

Particle spectra of parity-violating theories: A less radical approach and an upgrade of the particle spectrum for any tensor Lagrangian framework

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Due to computational barriers, parity violation effects have been grossly neglected in gravitational model-building, creating a significant gap in the landscape of existing models. We present a new algorithm for efficiently computing the particle spectrum for any parity-violating tensorial field theory. It allows us to extract conditions for the absence of massive ghosts without resorting to any manipulation of radicals in cases where the particle masses are irrational functions of the Lagrangian coupling coefficients. We test it against several examples, among which is the most general parity-indefinite Einstein-Cartan/Poincaré gravity that propagates two healthy massive scalars (in addition to the massless graviton). Furthermore, we upgrade the PSALTer software in the Wolfram Language to accommodate parity-violating theories. PSALTer is a contribution to the xAct project.

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I. INTRODUCTION

A. Motivation

In the metric-based approach to gravitation, at lowest order in the derivative expansion, there is only one diffeomorphism invariant operator of the metric field¹: the Ricci scalar. This is parity invariant, as are its quantum corrections [1–5]. This does not mean, however, that gravity necessarily preserves parity. Nor is its apparent parity invariance motivated by any physical principle or phenomenological observation. Rather, it is an artefact of the metrical formulation leading to a highly symmetric Riemann curvature tensor. Parity violation comes hand-in-hand with formulations of gravity where the metric and connection are independent from each other, such as in Einstein-Cartan (EC) theory or metric-affine gravity (MAG). In principle, this means that, starting at linear order in the derivative expansion, there exist in these theories

pseudoscalar operators whose *a priori* exclusion is not justified. Indeed, it is exactly the inclusion of such operators that can lead to good phenomenology: for instance, in EC theory and MAG, parity-breaking terms enable inflation of geometrical origin. Specifically, the inclusion of the pseudoscalar curvature is absolutely essential for the inflaton potential to have a plateau—see [6–8] for details, as well as [9–11] for generalizations. Moreover, the way gravity is formulated—and consequently whether or not it violates parity—also infiltrates and modifies the Standard Model of particle physics, as shown for instance in [12–21]. In spite of the nontrivial implications (e.g., Refs. [22–26]) for phenomenology, the field dynamics of parity-nonpreserving gravitational theories remain poorly studied, with the one exception being EC/Poincaré gravity, cf. [27–34] for a nonexhaustive list of references.

B. In this paper

We present two advances in the study of the particle content of parity-violating tensor field theories in the weak-field limit on four-dimensional Minkowski spacetime, with particular relevance to theories of gravity:

New software: We implement the parity-violating extension (for EC/Poincaré gravity, a first attempt was made in the 1980s [27], but was systematized much later in [32]) of the popular spin-projection operator (SPO) method [35–38] as an upgrade to the preexisting Wolfram Language framework for such calculations:

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¹To be precise, there is also the cosmological constant, but it is irrelevant for the discussions here.

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TABLE I. Indices and labels. Summation is assumed only over repeated spacetime indices.

Symbols	Values	Meaning
μ, ν	$\{0, 1, 2, 3\}$	Minkowski spacetime indices
$\dot{\mu}, \dot{\nu}$	$\{0, 1, 2, 3\}$	Curved spacetime indices
$\bar{\mu}, \bar{\nu}$	$\{0, 1, 2, 3\}$	Minkowski spacetime indices orthogonal to the momentum
X, Y	(Symbolic)	Distinct Lorentz-covariant tensor fields
μ_X, ν_X	$\{0, 1, 2, 3\}^{\mathbb{Z}^2}$	Collections of symmetrized spacetime indices associated with field X
J, J'	$\mathbb{Z}^{\geq}$	Spin
P, P'	$\{1, -1\}$	Parity
$i_{J^P}^X, j_{J^P}^X$	$\mathbb{Z}^>$	Multiple independent copies of a J^P state associated with field X
$s_J, s'J$	$\mathbb{Z}^>$	Slots for masses associated with J if P is not a quantum number
$s_{J^P}, s'J^P$	$\mathbb{Z}^>$	Slots for masses associated with a given J^P state
$\bar{\mu}_{J^P}, \bar{\nu}_{J^P}$	$\{0, 1, 2, 3\}^{\mathbb{Z}^2}$	Collections of momentum-orthogonal indices associated with J^P
$a_J, a'J$	$\mathbb{Z}^>$	Slots for the null eigenvectors (if any) of the J wave operator block O_J

particle spectrum for any tensor Lagrangian (PSALTer), a package for the *Mathematica* software system, first presented in [39]. It is a contribution to the open-source xAct tensor computer algebra project [40–46]. Earlier versions of PSALTer have been used in [47–52]. The PSALTer software can be obtained from the public GitHub repository [53]. The majority of the upgraded software was written by generative pretrained transformers, highlighting the growing application of artificial intelligence in theoretical physics.

New algorithm: We point out a novel way to efficiently derive conditions for the absence of ghosts, that does not involve any inversion of the wave operator nor the computation of residues of the propagator at massive poles. This is particularly useful for, but not limited to, parity-violating theories which propagate two or more massive modes within one spin sector, since it completely bypasses dealing with radicals. It will hopefully be implemented in PSALTer in the near future.

We reiterate in particular that the advances to the PSALTer software apply exclusively to field theories in *four* spacetime dimensions. By contrast, it is well known that parity violation in gauge theories has a natural setting in three dimensions, upon which we comment briefly in Secs. III A and III B. The SPO implementation, however, which involves the decomposition of higher-rank tensor fields under the spatial rotation group orthogonal to the time direction, has been most highly developed in the context of four-dimensional field theory, and this is the only setting in which PSALTer can presently be used.

C. Organization of the paper

This paper is organized as follows. In Sec. II, we briefly outline some basics of the SPO formalism and discuss the standard condition(s) for absence of tachyons, as well as our novel simplified way to obtain constraints for ghost-freedom. In Sec. III, the new features in PSALTer are illustrated by obtaining the particle spectrographs of various parity-violating theories. This section is divided into two parts: in the first,

we consider somewhat trivial toy-models involving p -forms, which provide a pedagogical introduction. In the second part, we use PSALTer to study the spectrum of the most general parity-indefinite EC theory that propagates the massless graviton and two scalars of gravitational origin—we explicitly show that this model is free from ghosts and tachyons. We conclude in Sec. IV. Various technicalities can be found in the Appendices.

D. Conventions

The conventions and notation are aligned as closely as possible with [39]; any departures will be explicitly noted. We work exclusively in four spacetime dimensions. We use the particle physics or mostly minus signature for the Minkowski metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and we take $\epsilon_{0123} = 1$ for the totally antisymmetric tensor. The various kinds of indices and labels introduced are summarized in Table I.

II. THEORETICAL DEVELOPMENT

A. Introducing the algorithm

Any tensorial field theory, which has been expanded quadratically around Minkowski spacetime is described by the free action

$$S = \int d^4x \sum_X \zeta_{\mu_X} \left[\sum_Y \mathcal{O}^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} - j^{\mu_X} \right], \quad (1)$$

where Eq. (1) has the following components:

- (1) The quantities ζ_{μ_X} are real tensor fields (they are never pseudotensors).² Different fields are distinguished by an index X , and have a collection of spacetime indices μ_X , with or without some sym-

²Note that the use of pseudotensor fields can always be exchanged for tensorial ζ_{μ_X} by altering the parity of various contributing terms in the wave operator.

metry. The symmetries implemented in PSALTer so far for any X are: scalar ζ ; vector ζ_μ ; the general tensor of second rank $\zeta_{\mu\nu}$ and the special cases of antisymmetric $\zeta_{\mu\nu} \equiv \zeta_{[\mu\nu]}$ or symmetric fields $\zeta_{(\mu\nu)}$; the general tensor of third rank $\zeta_{\mu\nu\sigma}$ and the special cases $\zeta_{\mu\nu\sigma} \equiv \zeta_{[\mu\nu]\sigma}$, $\zeta_{(\mu\nu)\sigma}$, $\zeta_{[\mu|\nu|\sigma]}$, $\zeta_{\mu[\nu|\sigma]}$, $\zeta_{(\mu\nu)\sigma}$, $\zeta_{(\mu|\nu|\sigma)}$, and $\zeta_{\mu(\nu\sigma)}$.

- (2) The quantity $\mathcal{O}^{\mu_X}_{\nu_Y}$ is the wave operator, which contains all the kinematical data. It is a real differential operator constructed from $\eta_{\mu\nu}$, ∂_μ and the totally antisymmetric tensor $\epsilon_{\mu\nu\sigma\lambda}$. The presence of the latter indicates that the model is parity-violating, and marks a departure from the assumptions of [39]. The wave operator in PSALTer must have a homogeneous, linear parametrization in terms of some collection of coupling coefficients, i.e., its every term must be linear in these couplings.
- (3) The quantities j^{μ_X} are real sources for the fields ζ_{μ_X} , which inherit their index symmetries. In the presence of gauge redundancies, the sources must obey the corresponding constraints.

So far, Eq. (1) has been written in coordinate space. Accordingly, we introduce the momentum k^μ and define (using x and k as shorthand) the Fourier transform and its inverse

$$\begin{aligned}\zeta_{\mu_X}(k) &\equiv \int d^4k \exp(-ik_\mu x^\mu) \zeta_{\mu_X}(x), \\ \zeta_{\mu_X}(x) &\equiv \frac{1}{(2\pi)^4} \int d^4k \exp(ik_\mu x^\mu) \zeta_{\mu_X}(k).\end{aligned}\quad (2)$$

By using Eq. (2) and the convolution theorem, it is possible to express Eq. (1) in momentum space as

$$\begin{aligned}S &= \frac{1}{(2\pi)^4} \int d^4k \sum_X \left[\zeta_{\mu_X}(-k) \sum_Y \mathcal{O}^{\mu_X}_{\nu_Y}(k) \zeta^{\nu_Y}(k) \right. \\ &\quad \left. - \frac{1}{2} (\zeta_{\mu_X}(-k) j^{\mu_X}(k) + \zeta_{\mu_X}(k) j^{\mu_X}(-k)) \right].\end{aligned}\quad (3)$$

For real fields and sources, it follows that $\zeta_{\mu_X}(-k) \equiv \zeta_{\mu_X}^*(k)$ and $j_{\mu_X}(-k) \equiv j_{\mu_X}^*(k)$. Accordingly, we will henceforth suppress the k -dependence, so that Eq. (3) becomes

$$S = \frac{1}{(2\pi)^4} \int d^4k \sum_X \left[\zeta_{\mu_X}^* \sum_Y \mathcal{O}^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} - \frac{1}{2} (\zeta_{\mu_X}^* j^{\mu_X} + \zeta^{\mu_X} j_{\mu_X}^*) \right], \quad (4)$$

and from the equations of motion, which follow from Eq. (4), we can immediately read off a formal definition for the (scalar-valued) saturated propagator

$$\Pi \equiv j_{\mu_X}^* (\mathcal{O}^{-1})^{\mu_X}_{\nu_Y} j^{\nu_Y}, \quad (5)$$

where in Eq. (5) the quantity $(\mathcal{O}^{-1})^{\mu_X}_{\nu_Y}$ is the inverse of the wave operator. This definition will need further careful treatment in Sec. II A, because gauge symmetries actually render the wave operator *noninvertible*. For the moment, Eq. (5) is safe heuristically because any singular parts of the inverse will be sandwiched between parts of the j^{ν_Y} which those same symmetries force to vanish. Thus, the only remaining parts of Π refer exclusively to (i.e., they are *saturated* by) the unconstrained parts of j^{ν_Y} , which are also the physical parts. Working in k -space, the pole structure of the propagator encodes all important information about the particle content. The squares of the particle masses can be read off from the positions of the poles; if these are real and positive, a theory is tachyon-free. Meanwhile, the positivity of the pole residues guarantees freedom from ghosts. Note, however, that extracting ghost-freedom constraints from Eq. (5) becomes quite involved for parity-violating theories with multiple massive particles of spin- J . We discuss how this difficulty can be fully bypassed in Sec. II C.

B. Spin-projection operators

1. Fully covariant approach

The simplest way to proceed when it comes to tensorial field theories is to work in a fully covariant manner by using SPOs. We will develop the general theory and conventions for SPOs in detail in Appendix A, and in Appendix B we provide explicit formulas for those SPOs which are relevant for the examples in Sec. III. Here, we will only briefly outline the main ideas and notation. As their name suggests, SPOs break tensorial fields down into their irreducible parts with respect to the three-dimensional rotation group $SO(3)$, i.e., into constituent parts of definite spin J and parity P , which we denote by J^P . The action of the SPOs, however, goes beyond mere decomposition: they constitute a complete basis for the possible ways in which the various J^P states from across all the fields interface with each other. Dealing first with simple decomposition, we use the labels $i_{J^P}^X$ to indicate the multiple independent states with a common J^P which are contained within the single field X . Then, the diagonal SPOs

$$\mathcal{P}(i_{J^P}^X, i_{J^P}^X)_{\mu_X}{}^{\nu_X}, \quad (6)$$

form a complete basis

$$\begin{aligned}\zeta_{\mu_X} &\equiv \sum_{J,P} \sum_{i_{J^P}^X} \mathcal{P}(i_{J^P}^X, i_{J^P}^X)_{\mu_X}{}^{\nu_X} \zeta_{\nu_X} \\ &\Rightarrow \sum_{J,P} \sum_{i_{J^P}^X} \mathcal{P}(i_{J^P}^X, i_{J^P}^X)_{\mu_X}{}^{\nu_X} = \Delta^{\nu_X}_{\mu_X},\end{aligned}\quad (7)$$

where $\Delta^{\nu_X}_{\mu_X}$ in Eq. (7) is the product of Kronecker symbols δ^ν_μ , with one Kronecker factor for each index, in order. The

diagonal SPOs are also positive-definite; recalling that parity is associated with having even or odd free spatial indices, and that spatial indices pick up a minus sign under contraction, this leads to the positivity condition

$$P \zeta_{\mu_X}^* \mathcal{P}(i_{j_P}^X, i_{j_P}^X)^{\mu_X} \zeta^{\nu_X} \geq 0. \quad (8)$$

Going beyond decomposition, the general notation

$$\mathcal{P}(i_{j_P}^X, i_{j_{P'}}^Y)^{\mu_X \nu_Y}, \quad (9)$$

encompasses off-diagonal SPOs in which different states $i_{j_P}^X$ and $j_{j_{P'}}^Y$ are interfaced.³ Note that for parity-violating tensorial field theories, the two states need only share a common J [27,32]. This requirement is broader than that considered in [39], where the SPOs were additionally required to have the same parity $P = P'$. The parity-violating SPOs, i.e., those off-diagonal SPOs for which $P \neq P'$, necessarily contain an odd power of the totally antisymmetric tensor. Together with the diagonal SPOs as a special case, all SPOs, in addition to completeness Eq. (7), satisfy the following properties:

$$\mathcal{P}(i_{j_P}^X, j_{j_{P'}}^Y)^{\mu_X \nu_Y} = \mathcal{P}(j_{j_{P'}}^Y, i_{j_P}^X)^{\nu_Y \mu_X}, \quad (10a)$$

$$\mathcal{P}(i_{j_P}^X, j_{j_{P'}}^Y)^{\mu_X \nu_Y} = P P' \mathcal{P}(j_{j_{P'}}^Y, i_{j_P}^X)^{\mu_X \nu_Y}, \quad (10b)$$

$$\begin{aligned} \mathcal{P}(i_{j_P}^X, i_{j_{P'}}^Y)^{\mu_X \nu_Y} \mathcal{P}(k_{j_{P''}}^Y, l_{j_{P'''}}^Z)^{\nu_Y \sigma_Z} \\ = \delta_{jk} \delta_{JJ'} \delta_{P'P''} \mathcal{P}(i_{j_P}^X, l_{j_{P'''}}^Z)^{\mu_X \sigma_Z}, \end{aligned} \quad (10c)$$

where Eqs. (10a) and (10c) encode symmetry and orthogonality, respectively. The condition in Eq. (10b) implies Hermiticity for parity-preserving SPOs, and skew-Hermiticity for parity-violating SPOs. Within each J sector, our convention is to collect all the $P = 1$ states together, followed by the $P = -1$ states. When the SPOs are arranged in this 2×2 block form, their (skew-) Hermiticity is confined to (off-)diagonal blocks. We refer to this property as *checker-Hermiticity*, and discuss its consequences for algebraic manipulations in Appendix C.

