contrast, the selection between linear and exponential parametrizations appears to have a more substantial impact on the scaling behavior of the renormalized flow near the non-Gaussian fixed point.

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I. INTRODUCTION

The determination of a non-Gaussian fixed point for the theory of gravitation, even in its basic Einstein-Hilbert formulation, is an essential step in the process of understanding not only the (nonperturbative) renormalizability of the underlying field theory but, more generally, the quantum nature of gravity. Although the presence of such a fixed point (FP), originally discovered in [1,2] for gravity in $d = 2 + \varepsilon$ dimensions, was later confirmed for the ($d = 4$)-dimensional case [3–12], a complete picture of the structure of the ultraviolet (UV) critical manifold in $d = 4$ still requires further investigation. On general grounds, the approach followed in [3] is based on the study of a renormalization group (RG) flow equation inspired by

the original work of Wilson and Kogut [13]. Then, the presence of a FP for the specific flow equation considered, distinct from the Gaussian noninteracting FP and characterized by a finite-dimensional UV critical surface (i.e. with a finite number of attractive eigendirections in the RG trajectory space), guarantees that the theory is nonperturbatively renormalizable and indicated as asymptotically safe. In this context the gravitational theory is treated as a field theory where the elementary degrees of freedom are played by the components of the metric tensor and, in addition, we impose the physical requirement that the action which characterizes the theory must be invariant under diffeomorphisms of the metric.

Already in the definition of the basic modes of the metric $g_{\mu\nu}$, the system is not univocally constrained. In fact, after necessarily introducing a fiducial background metric $\bar{g}_{\mu\nu}$, one is allowed to treat the fluctuations $h_{\mu\nu}$ around the background in different ways: in particular, the linear [14] and the exponential [15] are the most commonly adopted parametrizations and we shall consider both in the following. Then, in the path integral formulation of the problem, the diffeomorphism invariance of the functional integral can be implemented through the Faddeev-Popov method and this procedure brings in more freedom related to the

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selection of a particular gauge, which is essential to perform explicit calculations. Therefore, it is evident that both the freedom in the choice of the parametrization and of the gauge fixing function can lead to an unphysical dependence on some unconstrained parameters of various quantities extracted from the functional integration.

In principle one must distinguish between on-shell physical observables and off-shell quantities. In fact, while the former must be invariant under any kind of symmetry transformation that leaves the physical system unmodified, there is no such restriction concerning the latter. So, for instance, while the full effective action of a system is typically gauge dependent, physical quantities computed from the functional derivatives of the effective action computed at its minimum must not show any residual gauge dependence. However, the distinction between the two sets of variables can be spoiled by other approximations that must unavoidably be introduced in the analysis of the problem. This includes the particular truncation of the action selected and also the approximation scheme adopted to explicitly solve the RG flow equation and therefore we can expect residual gauge and parametrization dependence in the determination of any variable.

A thorough study of the residual gauge and parametrization dependence has already been carried out in [11] by focusing on the non-Gaussian FP of the Einstein-Hilbert action and on its UV critical manifold. In particular, in [11] the RG flow equation for the effective average action, introduced in [16–18], is used to perform this analysis and, with the help of the principle of minimum sensitivity, a weak parametrization dependence of the UV stable non-Gaussian FP is pointed out. However, as already noticed, even the details of the RG flow equation play a crucial role in introducing a redundant dependence on not physically detectable parameters, such as the dependence on the specific infrared regulator used to determine the flow [6].

Therefore, in this paper, we analyze the same problem addressed in [11], but instead of following the flow of the one particle irreducible effective average action, we focus on a Wilsonian flow, closer in spirit to the original work of Wilson and Kogut [13], subsequently developed in the work of Wegner and Houghton [19]. More specifically, we shall refer to the proper time (PT) version of the flow equation [20–24], which has been employed in many contexts, from the renormalization properties of scalar theories [25], including the determination of critical exponents for the 3D Ising system [26–28], to the analysis of the asymptotic safety of the conformal sector of quantum gravity [29–31], to precise determination of observables in quantum mechanics, of a strictly nonperturbative nature [32,33].

Although the PT equation does not belong to the class of exact flows for the effective average action [16,34,35], the possibility of interpreting the PT flow as a coarse grained

Wilsonian flow has recently been proposed by several authors [24,36,37] and further discussed in [38].

In fact, the PT flow equation turns out to be extremely accurate in the determination of observables (even at first- or second-order truncation in the derivative expansion), but it also has the feature of preserving specific symmetries of the action, as in the case of the gauge symmetry of Yang-Mills theories [39], due to the gauge preserving nature of the proper time regulator introduced in [40]. Because of these properties, we want to investigate to which extent the residual dependence on unphysical parameters coming from fluctuation parametrization and gauge fixing, in quantum gravity, does actually show up in the PT flow.

In this respect, it must be noticed that in a recent paper [41] the PT flow was considered as the possible realization of a particular variant of the dimensional regularization which deals with poles appearing at all dimensions d , and also field redefinitions are used to remove off-shell contributions to the RG equations, in the spirit of the essential renormalization group [42,43], where only the flow of the couplings which contribute to the scaling of physical observables is taken into account; within this approach the independence of the Newton's constant beta function from the parametrization is observed, to all orders in the scalar curvature.

Here, we shall simply limit ourselves to keep track of the persistence of physically relevant features of the quantum Einstein gravity, namely, the presence of a non-Gaussian fixed point with positive values of the Newton and cosmological constant couplings and with two relevant, i.e. UV-attractive, (real part of) eigenvalues that regulate the renormalizability of the theory, against changes in the parametrization of the fluctuations of the metric tensor and in the particular gauge choice. Concerning the latter issue we shall examine not only the usual renormalizable gauge [44] but also the results obtained by resorting to the physical gauge choice [45–50]. In addition, we cannot neglect, in any RG flow analysis, the freedom associated with the choice of the particular regulator used to define the intermediate steps of the flow at different values of the energy scale k . For the PT flow, we introduce a commonly used parametrization of the regulator in terms of an integer parameter m , and we shall also consider two slightly different ways of implementing the regulator in the PT flow, according to [22,36].

Section II is dedicated to the definition of the problem together with the introduction of all the unconstrained parameters entering our analysis. Then, in Sec. III as a simple test of our setup, the parametrization dependence of the non-Gaussian FP in $d = 2 + \varepsilon$ is considered, while Sec. IV is devoted to the complete analysis of the model in $d = 4$. Conclusions are reported in Sec. V, while the Appendixes contain some relevant details of the complete calculations.

II. FLOW EQUATION WITH DIFFERENT GAUGES AND PARAMETRIZATIONS

The main goal of a quantum theory of Einstein gravity in a d -dimensional Euclidean space-time is to evaluate the path integral

$$Z = \frac{1}{V_{\text{diff}}} \int Dg_{\mu\nu} e^{-S[g]} \quad (2.1)$$

over the metric $g_{\mu\nu}$ of a generic action $S[g]$ that is invariant under diffeomorphisms, with V_{diff} representing the redundancies of the symmetry.

The nonperturbative methods of the renormalization group are very useful for evaluating Eq. (2.1). In this framework, our description of the theory is encoded in an effective action, which describes the theory as effective up to a momentum scale Λ . The same theory can be described at lower momentum scales using a flow equation that connects the actions at different scales.

In this work, we use a flow equation for the Wilsonian effective action S_Λ , regularized via the proper time integral [24,36,51]

$$\Lambda \partial_\Lambda S_\Lambda = \frac{1}{2} \int_0^\infty \frac{ds}{s} r_\Lambda(s) \text{STr} e^{-s \tilde{S}_\Lambda^{(2)}}. \quad (2.2)$$

Here, $\tilde{S}_\Lambda^{(2)}$ is the Hessian of the gauge fixed Wilsonian effective action and of the ghost actions. The regularization is implemented through the regulating function $\rho_{k,\Lambda}(s)$, which depends on the IR scale k and the UV scale Λ , with

$$r_\Lambda(s) = \Lambda \partial_\Lambda \rho_{k,\Lambda}(s). \quad (2.3)$$

A standard approach to the problem concerns the use of the background field method, in which the metric $g_{\mu\nu}$ is decomposed into a classical background $\bar{g}_{\mu\nu}$ and a fluctuating metric $h_{\mu\nu}$ [44]. This splitting can be carried out using different parametrizations, and in this work we are interested in the linear parametrization [3]

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (2.4)$$

and in the exponential parametrization [10]

$$g_{\mu\nu} = \bar{g}_{\mu\rho} (e^h)^\rho{}_\nu. \quad (2.5)$$

It is useful to summarize both parametrizations through a dependence on the parameter τ [11],

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{\tau}{2} h_\mu^\rho h_{\rho\nu} + o(h^3), \quad (2.6)$$

that for $\tau = 0$ leads to the linear parametrization, while for $\tau = 1$ it corresponds to the second-order expansion in $h_{\mu\nu}$ of the exponential parametrization.

In general, to distinguish between objects evaluated on the full metric and those evaluated on the background, we denote the latter as barred objects.

We assume that we have a maximally symmetric background metric, such that the Riemann and Ricci tensors are expressed solely in terms of the background metric and the curvature scalar $\bar{R}$ [52],

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)} (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}) \bar{R}, \quad \bar{R}_{\mu\nu} = \frac{\bar{R}}{d} \bar{g}_{\mu\nu}. \quad (2.7)$$

The fluctuation is decomposed with the York decomposition [53,54], and the path integral is rewritten in terms of the fields $h_{\mu\nu}^T, \xi_\mu, \sigma, h$. A derivation of the York decomposition, together with the definitions of these fields, is given in Appendix A.

With the freedom to fix the gauge in different ways, we adopt two different procedures.

As the first option, we implement the Faddeev-Popov gauge fixing procedure, and choose the background field gauge (BFG) [3,44]

$$F_\mu[h; \bar{g}] = (\bar{g}^{\alpha\gamma} \delta_\mu^\beta \bar{D}_\gamma - \omega \bar{g}^{\alpha\beta} \bar{D}_\mu) h_{\alpha\beta}, \quad (2.8)$$

adding a gauge fixing term to the action

$$\mathcal{S}_{\text{gf}}[h; \bar{g}] = \frac{1}{\alpha} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu, \quad (2.9)$$

and the Faddeev-Popov determinant in the usual form as an action for the ghosts $\bar{C}^\mu$ and C^μ

$$\mathcal{S}_{\text{gh}}[h, C, \bar{C}; \bar{g}] = - \int d^d x \sqrt{\bar{g}} \bar{C}_\mu \frac{\delta F^\mu}{\delta \epsilon^\nu} C^\nu. \quad (2.10)$$

Similarly to the fluctuation, the ghost field is also decomposed into its transverse and longitudinal components [54],

$$C_\mu = \hat{C}_\mu + \bar{D}_\mu \frac{\eta}{\sqrt{-\bar{D}^2}}, \quad (2.11)$$

with a trivial Jacobian associated with $\{C_\mu\} \rightarrow \{\hat{C}_\mu, \eta\}$.