2. Wave operator

As a consequence of the properties of the SPO basis in Eqs. (7) and (10a)–(10c), the spectral analysis can be performed in a far more convenient matrix representation. For example, the wave operator introduced in Eq. (1) can be expressed as

$$\mathcal{O}^{\mu_X}_{\nu_Y} = \sum_{J,P,P'} \sum_{i_{j_P}^X, j_{j_{P'}}^Y} [\mathcal{O}_J]_{i_{j_P}^X, j_{j_{P'}}^Y} \mathcal{P}(i_{j_P}^X, j_{j_{P'}}^Y)^{\mu_X \nu_Y}, \quad (11)$$

³In this general notation, the first and second SPO arguments always share field labels X and Y with the first and second collections of Lorentz indices, respectively.

where, for each J , the checker-Hermitian wave operator coefficient matrix $\mathcal{O}_J$ is indexed by all the $i_{j_P}^X$ and $j_{j_{P'}}^Y$. As explained above, the different spins do not mix at the level of the free action, and so the total wave operator coefficient matrix $\mathcal{O}$ assumes a block-diagonal form in J -space

$$\begin{aligned} \mathcal{O} &\equiv \bigoplus_J \mathcal{O}_J, \\ [\mathcal{O}_J]_{\mathcal{P}(i_{j_P}^X, j_{j_{P'}}^Y)^{\mu_X \nu_Y}} \mathcal{P}(i_{j_P}^X, j_{j_{P'}}^Y)^{\mu_X \nu_Y} \\ &= \mathcal{P}(i_{j_P}^X, i_{j_P}^X)^{\mu_X \sigma_X} \mathcal{O}^{\sigma_X}_{\lambda_Y} \mathcal{P}(j_{j_{P'}}^Y, j_{j_{P'}}^Y)^{\lambda_Y \nu_Y}, \end{aligned} \quad (12)$$

where Eqs. (11) and (12) are mutually consistent due to the properties in Eqs. (7) and (10a)–(10c). Within each $\mathcal{O}_J$ block, however, various massive and massless particles can perfectly well coexist: this complicates the calculations, as we will see in Sec. II C.

3. Saturated propagator

Having determined the structure of the general wave operator in terms of SPOs, the next objective is to obtain the saturated propagator Π given in Eq. (5). Comparison of Eqs. (5) and (12) suggests that $\mathcal{O}^{-1}$ is the relevant quantity to compute. As anticipated, however, difficulties arise when one or more of the $\mathcal{O}_J$ are degenerate, so that $\mathcal{O}^{-1}$ may not exist. As explained in [39], the dimensionality of the kernel of each $\mathcal{O}_J$, multiplied by the multiplicity $2J + 1$, and summed over all J , yields the total number of gauge generators for the free theory in Eq. (1). It is also explained in [39] that the most elegant approach to inverting such a singular matrix is via the *Moore-Penrose pseudoinverse* [54,55]—see Appendix D.⁴ Denoting the pseudoinverse by $\mathcal{O}^+_{J+}$, we can replace Eq. (5) with

$$\begin{aligned} \Pi &= \sum_{X,Y} \sum_{J,P,P'} \sum_{i_{j_P}^X, j_{j_{P'}}^Y} [\mathcal{O}^+]_{i_{j_P}^X, j_{j_{P'}}^Y} j_{\mu_X}^* \mathcal{P}(i_{j_P}^X, j_{j_{P'}}^Y)^{\mu_X \nu_Y} j^{\nu_Y}, \\ \mathcal{O}^+_{J+} &= \begin{bmatrix} \mathcal{O}^+_{J+} & \mathcal{O}^+_{J\pm} \\ \mathcal{O}^+_{J\mp} & \mathcal{O}^+_{J-} \end{bmatrix}, \end{aligned} \quad (13)$$

Note that the Moore-Penrose pseudoinverse of a checker-Hermitian matrix is also checker-Hermitian. This result is developed across Appendices C–E, which uses the shading scheme in Eq. (13) to indicate a checker-Hermitian block structure. The relationship between Moore-Penrose pseudoinversion and gauge fixing is described in Appendix F.

⁴Pseudoinversion may be understood as a systematic approach to inverting the nonsingular parts of a matrix. In analyses prior to [39], a less controlled but physically valid procedure—effectively equivalent to pseudoinversion—was to simply invert the largest nonsingular submatrices of each $\mathcal{O}_J$.

C. Massless spectrum

1. No-ghost condition

When massless particles are involved, SPOs—irrespective of whether these are parity-preserving or parity-violating—should be employed with certain care. This is because they are constructed out of the usual transverse and longitudinal projectors, which are not well-defined in the massless limit. Specifically, all SPOs incorporate powers of k^μ into their tensor structures; these are necessarily accompanied by negative powers of k for reasons of normalization. These negative powers can lead to spurious massless poles in Π . The safest approach is to explicitly compute the full saturated propagator in Eq. (13) by expanding both the SPOs and the sources in terms of their tensorial components in some fiducial frame. Being artefacts of a “poor” choice of basis, spurious singularities cancel out and the limit on the light cone can be carefully taken. Absence of massless ghosts requires that the corresponding residue be positive

$$\text{Res}_{k^2 \rightarrow 0} \Pi > 0. \quad (14)$$

This brute-force procedure (see, e.g., Refs. [32,39,56]) is not especially elegant, and indeed the massless spectrum typically accounts for more wallclock time in PSALTer (which uses this method) than its massive counterpart, which we discuss next in Sec. II C. Moreover, the version of the algorithm presented in [39] does not require any particular modification when parity-violating operators are introduced.

D. Massive spectrum

1. No-tachyon condition

Consider the spin-sector J for which the coefficient matrix $\mathcal{O}_J$ is not block-diagonal, i.e., there is parity violation. This sector may have various simple zeroes⁵ in $k^2 \equiv k^\mu k_\mu$, at the positions of the square masses $M_{s_J}^2 \neq 0$, where s_J is a label for the various masses associated with J . These correspond to the zeroes of the determinant of the largest nondegenerate submatrix of $\mathcal{O}_J$ or, equivalently, the zeroes of its pseudodeterminant. In general, the formulas for these zeroes in terms of the Lagrangian coupling coefficients may not be expressible using rational functions, requiring either radicals or transcendental functions. We are interested in stable and nontachyonic states, i.e., the masses $M_{s_J}^2$ must be real and positive [32,34]

$$M_{s_J}^2 > 0 \quad \forall J, s_J. \quad (15)$$

Note that Eq. (15) is a generalization of the parity-preserving case in [39,56], for which masses are confined to a given J^P sector and labeled by s_{J^P} .

⁵As shown in [39], nonsimple zeroes always signal the presence of ghost modes.

2. No-ghost condition

The no-tachyon condition in Eq. (15) must be complemented with a no-ghost criterion as well. There exist two fully equivalent ways, explained shortly, to determine if the massive modes have healthy kinetic terms. Which one to use crucially depends on how many such modes are present in each spin sector. The well-known approach is to demand that the residues of the propagator in Eq. (13), evaluated at all the unique massive poles $M_{s_J}^2$, be positive. We show in Appendix G how the checker-Hermitian structure in Eq. (13) reduces this criterion to

$$\text{Res}_{k^2 \rightarrow M_{s_J}^2} (\text{tr} \mathcal{O}_{J+}^+ - \text{tr} \mathcal{O}_{J-}^+) > 0 \quad \forall J, s_J. \quad (16)$$

The formula in Eq. (16) was not known previously, though it is consistent with the known formula—see Eq. (G10)—in the parity-preserving case, and is fully equivalent to the condition used in [32]. Due to it being a straightforward extension of the parity-preserving criterion, it is precisely Eq. (16) which is implemented in the upgrade to PSALTer. In practice, however, this criterion becomes extremely difficult to apply in those cases where the massive spectrum comprises more than one field, and $M_{s_J}^2$ are expressible in terms of radicals or transcendental functions of the Lagrangian coupling coefficients. We will see in Sec. III B 2, for example, that this is exactly what happens in EC theory. In fact, the complications are due to the presence of massless modes and—more specifically—because of their kinetic mixings with the massive ones.⁶ We now propose a novel way for determining the no-ghost condition in such cases. By reordering rows and columns in $\mathcal{O}_J$ one can obtain a matrix with the canonical block structure⁷

$$\mathcal{O}_J = \begin{bmatrix} \mathcal{O}_{J_m} & \mathcal{O}_{J_{m\gamma}} \\ \mathcal{O}_{J_{\gamma m}} & \mathcal{O}_{J_\gamma} \end{bmatrix}, \quad (17)$$

such that massive modes of positive and negative parity occupy the upper-left submatrices of $\mathcal{O}_{J_m}$, the massless modes are contained in $\mathcal{O}_{J_\gamma}$, and the kinetic mixings between massive and massless modes are contained in $\mathcal{O}_{J_{m\gamma}}$ and $\mathcal{O}_{J_{\gamma m}}$. The key insight is that these mixings can be rotated away, i.e., $\mathcal{O}_J$ can be block-diagonalized by the transformation

⁶It can also happen that, due to gauge redundancies, a sector does not contain massless propagating particles; this is the situation with the spin-one fields of the general quadratic EC gravity [32,34], see Appendix I. Also in this case, it is simpler to not compute the residues of the propagator at the massive poles.

⁷Note that Eq. (17) is not shaded, because the reordered coefficient matrix will not, in general, be checker-Hermitian. However, we can always make the diagonal blocks $\mathcal{O}_{J_m}$ and $\mathcal{O}_{J_\gamma}$ checker-Hermitian, as we will see in Sec. III B 2.

$$\mathbf{o}_J = \begin{bmatrix} 1 & \mathbf{o}_{J_{m\gamma}} \mathbf{o}_{J_\gamma}^+ \\ 0 & 1 \end{bmatrix} \cdot \tilde{\mathbf{o}}_J \cdot \begin{bmatrix} 1 & 0 \\ \mathbf{o}_{J_\gamma}^+ \mathbf{o}_{J_{\gamma m}} & 1 \end{bmatrix}, \quad \tilde{\mathbf{o}}_J = \begin{bmatrix} \tilde{\mathbf{o}}_{J_m} & 0 \\ 0 & \mathbf{o}_{J_\gamma} \end{bmatrix}, \quad (18)$$

where the first block in Eq. (18)—which encodes all the information about the massive spectrum—decomposes as a polynomial in k according to⁸

$$\tilde{\mathbf{o}}_{J_m} = \mathbf{o}_{J_m} - \mathbf{o}_{J_{m\gamma}} \cdot \mathbf{o}_{J_\gamma}^+ \cdot \mathbf{o}_{J_{\gamma m}} \equiv \mathbf{K}_J k^2 + \mathbf{M}_J. \quad (19)$$

Here $\mathbf{K}_J$ and $\mathbf{M}_J$ in Eq. (19) are the kinetic- and mass-matrices of the massive modes, respectively, which depend exclusively on the Lagrangian coupling coefficients. The no-ghost criterion simply becomes that the kinetic matrix be negative-definite

$$\mathbf{K}_J < 0, \quad \forall J. \quad (20)$$

The condition in Eq. (20) is necessarily equivalent to that in Eq. (16). Note that if the kinetic terms are canonicalized, then the masses correspond to the eigenvalues of $\mathbf{L}_J \equiv -\mathbf{K}_J^+ \mathbf{M}_J$, which coincide with the $M_{s_J}^2$ as can be straightforwardly verified. Note that our new method is not yet implemented in PSALTER, but we illustrate its use in Sec. III B 2.

III. EXAMPLES WITH CODE

A. How to use this section

We now demonstrate the capabilities of the algorithm presented in Sec. II, when it is implemented in the latest

```
In[#] := Get["xAct`PSALTER`"];
```

During the loading process, several other *Mathematica* dependencies are loaded, including `xTensor` [40,42], `SymManipulator` [46], `xPerm` [41], `xCore` [44], `xTras` [43], and `xCoba` [45] from `xAct`. The PSALTER package will next predefine the geometric environment: a flat manifold **M4** with metric **G**, the totally antisymmetric tensor **epsilonG**, and the derivative on flat spacetime **CD**. The lower-case Latin alphabet **a, b, ..., z**, is fully reserved for Minkowski spacetime indices, which automatically format as Greek letters $\alpha, \beta, \dots, \zeta$. For example, **G****[-m, -n]** formats as $\eta_{\mu\nu}$ and **CD****[-m]**@ formats as ∂_μ , while **epsilonG****[-m, -n, -r, -s]** formats as $\epsilon_{\mu\nu\rho\sigma}$.⁹

⁸In the literature, $\tilde{\mathbf{o}}_{J_m}$ is called the *Schur complement*.

⁹Note how the nearest Greek equivalents to Latin counterparts are automatically used for rendering.

version of the PSALTER software. There are two types of examples we consider here. The first are various toy-models involving p -form fields, which were chosen simply to illustrate the versatility of the software. The other example we study is a specific subclass of Einstein-Cartan gravity that propagates two massive scalars—this has been carefully chosen not only because the PSALTER results can be cross-checked analytically, see Appendix H, but also because such a theory can have interesting applications in cosmology and particle physics. Note that the sources for these examples can be found in the supplement [57].

1. Syntax highlighting

From this point on, we will occasionally present code listings which have syntax highlighting, our conventions for which are as follows. Symbols belonging to the Wolfram Language (*Mathematica*) are monospace, those belonging to `xAct` are gray, those belonging to PSALTER are bold, and those which will be defined as part of the user session are italics. The start of each new input cell in a *Mathematica* notebook is denoted by **In**[#]:=, and the start of each output cell is denoted by **Out**[#]:= . Comments within the code appear in (*gray italics*), and strings (which are not symbols) are shown in black italics.

2. Loading the software

As explained in [39], the software is loaded via the `Get` command:

B. Various p -form toy models

In this section we will work with the simplest kinds of structured tensor fields, namely p -forms, or totally antisymmetric fields. In fact, such fields have a very natural association with parity violation in three-dimensional (though still Lorentzian) settings, through topological gauge theory in which the Chern-Simons term is present [58–63]. As discussed already in Sec. I, the various Chern-Simons-type theories have profound physical significance [64,65]. However, we will be unable to work with them directly. The reason for this is simply because PSALTER is designed to work exclusively in four dimensions. Nevertheless, there are still p -form theories in four dimensions which owe their free spectra to parity violation, and we will now present a few examples.

1. Definitions

Scalar field. First, we introduce a scalar (zero-form) ϕ , which we call *ScalarField*:

TABLE II. The declaration of *ScalarField*, which contains only a $J^P = 0^+$ mode. These definitions are used in Fig. 4.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
ϕ	(None)	$\phi_{0^+}^{\#1}$	ρ
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$\phi_{0^+}^{\#1}$	(None)	ϕ	$\rho_{0^+}^{\#1}$

```
In[#]:= DefField[ScalarField[], PrintAs -> "\[Phi]",
  ↪ PrintSourceAs -> "\[Rho]"];
```

The scalar source, denoted by ρ , is automatically defined by PSALTER. The output is shown in Table II; we see that the scalar field ϕ contains only a $J^P = 0^+$ mode.

Vector field. We next introduce a vector (one-form) $\mathcal{A}_\mu$, which we call *VectorField*:

```
In[#]:= DefField[VectorField[-m], PrintAs ->
  ↪ "\[ScriptCapitalA]", PrintSourceAs -> "\!\(\( \star
  ↪ SubscriptBox[\(\[ScriptCapitalJ])\),
  ↪ \((\[ScriptCapitalA])\))\)]");
```

The output is shown in Table III; we see that $J^P = 0^+$ and $J^P = 1^-$ modes are present in the field, and that the source $\mathcal{J}_{(A)}^\mu$ and its corresponding irreps are also defined.

Two-form field. A rank-two antisymmetric tensor (two-form) $\mathcal{B}_{\mu\nu}$ is defined as *TwoFormField*:

```
In[#]:= DefField[TwoFormField[-m, -n], Antisymmetric[{ -m, -n}],
  ↪ PrintAs -> "\[ScriptCapitalB]", PrintSourceAs -> "\!\(\( \star
  ↪ SubscriptBox[\(\[ScriptCapitalJ])\),
  ↪ \((\[ScriptCapitalB])\))\)]");
```

The output is shown in Table IV; we see that it carries $J^P = 1^+$ and $J^P = 1^-$ modes, and that the source $\mathcal{J}_{(B)}^{\mu\nu}$ and its irreps are also defined.