As we can see, this gauge fixing is parametrized by α and ω . The gauge fixing is not well defined for the singular value

$$\omega_{\text{sing}} = 1, \quad (2.12)$$

as observed in [11].

The second option regards the possibility to fix some degrees of freedom of the theory, with the use of the physical gauge [45–50]. If we fix $\xi_\mu = 0$, a ghost action appears, in terms of the real ghost field b_μ ,

$$\mathcal{S}_{\text{gh}}^{(\xi=0)} = \int d^d x \sqrt{\bar{g}} b^\mu \left(-\bar{D}^2 - \frac{\bar{R}}{d} + o(h) \right) b_\mu. \quad (2.13)$$

We can also fix $\sigma = 0$ or $h = 0$. The associated ghost actions are, respectively in terms of the scalar fields χ and ϕ ,

$$\mathcal{S}_{\text{gh}}^{(\sigma=0)} = \int d^d x \sqrt{\bar{g}} \chi \left(-\bar{D}^2 - \frac{\bar{R}}{d-1} + o(h) \right) \chi \quad (2.14)$$

and

$$\mathcal{S}_{\text{gh}}^{(h=0)} = \int d^d x \sqrt{\bar{g}} \phi \left(-\bar{D}^2 + o(h) \right) \phi. \quad (2.15)$$

The last two conditions cannot be imposed simultaneously; therefore, we use two possible gauge choices.

In the first case, we impose the conditions $\xi_\mu = 0$ and $\sigma = 0$, so the fluctuation is described only by $h_{\mu\nu}^T$ and h . We refer to this gauge choice as the “T-h gauge” (ThG).

The second gauge is defined by $\xi_\mu = 0$ and $h = 0$. The action remains with $h_{\mu\nu}^T$ and σ , and we refer to this second gauge choice as the “T- σ gauge” (T σ G).

The details concerning the gauge fixing procedures are shown in Appendix B.

Equation (2.2) describes the flow of all possible couplings for $\mathcal{S}_\Lambda[g]$. We take the so-called Einstein-Hilbert truncation, where the action is constrained to the form

$$\mathcal{S}_\Lambda = 2\kappa^2 Z_\Lambda \int d^d x \sqrt{\bar{g}} (-R + 2\tilde{\lambda}_\Lambda), \quad (2.16)$$

where $\kappa = (32\pi G)^{-1/2}$, with G the Newton constant, and $Z_\Lambda = G/G_\Lambda$. This truncation is basically the Einstein-Hilbert action, with the Newton constant G and the cosmological constant $\tilde{\lambda}$ being “promoted” to running coupling constants G_Λ and $\tilde{\lambda}_\Lambda$. Their values at low momentum scales must match the values of G and $\tilde{\lambda}$ in order to describe the physics of our Universe.

This “promotion” is also reflected in the gauge fixing term, where we add an overall scale-dependent factor $\kappa^2 Z_\Lambda$. The scale dependence of the ghost sector for the Einstein-Hilbert truncation was studied in [55]. One of the results of the work is that the wave-function renormalization of the ghosts exhibits an UV behavior similar to that of the graviton wave-function renormalization. Following this idea, we add Z_Λ as an overall factor in the ghost actions. The Hessian in the flow equation is evaluated in the background. In the BFG, expanding the gauge fixed action around the background, the quadratic term is

$$\begin{aligned} \tilde{\mathcal{S}}_{\text{BFG},\Lambda}^{\text{quadratic}} = & \frac{1}{2} \int d^d x \sqrt{\bar{g}} \left[h_{\mu\nu}^T \mathcal{S}_{TT} h^{T\mu\nu} + \xi_\mu \mathcal{S}_{\xi\xi} \xi^\mu \right. \\ & \left. + (\sigma \quad h) \begin{pmatrix} \mathcal{S}_{\sigma\sigma} & \mathcal{S}_{h\sigma} \\ \mathcal{S}_{h\sigma} & \mathcal{S}_{hh} \end{pmatrix} \begin{pmatrix} \sigma \\ h \end{pmatrix} \right], \end{aligned} \quad (2.17)$$

while the ghost action, already being quadratic in the ghost fields, is evaluated in the background metric

$$\mathcal{S}_{\text{BFG}}^{\text{gh}} = \int d^d x \sqrt{\bar{g}} [\tilde{\mathcal{C}}_\mu \mathcal{S}_{CC} \hat{C}^\mu + \bar{\eta} \mathcal{S}_{\eta\eta} \eta]. \quad (2.18)$$

This Hessian is almost diagonal, and we exploit the possibility to fix

$$\omega = \frac{2 + d\alpha - 2\alpha}{2d}, \quad (2.19)$$

a value for which $\mathcal{S}_{h\sigma} = 0$; therefore the Hessian is diagonal. With this approach our strategy differs from that adopted in [11]. The flow equation (2.2) requires the exponentiation of the Hessian and in our approach it is a trivial step. Moreover, each term of the Hessian has the form

$$\mathcal{S}_{ii} = -A_{\Lambda,i} \bar{D}^2 + B_{\Lambda,i} \tilde{\lambda}_\Lambda + C_{\Lambda,i} \bar{R}, \quad (2.20)$$

for $i \in \Omega_{\text{BFG}} \equiv \{T, \xi, \sigma, h, C, \eta\}$. With the constraint (2.19), the singular value (2.12) is reflected in

$$\alpha_{\text{sing}} = 2 \frac{d-1}{d-2}, \quad (2.21)$$

the coefficients $A_{\Lambda,\sigma}, A_{\Lambda,h}, A_{\Lambda,\eta}$ are null, and this yields a singularity in the flow equation.

In the ThG, the quadratic part of the action, expanded around the background, is

$$\tilde{\mathcal{S}}_{\text{ThG},\Lambda}^{\text{quadratic}} = \frac{1}{2} \int d^d x \sqrt{\bar{g}} [h_{\mu\nu}^T \mathcal{S}_{\bar{T}\bar{T}} h^{T\mu\nu} + h \mathcal{S}_{\bar{h}\bar{h}} h] \quad (2.22)$$

and the ghost action

$$\tilde{\mathcal{S}}_{\text{ThG},\Lambda}^{\text{gh}} = \frac{1}{2} \int d^d x \sqrt{\bar{g}} [b_\mu \mathcal{S}_{bb} b^\mu + \chi \mathcal{S}_{\chi\chi} \chi]. \quad (2.23)$$

Alternatively, in the T σ G, we find

$$\tilde{\mathcal{S}}_{\text{T}\sigma\text{G},\Lambda}^{\text{quadratic}} = \frac{1}{2} \int d^d x \sqrt{\bar{g}} [h_{\mu\nu}^T \mathcal{S}_{\bar{T}\bar{T}} h^{T\mu\nu} + \sigma \mathcal{S}_{\bar{\sigma}\bar{\sigma}} \sigma] \quad (2.24)$$

and

$$\tilde{\mathcal{S}}_{\text{T}\sigma\text{G},\Lambda}^{\text{gh}} = \frac{1}{2} \int d^d x \sqrt{\bar{g}} [b_\mu \mathcal{S}_{bb} b^\mu + \phi \mathcal{S}_{\phi\phi} \phi]. \quad (2.25)$$

With the physical gauges, the Hessian is automatically diagonal, and each term of the Hessian takes the form of (2.20), with $i \in \Omega_{\text{ThG}} \equiv \{\tilde{T}, \tilde{h}, b, \chi\}$ in the ThG, $i \in \Omega_{\text{T}\sigma\text{G}} \equiv \{\tilde{T}, \tilde{\sigma}, b, \phi\}$ in the T σ G.

The explicit form of the components of the Hessians is reported in Appendix C.

With the Hessians decomposed into pieces that can be written in the form in Eq. (2.20), the regularization is carried out straightforwardly. The function $\rho_{k,\Lambda}(s)$ defined in Eq. (2.3), provides an IR and an UV regularization, as it vanishes for both $s > 1/k^2$ and $s < 1/\Lambda^2$. We shall test the flow equation for different families of regulators, also parametrized by the integer $m > d/2$, as previously discussed in [36]. Namely, by taking

$$\rho_{k,\Lambda}(s; m) = \frac{\Gamma(m, msk^2) - \Gamma(m, ms\Lambda^2)}{\Gamma(m)} \quad (2.26)$$

and $r_\Lambda(s; m)$ according to Eq. (2.3),

$$r_\Lambda(s; m) = \frac{2}{\Gamma(m)} (ms\Lambda^2)^m e^{-ms\Lambda^2}, \quad (2.27)$$

the exponential term suppresses modes with $A_{\Lambda,i}p^2 > 1/s$. Then, since we want an effective suppression of the modes with $p^2 > 1/s$, we have to suitably rescale the regularization functions in Eqs. (2.26) and (2.27) by the factor $A_{\Lambda,i}$. Therefore we take

$$\rho_{k,\Lambda}(A_{\Lambda,i}s; m) = \frac{\Gamma(m, mA_{\Lambda,i}sk^2) - \Gamma(m, mA_{\Lambda,i}s\Lambda^2)}{\Gamma(m)} \quad (2.28)$$

and

$$r_\Lambda(A_{\Lambda,i}s; m) = \frac{2}{\Gamma(m)} \left(1 + \frac{\Lambda}{2} \partial_\Lambda \log A_{\Lambda,i} \right) \times (mA_{\Lambda,i}s\Lambda^2)^m e^{-mA_{\Lambda,i}s\Lambda^2}. \quad (2.29)$$

In the following, we refer to this regularization as the “B scheme”. Instead, if we insert the factors $A_{\Lambda,i}$ directly into the derivative, we have

$$r_\Lambda(A_{\Lambda,i}s; m) = \frac{2}{\Gamma(m)} (mA_{\Lambda,i}s\Lambda^2)^m e^{-mA_{\Lambda,i}s\Lambda^2}, \quad (2.30)$$

and we indicate the latter as the “C scheme.” For the B scheme, Eq. (2.2) becomes

$$\Lambda \partial_\Lambda \mathcal{S}_\Lambda[g] = \left(1 + \frac{\Lambda}{2} \partial_\Lambda \log Z_\Lambda \right) \sum_i \text{STr}_i \left(\frac{m\Lambda^2 A_{\Lambda,i}}{\mathcal{S}_{ii} + m\Lambda^2 A_{\Lambda,i}} \right)^m, \quad (2.31)$$

while for the C scheme we get

$$\Lambda \partial_\Lambda \mathcal{S}_\Lambda[g] = \sum_i \text{STr}_i \left(\frac{m\Lambda^2 A_{\Lambda,i}}{\mathcal{S}_{ii} + m\Lambda^2 A_{\Lambda,i}} \right)^m, \quad (2.32)$$

where STr_i is the supertrace over the degrees of freedom of the i th field. In our case, the supertrace is simply the trace with a factor of -2 for the complex ghosts and a factor of -1 for the real ghosts.