Three-form field. Finally, a rank-three totally antisymmetric tensor (three-form) $\mathcal{C}_{\mu\nu\sigma}$ is defined as *ThreeFormField*:

```
In[#]:= DefField[ThreeFormField[-m, -n, -r], Antisymmetric[{ -m,
  ↪ -n, -r}], PrintAs -> "\[ScriptCapitalC]", PrintSourceAs ->
  ↪ "\!\(\( \star SubscriptBox[\(\[ScriptCapitalJ])\),
  ↪ \((\[ScriptCapitalC])\))\)]");
```

TABLE III. The declaration of *VectorField*, with $J^P = 0^+$ and $J^P = 1^-$ modes. These definitions are used in Fig. 3.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
$\mathcal{A}_\alpha$	(None)	$\mathcal{A}_{1^- \alpha}^{\#1} + \mathcal{A}_{0^+ \alpha}^{\#1}$	$\mathcal{J}_{(A)\alpha}$
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$\mathcal{A}_{0^+}^{\#1}$	(None)	$\mathcal{A}^\alpha n_\alpha$	$\mathcal{J}_{(A)0^+}^{\#1}$
$\mathcal{A}_{1^- \alpha}^{\#1}$	(None)	$\mathcal{A}_\alpha - \mathcal{A}^\beta n_\alpha n_\beta$	$\mathcal{J}_{(A)1^- \alpha}^{\#1}$

TABLE IV. The declaration of *TwoFormField*, which contains $J^P = 1^+$ and $J^P = 1^-$ modes. These definitions are used in Figs. 1–3.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
$\mathcal{B}_{\alpha\beta}$	$\mathcal{B}_{\beta\alpha} == -\mathcal{B}_{\alpha\beta}$	$\mathcal{B}_{1^+ \alpha\beta}^{\#1} - \mathcal{B}_{1^- \beta\alpha}^{\#1} + \mathcal{B}_{1^- \alpha\beta}^{\#1}$	$\mathcal{J}_{(\mathcal{B})\alpha\beta}$
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$\mathcal{B}_{1^+ \alpha\beta}^{\#1}$	$\mathcal{B}_{1^+ \beta\alpha}^{\#1} == -\mathcal{B}_{1^+ \alpha\beta}^{\#1}$	$\mathcal{B}_{\alpha\beta} + \mathcal{B}_{\beta\chi} n_\alpha n^\chi - \mathcal{B}_{\alpha\chi} n_\beta n^\chi$	$\mathcal{J}_{(\mathcal{B})1^+ \alpha\beta}^{\#1}$
$\mathcal{B}_{1^- \alpha\beta}^{\#1}$	(None)	$\mathcal{B}_{\alpha\beta} n^\beta$	$\mathcal{J}_{(\mathcal{B})1^- \alpha\beta}^{\#1}$

TABLE V. The declaration of *ThreeFormField*, which contains $J^P = 0^-$ and $J^P = 1^+$ modes. These definitions are used in Fig. 4.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
$C_{\alpha\beta\chi}$	$C_{\alpha\chi\beta} == -C_{\alpha\beta\chi}$ & $C_{\beta\alpha\chi} == -C_{\alpha\beta\chi}$	$C_{0^- \alpha\beta\chi}^{\#1} + C_{1^+ \beta\chi\alpha}^{\#1} - C_{1^+ \alpha\chi\beta}^{\#1} + C_{1^+ \alpha\beta\chi}^{\#1}$	$\mathcal{J}(C)_{\alpha\beta\chi}$
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$C_{0^- \alpha\beta\chi}^{\#1}$	$C_{0^- \alpha\chi\beta}^{\#1} == -C_{0^- \alpha\beta\chi}^{\#1}$ & $C_{0^- \beta\alpha\chi}^{\#1} == -C_{0^- \alpha\beta\chi}^{\#1}$	$C_{\alpha\beta\chi} - C_{\beta\chi\delta} n_\alpha n^\delta + C_{\alpha\chi\delta} n_\beta n^\delta - C_{\alpha\beta\delta} n_\chi n^\delta$	$\mathcal{J}(C)_{0^- \alpha\beta\chi}^{\#1}$
$C_{1^+ \alpha\beta\chi}^{\#1}$	$C_{1^+ \beta\alpha\chi}^{\#1} == -C_{1^+ \alpha\beta\chi}^{\#1}$	$C_{\alpha\beta\chi} n^\chi$	$\mathcal{J}(C)_{1^+ \alpha\beta\chi}^{\#1}$

The output is shown in Table V; we see that it carries $J^P = 1^+$ and $J^P = 0^-$ irreps, and that the source $\mathcal{J}_{(C)}^{\mu\nu\sigma}$ and its irreps are also defined.

Lagrangian coupling coefficients. In what follows, we shall consider three-parameter models, with the three couplings α , β , and γ as named variables *Coupling1*, *Coupling2*, and *Coupling3*:

```
In[#]:= DefConstantSymbol[Coupling1, PrintAs -> "[Alpha]"];
```

```
In[#]:= DefConstantSymbol[Coupling2, PrintAs -> "[Beta]"];
```

```
In[#]:= DefConstantSymbol[Coupling3, PrintAs -> "[Gamma]"];
```

Note that PSALTer strictly requires all the operators in the Lagrangian density to be *linearly* parametrized by these Lagrangian coupling coefficients.

2. Spectroscopy

Parity-violating massive two-form. The simplest parity-violating model that we could think of involves a two-form $\mathcal{B}^{\mu\nu}$ with a purely parity-violating mass term

$$S = \int d^4x [\alpha \partial_{[\mu} \mathcal{B}_{\nu\rho]} \partial^{[\mu} \mathcal{B}^{\nu\rho]} + \gamma \epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\mu\nu} \mathcal{B}_{\rho\sigma}]. \quad (21)$$

Note that in Eq. (21) both terms are linearly parametrized by α and γ , and square brackets denote antisymmetrization of the enclosed indexes. Henceforth, we will always omit the source coupling, which in Eq. (21) would require adding the term $\int d^4x \mathcal{B}_{\mu\nu} \mathcal{J}_{(\mathcal{B})}^{\mu\nu}$. This is because the source $j^{\mu\nu}$ in Eq. (1) is introduced as a formal test field, useful only for the computations in Sec. II. The source-free model defined in Eq. (21) has precisely the same particle spectrum. In fact, PSALTer does not accept source couplings to be input by the user; they are automatically included when Eq. (21) is analyzed with the command:

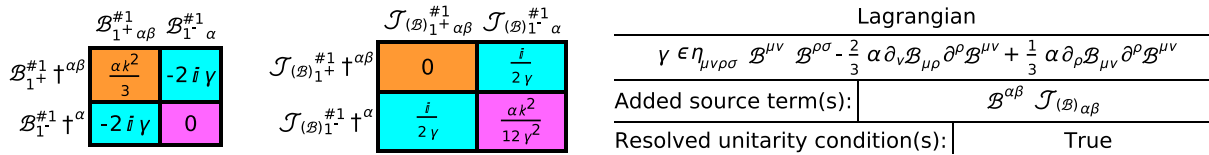


FIG. 1. The spectrograph of the parity-violating two-form model in Eq. (21).

```

In[#] := ParticleSpectrum[Coupling3 * epsilonG[-m, -n, -r, -s] *
  -> TwoFormField[m, n] * TwoFormField[r, s] - (2 * Coupling1 *
  -> CD[-n][TwoFormField[-m, -r]] * CD[r][TwoFormField[m, n]])/3
  -> + (Coupling1 * CD[-r][TwoFormField[-m, -n]] *
  -> CD[r][TwoFormField[m, n]])/3, TheoryName ->
  -> "ParityViolatingMassiveTwoFormTheory", MaxLaurentDepth -> 3];
    
```

The output of the PSALTER software is given in a few seconds; it is shown in Fig. 1. The result here is trivial: despite the rich appearance of Eq. (21), the theory does not contain any dynamical degrees of freedom (d.o.f.).

Parity-indefinite massive two-form. The next model under consideration is a direct generalization of Eq. (21), by allowing for both parity-preserving and parity-violating

mass terms

$$S = \int d^4x [\alpha \partial_{[\mu} \mathcal{B}_{\nu\rho]} \partial^{[\mu} \mathcal{B}^{\nu\rho]} + \beta \mathcal{B}_{\mu\nu} \mathcal{B}^{\mu\nu} + \gamma \epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\mu\nu} \mathcal{B}_{\rho\sigma}], \quad (22)$$

through the addition of the β coupling. The dynamics of Eq. (22) are probed using the following input:

```

In[#] := ParticleSpectrum[Coupling2 * TwoFormField[-m, -n] *
  -> TwoFormField[m, n] + Coupling3 * epsilonG[-m, -n, -r, -s] *
  -> TwoFormField[m, n] * TwoFormField[r, s] - (2 * Coupling1 *
  -> CD[-n][TwoFormField[-m, -r]] * CD[r][TwoFormField[m, n]])/3
  -> + (Coupling1 * CD[-r][TwoFormField[-m, -n]] *
  -> CD[r][TwoFormField[m, n]])/3, TheoryName ->
  -> "ParityIndefiniteMassiveTwoFormTheory", MaxLaurentDepth ->
  -> 3];
    
```

The output of the PSALTER software is shown in Fig. 2. If we adhere to the formal notation from Sec. II C, we say that the theory in Eq. (22) propagates a massive spin-one particle with square mass $M_{1_1}^2 = -3(\beta^2 + 4\gamma^2)/\alpha\beta$. The particle is not a ghost if $\alpha > 0$, and it is neither a ghost nor a tachyon if additionally $\beta < 0$.

One-by-two Cremmer-Scherk-Kalb-Ramond (CSKR) theory. Yet another parity-indefinite action, that in common with the parity-violating two-form theory propagates a single massive spin-one particle, also involves a two-form $\mathcal{B}_{\mu\nu}$ coupled appropriately to a vector field $\mathcal{A}_\mu$

(see, e.g., Ref. [66])

$$S = \int d^4x [\alpha \partial_{[\mu} \mathcal{A}_{\nu]} \partial^{[\mu} \mathcal{A}^{\nu]} + \beta \partial_{[\mu} \mathcal{B}_{\nu\rho]} \partial^{[\mu} \mathcal{B}^{\nu\rho]} + \gamma \epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\mu\nu} \partial_{[\rho} \mathcal{A}_{\sigma]}], \quad (23)$$

where the constants in Eq. (23) are not supposed to be consistent with those in the previous examples, not even up to the mass dimension¹⁰: we are simply reusing symbols

¹⁰Indeed, in Eq. (23) the constants are all dimensionless.

$$\begin{array}{c}
\mathcal{B}_{1^+}^{\#1} \dagger^{\alpha\beta} \quad \mathcal{B}_{1^+}^{\#1} \alpha \\
\begin{array}{|c|c|} \hline \beta + \frac{\alpha k^2}{3} & -2i\gamma \\ \hline -2i\gamma & \beta \\ \hline \end{array} \\
\mathcal{B}_{1^+}^{\#1} \dagger^\alpha
\end{array}
\quad
\begin{array}{c}
\mathcal{J}_{(\mathcal{B})1^+}^{\#1} \dagger^{\alpha\beta} \quad \mathcal{J}_{(\mathcal{B})1^+}^{\#1} \alpha \\
\begin{array}{|c|c|} \hline \frac{\beta}{\text{Det}(1)} & \frac{2i\gamma}{\text{Det}(1)} \\ \hline \frac{2i\gamma}{\text{Det}(1)} & \beta + \frac{\alpha k^2}{3} \\ \hline \end{array} \\
\mathcal{J}_{(\mathcal{B})1^+}^{\#1} \dagger^\alpha
\end{array}$$

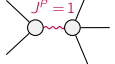
Abbreviations used in matrices			
Det(1) = $\beta^2 + 4\gamma^2 + \frac{1}{3}\alpha\beta k^2$			
Lagrangian			
$\beta \mathcal{B}_{\mu\nu} \mathcal{B}^{\mu\nu} + \gamma \epsilon_{\mu\nu\rho\sigma} \mathcal{B}^{\mu\nu} \mathcal{B}^{\rho\sigma} - \frac{2}{3}\alpha \partial_\nu \mathcal{B}_{\mu\rho} \partial^\rho \mathcal{B}^{\mu\nu} + \frac{1}{3}\alpha \partial_\rho \mathcal{B}_{\mu\nu} \partial^\rho \mathcal{B}^{\mu\nu}$			
Added source term(s):		$\mathcal{B}^{\alpha\beta} \mathcal{J}_{(\mathcal{B})\alpha\beta}$	
Resolved pole(s)	# polarization(s)	Square mass	Residue
	3	$-\frac{3(\beta^2 + 4\gamma^2)}{\alpha\beta}$	$\frac{3(\beta^2 + 4\gamma^2)}{\alpha\beta^2}$
Resolved unitarity condition(s):		$\beta < 0 \ \& \ \alpha > 0$	

FIG. 2. The spectrograph of the parity-violating two-form model in Eq. (22).

$$\begin{array}{c}
\mathcal{A}_{0^+}^{\#1} \dagger \\
0 \\
\mathcal{B}_{1^+}^{\#1} \dagger^{\alpha\beta} \quad \mathcal{B}_{1^+}^{\#1} \alpha \quad \mathcal{A}_{1^+}^{\#1} \alpha \\
\begin{array}{|c|c|c|} \hline \frac{\beta k^2}{3} & 0 & -\sqrt{2}\gamma k \\ \hline 0 & 0 & 0 \\ \hline \sqrt{2}\gamma k & 0 & 2\alpha k^2 \\ \hline \end{array} \\
\mathcal{A}_{1^+}^{\#1} \dagger^\alpha
\end{array}
\quad
\begin{array}{c}
\mathcal{J}_{(\mathcal{A})0^+}^{\#1} \dagger \\
0 \\
\mathcal{J}_{(\mathcal{B})1^+}^{\#1} \dagger^{\alpha\beta} \quad \mathcal{J}_{(\mathcal{B})1^+}^{\#1} \alpha \quad \mathcal{J}_{(\mathcal{A})1^+}^{\#1} \alpha \\
\begin{array}{|c|c|c|} \hline \frac{2\alpha k^2}{\text{Det}(1)} & 0 & \frac{\sqrt{2}\gamma k}{\text{Det}(1)} \\ \hline 0 & 0 & 0 \\ \hline \frac{\sqrt{2}\gamma k}{\text{Det}(1)} & 0 & \frac{\beta k^2}{3\text{Det}(1)} \\ \hline \end{array} \\
\mathcal{J}_{(\mathcal{A})1^+}^{\#1} \dagger^\alpha
\end{array}$$

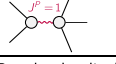
Abbreviations used in matrices			
Det(1) = $2\gamma^2 k^2 + \frac{2}{3}\alpha\beta k^4$			
Lagrangian			
$-2\gamma \epsilon_{\mu\nu\rho\sigma} \mathcal{B}^{\rho\sigma} \partial^\nu \mathcal{A}^\mu - 2\alpha \partial_\nu \mathcal{A}_\rho \partial^\rho \mathcal{A}^{\mu\nu} + 2\alpha \partial_\nu \mathcal{A}_\mu \partial^\nu \mathcal{A}^\mu - \frac{2}{3}\beta \partial_\nu \mathcal{B}_{\mu\rho} \partial^\rho \mathcal{B}^{\mu\nu} + \frac{1}{3}\beta \partial_\rho \mathcal{B}_{\mu\nu} \partial^\rho \mathcal{B}^{\mu\nu}$			
Added source term(s):		$\mathcal{A}^\alpha \mathcal{J}_{(\mathcal{A})\alpha} + \mathcal{B}^{\alpha\beta} \mathcal{J}_{(\mathcal{B})\alpha\beta}$	
Source constraint(s)	# constraint(s)	Covariant form	
$\mathcal{J}_{(\mathcal{A})0^+}^{\#1} = 0$	1	$\partial_\alpha \mathcal{J}_{(\mathcal{A})}^\alpha = 0$	
$\mathcal{J}_{(\mathcal{B})1^+}^{\#1\alpha} = 0$	3	$\partial_\beta \mathcal{J}_{(\mathcal{B})}^{\alpha\beta} = 0$	
Total # constraint(s):		4	
Resolved pole(s)	# polarization(s)	Square mass	Residue
	3	$-\frac{3\gamma^2}{\alpha\beta}$	$-\frac{1}{2\alpha} + \frac{3}{\beta}$
Resolved unitarity condition(s):		$\beta > 0 \ \& \ \alpha < 0 \ \& \ (\gamma < 0 \ \ \gamma > 0)$	

FIG. 3. The spectrograph of the “one-by-two” CSKR model in Eq. (23). All quantities are defined in Tables III and IV.

which have already been defined in the user session. We refer to this as the “one-by-two” CSKR model because it mixes one- and two-forms.¹¹ The model in Eq. (23) is probed using the following input:

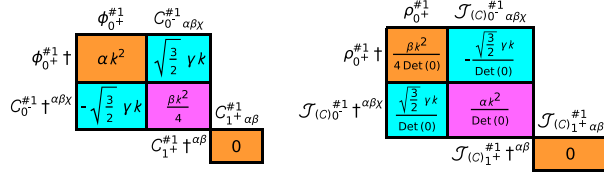
```

In[#]:= ParticleSpectrum[-2 * Coupling3 * epsilonG[-m, -n, -r,
  -> -s] * TwoFormField[r, s] * CD[n][VectorField[m]] - 2 *
  -> Coupling1 * CD[-m][VectorField[-n]] * CD[n][VectorField[m]]
  -> + 2 * Coupling1 * CD[-n][VectorField[-m]] *
  -> CD[n][VectorField[m]] - (2 * Coupling2 *
  -> CD[-n][TwoFormField[-m, -r]] * CD[r][TwoFormField[m, n]])/3
  -> + (Coupling2 * CD[-r][TwoFormField[-m, -n]] *
  -> CD[r][TwoFormField[m, n]])/3, TheoryName ->
  -> "OneByTwoCSKRTheory", MaxLaurentDepth -> 3];

```

The output is shown in Fig. 3. As anticipated, the particle content of Eq. (23) comprises a massive spin-one particle; its square mass is $M_{1_1}^2 = -3\gamma^2/\alpha\beta$, and the no-ghost and no-tachyon conditions are $\alpha < 0$ and $\beta > 0$. Note that if a theory possesses gauge invariances, PSALTER automatically identifies the associated source constraints; in the present case, the action is invariant under the independent transformations $\mathcal{B}_{\mu\nu} \mapsto \mathcal{B}_{\mu\nu} + 2\partial_{[\mu}\xi_{\nu]}$ and $\mathcal{A}_\mu \mapsto \mathcal{A}_\mu + \partial_\mu \xi$, where $\xi_\mu \equiv \xi_\mu(x)$ and $\xi \equiv \xi(x)$ are local gauge generators. As a consequence of these gauge invariances, the sources $\mathcal{J}_{(\mathcal{A})}^\mu$ and $\mathcal{J}_{(\mathcal{B})}^{\mu\nu}$ are

¹¹More generally, in $d = q + p + 1$ -dimensions, the p -by- q CSKR model mixes a p -form with a q -form; it is built from two gauge-invariant kinetic terms and a topological interaction. Evidently, in $d = 4$ there is exactly one further CSKR theory, which we discuss presently.