Equation (2.31) differs from Eq. (2.32) because of the factor $\frac{\Lambda}{2} \partial_\Lambda \log Z_\Lambda$. The former is a direct consequence of the truncation choice. With Z_Λ in the ghost sector, each term of the Hessian has the same structure, and the factor $\Lambda \partial_\Lambda \log Z_\Lambda$ appears overall. In contrast, in the latter scheme, the presence of Z_Λ in the ghost sector is not necessary to get the flow equation (2.32). It must be remarked that the C scheme should not be considered as a further approximation of the B scheme where some (but not all) dependence on the wave-function renormalization is neglected. Instead, it could be regarded as a different scheme, not produced by Eq. (2.28), but rather from another $\rho_{k,\Lambda}$ whose derivative gives Eq. (2.32).

The traces in the two flow equations are over the fields defined by the York decomposition and they can be handled by using the heat kernel expansion, as shown in [5]. On the right-hand side of these equations, the flow of $\tilde{\lambda}_\Lambda/G_\Lambda$ is the coefficient of $(1/8\pi) \int d^d x \sqrt{g}$, and the flow of $1/G_\Lambda$ is the coefficient of $(-1/16\pi) \int d^d x \sqrt{g} R$. It is convenient to write the expressions in terms of the dimensionless quantities $g_\Lambda = \Lambda^{d-2} G_\Lambda$, $\lambda_\Lambda = \Lambda^{-2} \tilde{\lambda}_\Lambda$. In this way, the two beta functions for the dimensionless coupling constants are

$$\begin{aligned} \Lambda \partial_\Lambda g_\Lambda &= \beta_g(g_\Lambda, \lambda_\Lambda; d, \text{gauge}, \alpha, \text{parametrization}, \text{scheme}, m) \\ &\equiv (d-2+\eta_N)g_\Lambda, \end{aligned} \quad (2.33)$$

$$\begin{aligned} \Lambda \partial_\Lambda \lambda_\Lambda &= \beta_\lambda(g_\Lambda, \lambda_\Lambda; d, \text{gauge}, \alpha, \text{parametrization}, \text{scheme}, m) \\ &\equiv (\eta_N-2)\lambda_\Lambda + \Phi, \end{aligned} \quad (2.34)$$

where $\eta_N \equiv -\Lambda \partial_\Lambda \log Z_\Lambda$. The beta functions show a dependence on the dimension, the gauge choice, the parametrization, and the regulating scheme, and they can be written in terms of η_N and Φ . Their values are shown in Appendix D for the scenarios discussed below: in the BFG, ThG, or T σ G; with the linear or exponential parametrization [respectively $\tau = 0$ or $\tau = 1$ in Eq. (2.6)]; and finally with the C or B regulating scheme.

III. RUNNING OF g_Λ IN $d=2+\epsilon$

We start by checking our setup, and more specifically Eq. (2.33) with $\lambda_\Lambda = 0$ fixed, against the well-settled case of the fixed point in $d = 2 + \epsilon$, (see [1,2]). By performing an expansion around $g = 0$ and $\epsilon = 0$, we get

$$\beta_g = (\varepsilon + \eta_N)g_\Lambda \equiv \varepsilon g_\Lambda + A g_\Lambda^2 + o(\varepsilon g_\Lambda) + o(\varepsilon^2) + o(g_\Lambda^2). \quad (3.1)$$

The value of A can be computed analytically and it turns out to be independent of the regulating scheme, yet it shows a dependence on the gauge and parametrization choices. For the BFG and the ThG we find $A = -\frac{38}{3} - 4\tau$, while for the T σ G, $A = -\frac{26}{3} - 4\tau$. A curious remark is that the beta function has the same value for the first two gauges in the linear parametrization ($\tau = 0$) and for the last gauge in the exponential parametrization ($\tau = 1$).

It is clear that the different results are due to the Hessian of the effective action. In fact, the terms that make it up show a sensitivity to the parametrization, and furthermore, different terms appear when different gauges are used. A deeper analysis of this result consists in comparing the various contributions, obtained by splitting the flow equation into pieces generated by the different terms in the Hessian as, for instance, the traceless-transverse contributions in the ThG and C scheme,

$$\text{traceless-transverse contribution} \equiv \text{STr}_{\tilde{T}} \left(\frac{m\Lambda^2 A_{\Lambda, \tilde{T}}}{\mathcal{S}_{\tilde{T}\tilde{T}} + m\Lambda^2 A_{\Lambda, \tilde{T}}} \right)^m, \quad (3.2)$$

and the analogous contributions coming from the scalar h , from the vector ghost and from the scalar ghost. Their sum gives back the full flow. The same procedure is applied for each scenario and the results are separately reported in Table I. In the data of Table I, we find no dependence either on the parameter m or on the scheme (B or C), while they are dependent on τ and on the gauge choice.

We notice that the dependence on the parametrization is entirely carried by the scalar parts in all gauges. Then, if we compare the BFG and the ThG, we see that the differences arise in the transverse vector and scalar parts, including the ghosts. But, as shown in Table II, if we take the sum of the transverse vector and the transverse vector ghost part, the result in the two gauges is the same, and the same happens for the scalar and the scalar ghost part. The different value of A

TABLE I. Contribution to A in Eq. (2.20) from different sectors of the Hessian, for the three different gauge choices.

	BFG	ThG	T σ G
Traceless-transverse	-12	-12	-12
Transverse vector	$\frac{2}{3}$		
Scalar σ	$\frac{14}{3}$		$\frac{14}{3} - 4\tau$
Scalar h	$\frac{14}{3} - 4\tau$	$\frac{14}{3} - 4\tau$	
Transverse vector ghost	$-\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
Scalar ghost	$-\frac{28}{3}$	$-\frac{14}{3}$	$-\frac{2}{3}$
Total	$-\frac{38}{3} - 4\tau$	$-\frac{38}{3} - 4\tau$	$-\frac{26}{3} - 4\tau$

TABLE II. Contribution to A in Eq. (2.20) from different sectors of the Hessian, grouped by their spin.

	BFG	ThG	T σ G
Spin-2	-12	-12	-12
Spin-1	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
Spin-0	-4τ	-4τ	$4 - 4\tau$
Total	$-\frac{38}{3} - 4\tau$	$-\frac{38}{3} - 4\tau$	$-\frac{26}{3} - 4\tau$

for the T σ G, when compared to the ThG, is entirely due to the spin-0 part, with a difference of 4, the same difference between the results for the linear and exponential parametrizations.

IV. NON-GAUSSIAN FIXED POINT IN $d=4$

We are now able to search for the non-Gaussian fixed point (NGFP) $(\lambda_\Lambda^*, g_\Lambda^*) \neq (0, 0)$ of the two beta functions (2.33) and (2.34), and analyze its stability in $d=4$ for the various scenarios introduced above and therefore, in addition to the regulation scheme (B or C) and to the specific gauge choice, we must take into account the dependence on the parameter m in the regulator function, and also (for the BFG gauge only) on the gauge fixing parameter α . In the following we shall select three representative values of m , namely, $m=3, 4, 8$; in fact the first two values are the smallest integers allowed for this parameter, according to the constraint $m > d/2$, and $m=8$ essentially corresponds to the value which provides the best determination of the universal index ν for the critical 3D Ising model, at least to the lowest-order truncation of the derivative expansion [28].

The gauge fixing requires non-negative values of α . Nevertheless, we investigate the dependence on this parameter in the interval $(-10, 10)$, by analytically continuing the expressions to negative α .

A. Background field gauge

Let us begin by analyzing the scenario with the BFG and linear parametrization ($\tau=0$). In both the B and C schemes, we find the NGFP for different values of m and α with the exception of $\alpha=3$ [obtained from Eq. (2.21) for $d=4$] where, as mentioned above, the flow shows a singularity. The fixed point position on the plane depends on the scenarios and the parameters but is always located in the first quadrant as shown in Fig. 1.

Concerning the stability issue, we examine the real part of the eigenvalues θ of the stability matrix. Away from the singular point, $\text{Re}\theta$ is negative, independently of the parameters and the scheme. Thus, the fixed point is UV attractive, as shown in Fig. 2 (note that our convention on the sign of the critical exponents is opposite to the one adopted in [11]). The eigenvalues in the B scheme are always a pair of complex conjugates, while in the C scheme

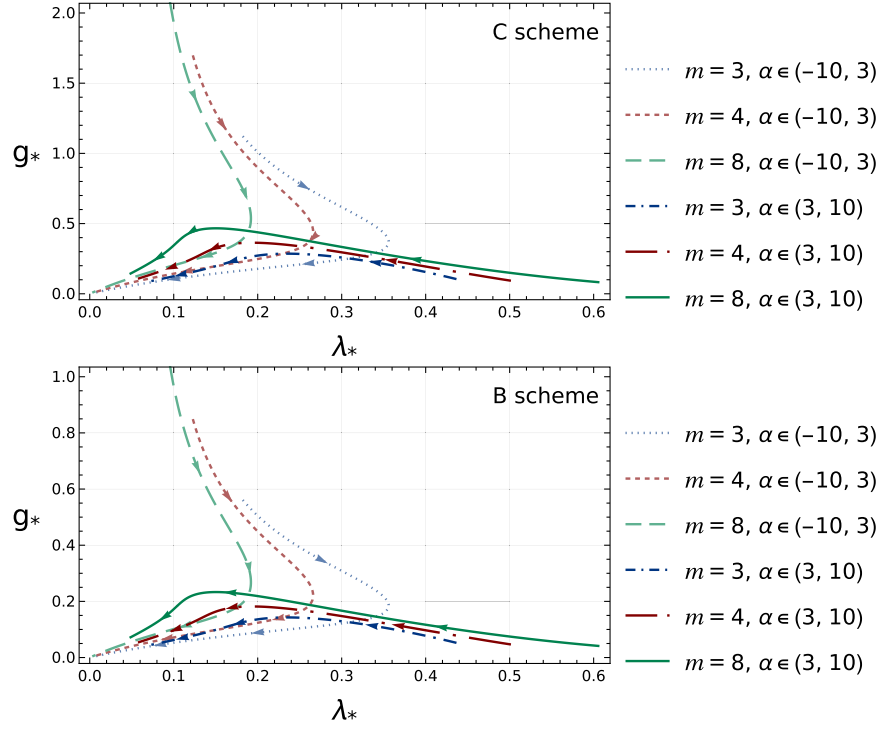


FIG. 1. NGFP in the BFG and linear parametrization for the two regularization schemes B (lower panel) and C (upper panel). Each line corresponds to a value of m and each dot on a line corresponds to one particular value of α . α increases, following the arrow, from -10 to 3 below the singularity $\alpha = 3$ and it grows from 3 to 10 above the singularity.