Abbreviations used in matrices			
Det(0) = $\frac{1}{4}(6\gamma^2 k^2 + \alpha\beta k^4)$			
Lagrangian			
$\gamma \epsilon^{\mu\nu\rho\sigma} C^{\nu\rho\sigma} \partial^\mu \phi + \alpha \partial_\mu \phi \partial^\mu \phi - \frac{3}{4} \beta \partial_\rho C_{\mu\nu\sigma} \partial^\sigma C^{\mu\nu\rho} + \frac{1}{4} \beta \partial_\sigma C_{\mu\nu\rho} \partial^\sigma C^{\mu\nu\rho}$			
Added source term(s):	$\phi \rho + C^{a\beta\chi} \mathcal{J}_{(C)a\beta\chi}$		
Source constraint(s)	# constraint(s)	Covariant form	
$\mathcal{J}_{(C)1^+}^{a\beta} = 0$	3	$\partial_\chi \mathcal{J}_{(C)a\beta\chi} = 0$	
Total # constraint(s):	3		
Resolved pole(s)	# polarization(s)	Square mass	Residue
	1	$-\frac{6\gamma^2}{\alpha\beta}$	$\frac{1}{\alpha} - \frac{4}{\beta}$
Resolved unitarity condition(s):		$ \beta < 0 \ \&\& \ \alpha > 0 \ \&\& \ (\gamma < 0 \ \ \gamma > 0)$	

FIG. 4. The spectrograph of the zero-by-three CSKR model in Eq. (24). All quantities are defined in Tables II and V.

subject to the constraints $\partial_\mu \mathcal{J}_{(A)}^\mu = 0$ and $\partial_\mu \mathcal{J}_{(B)}^{\mu\nu} = 0$, which are automatically identified and taken account of in the PSALT_{er} code.

Zero-by-three CSKR theory. The other four-dimensional CSKR theory is the zero-by-three model, which mixes a scalar with a three-form. The action reads (see, e.g., Ref. [66])

$$S = \int d^4x [\alpha \partial_\mu \phi \partial^\mu \phi + \beta \partial_{[\mu} C_{\nu\rho\sigma]} \partial^{\mu} C^{\nu\rho\sigma}] + \gamma \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} \partial_\sigma \phi, \quad (24)$$

and the dynamics of Eq. (24) are probed using the following input:

```
In[#]:= ParticleSpectrum[Coupling3 * epsilonG[-m, -n, -r, -s] *
  -> ThreeFormField[n, r, s] * CD[m][ScalarField[]] + Coupling1 *
  -> CD[-m][ScalarField[]] * CD[m][ScalarField[]] - (3 *
  -> Coupling2 * CD[-r][ThreeFormField[-m, -n, -s]] *
  -> CD[s][ThreeFormField[m, n, r]])/4 + (Coupling2 *
  -> CD[-s][ThreeFormField[-m, -n, -r]] * CD[s][ThreeFormField[m,
  -> n, r]])/4, TheoryName -> "ZeroByThreeCSKRTheory",
  -> MaxLaurentDepth -> 3];
```

The output is shown in Fig. 4, from which we see that for $\alpha > 0$ and $\beta < 0$ the theory in Eq. (24) propagates a healthy massive spin-zero mode with mass $M_{1_0}^2 = -6\gamma^2/\alpha\beta$. Since the action in Eq. (24) is invariant under $C_{\mu\nu\rho} \mapsto C_{\mu\nu\rho} + \partial_{[\mu} \xi_{\nu\rho]}$ for local gauge generator $\xi_{\mu\nu} \equiv \xi_{[\mu\nu]} \equiv \xi_{\mu\nu}(x)$, there is an associated source constraint which is given by $\partial_\mu \mathcal{J}_{(C)}^{\mu\nu\rho} = 0$.

C. Einstein-Cartan gravity

1. Localizing the Poincaré group

Gravitational interactions emerge organically by gauging the Lorentz group; i.e., promoting the global symmetry under Lorentz rotations—which is an established feature of all laboratory physics—to a local symmetry. By combining this with the local symmetry of translations—which in general relativity is already manifest as general covariance—one arrives at a gauge theory of the whole Poincaré group [67–69].

This calls for the introduction of two gauge fields: the (co) tetrad e^α_μ and spin connection $\omega^{\alpha\beta}_\mu \equiv \omega^{[\alpha\beta]}_\mu$, resulting in the Einstein-Cartan (EC) formulation of gravity.¹² To be consistent with the notation used so far, Greek letters such as α

¹²Extra pedantry may avoid confusion resulting from our choice of words here. Strictly speaking, *EC gravity* refers exclusively to the framing of the Einstein-Hilbert action in this formulation, and then only when it is combined with the geometric interpretation in which the spacetime genuinely has curvature and torsion [70–76]. Physics in general, and particle physics in particular, are utterly oblivious to our interpretation of the geometry. It comes as no surprise, therefore, that one can also gauge the Poincaré group entirely within Minkowski spacetime [67–69] (see also [47,77–82]). Properly, only this Minkowski formulation is referred to as Poincaré gauge theory (PGT). As with EC gravity, PGT has also become associated for historical reasons with a *theory*, not just with a formulation. PGT extends the Einstein-Hilbert action to include all other scalar invariants which are quadratic in the translational and rotational field strength tensors [geometrically, the torsion and the curvature in Eq. (25)]. We deal with this much larger model in Appendix I.

TABLE VI. The declaration of *TetradPerturbation*. These definitions are used in Figs. 5 and 6.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
$f_{\alpha\beta}$	(None)	$\frac{1}{3}\eta_{\alpha\beta}f_{0^+}^{\#1} + f_{1^+ \alpha\beta}^{\#1} + f_{2^+ \alpha\beta}^{\#1} + f_{1^- \beta}^{\#1}n_\alpha + f_{1^- \alpha}^{\#2}n_\beta - \frac{1}{3}f_{0^+}^{\#1}n_\alpha n_\beta + f_{0^+}^{\#2}n_\alpha n_\beta$	$\tau_{\alpha\beta}$
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$f_{0^+}^{\#1}$	(None)	$f_\alpha^\alpha - f^{\alpha\beta}n_\alpha n_\beta$	$\tau_{0^+}^{\#1}$
$f_{0^+}^{\#2}$	(None)	$f^{\alpha\beta}n_\alpha n_\beta$	$\tau_{0^+}^{\#2}$
$f_{1^+ \alpha\beta}^{\#1}$	$f_{1^+ \beta\alpha}^{\#1} = -f_{1^+ \alpha\beta}^{\#1}$	$\frac{f_{\alpha\beta}}{2} - \frac{f_{\beta\alpha}}{2} + \frac{1}{2}f_\beta^\xi n_\alpha n_\xi - \frac{1}{2}f_\beta^\xi n_\alpha n_\xi - \frac{1}{2}f_\alpha^\xi n_\beta n_\xi + \frac{1}{2}f_\alpha^\xi n_\beta n_\xi$	$\tau_{1^+ \alpha\beta}^{\#1}$
$f_{1^- \alpha}^{\#1}$	(None)	$f_\alpha^\beta n_\beta - f^{\beta\chi}n_\alpha n_\beta n_\chi$	$\tau_{1^- \alpha}^{\#1}$
$f_{1^- \alpha}^{\#2}$	(None)	$f_\alpha^\beta n_\beta - f^{\beta\chi}n_\alpha n_\beta n_\chi$	$\tau_{1^- \alpha}^{\#2}$
$f_{2^+ \alpha\beta}^{\#1}$	$f_{2^+ \beta\alpha}^{\#1} = f_{2^+ \alpha\beta}^{\#1}$	$\frac{f_{\alpha\beta}}{2} + \frac{f_{\beta\alpha}}{2} - \frac{1}{3}n_{\alpha\beta}f_\chi^\chi + \frac{1}{3}f_\chi^\chi n_\alpha n_\beta - \frac{1}{2}f_\beta^\chi n_\alpha n_\chi - \frac{1}{2}f_\beta^\chi n_\alpha n_\chi - \frac{1}{2}f_\alpha^\chi n_\beta n_\chi - \frac{1}{2}f_\alpha^\chi n_\beta n_\chi - \frac{1}{2}f_\alpha^\chi n_\beta n_\chi - \frac{1}{2}f_\alpha^\chi n_\beta n_\chi + \frac{1}{3}n_{\alpha\beta}f_\chi^\chi n_\delta + \frac{2}{3}f_\chi^\delta n_\alpha n_\beta n_\chi n_\delta$	$\tau_{2^+ \alpha\beta}^{\#1}$

and β here continue to stand for internal Lorentz indexes, as manipulated with the Minkowski metric, whereas accented Greek letters such as $\acute{\mu}$ and $\acute{\nu}$ stand for curved spacetime indexes. The curved-space metric is $g_{\acute{\mu}\acute{\nu}} = e^\alpha_{\acute{\mu}} e^\beta_{\acute{\nu}} \eta_{\alpha\beta}$, and we assume the auxiliary identities $e^\alpha_{\acute{\mu}} e_\alpha^{\acute{\nu}} = \delta_{\acute{\mu}}^{\acute{\nu}}$ and $e^\alpha_{\acute{\mu}} e^{\acute{\mu}}_\beta = \delta_\beta^\alpha$ which function as extra kinematic restrictions. The associated field strength tensors, out of which the action is built, are torsion and curvature

$$\begin{aligned} \mathcal{T}^\alpha_{\acute{\mu}\acute{\nu}} &\equiv \partial_{\acute{\mu}} e^\alpha_{\acute{\nu}} - \partial_{\acute{\nu}} e^\alpha_{\acute{\mu}} + \omega^\alpha_{\beta\acute{\mu}} e^\beta_{\acute{\nu}} - \omega^\alpha_{\beta\acute{\nu}} e^\beta_{\acute{\mu}}, \\ \mathcal{R}^{\alpha\beta}_{\acute{\mu}\acute{\nu}} &\equiv \partial_{\acute{\mu}} \omega^{\alpha\beta}_{\acute{\nu}} - \partial_{\acute{\nu}} \omega^{\alpha\beta}_{\acute{\mu}} + \omega^\alpha_{\gamma\acute{\mu}} \omega^{\gamma\beta}_{\acute{\nu}} - \omega^\alpha_{\gamma\acute{\nu}} \omega^{\gamma\beta}_{\acute{\mu}}. \end{aligned} \quad (25)$$

To extract the flat particle content of a theory—provided that Minkowski spacetime is an admissible perturbative background in that no accidental gauge symmetries are present¹³—the tetrad is perturbed around the Kronecker

choice of vacuum (see alternative vacua in [47,85,86]) so that

$$e^\alpha_{\acute{\mu}} \equiv \delta^\alpha_{\acute{\mu}} + f^\alpha_{\acute{\mu}}, \quad e^\alpha_{\acute{\mu}} \equiv \delta^\alpha_{\acute{\mu}} - f^\alpha_{\acute{\mu}} + \mathcal{O}(f^2), \quad (26)$$

Equation (26) defines a concrete perturbation scheme in $f^\alpha_{\acute{\mu}}$. This makes it evident that, at the quadratic order of the free theory in Eq. (1), Greek and accented Greek indices can be freely exchanged. To complete our setup of the weak-field regime, we assume that $\omega^{\alpha\beta}_{\acute{\mu}}$ is inherently perturbative.

2. Definitions

Tetrad perturbation. We define the asymmetric rank-two *TetradPerturbation*:

```
In[#]:= DefField[TetradPerturbation[-a, -b], PrintAs ->
  ↳ "[ScriptF]", PrintSourceAs -> "[Tau]"];
```

The output is shown in Table VI. Neglecting the (higher-order) distinction between indices, the field $f^{\alpha\beta}$ is found to contain 2^+ , 1^- , 1^+ , and two 0^+ modes. The conjugate source $\tau_{\alpha\beta}$ has a physical interpretation as the asymmetric stress-energy tensor, and it is automatically defined by PSALter.

Spin connection. We define the pair-antisymmetric *SpinConnection* as:

```
In[#]:= DefField[SpinConnection[-a, -b, -c], Antisymmetric[{ -b,
  ↳ -c}], PrintAs -> "[Omega]", PrintSourceAs -> "[Sigma]"];
```

The output is shown in Table VII. Working again exclusively with the Lorentz indices at lowest order, the field $\omega^{\alpha\beta}_\gamma$ is found to contain 2^+ , 2^- , two 1^- , two 1^+ , 0^+ , and 0^- modes. The conjugate source $\sigma_{\alpha\beta}^\gamma$ has a physical interpretation as the spin current, and it is automatically defined by PSALter.

¹³Any gauge symmetry which is either broken nonlinearly or absent on nonflat backgrounds is termed accidental [38]. This feature necessarily signals a pathology [83,84].

TABLE VII. The declaration of *SpinConnection*. These definitions are used in Figs. 5 and 6.