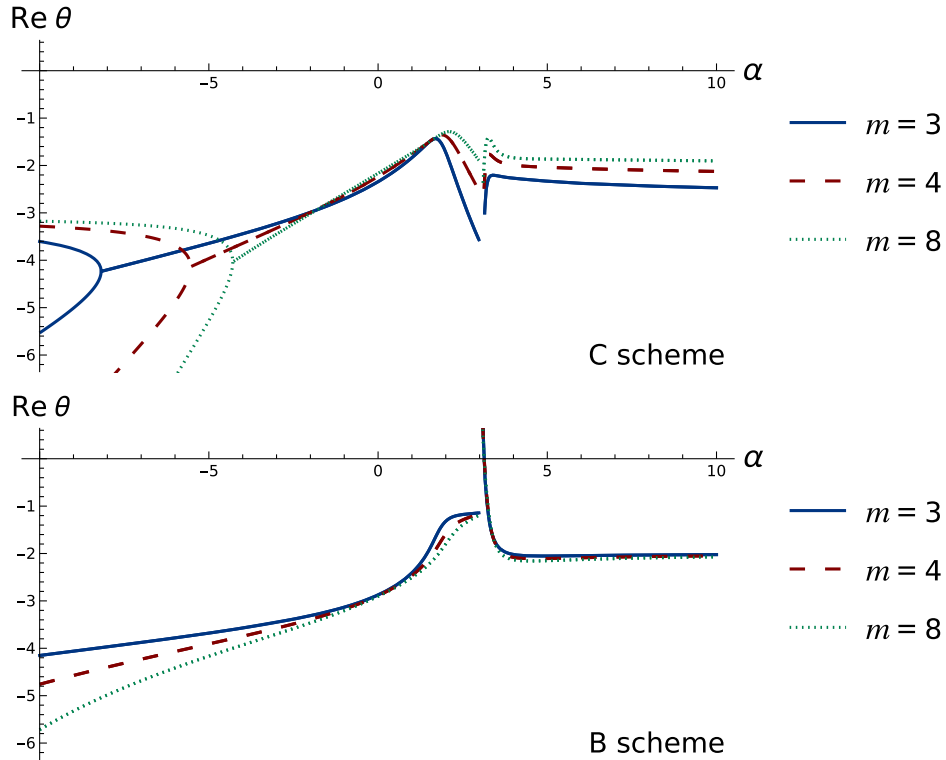


FIG. 2. The real part of the eigenvalues θ of the stability matrix evaluated at the NGFP with the BFG and the linear parametrization for the two regularization schemes B (lower panel) and C (upper panel). As in Fig. 1, each line is parametrized by the gauge fixing coefficient α and corresponds to a particular value of m . The two plots show the effects of the singularity at $\alpha = 3$, and for the C scheme, the presence of a bifurcation, illustrating how the eigenvalues change from real to complex.

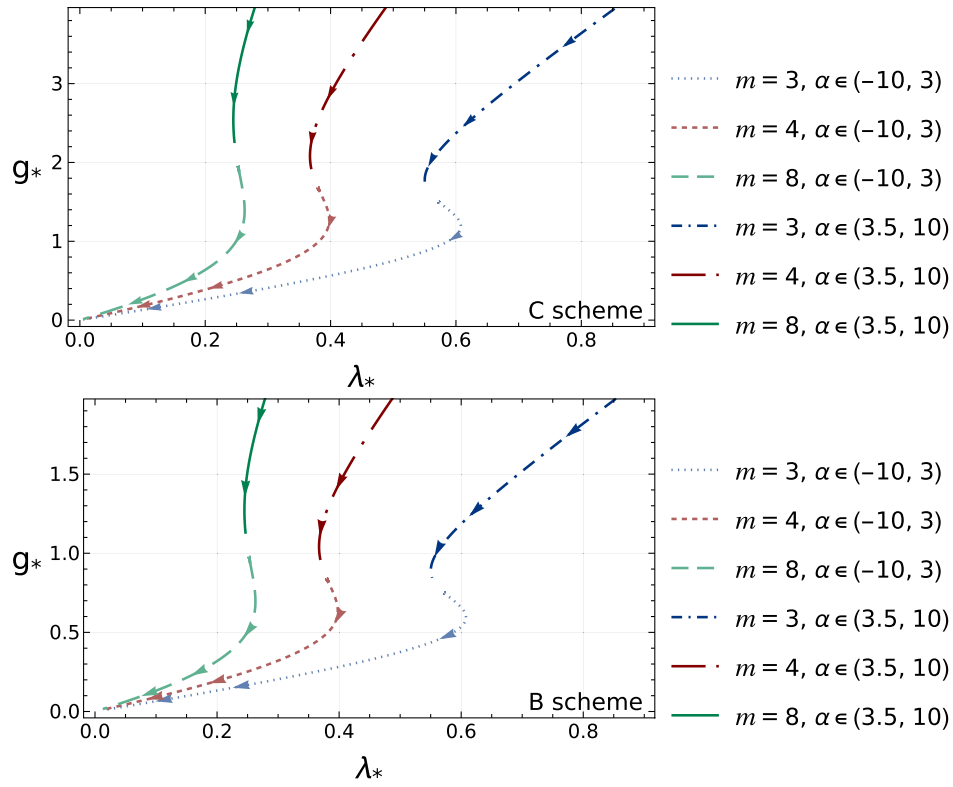


FIG. 3. NGFP with the BFG and exponential parametrization for the two regulating schemes B (lower panel) and C (upper panel). Coding is as in Fig. 1. Values of α within interval $3 < \alpha \lesssim 3.5$ are not plotted, as in this region the NGFP solution is not found.

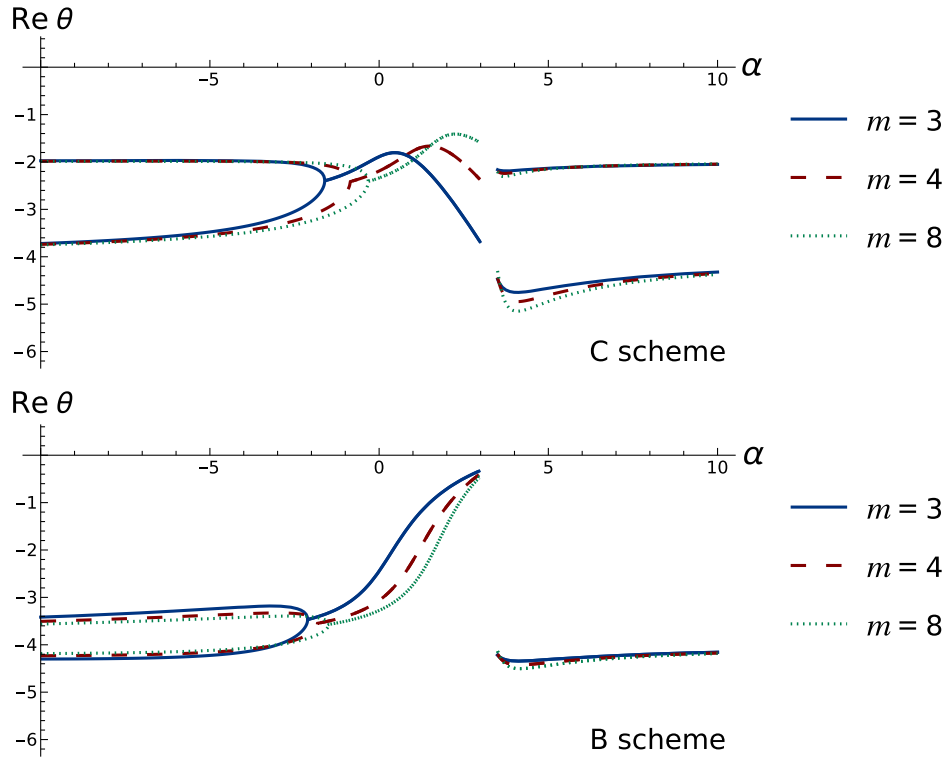


FIG. 4. Real part of the eigenvalues θ of the stability matrix evaluated at the NGFP, with the BFG and exponential parametrization, for the two regulating schemes B (lower panel) and C (upper panel). Coding is as in Fig. 2. As in the case of the upper panel of Fig. 2, the presence of bifurcations, corresponding to the change of the eigenvalues from real to complex, is observed.

they bifurcate for negative α and become a pair of real eigenvalues. In proximity of the singularity, the stability of the fixed point strongly depends on the scheme. In fact, in the B scheme, the NGFP is attractive when the singularity is approached from the left, while it becomes repulsive when it is approached from the right. In contrast, in the C scheme, the NGFP remains attractive in both regions.

By switching to the scenario in the BFG and exponential parametrization, we find the NGFP for different values of m and α in both schemes, with the exception of the region close to the singularity, where we find the NGFP on the left of the singularity, while on the right side it disappears in the interval $3 < \alpha \lesssim 3.5$. The fixed point is located in the first quadrant and its location depends on both the scheme and the parameters (see Fig. 3).

The NGFP (when it exists) is UV attractive independently of the scenario. The dependence shows up only in the values of the eigenvalues as illustrated in Fig. 4. With a bifurcation, they go from real to complex and, for $\alpha > 3.5$, they are real in the C scheme and complex in the B scheme.

B. Physical gauge

Within the physical gauge, there is no singularity in the flow, and we find the NGFP in the first quadrant and, as before, its location depends on the specific scenario. The eigenvalues are complex in the ThG, while in the T σ G, the eigenvalues are complex for the linear parametrization but are real for the exponential one. In all scenarios, the NGFP is UV attractive, as shown in Table III.

For the T σ G in the exponential parametrization, the eigenvalues are independent of the parameter m . To understand why this scenario generates this property, we must return to (C8), (C12), (C10), and (C13), which, for $\tau = 1$, are independent of $\tilde{\lambda}_\Lambda$, and Z_Λ appears only as a multiplicative factor. After the integration over s , in the C scheme, it is clear that the rhs of the flow equation does not depend of the coupling constants. Thus the flow equation is simply

$$\Lambda \partial_\Lambda \mathcal{S}_\Lambda = \int d^d x \sqrt{g} (\Omega_0(d, m) \Lambda^d + \Omega_1(d, m) R \Lambda^{d-2} + o(R^2)), \quad (4.1)$$

where we included all the dependencies of the regulator in the coefficients $\Omega_0(d, m)$ and $\Omega_1(d, m)$.