Fundamental field	Symmetries	Decomposition into SO(3) irrep(s)	Source
$\omega_{\alpha\beta\chi}$	$\omega_{\alpha\chi\beta} == -\omega_{\alpha\beta\chi}$	$\omega_{0^+ \alpha\beta\chi}^{\#1} - \frac{1}{2}\eta_{\alpha\chi}\omega_{1^+ \beta}^{\#1} + \frac{1}{2}\eta_{\alpha\beta}\omega_{1^+ \chi}^{\#1} + \frac{4}{3}\omega_{2^+ \beta\chi\alpha}^{\#1} + \omega_{1^+ \beta\chi}^{\#2}n_\alpha$ $+ \frac{1}{3}\eta_{\alpha\chi}\omega_{0^+ \beta}^{\#1}n_\beta - \omega_{1^+ \alpha\chi}^{\#1}n_\beta + \omega_{2^+ \alpha\chi}^{\#1}n_\beta - \frac{1}{2}\omega_{1^+ \chi}^{\#1}n_\alpha n_\beta - \omega_{1^+ \beta\chi}^{\#2}n_\alpha n_\beta$ $- \frac{1}{3}\eta_{\alpha\beta}\omega_{0^+ \chi}^{\#1}n_\chi + \omega_{1^+ \alpha\beta}^{\#1}n_\chi - \omega_{2^+ \alpha\beta}^{\#1}n_\chi + \frac{1}{2}\omega_{1^+ \beta}^{\#1}n_\alpha n_\chi + \omega_{1^+ \beta\chi}^{\#2}n_\alpha n_\chi$	$\sigma_{\alpha\beta\chi}$
SO(3) irrep	Symmetries	Expansion in terms of the fundamental field	Source SO(3) irrep
$\omega_{0^+}^{\#1}$	(None)	$\omega_{\alpha\beta}^{\beta}n^\alpha$	$\sigma_{0^+}^{\#1}$
$\omega_{0^+ \alpha\beta\chi}^{\#1}$	$\omega_{0^+ \alpha\chi\beta}^{\#1} == -\omega_{0^+ \alpha\beta\chi}^{\#1}$ & $\omega_{0^+ \beta\alpha\chi}^{\#1} == -\omega_{0^+ \alpha\beta\chi}^{\#1}$	$\frac{1}{3}\omega_{\alpha\beta\chi} - \frac{1}{3}\omega_{\beta\alpha\chi} + \frac{1}{3}\omega_{\chi\alpha\beta} - \frac{1}{3}\omega_{\beta\chi\delta}n_\alpha n^\delta + \frac{1}{3}\omega_{\chi\beta\delta}n_\alpha n^\delta$ $- \frac{1}{3}\omega_{\delta\beta\chi}n_\alpha n^\delta + \frac{1}{3}\omega_{\alpha\chi\delta}n_\beta n^\delta - \frac{1}{3}\omega_{\chi\alpha\delta}n_\beta n^\delta + \frac{1}{3}\omega_{\delta\alpha\chi}n_\beta n^\delta$ $- \frac{1}{3}\omega_{\alpha\beta\chi}n_\chi n^\delta + \frac{1}{3}\omega_{\beta\alpha\chi}n_\chi n^\delta - \frac{1}{3}\omega_{\delta\alpha\beta}n_\chi n^\delta$	$\sigma_{0^+ \alpha\beta\chi}^{\#1}$
$\omega_{1^+ \alpha\beta}^{\#1}$	$\omega_{1^+ \beta\alpha}^{\#1} = -\omega_{1^+ \alpha\beta}^{\#1}$	$\frac{1}{2}\omega_{\alpha\beta\chi}n^\chi - \frac{1}{2}\omega_{\beta\alpha\chi}n^\chi - \frac{1}{2}\omega_{\chi\beta\delta}n_\alpha n^\delta + \frac{1}{2}\omega_{\chi\alpha\delta}n_\beta n^\delta$	$\sigma_{1^+ \alpha\beta}^{\#1}$
$\omega_{1^+ \alpha\beta}^{\#2}$	$\omega_{1^+ \beta\alpha}^{\#2} = -\omega_{1^+ \alpha\beta}^{\#2}$	$\omega_{\chi\alpha\beta}n^\chi + \omega_{\chi\beta\delta}n_\alpha n^\delta - \omega_{\chi\alpha\delta}n_\beta n^\delta$	$\sigma_{1^+ \alpha\beta}^{\#2}$
$\omega_{1^+ \alpha}^{\#1}$	(None)	$-\omega_{\alpha\beta}^{\beta} + \omega_{\beta\chi}^{\chi}n_\alpha n^\beta + \omega_{\beta\alpha\chi}n^\beta n^\chi$	$\sigma_{1^+ \alpha}^{\#1}$
$\omega_{1^+ \alpha}^{\#2}$	(None)	$\omega_{\beta\alpha\chi}n^\beta n^\chi$	$\sigma_{1^+ \alpha}^{\#2}$
$\omega_{2^+ \alpha\beta}^{\#1}$	$\omega_{2^+ \beta\alpha}^{\#1} = -\omega_{2^+ \alpha\beta}^{\#1}$	$-\frac{1}{2}\omega_{\alpha\beta\chi}n^\chi - \frac{1}{2}\omega_{\beta\alpha\chi}n^\chi - \frac{1}{3}\eta_{\alpha\beta}\omega_{\chi\delta}n^\delta + \frac{1}{3}\omega_{\chi\delta}n_\alpha n_\beta n^\chi$ $+ \frac{1}{2}\omega_{\chi\beta\delta}n_\alpha n^\delta + \frac{1}{2}\omega_{\chi\alpha\delta}n_\beta n^\delta$	$\sigma_{2^+ \alpha\beta}^{\#1}$
$\omega_{2^+ \alpha\beta\chi}^{\#1}$	$\omega_{2^+ \beta\alpha\chi}^{\#1} == -\omega_{2^+ \alpha\beta\chi}^{\#1}$	$-\frac{1}{4}\omega_{\alpha\beta\chi} + \frac{1}{4}\omega_{\beta\alpha\chi} + \frac{1}{2}\omega_{\chi\alpha\beta} - \frac{3}{8}\eta_{\beta\chi}\omega_{\alpha\delta}^\delta + \frac{3}{8}\eta_{\alpha\chi}\omega_{\beta\delta}^\delta$ $-\frac{3}{8}\omega_{\beta\delta}^\delta n_\alpha n_\chi + \frac{3}{8}\omega_{\alpha\delta}^\delta n_\beta n_\chi$ $+ \frac{1}{4}\omega_{\beta\chi\delta}n_\alpha n^\delta + \frac{1}{2}\omega_{\chi\beta\delta}n_\alpha n^\delta + \frac{1}{4}\omega_{\delta\beta\chi}n_\alpha n^\delta$ $+ \frac{3}{8}n_{\beta\chi}\omega_{\delta\epsilon}^\epsilon n_\alpha n^\delta - \frac{1}{4}\omega_{\alpha\chi\delta}n_\beta n^\delta$ $-\frac{1}{2}\omega_{\chi\alpha\delta}n_\beta n^\delta - \frac{1}{4}\omega_{\delta\alpha\chi}n_\beta n^\delta - \frac{3}{8}n_{\alpha\chi}\omega_{\delta\epsilon}^\epsilon n_\beta n^\delta$ $+ \frac{1}{4}\omega_{\alpha\beta\delta}n_\chi n^\delta - \frac{1}{4}\omega_{\beta\alpha\delta}n_\chi n^\delta - \frac{1}{2}\omega_{\delta\alpha\beta}n_\chi n^\delta$ $+ \frac{3}{8}n_{\beta\chi}\omega_{\delta\alpha\epsilon}^\epsilon n^\delta n^\epsilon - \frac{3}{8}n_{\alpha\chi}\omega_{\delta\beta\epsilon}^\epsilon n^\delta n^\epsilon$ $-\frac{3}{8}\omega_{\delta\beta\epsilon}^\epsilon n_\alpha n_\chi n^\delta n^\epsilon + \frac{3}{8}\omega_{\delta\alpha\epsilon}^\epsilon n_\beta n_\chi n^\delta n^\epsilon$	$\sigma_{2^+ \alpha\beta\chi}^{\#1}$

Lagrangian coupling coefficients. As with our analysis in Fig. 4, we also define *C1* so as to denote the coupling c_1 :

```
In[#]:= DefConstantSymbol[C1, PrintAs -> "\! \! (\ *
\rightarrow SubscriptBox[\! (\! [ScriptC]\! ), \! (1\! )\! ]\! )"];
```

For the following considerations, there are ten Lagrangian coupling coefficients that need to be introduced, ranging up to *C10* which denotes c_{10} . Of these, the first five are dimensionless, and the final five are of mass dimension two.

3. Spectroscopy

Parity-indefinite Einstein-Cartan gravity. The tetrad and spin connection have in total 40 components (16 are in $e^\alpha{}_\mu$ and 24 in the pair-antisymmetric $\omega^{\alpha\beta}{}_\mu$), and if the action is taken to contain terms which are at most quadratic in the derivatives, then the theory can accommodate up to 20 propagating d.o.f.¹⁴ It is well known [32,56,87] that even linearly, not all of these can harmoniously coexist.

¹⁴These are the two massless graviton polarizations, and three pairs of massive particles of spin two (two times five d.o.f), spin one (two times three d.o.f), and spin zero (two times one d.o.f).

Although there are choices of parameters for which particles carrying spin $J \geq 1$ may be healthy at the linearized level, it is widely believed that when it comes to (at least parity-preserving) EC gravity, the only non-linearly viable models propagate exclusively scalar modes [88–91].¹⁵ The most general parity-indefinite action that propagates the graviton and scalars only, comprises all invariants which are at most quadratic in torsion and the scalar $\mathcal{R}$ and pseudoscalar $\tilde{\mathcal{R}}$ curvatures, whose definitions are

$$\mathcal{R} \equiv e_\alpha{}^\mu e_\beta{}^\nu \mathcal{R}^{\alpha\beta}{}_{\mu\nu}, \quad \tilde{\mathcal{R}} \equiv \epsilon^{\alpha\beta\kappa\lambda} \eta_{\alpha\gamma} \eta_{\beta\delta} e_\kappa{}^\mu e_\lambda{}^\nu \mathcal{R}^{\gamma\delta}{}_{\mu\nu}. \quad (27)$$

To accompany Eq. (27) we define $e \equiv \det(e^\alpha{}_\mu)$, and the Lorentz-indexed torsion $\mathcal{T}_{\mu\nu\rho} \equiv \eta_{\mu\sigma} e_\nu{}^\mu e_\rho{}^\nu \mathcal{T}^\sigma{}_{\mu\nu}$, along

¹⁵See also [29,49,92–102] for applications and [78,103–106] for reviews.

with its trace $\mathcal{T}_\nu \equiv \eta^{\mu\rho} \mathcal{T}_{\mu\nu\rho}$. Accounting for all possible combinations, this action reads¹⁶

$$S = \int d^4x e [c_1 \mathcal{R} + c_2 \tilde{\mathcal{R}} + c_3 \mathcal{R}^2 + c_4 \mathcal{R} \tilde{\mathcal{R}} + c_5 \tilde{\mathcal{R}}^2 + c_6 \mathcal{T}_{\mu\nu\rho} \mathcal{T}^{\mu\nu\rho} + c_7 \mathcal{T}_{\mu\nu\rho} \mathcal{T}^{\nu\rho\mu} + c_8 \mathcal{T}_\mu \mathcal{T}^\mu + c_9 \epsilon^{\mu\nu\rho\sigma} \mathcal{T}_{\lambda\mu\nu} \mathcal{T}^\lambda_{\rho\sigma} + c_{10} \epsilon^{\mu\nu\rho\sigma} \mathcal{T}_{\mu\nu\lambda} \mathcal{T}_{\rho\sigma}{}^\lambda]. \quad (29)$$

The theory in Eq. (29) is an extension of that proposed in [19] (see also [8,107]), which uses terms solely quadratic in $\mathcal{R}$ and $\tilde{\mathcal{R}}$. It was shown in [83,84] that, in isolation, the square of the scalar curvature propagates the Einstein graviton on a de Sitter background, but that this species becomes strongly coupled on Minkowski spacetime. The inclusion of the Einstein-Hilbert term guarantees, however, that no accidental symmetries arise and that perturbation theory makes sense also on flat backgrounds. Then, $\mathcal{R}$ and $\mathcal{R}^2$ propagate a positive-parity scalar mode *in addition to* the massless graviton.¹⁷ By contrast, $\tilde{\mathcal{R}}^2$ propagates a negative-parity scalar mode [88], emerging from the axial vector part of torsion. Given its nonlinear consistency, the evident lack of accidental gauge symmetries, and the interesting roles the scalars of gravitational origin can play in particle physics and cosmology [8,19–21,109], the model defined by Eq. (29) stands out and certainly deserves further scrutiny. To study the general case of Eq. (29) we input:

```
In[#]:= ParticleSpectrum[(2 * C6 + C7) * SpinConnection[-a, -b,
  ↳ -c] * SpinConnection[a, b, c] + C10 * epsilonG[-b, -c, -d,
  ↳ -i] * SpinConnection[-a, d, i] * SpinConnection[a, b, c] +
  ↳ (C1 - 2 * C6 - 3 * C7) * SpinConnection[a, b, c] *
  ↳ SpinConnection[-b, -a, -c] - 2 * C10 * epsilonG[-a, -c, -d,
  ↳ -i] * (*omitted 3222 characters for brevity*) -d, -i] *
  ↳ CD[c][TetradPerturbation[a, b]] *
  ↳ CD[i][TetradPerturbation[d, -b]] + 4 * C4 * epsilonG[-c, -d,
  ↳ -i, -j] * CD[-b][SpinConnection[a, -a, b]] *
  ↳ CD[j][SpinConnection[c, d, i]], TheoryName ->
  ↳ "ScalarParityViolatingPGT", MaxLaurentDepth -> 1,
  ↳ AspectRatio -> Portrait, ShowPropagator -> False];
```

The resulting particle spectrum is shown in Fig. 5. The graviton is, as expected, present since it is the natural companion of the Einstein-Hilbert term. It is healthy as long as

$$c_1 < 0. \quad (30)$$

We also notice that the particle content contains two scalars which are dynamical. The current functionality in PSALTER does not yet allow these states to be resolved, due to the fact that their square masses $M_{1_0}^2$ and $M_{2_0}^2$ are not, in general, rational functions of the Lagrangian coupling coefficients in Eq. (29). Let us therefore discuss in detail the no-ghost and no-tachyon constraints for these scalars, by applying manually the improved techniques of Sec. II C. We start from the 4×4 coefficient matrix of the scalar $J = 0$ sector. By comparing Eq. (17) and Fig. 5 we find

¹⁶Note that up to a total derivative

$$\mathcal{R} \propto \epsilon^{\mu\nu\rho\sigma} \mathcal{T}_{\mu\nu\lambda} \mathcal{T}_{\rho\sigma}{}^\lambda, \quad (28)$$

so it is not really necessary to include the c_2 -term in the action.

¹⁷Even when the geometry is taken to be torsion-free, this mode persists and is instead associated with the scalar mode of the metric tensor. This is the well-known Starobinsky scalaron [108].

	$\omega_{0+}^{\#1}$	$f_{0+}^{\#1}$	$f_{0+}^{\#2}$	$\omega_{0+}^{\#1\alpha\beta\chi}$									
$\omega_{0+}^{\#1} \dagger$	$\frac{1}{2}(-Y_1 + 12c_3k^2)$	$-\frac{iY_1k}{\sqrt{2}}$	0	$2iY_2 + 6ic_4k^2$									
$f_{0+}^{\#1} \dagger$	$\frac{iY_1k}{\sqrt{2}}$	Y_3k^2	0	$2\sqrt{2}Y_2k$									
$f_{0+}^{\#2} \dagger$	0	0	0	0									
$\omega_{0+}^{\#1} \dagger^{\alpha\beta\chi}$	$2iY_2 + 6ic_4k^2$	$-2\sqrt{2}Y_2k$	0	$-Y_4 - 24c_5k^2$	$\omega_{1+}^{\#1\alpha\beta}$	$\omega_{1+}^{\#2\alpha\beta}$	$f_{1+}^{\#1\alpha\beta}$	$\omega_{1+}^{\#1\alpha}$	$\omega_{1+}^{\#2\alpha}$	$f_{1+}^{\#1\alpha}$	$f_{1+}^{\#2\alpha}$		
$\omega_{1+}^{\#1} \dagger^{\alpha\beta}$	$-\frac{Y_5}{2}$	$-\frac{Y_6}{\sqrt{2}}$	$\frac{iY_6k}{\sqrt{2}}$	$\frac{iY_7}{2}$	$-\frac{iY_8}{\sqrt{2}}$	0	$-Y_8k$						
$\omega_{1+}^{\#2} \dagger^{\alpha\beta}$	$-\frac{Y_6}{\sqrt{2}}$	Y_9	$-iY_9k$	$\frac{iY_8}{\sqrt{2}}$	$-ic_{10}$	0	$-\sqrt{2}c_{10}k$						
$f_{1+}^{\#1} \dagger^{\alpha\beta}$	$-\frac{iY_6k}{\sqrt{2}}$	iY_9k	Y_9k^2	$-\frac{Y_8k}{\sqrt{2}}$	$c_{10}k$	0	$-i\sqrt{2}c_{10}k^2$						
$\omega_{1+}^{\#1} \dagger^{\alpha}$	$\frac{iY_7}{2}$	$\frac{iY_8}{\sqrt{2}}$	$\frac{Y_8k}{\sqrt{2}}$	$-\frac{Y_{10}}{2}$	$\frac{Y_{11}}{\sqrt{2}}$	0	$-iY_{11}k$						
$\omega_{1+}^{\#2} \dagger^{\alpha}$	$-\frac{iY_8}{\sqrt{2}}$	$-ic_{10}$	$-c_{10}k$	$\frac{Y_{11}}{\sqrt{2}}$	$\frac{Y_{12}}{2}$	0	$-\frac{iY_{12}k}{\sqrt{2}}$						
$f_{1+}^{\#1} \dagger^{\alpha}$	0	0	0	0	0	0	0						
$f_{1+}^{\#2} \dagger^{\alpha}$	Y_8k	$\sqrt{2}c_{10}k$	$-i\sqrt{2}c_{10}k^2$	$iY_{11}k$	$\frac{iY_{12}k}{\sqrt{2}}$	0	$Y_{12}k^2$	$\omega_{2+}^{\#1\alpha\beta}$	$f_{2+}^{\#1\alpha\beta}$	$\omega_{2+}^{\#1\alpha\beta\chi}$			
								$\omega_{2+}^{\#1} \dagger^{\alpha\beta}$	$\frac{Y_{13}}{2}$	$\frac{iY_{13}k}{\sqrt{2}}$	$\frac{iY_{14}}{2}$		
								$f_{2+}^{\#1} \dagger^{\alpha\beta}$	$-\frac{iY_{13}k}{\sqrt{2}}$	$Y_{15}k^2$	$\frac{Y_{14}k}{\sqrt{2}}$		
								$\omega_{2+}^{\#1} \dagger^{\alpha\beta\chi}$	$\frac{iY_{14}}{2}$	$-\frac{Y_{14}k}{\sqrt{2}}$	$\frac{Y_{13}}{2}$		