Straightforwardly the beta functions are (2.33) and (2.34), with $\eta_N = 16\pi\Omega_1(d, m)g_\Lambda$ and $\Phi = 8\pi\Omega_0(d, m)g_\Lambda$. These beta functions are quadratic in the couplings, making it possible to analytically determine the NGFP,

$$(\lambda_\Lambda^*, g_\Lambda^*) = \left(\frac{\Omega_0(d, m)}{\Omega_1(d, m)} \frac{2-d}{2d}, \frac{2-d}{16\pi\Omega_1(d, m)} \right). \quad (4.2)$$

Clearly, the NGFP depends on the regulating parameter m , while for the critical exponents, i.e. the eigenvalues of the stability matrix, this dependence cancels out, with the result

TABLE III. NGFP and critical exponents in the physical gauge.

ThG—Linear parametrization					
Scheme	m	λ_Λ^*	g_Λ^*	$\text{Re}\theta$	$\text{Im}\theta$
C	3	0.3310	0.5152	−2.343	1.325
C	4	0.2474	0.6303	−2.234	1.148
C	8	0.1792	0.7813	−2.165	1.009
B	3	0.3310	0.2576	−2.875	2.494
B	4	0.2474	0.3152	−2.891	2.063
B	8	0.1792	0.3907	−2.904	1.727
ThG—Exponential parametrization					
Scheme	m	λ_Λ^*	g_Λ^*	$\text{Re}\theta$	$\text{Im}\theta$
C	3	0.5992	1.066	−1.889	1.326
C	4	0.3975	1.3140	−2.190	0.686
C	8	0.2610	1.5790	−2.336	0.359
B	3	0.5992	0.5330	−2.447	2.161
B	4	0.3975	0.6570	−3.064	1.072
B	8	0.2610	0.7895	−3.276	0.665
T σ G—Linear parametrization					
Scheme	m	λ_Λ^*	g_Λ^*	$\text{Re}\theta$	$\text{Im}\theta$
C	3	0.3133	0.5783	−2.567	1.345
C	4	0.2325	0.7161	−2.477	1.198
C	8	0.1677	0.8975	−2.424	1.086
B	3	0.3133	0.2892	−3.035	2.754
B	4	0.2325	0.3580	−3.061	2.402
B	8	0.1677	0.4487	−3.084	2.145
T σ G—Exponential parametrization					
Scheme	m	λ_Λ^*	g_Λ^*	θ_1	θ_2
C	3	0.5625	1.5708	−4	−2
C	4	0.375	1.76715	−4	−2
C	8	0.25	2.06167	−4	−2
B	3	0.5625	0.785398	−4	−4
B	4	0.375	0.883573	−4	−4
B	8	0.25	1.03084	−4	−4

$$\theta_1 = -d, \quad \theta_2 = 2 - d. \quad (4.3)$$

In the B scheme the flow equation is the same with only a multiplicative factor overall,

$$\Lambda \partial_\Lambda \mathcal{S}_\Lambda = \int d^d x \sqrt{g} (\Omega_0(d, m) \Lambda^d + \Omega_1(d, m) R \Lambda^{d-2} + o(R^2)) \left(1 - \frac{\eta}{2} \right), \quad (4.4)$$

and the beta functions are the previous with the prescriptions (D12) and (D13).

The NGFP is

$$(\lambda_\Lambda^*, g_\Lambda^*) = \left(\frac{\Omega_0(d, m)}{\Omega_1(d, m)} \frac{2-d}{2d}, \frac{2-d}{8\pi d \Omega_1(d, m)} \right). \quad (4.5)$$

It shows dependence on m , whereas the critical exponents are

$$\theta_1 = -d, \quad \theta_2 = \frac{1}{2}d(2-d). \quad (4.6)$$

So, in both schemes the position of the fixed point is not universal, while the critical exponents are independent of m . We verified that the second eigenvalue is different in the two schemes due to the presence of the factor $(1 - \frac{\eta}{2})$ in the B scheme that originates from the factor $\frac{\Lambda}{2} \partial_\Lambda \log Z_\Lambda$ in Eq. (2.31), unlike in the C scheme where such terms are absent, as noticed before.

The same results can also be obtained within a different approach, i.e. by using the renormalization group flow equation for the effective average action Γ_k [16–18]

$$\partial_t \Gamma_k = \frac{1}{2} \text{STr} \left[\frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k} \right] = \frac{1}{2} \sum_i \text{STr} \left[\frac{\partial_t R_{k,i}}{\Gamma_{ii} + R_{k,i}} \right], \quad (4.7)$$

where, in the last step, the same notation used for the Wilsonian effective action is adopted, with Γ_{ii} identical to $\mathcal{S}_{ii}$. $R_{k,i}$ is the regulator that can be written as $R_{k,i} = A_{k,i} k^2 R^{(0)}(\frac{-\bar{D}^2}{k^2})$, with $R^{(0)}(z)$ an appropriate regulating function. Then,

$$\begin{aligned} \partial_t R_{k,i} = & \partial_t A_{k,i} k^2 R^{(0)}\left(\frac{-\bar{D}^2}{k^2}\right) + 2A_{k,i} k^2 R^{(0)}\left(\frac{-\bar{D}^2}{k^2}\right) \\ & - 2A_{k,i} k^2 \left(\frac{-\bar{D}^2}{k^2}\right) \left[R^{(0)}\left(\frac{-\bar{D}^2}{k^2}\right) \right]' \end{aligned} \quad (4.8)$$

and, if we neglect the $\partial_t A_{k,i}$ in Eq. (4.8), $A_{k,i}$ is an overall factor in $\partial_t R_{k,i}$, thus the rhs of Eq. (4.7) is independent of the coupling constants. This leads us directly to Eq. (4.1), where the two coefficients now depend of the regulating function instead of m .

The NGFP is then (4.2) and the critical exponents are (4.3).

On the other hand, if we neglect $[R^{(0)}(\frac{-\bar{D}^2}{k^2})]'$ in Eq. (4.8), it can be written as

$$\partial_t R_{k,i} = 2A_{k,i} k^2 R^{(0)}\left(\frac{-\bar{D}^2}{k^2}\right) \left(1 - \frac{\eta}{2}\right). \quad (4.9)$$

Then Eq. (4.7) behaves as Eq. (4.4); thus the NGFP is (4.5) and the critical exponents are (4.6).

V. CONCLUSIONS

The nonperturbative renormalizability of quantum Einstein gravity (QEG) is a direct consequence of the existence of a non-Gaussian FP, characterized by a finite-dimensional UV-attractive critical manifold for the RG flow of the associated running action. Therefore, the goal of the present analysis is to provide further evidence to the already existing research, that residual dependence on nonphysical parametrizations, necessary to define the flow, disappear (at least partially) when one focuses on the physically relevant properties of such a critical manifold.

In fact, it is known that, for a truncated effective action of a diffeomorphism-invariant theory of gravity, quantities such as beta functions depend not only on the details of the regularization but also on field parametrization and gauge fixing choice. However, in a physically meaningful picture, it is desirable that the residual unphysical dependence does actually disappear when establishing the existence of a nontrivial solution of the FP equation (we recall here that its specific location—unlike its existence—is not a universal property) and the sign of the eigenvalues of the linearized flow in proximity of the FP.

To this aim, in our analysis, we selected the Einstein-Hilbert truncation and used either the linear or the exponential parametrization, the background field gauge or the physical gauge and, when resorting to the PT flow equation, we retained two suitable families of parametrized regulators. With this setup, the search for nonperturbative FP in $d = 2 + \epsilon$ dimensions shows that such a solution is always guaranteed but its specific value shows the residual parametrization dependence, which is entirely carried by the spin-0 degrees of freedom, while the spin-2 and spin-1 components remain independent. Furthermore, the results for the Newton and cosmological constants in $d = 4$ dimensions confirm the presence of such residual dependence.

More specifically, when resorting to the background gauge, even though the (real part of the) eigenvalues plotted in Figs. 2 and 4 show, at least qualitatively, a weak dependence on the regulator parameter m (see for instance the bifurcation of the eigenvalue solution occurring for negative α in all plots, with the exception of the lower panel of Fig. 2), the dependence on the regulator scheme (B or C) and on the linear or exponential parametrization is significant. On the other hand, the dependence on the gauge parameter α , quantitatively evident for $\alpha \leq 4$, practically vanishes for larger α .

When switching to the physical gauge, the values of the real part of the eigenvectors in Table III indicate a mild dependence both on the parametrization and on the parameter m , and a stronger dependence on the scheme B or C, with $|\text{Re}\theta|$ slightly larger in the former case. In this framework, it is noteworthy that the absence of the cosmological constant in the Hessian, obtained with exponential parametrization and with “T- σ gauge,” directly implies m -parameter independent eigenvalues.

Finally a comparison between the results in the two gauges is not particularly significant, due to the large numbers of parameters involved, and one can only notice that the changes in $\text{Re}\theta$ are reasonably limited, as long as the parameter α does not approach the singular value $\alpha = 3$. In any case, for almost all configurations analyzed, independently of the adopted field parametrization, gauge choice, and regulator parametrization, we always find a UV-attractive non-Gaussian fixed point with positive values of the coupling constants, thus confirming an asymptotically safe scenario even for the PT flow.