Abbreviations used in matrices

$Y_1 = 2c_1 - 2c_6 + c_7 - 3c_8$ & $Y_2 = c_{10} - c_2 - 2c_9$ & $Y_3 = 2c_6 - c_7 + 3c_8$ &
 $Y_4 = c_1 - 4(c_6 + c_7)$ & $Y_5 = c_1 - 6c_6 - 5c_7$ & $Y_6 = c_1 - 2c_6 - 3c_7$ &
 $Y_7 = 3c_{10} - 2(c_2 + 2c_9)$ & $Y_8 = c_{10} - 2(c_2 + 2c_9)$ & $Y_9 = 2c_6 + c_7$ &
 $Y_{10} = c_1 - 2c_6 + c_7 - 2c_8$ & $Y_{11} = c_1 - c_8$ & $Y_{12} = 2c_6 - c_7 + c_8$ &
 $Y_{13} = c_1 + 2c_6 - c_7$ & $Y_{14} = c_{10} + 2c_2 + 4c_9$ & $Y_{15} = 2c_6 - c_7$ &
 $\text{Det}(0) = k^2(2c_1^3 + 4c_1((c_6 + c_7)(2c_6 - c_7 + 3c_8) + 2(-c_{10} + c_2 + 2c_9)^2) +$
 $6c_1(8c_{10}c_4 - (c_3 + 4c_5)(2c_6 - c_7 + 3c_8) - 8c_4(c_2 + 2c_9))k^2 +$
 $24c_3((c_6 + c_7)(2c_6 - c_7 + 3c_8) + 2(-c_{10} + c_2 + 2c_9)^2)k^2 +$
 $36(c_4^2 - 4c_3c_5)(2c_6 - c_7 + 3c_8)k^4 - c_1^2(10c_6 + 7c_7 + 3c_8 - 48c_5k^2))$ &
 $\text{Det}(1) = \frac{1}{8}((c_1 - 4(c_6 + c_7))(2c_1 - 2c_6 + c_7 - 3c_8) + 8(-c_{10} + c_2 + 2c_9)^2)$
 $((c_1 + 2c_6 - c_7)^2 + (c_{10} + 2c_2 + 4c_9)^2)(1 + 3k^2 + 2k^4)$ &
 $\text{Det}(2) = -\frac{1}{4}c_1((c_1 + 2c_6 - c_7)^2 + (c_{10} + 2c_2 + 4c_9)^2)k^2$

Added source term(s):		$f^{\alpha\beta} \tau_{\alpha\beta} + \omega^{\alpha\beta\chi} \sigma_{\alpha\beta\chi}$	
Source constraint(s)	# constraint(s)	Covariant form	
$\tau_{0+}^{\#2} == 0$	1	$\partial_\beta \partial_\alpha \tau^{\alpha\beta} == 0$	
$\tau_{1+}^{\#2\alpha} - 2ik \sigma_{1+}^{\#2\alpha} == 0$	3	$\partial_\chi \partial_\beta \partial^\alpha \tau^{\beta\chi} + 2 \partial_\beta \partial^\delta \partial_\chi \partial_\beta \sigma^{\beta\alpha\chi} == \partial_\chi \partial^\chi \partial_\beta \tau^{\alpha\beta}$	
$\tau_{1+}^{\#1\alpha} == 0$	3	$\partial_\chi \partial_\beta \partial^\alpha \tau^{\beta\chi} == \partial_\chi \partial^\chi \partial_\beta \tau^{\beta\alpha}$	
$\tau_{1+}^{\#1\alpha\beta} - ik \sigma_{1+}^{\#2\alpha\beta} == 0$	3	$\partial_\chi \partial^\alpha \tau^{\beta\chi} + \partial_\chi \partial^\beta \tau^{\alpha\chi} + \partial_\chi \partial^\chi \tau^{\alpha\beta} + 2 \partial_\beta \partial_\chi \partial^\beta \sigma^{\chi\alpha\delta} ==$ $\partial_\chi \partial^\alpha \tau^{\chi\beta} + \partial_\chi \partial^\beta \tau^{\alpha\chi} + \partial_\chi \partial^\chi \tau^{\beta\alpha} + 2 \partial_\beta \partial_\chi \partial^\alpha \sigma^{\chi\beta\delta} + 2 \partial_\beta \partial_\delta \partial_\chi \sigma^{\chi\alpha\beta}$	
Total # constraint(s):	10		
Unresolved pole(s)	# d.o.f.	Pole structure(s)	
	2	$-2c_1^3 - 4c_1((c_6 + c_7)(2c_6 - c_7 + 3c_8) + 2(-c_{10} + c_2 + 2c_9)^2) -$ $6c_1(8c_{10}c_4 - (c_3 + 4c_5)(2c_6 - c_7 + 3c_8) - 8c_4(c_2 + 2c_9))k^2 -$ $24c_3((c_6 + c_7)(2c_6 - c_7 + 3c_8) + 2(-c_{10} + c_2 + 2c_9)^2)k^2 -$ $36(c_4^2 - 4c_3c_5)(2c_6 - c_7 + 3c_8)k^4 + c_1^2(10c_6 + 7c_7 + 3c_8 - 48c_5k^2)$	
Resolved pole(s)	# polarization(s)	Square mass	Residue
	2	0	$-\frac{1+8p^2}{c_1}$
Resolved unitarity condition(s):		$c_1 < 0$	

FIG. 5. The spectrograph of the theory defined by Eq. (29). In the current version of PSALTER, the analysis is halted once it has been determined that the square masses are not rational functions of the Lagrangian coupling coefficients. Thus, the only resolved unitarity condition corresponds to the graviton in Eq. (30). The existence of two unresolved poles is indicated. Our improved algorithm, not yet implemented in PSALTER, recovers the associated conditions Eqs. (33) and (38). Note also the appearance of ten gauge generators—owing to the Poincaré gauge redundancy—ensuring the absence of accidental symmetries. All quantities are defined in Tables VI and VII.

$$\begin{aligned}
\mathbf{o}_{0_m} &= \begin{bmatrix} -24c_5k^2 - \Upsilon_4 & 2i(3c_4k^2 + \Upsilon_2) \\ 2i(3c_4k^2 + \Upsilon_2) & \frac{1}{2}(12c_3k^2 - \Upsilon_1) \end{bmatrix}, & \mathbf{o}_{0_{m\gamma}} &= \begin{bmatrix} -2\sqrt{2}\Upsilon_2k & 0 \\ -i\frac{\Upsilon_1}{\sqrt{2}}k & 0 \end{bmatrix}, \\
\mathbf{o}_{0_{rm}} &= \begin{bmatrix} -2\sqrt{2}\Upsilon_2k & i\frac{\Upsilon_1}{\sqrt{2}}k \\ 0 & 0 \end{bmatrix}, & \mathbf{o}_{0_\gamma} &= \begin{bmatrix} \Upsilon_3k^2 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned} \tag{31}$$

where the definitions of the coupling abbreviations in Eq. (31) can also be found in Fig. 5. Then, from Eq. (31) we see that the kinetic- and mass-matrices in Eq. (19) are

$$\mathbf{K}_0 = 6 \begin{bmatrix} -4c_5 & ic_4 \\ ic_4 & c_3 \end{bmatrix}, \quad \mathbf{M}_0 = \frac{1}{\Upsilon_3} \begin{bmatrix} 8\Upsilon_2^2 - \Upsilon_3\Upsilon_4 & 2i\Upsilon_2(\Upsilon_1 + \Upsilon_3) \\ 2i\Upsilon_2(\Upsilon_1 + \Upsilon_3) & -\frac{\Upsilon_1}{2}(\Upsilon_1 + \Upsilon_3) \end{bmatrix}. \tag{32}$$

According to Eq. (20), the theory is ghost-free provided that $\mathbf{K}_0$ is negative-definite, i.e.

$$[c_5 > 0] \wedge [4c_3c_5 - c_4^2 > 0], \tag{33}$$

and from Eq. (33) it follows that $c_3 > 0$. Since $\mathbf{M}_0$ is a 2×2 matrix, tachyon-freedom further requires [34]

$$[\text{tr}(\mathbf{L}_0)^2 - 4\det(\mathbf{L}_0) > 0] \wedge [\text{tr}(\mathbf{L}_0) > 0] \wedge [\det(\mathbf{L}_0) > 0], \quad \mathbf{L}_0 \equiv -\mathbf{K}_0^+\mathbf{M}_0. \tag{34}$$

It can be easily checked that the second and third inequalities in Eq. (34) translate into

$$\left[\frac{c_3(\Upsilon_3\Upsilon_4 - 8\Upsilon_2^2) - 2(\Upsilon_1 + \Upsilon_3)(2c_4\Upsilon_2 + c_5\Upsilon_1)}{\Upsilon_3(c_4^2 - 4c_3c_5)} > 0 \right] \wedge \left[\frac{(\Upsilon_1 + \Upsilon_3)(\Upsilon_1\Upsilon_4 + 8\Upsilon_2^2)}{\Upsilon_3(c_4^2 - 4c_3c_5)} > 0 \right], \tag{35}$$

respectively. The latter inequality in Eq. (35), once the massless equation (30) and massive equation (33) no-ghost conditions are taken into account, together with $\Upsilon_1 + \Upsilon_3 = 2c_1$, gives

$$\Upsilon_3(\Upsilon_1\Upsilon_4 + 8\Upsilon_2^2) > 0, \tag{36}$$

and from the former—following more-or-less verbatim the procedure spelled out in [34] for simplifying such expressions—one finds that

$$\Upsilon_1\Upsilon_3 < 0. \tag{37}$$

In terms of the Lagrangian coupling coefficients, the conditions in Eqs. (36) and (37) read

$$\begin{aligned}
& [(2c_6 - c_7 + 3c_8)[(2c_1 - 2c_6 + c_7 - 3c_8)(c_1 - 4(c_6 + c_7)) + 8(c_2 + c_{10} - 2c_9)^2] > 0] \\
& \wedge [(2c_6 - c_7 + 3c_8)(2c_1 - 2c_6 + c_7 - 3c_8) < 0],
\end{aligned} \tag{38}$$

which are the well-known constraints [34], albeit written in a different notation. Accordingly, the theory in Eq. (29) is unitary if Eqs. (30), (33), and (38) are satisfied.

IV. CONCLUSIONS

A. Results of this paper

The spin-projection operator algorithm for parity-violating tensorial field theories, with its accompanying conventions, was implemented as part of the particle spectrum for any tensor Lagrangian (PSALTer) software initiative. PSALTer is an open-source package contribution to the xAct

project, designed for use with *Mathematica* (see [53] for instructions on how to obtain and install the software). As an illustration, the new functionality in PSALTer was calibrated against a number of examples, confirming the analytic results. As a byproduct, we suggested a simplified way to obtain no-ghost conditions. This does not involve the computation of any propagator residues over massive poles, which greatly facilitates the analysis.

B. Further work

The techniques developed in this paper open the door to a systematic and comprehensive investigation into the

particle content of parity-violating gravitational theories, and in particular metric-affine gravity (see [38] for the most comprehensive study of parity-preserving MAG), which is completely unexplored.

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Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or European Research Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

DATA AVAILABILITY

Note that the sources for these examples can be found in the public GitHub repository [57].

APPENDIX A: CONSTRUCTION OF OPERATORS

1. The spatial hypersurface

In this appendix we bootstrap the construction of SPOs, including the new parity-violating SPOs, and set out the precise conventions that are implemented in PSALTER. Projection by spin J and parity P is facilitated by choosing a preferred frame. We can derive this frame from the particle four-momentum k^μ , which can be chosen to be either timelike or null for massive and massless particles, respectively. In the timelike case, there is a unit vector $n^\mu \equiv k^\mu/k$, where $k^2 \equiv k^\mu k_\mu$, so that n^μ coincides with the preferred observer's four-velocity

$$[n^\mu]^T = [1 \quad 0 \quad 0 \quad 0]. \quad (\text{A1})$$

The usual transverse and longitudinal projectors, parallel and perpendicular to the spatial hypersurface, are

$$\Theta_\mu^\nu \equiv \delta_\mu^\nu - n_\mu n^\mu, \quad \Omega_\mu^\nu \equiv n_\mu n^\mu, \quad (\text{A2})$$

respectively. We also introduce the overbar notation for indices, such as $V^{\bar{\mu}} = \Theta_\mu^\mu V^\mu$, for a generic vector V^μ .

2. Parity-preserving SPOs

The parity-preserving SPOs include both diagonal and off-diagonal SPOs, and are constructed exclusively from Θ_μ^ν and Ω_μ^ν in Eq. (A2). They satisfy the following identities:

$$\mathcal{P}(i_{J^P}^X, j_{J^P}^Y)_{\mu_X}{}^{\nu_Y} \equiv \mathcal{P}(j_{J^P}^Y, i_{J^P}^X)_{\mu_X}{}^{\nu_Y}, \quad (\text{A3a})$$

$$\sum_{J,P} \sum_{i_{J^P}^X} \mathcal{P}(i_{J^P}^X, i_{J^P}^X)_{\mu_X}{}^{\nu_X} \equiv \Delta_{\mu_X}^{\nu_X}, \quad (\text{A3b})$$

$$\mathcal{P}(i_{J^P}^X, j_{J^P}^Y)_{\mu_X}{}^{\nu_Y} \mathcal{P}(k_{J^{P'}}^Y, l_{J^{P'}}^Z)_{\nu_Y}{}^{\sigma_Z} \equiv \delta_{jk} \delta_{JJ'} \delta_{PP'} \mathcal{P}(i_{J^P}^X, l_{J^P}^Z)_{\mu_X}{}^{\sigma_Z}, \quad (\text{A3c})$$

$$P \zeta_{\mu_X}^* \mathcal{P}(i_{J^P}^X, i_{J^P}^X)_{\mu_X}{}^{\nu_X} \zeta^{\nu_X} \geq 0, \quad (\text{A3d})$$

where Eqs. (A3a)–(A3d) encode symmetry,¹⁸ completeness, orthonormality,¹⁹ and positivity, respectively.

3. Reduced-index SPOs

In order to actually *construct* these parity-preserving SPOs, we may proceed as follows. By the usual methods of Young tableaux and trace-free decomposition, the *reduced-index* J^P states may be extracted manually so as to define the reduced-index SPOs

$$\zeta(i_{J^P}^X)_{\bar{\mu}_{J^P}} \equiv \mathcal{P}(i_{J^P}^X)_{\bar{\mu}_{J^P}}{}^{\nu_X} \zeta_{\nu_X}, \quad (\text{A4})$$

where $\bar{\mu}_{J^P}$ is a reduced collection of parallel indices specific to the J^P state. The term *reduced* here means that there may be fewer indices than in μ_X for some or all of the fields X which contain states with this J^P . For example, a high-rank tensor can contain many scalar states, however scalars do not require indices. The reduced-index states vanish upon contraction of any parallel indices (i.e., they are trace-free), and these indices will also carry further symmetry properties. As a consequence of these constraints, each reduced-index state has only $2J + 1$ independent components, corresponding to the spin multiplicity. The reduced-index

¹⁸Note that in [39] the property of Hermiticity was used instead of symmetry: this is because we now assume the parity-preserving SPOs to be real.

¹⁹Note that in [39] the more restrictive relation $\mathcal{P}(i_{J^P}^X, j_{J^P}^X)_{\mu_X}{}^{\nu_X} \mathcal{P}(k_{J^{P'}}^X, l_{J^{P'}}^X)_{\nu_X}{}^{\sigma_X} = \delta_{jk} \delta_{JJ'} \delta_{PP'} \mathcal{P}(i_{J^P}^X, l_{J^P}^X)_{\mu_X}{}^{\sigma_X}$ was used instead of Eq. (A3c), though the latter was in fact also true.

SPOs in Eq. (A4) are real projection operators constructed—once again—exclusively from Θ_μ^ν and Ω_μ^ν . They need not be normalized in any sense, and their definitions may vary according to conventions. Here, for example, we contrast with [39], in that we will no longer allow $\epsilon_{\mu\nu\sigma\lambda}$ to be used in the definition of reduced-index SPOs: this is because we wish to closely track the parity of the projected states. Consequently, the reduced-index states are always tensors (in the sense that they are never pseudotensors), and if there are N reduced indices then the parity is always $P \equiv (-1)^N$. The parity-preserving SPOs are typically more cumbersome than their reduced-index counterparts, but their properties in Eqs. (A3a)–(A3d) make them formally useful. Within a given J^P sector they are given by

$$\mathcal{P}(i_{J^P}^X, j_{J^P}^Y)_{\mu_X}^{\nu_Y} \equiv c(i_{J^P}^X) c(j_{J^P}^Y) \mathcal{P}(i_{J^P}^X)_{\bar{\mu}_X}^{\bar{\nu}_X} \mathcal{P}(j_{J^P}^Y)_{\bar{\sigma}_Y}^{\bar{\nu}_Y}, \quad (\text{A5})$$

where, for any given choice of unnormalized reduced-index SPOs, the nonzero $c(i_{J^P}^X) \in \mathbb{R}$ are fixed (each up to a sign) by the requirements of Eq. (A3b).²⁰ The construction in Eq. (A5) evidently ensures Eq. (A3a). By moving to the frame n^μ it is clear for finite fields that $P \equiv \text{sgn}(\zeta(i_{J^P}^X)_{\bar{\mu}_X}^{\bar{\nu}_X} \zeta^*(i_{J^P}^X)_{\bar{\mu}_X}^{\bar{\nu}_X})$ due to our choice of signature, and so Eq. (A5) also implies Eq. (A3d). The property in Eq. (A3c) already follows from the fact that the extraction of J^P states is an irreducible decomposition. In fact, we may conclude that

$$\mathcal{P}(i_{J^P}^X)_{\bar{\mu}_X}^{\bar{\nu}_X} \mathcal{P}(j_{J^{P'}}^X)_{\bar{\sigma}_X}^{\bar{\nu}_X} \equiv \frac{\delta_{ij} \delta_{JJ'} \delta_{PP'}}{c(i_{J^P}^X) c(j_{J^{P'}}^X)} \Delta_{\bar{\mu}_X}^{\bar{\nu}_X}, \quad (\text{A6})$$

where we use the same notation as in Eq. (7), so that Eqs. (A5) and (A6) together imply Eq. (A3c).

4. Parity-violating SPOs

We will now extend the above considerations to the case of parity-violating SPOs, comprising only off-diagonal projectors. We define

$$\epsilon_{\mu\nu\sigma} \equiv \epsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}} \equiv \epsilon_{\mu\nu\sigma\rho} n^\rho, \quad (\text{A7})$$

so that from Eq. (A7) a natural definition for $P' \neq P$ is

$$\begin{aligned} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)_{\mu_X}^{\nu_Y} &\equiv q_J c(i_{J^P}^X) c(j_{J^{P'}}^Y) \mathcal{P}(i_{J^P}^X)_{\bar{\mu}_X}^{\bar{\nu}_X} \mathcal{P}(j_{J^{P'}}^Y)_{\bar{\sigma}_Y}^{\bar{\nu}_Y} \\ &\times \mathcal{P}(j_{J^{P'}}^Y)_{\bar{\rho}_Y}^{\bar{\nu}_Y} \quad \forall J < 2, \end{aligned} \quad (\text{A8})$$

where q_J is a new normalization constant, which will be explained in a moment. In practice, the convention in

Eq. (A8) works well for $J < 2$ because the 0^+ and 0^- states have zero and three indices, respectively, while the 1^+ and 1^- states have two and one indices, respectively: these add up to the three indices of Eq. (A7). The even-odd partitioning leads to a cancellation of signs, so that Eq. (A8) extends Eq. (A3a) to the more general property

$$\mathcal{P}(i_{J^P}^X, i_{J^{P'}}^Y)_{\mu_X}^{\nu_Y} \equiv \mathcal{P}(j_{J^{P'}}^Y, i_{J^P}^X)_{\mu_X}^{\nu_Y}. \quad (\text{A9})$$

Note that Eq. (A9) would only imply Hermiticity for $q_J \in \mathbb{R}$. It will now be shown that the q_J (and by extension the parity-violating SPOs) are in fact imaginary, so that Eq. (A9) instead implies Hermiticity or skew-Hermiticity depending on P and P' .

5. Hermiticity and orthonormality

When allowing for parity violation, we must extend the orthonormality condition in Eq. (A3c) to

$$\begin{aligned} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)_{\mu_X}^{\nu_Y} \mathcal{P}(k_{J^{P''}}^Y, l_{J^{P''}}^Z)_{\nu_Y}^{\sigma_Z} \\ \equiv \delta_{jk} \delta_{JJ'} \delta_{P'P''} \mathcal{P}(i_{J^P}^X, l_{J^{P''}}^Z)_{\mu_X}^{\sigma_Z}. \end{aligned} \quad (\text{A10})$$

Note that Eq. (A10) is where we expect q_J to become important: its value will not depend on the (arbitrary) way in which the reduced-index SPOs are weighted, but rather it will be determined by the combinatoric properties of the totally antisymmetric tensor in Eq. (A7). First, consider the easy cases $P = P' = P'' \neq P'''$ or $P \neq P' = P'' = P'''$, i.e., the products of parity-preserving and parity-violating SPOs, and vice versa. For these cases, the definitions in Eqs. (A5), (A6), and (A8) automatically imply Eq. (A10) for any values of q_J . The only other case that can arise is the product of two parity-violating SPOs, i.e., $P'' = P' \neq P''' = P$. This case yields (for the only non-vanishing products in which $J' = J$ and $k_{J^{P''}}^Y = j_{J^{P'}}^Y$)

$$\begin{aligned} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)_{\mu_X}^{\nu_Y} \mathcal{P}(j_{J^{P'}}^X, l_{J^P}^Z)_{\nu_Y}^{\sigma_Z} \\ \equiv q_J^2 c(i_{J^P}^X) c(l_{J^P}^Z) \mathcal{P}(i_{J^P}^X)_{\bar{\mu}_X}^{\bar{\nu}_X} \epsilon_{\bar{\sigma}_Y}^{\bar{\nu}_Y} \epsilon_{\bar{\rho}_Y}^{\bar{\nu}_Y} \epsilon_{\bar{\rho}_Y}^{\bar{\nu}_Y} \mathcal{P}(l_{J^P}^Z)_{\bar{\sigma}_Y}^{\bar{\nu}_Y} \\ \forall J < 2. \end{aligned} \quad (\text{A11})$$

We thus see how Eq. (A11) makes it clear why q_J is a J -dependent factor. By comparing Eq. (A11) with Eq. (A10) the criterion for q_J to satisfy Eq. (A10) is determined to be

$$q_J^2 \epsilon_{\bar{\sigma}_Y}^{\bar{\nu}_Y} \epsilon_{\bar{\rho}_Y}^{\bar{\nu}_Y} \epsilon_{\bar{\rho}_Y}^{\bar{\nu}_Y} = \Delta_{\bar{\sigma}_Y}^{\bar{\nu}_Y}. \quad (\text{A12})$$

The values of q_J thus depend on the totally antisymmetric tensor in Eq. (A7). The relevant identities are

$$\epsilon^{\bar{\mu}\bar{\nu}\bar{\sigma}} \epsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}} = -6, \quad \epsilon_{\bar{\mu}}^{\bar{\nu}\bar{\sigma}} \epsilon_{\bar{\nu}\bar{\sigma}}^{\bar{\lambda}} = -2\delta_{\bar{\mu}}^{\bar{\lambda}}, \quad \epsilon_{\bar{\mu}\bar{\nu}}^{\bar{\sigma}} \epsilon_{\bar{\sigma}}^{\bar{\lambda}\bar{\rho}} = -2\delta_{[\bar{\mu}}^{\bar{\lambda}} \delta_{\bar{\nu}}^{\bar{\rho}]}, \quad (\text{A13})$$

²⁰Note that in [39] the use of $\epsilon_{\mu\nu\sigma\lambda}$ in some of the reduced-index SPOs leads instead to $c(i_{J^P}^X) \in \mathbb{C}$.

and these signal a potential problem: Eqs. (A12) and (A13) are not consistent with $q_J \in \mathbb{R}$. If the solutions

$$q_0 \equiv i/\sqrt{6}, \quad q_1 \equiv i/\sqrt{2}, \quad (\text{A14})$$

are allowed, then the symmetry condition in Eq. (A10) is no longer consistent with the more basic Hermiticity condition. This leads to a “catch-22” whereby, if one enforces Hermiticity, then the factor q_J^2 in Eq. (A12) becomes $q_J^* q_J > 0$ so that even imaginary solutions fail to satisfy Eqs. (A12) and (A13). We thus arrive at an interesting conclusion: Hermiticity and orthonormality are mutually exclusive properties of parity-violating spin-projection operators, as pointed out in [32]. It will be argued presently that orthonormality is a more convenient property than Hermiticity. For this paper, therefore, the parity-preserving SPOs will be real (and Hermitian), while the parity-violating SPOs will be imaginary (and skew-Hermitian).²¹ Accordingly, Eq. (A9) may be supplemented by the relation

$$\mathcal{P}(i_{JP}^X, j_{JP'}^Y)^{\mu_X}_{\nu_Y} \equiv PP' \mathcal{P}(j_{JP'}^Y, i_{JP}^X)^{\nu_Y}_{\mu_X}. \quad (\text{A15})$$

When the SPOs are arranged in a block structure, Eqs. (A9) and (A15) lead to matrix representations with the property of *checker*-Hermiticity. Checker-Hermiticity is discussed in detail in Appendix C.

6. Higher-spin cases

For $J \geq 2$, an alternative to the convention in Eq. (A8) must be agreed upon, so that Eqs. (A9), (A10), and (A15) are preserved. The current implementation in PSALTER does not support tensors above the third rank: this means that $J \leq 3$, and since there is no third-rank $J^P = 3^+$ representation to allow for parity violation in the $J = 3$ sector, new conventions are only needed for $J = 2$. Accordingly, these conventions will be

$$\mathcal{P}(i_{2+}^X, j_{2-}^Y)^{\mu_X}_{\nu_Y} \equiv q_2 c(i_{2+}^X) c(j_{2-}^Y) \mathcal{P}(i_{2+}^X)^{\overline{\sigma\pi}}_{\mu_X} \varepsilon_{\overline{\sigma}\overline{\rho\kappa}} \mathcal{P}(j_{2-}^Y)^{\nu_Y}_{\overline{\rho\kappa}}, \quad (\text{A16})$$

where the PSALTER conventions for the reduced-index SPOs are

$$\mathcal{P}(i_{2+}^X)^{[\overline{\sigma\pi}]}_{\mu_X} \equiv \mathcal{P}(j_{2-}^Y)^{\nu_Y}_{(\overline{\rho\kappa})\overline{\pi}} \equiv \varepsilon^{\overline{\rho\kappa}\pi} \mathcal{P}(j_{2-}^Y)^{\nu_Y}_{\overline{\rho\kappa}\pi} \equiv 0. \quad (\text{A17})$$

²¹In fact, (skew-)Hermiticity does not have to be tied to the real or imaginary character of SPOs. In our case, Eq. (A3a) implies that we are selecting a convention in PSALTER where the parity-preserving SPOs are real, and our ansatz in Eq. (A8) then forces the parity-violating SPOs to be imaginary. One can, in principle, construct an alternative to Eq. (A8) which is not linear in q_J but rather *bilinear* in some other parameter. This would lead to *all* SPOs being real and orthonormal [32].

TABLE VIII. The matrix of spin-projection operators associated with the spin-zero sector of zero-by-three CSKR theory. The row-rank ordering of this matrix corresponds exactly to the first diagonal block in Fig. 4. See Eqs. (A1) and (A2) for definitions of quantities. Indices to be contracted with the complex conjugate fields $\zeta_{\mu_X}^*$ (if any) are drawn from the set $\{\xi, \varphi, \zeta\}$ in order; those to be contracted with ζ^{μ_X} are drawn from $\{v, \psi, \omega\}$.

1	$\bar{i}\varepsilon\eta_\alpha^{\nu\psi\omega} n^\alpha$
$\bar{i}\varepsilon\eta_\alpha^{\xi\varphi\zeta} n^\alpha$	$-2\Omega^{\nu\psi}\Omega^{\omega\xi}\Omega^{\varphi\zeta} + \Omega^{\psi\omega}\Omega^{\varphi\xi}\Omega^{\nu\zeta} -$ $2\Omega^{\nu\omega}\Omega^{\xi\zeta}\Theta^{\psi\varphi} + \Omega^{\omega\xi}\Theta^{\nu\xi}\Theta^{\psi\varphi} -$ $2\Omega^{\nu\psi}\Omega^{\xi\varphi}\Theta^{\omega\zeta} + \Omega^{\psi\varphi}\Theta^{\nu\xi}\Theta^{\omega\zeta} -$ $2\Omega^{\nu\xi}\Theta^{\psi\varphi}\Theta^{\omega\zeta} + \Theta^{\nu\xi}\Theta^{\psi\varphi}\Theta^{\omega\zeta}$

TABLE IX. The matrix of spin-projection operators associated with the spin-one sector of zero-by-three CSKR theory. The row-rank ordering of this matrix corresponds exactly to the second diagonal block in Fig. 4. See Eqs. (A1) and (A2) for definitions of quantities. Indices to be contracted with the complex conjugate fields $\zeta_{\mu_X}^*$ (if any) are drawn from the set $\{\xi, \varphi, \zeta\}$ in order; those to be contracted with ζ^{μ_X} are drawn from $\{v, \psi, \omega\}$.

$\Omega^{\nu\psi}\Omega^{\omega\xi}\Omega^{\varphi\zeta} + \Omega^{\nu\omega}\Omega^{\xi\zeta}\Theta^{\psi\varphi} +$ $\Omega^{\nu\psi}\Omega^{\xi\varphi}\Theta^{\omega\zeta} + \Omega^{\nu\xi}\Theta^{\psi\varphi}\Theta^{\omega\zeta}$
--

From Eqs. (A12), (A13), (A16), and (A17) it follows that Eq. (A14) is extended by

$$q_2 \equiv i/\sqrt{2}. \quad (\text{A18})$$

With the final determination in Eq. (A18), all the parity-violating SPOs have been constructed.

APPENDIX B: EXPLICIT OPERATOR FORMULAS

1. Spin-projection operator tables

In this appendix we provide explicit formulas for the SPOs which are created automatically by PSALTER at runtime. The production of these formulas is not part of the PSALTER functionality, as it was pointed out in [39] that PSALTER itself makes the tabulation of SPOs redundant when presenting future spectroscopy results.²² The current paper, however, makes some significant developments in the theory and conventions of the SPOs themselves: explicit formulas may therefore be a useful companion to Appendix A. The SPOs corresponding to the massive two-form and one-by-two CSKR theory are somewhat trivial, and these we omit. We provide in Tables VIII and IX the SPOs corresponding to the analyses of zero-by-three CSKR theory. We also provide in

²²This is because the field kinematics presented in Tables II–VII already provides an implicit statement of all the SPOs used, and their explicit formulas may be recovered from these by an application of Eqs. (A4), (A5), (A8), and (A16).

TABLE X. The matrix of spin-projection operators associated with the spin-zero sector of Poincaré gauge theory. The row-rank ordering of this matrix corresponds exactly to the first diagonal block in Figs. 5 and 6. See Eqs. (A1) and (A2) for definitions of quantities. Indices to be contracted with the complex conjugate fields $\zeta_{\mu_X}^*$ (if any) are drawn from the set $\{\xi, \varphi, \zeta\}$ in order; those to be contracted with ζ^{μ_X} are drawn from $\{v, \psi, \omega\}$.

$\Omega^{\nu\psi}\Omega^{\omega\xi}\Omega^{\varphi\zeta} + \Omega^{\psi\xi}\Omega^{\varphi\zeta}\Theta^{\nu\omega}$ $+ \Omega^{\nu\psi}\Omega^{\omega\varphi}\Theta^{\xi\zeta} + \Omega^{\psi\varphi}\Theta^{\nu\omega}\Theta^{\xi\zeta}$	$\Theta^{\psi\omega}\Theta^{\xi\zeta}n^\varphi + \Omega^{\xi\varphi}\Theta^{\psi\omega}n^\zeta$	$\Omega^{\psi\omega}\Theta^{\xi\zeta}n^\varphi + \Omega^{\psi\omega}\Omega^{\xi\varphi}n^\zeta$	$\bar{i}\epsilon\eta_\beta^{\nu\psi\omega}\Omega^{\xi\beta}\Omega^{\varphi\zeta} + \bar{i}\epsilon\eta_\beta^{\nu\psi\omega}\Omega^{\varphi\beta}\Theta^{\xi\zeta}$
$\Theta^{\nu\omega}\Theta^{\varphi\zeta}n^\psi + \Omega^{\nu\psi}\Theta^{\varphi\zeta}n^\omega$	$\Theta^{\psi\omega}\Theta^{\varphi\zeta}$	$\Omega^{\psi\omega}\Theta^{\varphi\zeta}$	$\bar{i}\epsilon\eta_\alpha^{\nu\psi\omega}\Theta^{\varphi\zeta}n^\alpha$
$\Omega^{\nu\varphi}\Omega^{\omega\varphi}n^\zeta + \Omega^{\psi\varphi}\Theta^{\nu\omega}n^\zeta$	$\Omega^{\varphi\zeta}\Theta^{\psi\omega}$	$\Omega^{\psi\omega}\Omega^{\varphi\zeta}$	$\bar{i}\epsilon\eta_\chi^{\nu\psi\omega}\Omega^{\varphi\chi}n^\zeta$
$\bar{i}\epsilon\eta_\chi^{\xi\varphi\zeta}\Omega^{\nu\chi}\Omega^{\psi\omega} + \bar{i}\epsilon\eta_\chi^{\xi\varphi\zeta}\Omega^{\psi\chi}\Theta^{\nu\omega}$	$\bar{i}\epsilon\eta_\alpha^{\xi\varphi\zeta}\Theta^{\psi\omega}n^\alpha$	$\bar{i}\epsilon\eta_\chi^{\xi\varphi\zeta}\Omega^{\psi\chi}n^\omega$	$\frac{2}{3}\Omega^{\nu\psi}\Omega^{\omega\xi}\Omega^{\varphi\zeta} - \frac{1}{3}\Omega^{\psi\omega}\Omega^{\varphi\zeta}\Theta^{\nu\xi} + \frac{2}{3}\Omega^{\psi\omega}\Omega^{\xi\varphi}\Theta^{\nu\zeta}$ $- \frac{2}{3}\Omega^{\omega\zeta}\Theta^{\nu\varphi}\Theta^{\psi\xi} + \frac{1}{3}\Omega^{\omega\zeta}\Theta^{\nu\xi}\Theta^{\psi\varphi}$ $+ \frac{2}{3}\Omega^{\nu\psi}\Omega^{\varphi\zeta}\Theta^{\omega\xi} + \frac{2}{3}\Omega^{\psi\varphi}\Theta^{\nu\zeta}\Theta^{\omega\xi} - \frac{1}{3}\Omega^{\psi\varphi}\Theta^{\nu\xi}\Theta^{\omega\zeta}$ $- \frac{2}{3}\Theta^{\nu\varphi}\Theta^{\psi\xi}\Theta^{\omega\zeta} + \frac{1}{3}\Theta^{\nu\xi}\Theta^{\psi\varphi}\Theta^{\omega\zeta}$

Table X–XII the SPOs corresponding to the analyses of EC theory (i.e., Poincaré gauge theory).