As already mentioned, the gauge and parametrization dependence was already analyzed in [11]. When confronting the approach followed in this work with ours, it is necessary to remark the difference both in the RG flow equations adopted and in the implementation of the background field gauge fixing. In particular, they do not impose the constraint (2.19) and investigate the dependence on $\beta = d\omega - 1$ for fixed values of α . Thus, a direct comparison between the two results is not possible, but it is nevertheless interesting that these different approaches reach the same conclusion about the robustness of the asymptotic safety hypothesis and, in addition, the critical exponents in the two cases share similar properties, such as bifurcations from real to complex values. Finally, it would be interesting to investigate whether a suitable restriction to the flow of essential operators only, according to the spirit of [41], could further suppress the residual unphysical parametrization dependence in the beta functions of the theory.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: YORK DECOMPOSITION

The fluctuating metric, being a symmetric 2-tensor, can be decomposed into a spin-2, a spin-1, and two spin-0 components. In fact, we can separate it into its traceless and trace parts

$$h_{\mu\nu} = h_{\mu\nu}^{\text{Traceless}} + \frac{1}{d} \bar{g}_{\mu\nu} h, \quad (\text{A1})$$

with $h = \bar{g}_{\mu\nu} h^{\mu\nu}$ the first spin-0 component and $\bar{g}^{\mu\nu} h_{\mu\nu}^{\text{Traceless}} = 0$. From the traceless part, we isolate its transverse component $h_{\mu\nu}^T$, which satisfies the condition $\bar{D}^\mu h_{\mu\nu}^T = 0$, while the remaining degrees of freedom are

described by a vector v^μ :

$$h_{\mu\nu}^{\text{Traceless}} = h_{\mu\nu}^T + \bar{D}_\mu v_\nu + \bar{D}_\nu v_\mu - \frac{2}{d} \bar{g}_{\mu\nu} \bar{D}^\lambda v_\lambda. \quad (\text{A2})$$

$h_{\mu\nu}^T$ is the spin-2 tensor, while v_μ is decomposed into a transverse and a longitudinal part, respectively the spin-1 and spin-0 components:

$$v_\mu = \hat{\xi}_\mu + \bar{D}_\mu \frac{\hat{\sigma}}{2}, \quad (\text{A3})$$

with $\bar{D}^\mu \hat{\xi}_\mu = 0$. In this way we arrive at the York decomposition:

$$h_{\mu\nu} = h_{\mu\nu}^T + \bar{D}_\mu \hat{\xi}_\nu + \bar{D}_\nu \hat{\xi}_\mu + \left(\bar{D}_\mu \bar{D}_\nu - \frac{1}{d} \bar{g}_{\mu\nu} \bar{D}^2 \right) \hat{\sigma} + \frac{1}{d} \bar{g}_{\mu\nu} h. \quad (\text{A4})$$

This decomposition carries a nontrivial Jacobian in the path integral. We rescale

$$\begin{aligned} \hat{\xi}_\mu &= \left(-\bar{D}^2 - \frac{\bar{R}}{d} \right)^{-\frac{1}{2}} \xi_\mu \quad \text{and} \\ \hat{\sigma} &= (-\bar{D}^2)^{-\frac{1}{2}} \left(-\bar{D}^2 - \frac{\bar{R}}{d-1} \right)^{-\frac{1}{2}} \sigma, \end{aligned} \quad (\text{A5})$$

and the path integral in terms of $\{h_{\mu\nu}^T, \xi_\mu, \sigma, h\}$ has a trivial Jacobian.

APPENDIX B: GAUGE FIXING IN QEG

An infinitesimal diffeomorphism ϵ is described by the Lie derivative of the metric:

$$\delta g_{\mu\nu} = L_\epsilon g_{\mu\nu} \equiv \epsilon^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \epsilon^\rho + g_{\rho\nu} \partial_\mu \epsilon^\rho = D_\mu \epsilon_\nu + D_\nu \epsilon_\mu, \quad (\text{B1})$$

under which the action $\mathcal{S}_\Lambda[g]$ remains invariant.

With the metric split into background and fluctuation components, we must define their respective transformations. One possibility, dubbed background gauge transformation, the background metric transforms as

$$\delta_B \bar{g}_{\mu\nu} = L_\epsilon \bar{g}_{\mu\nu} \quad (\text{B2})$$

and thus for both the linear and the exponential parametrization

$$\delta_B h_{\mu\nu} = L_\epsilon h_{\mu\nu}. \quad (\text{B3})$$

Another option, indicated as quantum gauge transformation, corresponds to no background transformation:

$$\delta_Q \bar{g}_{\mu\nu} = 0, \quad (\text{B4})$$

and therefore, in the linear parametrization, we have

$$\delta_Q h_{\mu\nu} = L_\epsilon g_{\mu\nu}, \quad (\text{B5})$$

while with the exponential parametrization

$$\bar{g}_{\mu\rho} \delta_Q (e^h)_\nu^\rho = L_\epsilon g_{\mu\nu}. \quad (\text{B6})$$

For our purpose, after the expansion of Eq. (B6) in powers of h_ν^ρ , we find

$$\bar{g}_{\mu\rho} (e^h)_\alpha^\rho \left(\delta_Q h^\alpha{}_\nu + \frac{1}{2} [\delta_Q h, h]^\alpha{}_\nu \right) + o(h^2) = L_\epsilon g_{\mu\nu}. \quad (\text{B7})$$

Then, since we are interested in $\delta_Q h_{\mu\nu} = \delta_Q (g_{\mu\nu} h^\gamma{}_\nu)$, the transformation with the exponential parametrization is

$$\delta_Q h_{\mu\nu} = L_\epsilon g_{\alpha\beta} \left(\delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{2} \delta_\mu^\alpha h^\beta{}_\nu + \frac{1}{2} h^\alpha{}_\mu \delta_\nu^\beta \right) + o(h^2). \quad (\text{B8})$$

The quantum transformations for the fluctuation are clearly different for the two parametrizations, but they yield the same value when $h = 0$. Then, for both parametrizations, the transformation can be expressed as

$$\delta_Q h_{\mu\nu} = L_\epsilon \bar{g}_{\mu\nu} + o(h). \quad (\text{B9})$$

1. Background field gauge

We fix the gauge by using the quantum gauge transformations, imposing the condition

$$F_\mu[h; \bar{g}] = (\bar{g}^{\alpha\gamma} \delta_\mu^\beta \bar{D}_\gamma - \omega \bar{g}^{\alpha\beta} \bar{D}_\mu) h_{\alpha\beta}. \quad (\text{B10})$$

With the Faddeev-Popov procedure, we remove the redundancies in the path integral by inserting the gauge fixing action

$$\mathcal{S}_{\text{gf}}[h; \bar{g}] = \frac{1}{\alpha} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu, \quad (\text{B11})$$

and the Faddeev-Popov determinant $\det(\frac{\delta F_\mu}{\delta \epsilon^\nu})$, which is written as a path integral over the ghost fields $C_\mu, \bar{C}_\mu$ of the action

$$\mathcal{S}_{\text{gh}}[h, C, \bar{C}; \bar{g}] = - \int d^d x \sqrt{\bar{g}} \bar{C}_\mu \frac{\delta F^\mu}{\delta \epsilon^\nu} C^\nu. \quad (\text{B12})$$

Then, Eq. (2.1) is now

$$Z = \int D h_{\mu\nu}^T D \xi_\mu D \sigma D h D C_\mu D \bar{C}_\mu e^{-S[g] - S_{\text{gf}}[h; \bar{g}] - S_{\text{gh}}[h, C, \bar{C}; \bar{g}]}. \quad (\text{B13})$$

This gauge fixing procedure exhibits a dependence on the parametrization. While (B10) has been chosen independently of it, the Faddeev-Popov determinant, which depends on the quantum gauge transformations, differs in the two parametrizations. In fact, in the linear parametrization

$$\frac{\delta F_\mu}{\delta \epsilon^\nu} = (\bar{g}^{\alpha\gamma} \delta_\mu^\beta \bar{D}_\gamma - \omega \bar{g}^{\alpha\beta} \bar{D}_\mu) (g_{\nu\delta} D_\alpha + g_{\nu\alpha} D_\beta), \quad (\text{B14})$$

while in the exponential parametrization

$$\begin{aligned} \frac{\delta F_\mu}{\delta \epsilon^\nu} &= (\bar{g}^{\alpha\gamma} \delta_\mu^\beta \bar{D}_\gamma - \omega \bar{g}^{\alpha\beta} \bar{D}_\mu) (g_{\nu\delta} D_\rho + g_{\nu\rho} D_\delta) \\ &\times \left(\delta_\alpha^\rho \delta_\beta^\delta + \frac{1}{2} \delta_\alpha^\rho h^\delta{}_\beta + \frac{1}{2} h^\rho{}_\alpha \delta_\beta^\delta \right) + o(h^2). \end{aligned} \quad (\text{B15})$$

In the flow equation, we are interested in the truncation at $h = 0$; thus, the ghost action results the same in the two parametrizations.

2. Physical gauge

The idea of the physical gauge is to strongly constrain the redundant degrees of freedom in the path integral. If we decompose the vector generating the infinitesimal diffeomorphism into its transverse and longitudinal components

$$\epsilon_\mu = \epsilon_\mu^T + \bar{D}_\mu \frac{\psi}{\sqrt{-\bar{D}^2}}, \quad \text{with} \quad \bar{D}^\mu \epsilon_\mu^T = 0, \quad (\text{B16})$$

Eq. (B9) is

$$\delta h_{\mu\nu} = \bar{D}_\mu \epsilon_\nu^T + \bar{D}_\nu \epsilon_\mu^T + \bar{D}_\mu \bar{D}_\nu \frac{2\psi}{\sqrt{-\bar{D}^2}} + o(h). \quad (\text{B17})$$

This transformation must be implemented in the many fields of the York decomposition

$$\begin{aligned} \delta h_{\mu\nu} &= \delta h_{\mu\nu}^T + (\bar{D}_\mu \delta_{\nu}^{\hat{\xi}} + \bar{D}_\nu \delta_{\mu}^{\hat{\xi}}) \\ &+ \left(\bar{D}_\mu \bar{D}_\nu - \frac{1}{d} \bar{g}_{\mu\nu} \bar{D}^2 \right) \delta \hat{\sigma} + \frac{1}{d} \bar{g}_{\mu\nu} \delta h. \end{aligned} \quad (\text{B18})$$

The trace of the two equations gives the transformation

$$\delta h = -2 \sqrt{-\bar{D}^2} \psi + o(h). \quad (\text{B19})$$

We isolate the transverse and longitudinal components of the transformation

$$\delta \hat{\sigma} = \frac{2\psi}{\sqrt{-\bar{D}^2}} + o(h), \quad (\text{B20})$$

$$\delta \hat{\xi}_\mu = \epsilon_\mu^T + o(h), \quad (\text{B21})$$

$$\delta h_{\mu\nu}^T = o(h), \quad (\text{B22})$$

and write the transformations of $\hat{\sigma}$ and $\hat{\xi}_\mu$ in term of the rescaled fields σ and ξ_μ :

$$\delta \sigma = 2\sqrt{-\bar{D}^2 - \frac{\bar{R}}{d-1}}\psi + o(h), \quad (\text{B23})$$

$$\delta \xi_\mu = \sqrt{-\bar{D}^2 - \frac{\bar{R}}{d}}\epsilon_\mu^T + o(h). \quad (\text{B24})$$

Up to $o(h)$ terms, $h_{\mu\nu}^T$ is invariant, ξ_μ transforms with ϵ_μ^T , while σ and h transform with ψ . We can use these transformations to fix the gauge. ϵ^T can be fixed to have $\xi_\mu = 0$, yielding a determinant in the path integral, which can be written as an integration over the real ghost field b_μ , a transverse vector:

$$\mathcal{S}_{\text{gh}}^{(\xi=0)} = \int d^d x \sqrt{\bar{g}} b^\mu \left(-\bar{D}^2 - \frac{\bar{R}}{d} + o(h) \right) b_\mu. \quad (\text{B25})$$

Similarly, ψ can be fixed to have $h = 0$ or $\sigma = 0$. Here as well, there is a determinant, which we write as an integration over the real ghost fields ϕ and χ , respectively:

$$\mathcal{S}_{\text{gh}}^{(h=0)} = \int d^d x \sqrt{\bar{g}} \phi (-\bar{D}^2 + o(h)) \phi \quad (\text{B26})$$

and

$$\mathcal{S}_{\text{gh}}^{(\sigma=0)} = \int d^d x \sqrt{\bar{g}} \chi \left(-\bar{D}^2 - \frac{\bar{R}}{d-1} + o(h) \right) \chi. \quad (\text{B27})$$

If we impose the conditions $\xi_\mu = 0$ and $\sigma = 0$, the fluctuation is described only by $h_{\mu\nu}^T$ and h . We refer to this gauge choice as the T-h gauge (ThG):

$$Z = \int D h_{\mu\nu}^T D h D b_\mu D \chi e^{-\mathcal{S}[g]|_{\xi_\mu=\sigma=0} - \mathcal{S}_{\text{gh}}^{(\xi=0)} - \mathcal{S}_{\text{gh}}^{(\sigma=0)}}. \quad (\text{B28})$$

If we impose the conditions $\xi_\mu = 0$ and $h = 0$, the fluctuation is described only by $h_{\mu\nu}^T$ and σ . We refer to this gauge choice as the T- σ gauge (T σ G):

$$Z = \int D h_{\mu\nu}^T D \sigma D b_\mu D \phi e^{-\mathcal{S}[g]|_{\xi_\mu=h=0} - \mathcal{S}_{\text{gh}}^{(\xi=0)} - \mathcal{S}_{\text{gh}}^{(h=0)}}. \quad (\text{B29})$$

For the linear and exponential parametrizations, the ghost actions differ in the $o(h)$ term. When we consider the truncation in our flow equation, $o(h)$ vanishes, and the two parametrizations yield the same ghost actions.

APPENDIX C: COMPONENTS OF THE HESSIAN

1. Background field gauge

The traceless-transverse part is

$$\mathcal{S}_{TT} = 2\kappa^2 Z_\Lambda \left[-\frac{1}{2} \bar{D}^2 + (\tau - 1) \tilde{\lambda}_\Lambda + \frac{d^2 - 3d + 4 - \tau(d^2 - 3d + 2)}{2d(d-1)} \bar{R} \right], \quad (\text{C1})$$

the transverse vector parts are

$$\mathcal{S}_{\xi\xi} = 2\kappa^2 Z_\Lambda \left[-\frac{1}{\alpha} \bar{D}^2 + (\tau - 1) 2\tilde{\lambda}_\Lambda + \frac{\alpha(d-2) - 1 - \tau\alpha(d-2)}{\alpha d} \bar{R} \right], \quad (\text{C2})$$

$$\mathcal{S}_{CC} = Z_\Lambda \left[-\bar{D}^2 - \frac{\bar{R}}{d} \right], \quad (\text{C3})$$

and the scalar parts are

$$\mathcal{S}_{\sigma\sigma} = 2\kappa^2 Z_\Lambda \left[\frac{(d-1)(2 + d(\alpha-2) - 2\alpha)}{2\alpha d^2} \bar{D}^2 + \frac{d-1}{d} (\tau-1) \tilde{\lambda}_\Lambda + \frac{\alpha(1-\tau)(d^2 - 3d + 2) + 2 - 2d}{2\alpha d^2} \bar{R} \right], \quad (\text{C4})$$

$$\mathcal{S}_{h\sigma} = \kappa^2 Z_\Lambda \frac{(1-d)(\alpha(d-2) - 2d\omega + 2)}{\alpha d^2} \sqrt{-\bar{D}^2 \left(-\bar{D}^2 - \frac{\bar{R}}{d-1} \right)}, \quad (\text{C5})$$

$$\mathcal{S}_{hh} = 2\kappa^2 Z_\Lambda \left[\frac{\alpha(d^2 - 3d + 2) - 2(d\omega - 1)^2}{2\alpha d^2} \bar{D}^2 + \frac{d + 2(\tau - 1)}{2d} \tilde{\lambda}_\Lambda + \frac{-d^2 + 6d - 8 - 2\tau(d - 2)}{4d^2} \bar{R} \right], \quad (\text{C6})$$

$$\mathcal{S}_{\eta\eta} = 2Z_\Lambda(1 - \omega) \left[-\bar{D}^2 - \frac{\bar{R}}{d(1 - \omega)} \right]. \quad (\text{C7})$$

2. Physical gauge

For the ThG, in the Hessian we find the traceless-transverse part

$$\mathcal{S}_{\hat{T}\hat{T}} = 2\kappa^2 Z_\Lambda \left[-\frac{\bar{D}^2}{2} + (\tau - 1)\tilde{\lambda}_\Lambda + \frac{2 - (\tau - 1)(d^2 - 3d + 2)}{2d(d - 1)} \bar{R} \right], \quad (\text{C8})$$

the scalar part

$$\mathcal{S}_{\hat{h}\hat{h}} = 2\kappa^2 Z_\Lambda \left[\frac{(d - 1)(d - 2)}{2d^2} \bar{D}^2 + \frac{d + 2(\tau - 1)}{2d} \tilde{\lambda}_\Lambda - \frac{(d - 2)(d + 2\tau - 4)}{4d^2} \bar{R} \right], \quad (\text{C9})$$

the ghost transverse vector part

$$\mathcal{S}_{bb} = 2Z_\Lambda \left[-\bar{D}^2 - \frac{\bar{R}}{d} \right], \quad (\text{C10})$$

and the ghost scalar part

$$\mathcal{S}_{\chi\chi} = 2Z_\Lambda \left[-\bar{D}^2 - \frac{\bar{R}}{d - 1} \right]. \quad (\text{C11})$$

Differently, for the TσG the Hessian is composed by traceless-transverse part (C8), the scalar part

$$\mathcal{S}_{\hat{\sigma}\hat{\sigma}} = 2\kappa^2 Z_\Lambda \left[\frac{(d - 1)(d - 2)}{2d^2} \bar{D}^2 + (\tau - 1) \frac{d - 1}{d} \tilde{\lambda}_\Lambda - (\tau - 1) \frac{d^2 - 3d + 2}{2d^2} \bar{R} \right], \quad (\text{C12})$$

the transverse ghost part (C10), and the scalar ghost part

$$\mathcal{S}_{\phi\phi} = -2Z_\Lambda \bar{D}^2. \quad (\text{C13})$$

APPENDIX D: η_N AND Φ FOR VARIOUS SCENARIOS

We start by displaying η_N and Φ in the BFG with linear parametrization and C scheme of regularization:

$$\begin{aligned} \eta_N^{\text{lin,BFG}} = & 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(-\frac{d}{2} + m + 1)}{3\Gamma(m)} \left\{ \frac{2m^m}{d} [-12(\alpha + 1) + (1 - 6\alpha)d^2 + (18\alpha + 5)d] (m - 2\alpha\lambda_\Lambda)^{\frac{d}{2}-m-1} \right. \\ & - \frac{d + 1}{d} (5d^2 - 22d + 48) m^m (m - 2\lambda_\Lambda)^{\frac{d}{2}-m-1} + \left(\frac{48}{-2\alpha + (\alpha - 2)d + 2} - 4d + \frac{48}{d} - 24 \right) m^{\frac{d}{2}-1} \\ & + \frac{2m^m [(\alpha + 4)d - 2(\alpha + 11)]}{2d\lambda_\Lambda + m[-2\alpha + (\alpha - 2)d + 2]} \left(\frac{2d\lambda_\Lambda}{-2\alpha + (\alpha - 2)d + 2} + m \right)^{\frac{d}{2}-m} \\ & \left. + \frac{2m^m [(7\alpha - 2)d - 2(7\alpha + 5)]}{2\alpha d\lambda_\Lambda + m[-2\alpha + (\alpha - 2)d + 2]} \left(\frac{2\alpha d\lambda_\Lambda}{-2\alpha + (\alpha - 2)d + 2} + m \right)^{\frac{d}{2}-m} \right\} g_\Lambda, \quad (\text{D1}) \end{aligned}$$

$$\begin{aligned}\Phi^{\text{lin,BFG}} = & -\frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} 2^{2-d} \pi^{1-\frac{d}{2}} \left\{ 4dm^{d/2} - 2m^m \left(\frac{2d\lambda_\Lambda}{-2\alpha + (\alpha-2)d+2} + m \right)^{\frac{d}{2}-m} \right. \\ & - 2m^m \left(\frac{2\alpha d\lambda_\Lambda}{-2\alpha + (\alpha-2)d+2} + m \right)^{\frac{d}{2}-m} - 2(d-1)m^m(m-2\alpha\lambda_\Lambda)^{\frac{d}{2}-m} \\ & \left. - (d-2)(d+1)m^m(m-2\lambda_\Lambda)^{\frac{d}{2}-m} \right\} g_\Lambda.\end{aligned}\quad (\text{D2})$$

Then, in the BFG with exponential parametrization and C scheme of regularization, η_N and Φ are

$$\begin{aligned}\eta_N^{\text{exp,BFG}} = & 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(-\frac{d}{2}+m+1)}{3\Gamma(m)} \left\{ [d(d^2+d-16)-48] \frac{m^{\frac{d}{2}-1}}{d} \right. \\ & + \frac{2(d-2)[(\alpha+4)d-2(\alpha+5)]m^m}{2d^2\lambda_\Lambda + (d-2)[-2\alpha + (\alpha-2)d+2]m} \left(\frac{2d^2\lambda_\Lambda}{(d-2)[-2\alpha + (\alpha-2)d+2]} + m \right)^{\frac{d}{2}-m} \\ & \left. + \frac{[2(\alpha-2)d-4(\alpha+5)]m^{\frac{d}{2}-1}}{-2\alpha + (\alpha-2)d+2} + \left(\frac{48}{-2\alpha + (\alpha-2)d+2} - 4d + \frac{48}{d} - 24 \right) m^{\frac{d}{2}-1} \right\} g_\Lambda,\end{aligned}\quad (\text{D3})$$

$$\Phi^{\text{exp,BFG}} = 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} \left\{ (d^2-3d-2)m^{d/2} + 2m^m \left(\frac{2d^2\lambda_\Lambda}{(d-2)[-2\alpha + (\alpha-2)d+2]} + m \right)^{\frac{d}{2}-m} \right\} g_\Lambda. \quad (\text{D4})$$

In the ThG with linear parametrization and C scheme of regularization, η_N and Φ are

$$\begin{aligned}\eta_N^{\text{lin,ThG}} = & 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(-\frac{d}{2}+m+1)}{3\Gamma(m)} \left\{ \frac{12-12d+5d^2+d^3}{d(1-d)} 2m^{\frac{d}{2}-1} \right. \\ & \left. - \frac{1+d}{d} (48-22d+5d^2)m^m(m-2\lambda_\Lambda)^{\frac{d}{2}-m-1} + 2 \frac{2d-11}{1-d} m^m \left(m - \frac{d\lambda_\Lambda}{d-1} \right)^{\frac{d}{2}-m-1} \right\} g_\Lambda,\end{aligned}\quad (\text{D5})$$

$$\Phi^{\text{lin,ThG}} = 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} \left\{ -2dm^{\frac{d}{2}} + 2m^m \left(m - \frac{d\lambda_\Lambda}{d-1} \right)^{\frac{d}{2}-m} + (d-2)(d+1)m^m(m-2\lambda_\Lambda)^{\frac{d}{2}-m} \right\} g_\Lambda. \quad (\text{D6})$$

In the ThG with exponential parametrization and C scheme of regularization, η_N and Φ are

$$\begin{aligned}\eta_N^{\text{exp,ThG}} = & 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(-\frac{d}{2}+m+1)}{3\Gamma(m)} \left\{ \frac{(d^3-4d^2-35d+26)}{d-1} m^{\frac{d}{2}-1} \right. \\ & \left. - 2 \frac{2d-5}{d-1} m^m \left(m - \frac{d^2\lambda}{d^2-3d+2} \right)^{\frac{d}{2}-m-1} \right\} g_\Lambda,\end{aligned}$$

$$\Phi^{\text{exp,ThG}} = 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} \left\{ (d^2-3d-2)m^{d/2} + 2m^m \left(m - \frac{d^2\lambda}{d^2-3d+2} \right)^{\frac{d}{2}-m} \right\} g_\Lambda. \quad (\text{D7})$$

In the T σ G with linear parametrization and C scheme of regularization, η_N and Φ are

$$\begin{aligned}\eta_N^{\text{lin,T}\sigma\text{G}} = & -2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(-\frac{d}{2}+m+1)}{3\Gamma(m)} \left\{ \frac{2}{d} (d^2+6d-12)m^{\frac{d}{2}-1} - 14m^m \left(\frac{2d\lambda_\Lambda}{d-2} + m \right)^{\frac{d}{2}-m-1} \right. \\ & \left. + \frac{1+d}{d} (5d^2-22d+48)m^m(m-2\lambda_\Lambda)^{\frac{d}{2}-m-1} \right\} g_\Lambda,\end{aligned}\quad (\text{D8})$$

$$\Phi^{\text{lin,T}\sigma\text{G}} = 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} \left\{ -2dm^{d/2} + 2m^m \left(\frac{2d\lambda_\Lambda}{d-2} + m \right)^{\frac{d}{2}-m} + (d-2)(d+1)m^m(m-2\lambda_\Lambda)^{\frac{d}{2}-m} \right\} g_\Lambda, \quad (\text{D9})$$

while in the T σ G with exponential parametrization and C scheme of regularization

$$\eta_N^{\text{exp, T}\sigma\text{G}} = 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(-\frac{d}{2} + m + 1)}{3\Gamma(m)} (d^2 - 3d - 36) m^{\frac{d}{2}-1} g_\Lambda, \quad (\text{D10})$$

$$\Phi^{\text{exp, T}\sigma\text{G}} = 2^{2-d} \pi^{1-\frac{d}{2}} \frac{\Gamma(m - \frac{d}{2})}{\Gamma(m)} m^{\frac{d}{2}} d(d-3) g_\Lambda. \quad (\text{D11})$$

Finally, in the B scheme of regularization η_N and Φ can be expressed in terms of their counterparts in the C scheme for every gauge and parametrization:

$$\eta_N^{\text{B}} = \frac{\eta_N^{\text{C}}}{1 + \frac{\eta_N^{\text{C}}}{2}}, \quad (\text{D12})$$

$$\Phi^{\text{B}} = \left(1 - \frac{\eta_N^{\text{B}}}{2}\right) \Phi^{\text{C}}. \quad (\text{D13})$$

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- [1] S. Weinberg, Critical phenomena for field theorists, in *Proceedings of the 14th International School of Subnuclear Physics: Understanding the Fundamental Constituents of Matter, Erice, Italy, 1976*, 10.1007/978-1-4684-0931-4_1.
- [2] S. Weinberg, Ultraviolet divergences in quantum theories of gravitation, in *General Relativity: An Einstein Centenary Survey* (Cambridge University Press, Cambridge, England, 1980), pp. 790–831.
- [3] M. Reuter, *Phys. Rev. D* **57**, 971 (1998).
- [4] S. Falkenberg and S. D. Odintsov, *Int. J. Mod. Phys. A* **13**, 607 (1998).
- [5] O. Lauscher and M. Reuter, *Phys. Rev. D* **65**, 025013 (2002).
- [6] M. Reuter and F. Saueressig, *Phys. Rev. D* **65**, 065016 (2002).
- [7] A. Codello and R. Percacci, *Phys. Rev. Lett.* **97**, 221301 (2006).
- [8] A. Codello, R. Percacci, and C. Rahmede, *Int. J. Mod. Phys. A* **23**, 143 (2008).
- [9] A. Codello, R. Percacci, and C. Rahmede, *Ann. Phys. (Amsterdam)* **324**, 414 (2009).
- [10] A. Nink, *Phys. Rev. D* **91**, 044030 (2015).
- [11] H. Gies, B. Knorr, and S. Lippoldt, *Phys. Rev. D* **92**, 084020 (2015).
- [12] G. P. De Brito, N. Ohta, A. D. Pereira, A. A. Tomaz, and M. Yamada, *Phys. Rev. D* **98**, 026027 (2018).
- [13] K. G. Wilson and J. B. Kogut, *Phys. Rep.* **12**, 75 (1974).
- [14] B. S. DeWitt, *The Global Approach to Quantum Field Theory. Vol. 1, 2* (Oxford University Press, Oxford, UK, 2003), Vol. 114.
- [15] H. Kawai, Y. Kitazawa, and M. Ninomiya, *Prog. Theor. Phys. Suppl.* **114**, 149 (1993).
- [16] C. Wetterich, *Phys. Lett. B* **301**, 90 (1993).
- [17] T. R. Morris, *Int. J. Mod. Phys. A* **09**, 2411 (1994).
- [18] J. Berges, N. Tetradis, and C. Wetterich, *Phys. Rep.* **363**, 223 (2002).
- [19] F. J. Wegner and A. Houghton, *Phys. Rev. A* **8**, 401 (1973).
- [20] M. Oleszczuk, *Z. Phys. C* **64**, 533 (1994).
- [21] R. Floreanini and R. Percacci, *Phys. Lett. B* **356**, 205 (1995).
- [22] S.-B. Liao, *Phys. Rev. D* **53**, 2020 (1996).
- [23] O. Bohr, B. J. Schaefer, and J. Wambach, *Int. J. Mod. Phys. A* **16**, 3823 (2001).
- [24] S. P. de Alwis, *J. High Energy Phys.* 03 (2018) 118.
- [25] A. Bonanno and G. Lacagnina, *Nucl. Phys. B* **693**, 36 (2004).
- [26] A. Bonanno and D. Zappala, *Phys. Lett. B* **504**, 181 (2001).
- [27] M. Mazza and D. Zappala, *Phys. Rev. D* **64**, 105013 (2001).
- [28] D. F. Litim and D. Zappala, *Phys. Rev. D* **83**, 085009 (2011).
- [29] A. Bonanno and F. Guarnieri, *Phys. Rev. D* **86**, 105027 (2012).
- [30] A. Bonanno, M. Conti, and D. Zappalà, *Phys. Lett. B* **847**, 138311 (2023).
- [31] A. M. Bonanno, M. Conti, and S. L. Cacciatori, *Phys. Rev. D* **108**, 026008 (2023).
- [32] D. Zappalà, *Phys. Lett. A* **290**, 35 (2001).
- [33] A. Bonanno, A. Codello, and D. Zappalà, *Ann. Phys. (Amsterdam)* **445**, 169090 (2022).
- [34] D. F. Litim and J. M. Pawłowski, *Phys. Rev. D* **65**, 081701 (2002).
- [35] D. F. Litim and J. M. Pawłowski, *Phys. Rev. D* **66**, 025030 (2002).
- [36] A. Bonanno, S. Lippoldt, R. Percacci, and G. P. Vacca, *Eur. Phys. J. C* **80**, 249 (2020).
- [37] S. Abel and L. Heurtier, arXiv:2311.12102.
- [38] C. Wetterich, *Phys. Lett. B* **864**, 139435 (2025).
- [39] S.-B. Liao, *Phys. Rev. D* **56**, 5008 (1997).
- [40] J. S. Schwinger, *Phys. Rev.* **82**, 664 (1951).
- [41] K. Falls and R. Ferrero, arXiv:2411.00938.
- [42] K. Falls, *Phys. Rev. D* **96**, 126016 (2017).
- [43] A. Baldazzi, R. B. A. Zinati, and K. Falls, *SciPost Phys.* **13**, 085 (2022).
- [44] L. F. Abbott, *Nucl. Phys. B* **185**, 189 (1981).

- [45] R. Percacci and G. P. Vacca, [Eur. Phys. J. C **75**, 188 \(2015\)](#).
- [46] M. R. Gaberdiel, D. Grumiller, and D. Vassilevich, [J. High Energy Phys. **11** \(2010\) 094](#).
- [47] H.-b. Zhang and X. Zhang, [Classical Quantum Gravity **29**, 145013 \(2012\)](#).
- [48] J. Alexandre, N. Houston, and N. E. Mavromatos, [Phys. Rev. D **88**, 125017 \(2013\)](#).
- [49] R. Percacci, M. J. Perry, C. N. Pope, and E. Sezgin, [J. High Energy Phys. **03** \(2014\) 083](#).
- [50] P. Labus, R. Percacci, and G. P. Vacca, [Phys. Lett. B **753**, 274 \(2016\)](#).
- [51] A. Bonanno and M. Reuter, [J. High Energy Phys. **02** \(2005\) 035](#).
- [52] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).
- [53] J. W. York, Jr., [J. Math. Phys. \(N.Y.\) **14**, 456 \(1973\)](#).
- [54] D. Dou and R. Percacci, [Classical Quantum Gravity **15**, 3449 \(1998\)](#).
- [55] K. Groh and F. Saueressig, [J. Phys. A **43**, 365403 \(2010\)](#).