APPENDIX C: CHECKER-HERMITICITY

1. A new kind of structured matrix

In this appendix we introduce the concept of *checker-Hermiticity* as a generalization of Hermiticity which becomes physically relevant in the analysis of parity-violating particle spectra. A checker-Hermitian matrix admits a 2×2 block structure in which the diagonal blocks are Hermitian, but the off-diagonal blocks are skew-Hermitian; thus, it has the structure of a checkerboard. The spectral theorem applies only to the Hermitian or skew-Hermitian limits of a checker-Hermitian matrix. Nonetheless, we observe that checker-Hermiticity leads to some remarkably convenient properties.

Definition C.1. We define the *checker-Hermitian conjugate* $M^\ddagger$ of an arbitrary complex square matrix M as resulting from the following operation on its 2×2 block structure

$$M \equiv \begin{bmatrix} M_+ & M_\pm \\ M_\mp & M_- \end{bmatrix} \Rightarrow M^\ddagger \equiv \begin{bmatrix} M_+^\dagger & -M_\mp^\dagger \\ -M_\pm^\dagger & M_-^\dagger \end{bmatrix}, \quad (C1)$$

where M_+ and M_- are square complex matrices, and $M_\pm$ and $M_\mp$ are (possibly rectangular) complex matrices which are conformable for operations such as $M_\pm M_\mp$ and $M_\mp M_+$ etc. The dimensions of these matrices are determined by the context of the problem.

Remark. The notation in Eq. (C1) is chosen to reflect the parity indices of different kinds of SPOs. Thus (+) and (−) denote parity-preserving sectors, and (±) and (∓) denote parity-violating sectors. Within this paper, parity is the context which will provide the dimensions of the 2×2 division in checker-Hermitian conjugation.

Definition C.2. We say C is *checker-Hermitian* when $C \equiv C^\ddagger$.

Corollary C.3. Let C be checker-Hermitian, then it has the 2×2 block structure

$$C \equiv C^\ddagger \Rightarrow C \equiv \begin{bmatrix} C_+ & C_\pm \\ -C_\pm^\dagger & C_- \end{bmatrix}, \quad C_+ \equiv C_+^\dagger, \quad C_- \equiv C_-^\dagger. \quad (C2)$$

Proof. This follows immediately from Eq. (C1). ■

Corollary C.4. Let C be checker-Hermitian and invertible, then C^{-1} is also checker-Hermitian.

Proof. The proof follows immediately from the well-known formula for the inverse of a block matrix. Alternatively, we can infer the result as follows. Let $C \equiv C_{dh} + C_{os}$ be decomposed into the diagonal (d) and off-diagonal (o) block parts which must be respectively Hermitian (h) and skew-Hermitian (s) by the property $C \equiv C^\ddagger$. Without assuming checker-Hermiticity of the inverse, we have $C^{-1} \equiv C_{dh}^{-1} + C_{oh}^{-1} + C_{ds}^{-1} + C_{os}^{-1}$. By letting C^{-1} act as a left inverse and taking the Hermitian conjugate we have

$$\begin{aligned} C_{dh}^{-1} \cdot C_{dh} + C_{oh}^{-1} \cdot C_{os} + C_{ds}^{-1} \cdot C_{dh} + C_{os}^{-1} \cdot C_{os} \\ \equiv C_{dh} \cdot C_{dh}^{-1} + C_{os} \cdot C_{os}^{-1} - C_{dh} \cdot C_{ds}^{-1} - C_{os} \cdot C_{oh}^{-1} \equiv 1, \end{aligned} \quad (C3a)$$

$$\begin{aligned} C_{oh}^{-1} \cdot C_{dh} + C_{dh}^{-1} \cdot C_{os} + C_{os}^{-1} \cdot C_{dh} + C_{ds}^{-1} \cdot C_{os} \\ \equiv C_{dh} \cdot C_{os}^{-1} + C_{os} \cdot C_{dh}^{-1} - C_{dh} \cdot C_{oh}^{-1} - C_{os} \cdot C_{ds}^{-1} \equiv 0, \end{aligned} \quad (C3b)$$

and from Eqs. (C3a) and (C3b) we deduce that $C_{dh}^{-1} + C_{os}^{-1} - C_{oh}^{-1} - C_{ds}^{-1}$ is the right inverse of C . The uniqueness of the inverse for square C implies that $C_{oh}^{-1} \equiv C_{ds}^{-1} \equiv 0$ so that $C^{-1} \equiv (C^{-1})^\ddagger$ as required. ■

Theorem C.5. Let C be checker-Hermitian and singular, with an orthonormal set of complex right null eigenvectors $\{v_i\}$, then an orthonormal set of left null eigenvectors $\{u_i\}$

TABLE XII. The matrix of spin-projection operators associated with the spin-two sector of Poincaré gauge theory. The row-rank ordering of this matrix corresponds exactly to the third diagonal block in Figs. 5 and 6. See Eqs. (A1) and (A2) for definitions of quantities. Indices to be contracted with the complex conjugate fields $\zeta_{\mu_X}^*$ (if any) are drawn from the set $\{\xi, \varphi, \zeta\}$ in order; those to be contracted with ζ^{μ_X} are drawn from $\{v, \psi, \omega\}$.

$\frac{1}{6}\Omega^{\nu\psi}\Omega^{\omega\xi}\Omega^{\varphi\zeta} - \frac{1}{3}\Omega^{\psi\xi}\Omega^{\varphi\zeta}\Theta^{\nu\omega}$ $+ \frac{1}{2}\Omega^{\psi\omega}\Omega^{\varphi\zeta}\Theta^{\nu\xi} + \frac{1}{2}\Omega^{\psi\omega}\Omega^{\xi\varphi}\Theta^{\nu\zeta}$ $+ \frac{1}{2}\Omega^{\nu\psi}\Omega^{\varphi\zeta}\Theta^{\omega\xi} + \frac{1}{2}\Omega^{\psi\varphi}\Theta^{\nu\zeta}\Theta^{\omega\xi}$ $+ \frac{1}{2}\Omega^{\psi\varphi}\Theta^{\nu\xi}\Theta^{\omega\zeta}$ $- \frac{1}{3}\Omega^{\nu\psi}\Omega^{\omega\varphi}\Theta^{\xi\zeta} - \frac{1}{3}\Omega^{\psi\varphi}\Theta^{\nu\omega}\Theta^{\xi\zeta}$ $- \frac{1}{3}\Theta^{\nu\omega}\Theta^{\varphi\zeta}\Theta^{\psi\xi} - \frac{1}{2}\Theta^{\nu\zeta}\Omega^{\psi\varphi}\Theta^{\omega\xi}$ $- \frac{1}{2}\Theta^{\nu\varphi}\Theta^{\psi\zeta}\Theta^{\omega\xi} - \frac{1}{3}\Omega^{\nu\psi}\Theta^{\varphi\zeta}\Theta^{\omega\xi}$ $- \frac{1}{2}\Omega^{\psi\omega}\Theta^{\nu\zeta}\Theta^{\omega\xi} - \frac{1}{2}\Omega^{\psi\omega}\Theta^{\nu\varphi}\Theta^{\omega\xi}$	$-\frac{1}{3}\Theta^{\psi\omega}\Theta^{\xi\zeta}\Theta^{\nu\varphi} - \frac{1}{3}\Omega^{\xi\varphi}\Theta^{\psi\omega}\Theta^{\nu\zeta}$ $-\frac{1}{2}\Omega^{\omega\varphi}\Theta^{\psi\xi}\Theta^{\nu\zeta} - \frac{1}{2}\Omega^{\psi\varphi}\Theta^{\omega\xi}\Theta^{\nu\zeta}$ $-\frac{1}{2}\Theta^{\psi\varphi}\Theta^{\omega\xi}\Theta^{\nu\zeta} - \frac{1}{2}\Theta^{\psi\xi}\Theta^{\omega\varphi}\Theta^{\nu\zeta}$	$\frac{1}{4}i\eta_{\beta}^{\psi\omega\xi}\Omega^{\nu\beta}\Omega^{\varphi\zeta} + \frac{1}{4}i\eta_{\beta}^{\nu\omega\xi}\Omega^{\psi\beta}\Omega^{\varphi\zeta}$ $+ \frac{1}{4}i\eta_{\beta}^{\psi\omega\zeta}\Omega^{\varphi\beta}\Theta^{\nu\xi} + \frac{1}{4}i\eta_{\beta}^{\psi\omega\xi}\Omega^{\varphi\beta}\Theta^{\nu\zeta}$ $+ \frac{1}{4}i\eta_{\beta}^{\nu\omega\zeta}\Omega^{\varphi\beta}\Theta^{\psi\xi} + \frac{1}{4}i\eta_{\beta}^{\nu\omega\xi}\Omega^{\varphi\beta}\Theta^{\psi\zeta}$
$\frac{1}{4}i\eta_{\beta}^{\nu\varphi\zeta}\Omega^{\psi\beta}\Omega^{\omega\xi} + \frac{1}{4}i\eta_{\beta}^{\nu\zeta\zeta}\Omega^{\psi\beta}\Omega^{\omega\varphi}$ $+ \frac{1}{4}i\eta_{\beta}^{\omega\varphi\zeta}\Omega^{\psi\beta}\Theta^{\nu\xi} + \frac{1}{4}i\eta_{\beta}^{\omega\xi\zeta}\Omega^{\psi\beta}\Theta^{\nu\varphi}$ $+ \frac{1}{4}i\eta_{\beta}^{\nu\varphi\zeta}\Omega^{\psi\beta}\Theta^{\omega\xi} + \frac{1}{4}i\eta_{\beta}^{\nu\xi\zeta}\Omega^{\psi\beta}\Theta^{\omega\varphi}$	$\frac{1}{4}i\in\eta_{\alpha}^{\omega\varphi\zeta}\Theta^{\psi\xi}\Theta^{\nu\alpha} + \frac{1}{4}i\in\eta_{\alpha}^{\omega\xi\zeta}\Theta^{\psi\varphi}\Theta^{\nu\alpha}$ $+ \frac{1}{4}i\in\eta_{\alpha}^{\psi\varphi\zeta}\Theta^{\omega\xi}\Theta^{\nu\alpha} + \frac{1}{4}i\in\eta_{\alpha}^{\psi\xi\zeta}\Theta^{\omega\varphi}\Theta^{\nu\alpha}$	$\frac{3}{16}\Omega^{\nu\psi}\Omega^{\omega\xi}\Omega^{\varphi\zeta} + \frac{9}{8}\Omega^{\omega\xi}\Omega^{\varphi\zeta}\Theta^{\nu\psi} + \frac{9}{16}\Omega^{\psi\xi}\Omega^{\varphi\zeta}\Theta^{\nu\omega}$ $- \frac{3}{8}\Omega^{\psi\omega}\Omega^{\varphi\zeta}\Theta^{\nu\xi} - \frac{3}{8}\Omega^{\psi\omega}\Omega^{\xi\varphi}\Theta^{\nu\zeta}$ $+ \frac{1}{8}\Omega^{\omega\zeta}\Theta^{\nu\psi}\Theta^{\psi\xi} + \frac{3}{8}\Omega^{\omega\zeta}\Theta^{\nu\xi}\Theta^{\psi\varphi}$ $- \frac{1}{8}\Omega^{\nu\psi}\Omega^{\varphi\zeta}\Theta^{\omega\xi} - \frac{3}{8}\Omega^{\psi\varphi}\Theta^{\nu\zeta}\Theta^{\omega\xi}$ $+ \frac{9}{16}\Omega^{\xi\zeta}\Theta^{\nu\psi}\Theta^{\omega\varphi} + \frac{9}{16}\Omega^{\xi\varphi}\Theta^{\nu\psi}\Theta^{\omega\zeta}$ $- \frac{3}{8}\Omega^{\psi\varphi}\Theta^{\nu\xi}\Theta^{\omega\zeta} + \frac{3}{8}\Theta^{\nu\varphi}\Theta^{\psi\xi}\Theta^{\omega\zeta}$ $+ \frac{3}{8}\Theta^{\nu\xi}\Theta^{\psi\varphi}\Theta^{\omega\zeta}$ $+ \frac{9}{16}\Omega^{\nu\psi}\Omega^{\omega\varphi}\Theta^{\xi\zeta} + \frac{9}{16}\Omega^{\omega\varphi}\Theta^{\nu\psi}\Theta^{\xi\zeta}$ $+ \frac{9}{16}\Omega^{\psi\varphi}\Theta^{\nu\omega}\Theta^{\xi\zeta} + \frac{9}{16}\Theta^{\nu\psi}\Theta^{\omega\varphi}\Theta^{\xi\zeta}$

Corollary D.4. Let $\mathbf{H}$ be Hermitian and singular, then $\mathbf{H}^+$ is also Hermitian.

Proof. This follows immediately from Eq. (D3). ■

Theorem D.5. Let $\mathbf{C}$ be checker-Hermitian and singular, with orthonormal sets of complex right and left null eigenvectors $\{\mathbf{v}_i\}$ and $\{\mathbf{u}_i\}$ respectively, then the Moore-Penrose pseudoinverse $\mathbf{C}^+$ is given by

$$\mathbf{C}^+ \equiv \left(1 - \sum_i \mathbf{v}_i \cdot \mathbf{v}_i^\dagger\right) \cdot \left(\mathbf{C} + \sum_j \mathbf{u}_j \cdot \mathbf{v}_j^\dagger\right)^{-1} \cdot \left(1 - \sum_k \mathbf{u}_k \cdot \mathbf{u}_k^\dagger\right). \quad (\text{D4})$$

Proof. This formula is verified by substituting Eq. (D4) into Eq. (D1). Alternatively, it may be deduced directly from Eq. (D2) by noting that $\mathbf{C}^\dagger \cdot \mathbf{C}$ and $\mathbf{C} \cdot \mathbf{C}^\dagger$ are Hermitian and singular, and hence pseudoinvertible via the formula in Eq. (D3). ■

Remark. Equations (C4) and (D4) are useful because they allow $\mathbf{C}^+$ to be computed from the (right) null eigenvectors of $\mathbf{C}$. Note that Eq. (D4) is consistent with (and is a minimal modification of) the formula in Eq. (D3) for the Moore-Penrose pseudoinverse of a Hermitian matrix.

Corollary D.6. Let $\mathbf{C}$ be checker-Hermitian and singular, then the Moore-Penrose pseudoinverse $\mathbf{C}^+$ is also checker-Hermitian.

Proof. This follows immediately from Eq. (D4). ■

APPENDIX E: OPERATOR COEFFICIENT MATRICES

1. Physicality equals checker-Hermiticity

In this appendix we show that the operator coefficient matrices used in Sec. II and Appendix G have a checker-Hermitian structure. Despite this fact, we particularly emphasise that *the operators themselves are always Hermitian* [32]—this is in line with the basic requirements of physicality. The apparent discrepancy arises only because of our discussion in Appendix A, in which a checker-Hermitian but orthonormal basis of SPOs was selected. To begin, notice how Eq. (11) implies that the block-constituents $\mathbf{O}_J$ of the wave operator coefficient matrix $\mathbf{O}$ defined in Eq. (12) are structured matrices. To see this, we substitute Eq. (11) into Eq. (4) so that the quadratic theory (with sources suppressed) becomes

$$S = \frac{1}{(2\pi)^4} \int d^4k \sum_X \left[\zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathbf{O}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} \right]. \quad (\text{E1})$$

Since S must be real, it is possible to take the complex conjugate of Eq. (E1) and use Eqs. (10a) and (10b) to show

$$S = \frac{1}{(2\pi)^4} \int d^4k \sum_X \left[\zeta^{\mu_X} \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(j_{J^{P'}}^Y, i_{J^P}^X)^{\nu_Y}_{\mu_X} \zeta^{\nu_Y} \right]. \quad (\text{E2})$$

By comparing Eqs. (E1) and (E2) it follows that

$$[\mathcal{O}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} = [\mathcal{O}_J^\dagger]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y}, \quad (\text{E3})$$

where we use the notation defined in Eq. (C1). This means that $\mathcal{O}_J$ and $\mathcal{O}_J^\dagger$ serve equally well as the coefficient matrix representation of the wave operator. Moving forward, therefore, it is always safe to assume

$$\mathcal{O}_J = \mathcal{O}_J^\dagger \quad \forall J \Leftrightarrow \mathcal{O} = \mathcal{O}^\dagger, \quad (\text{E4})$$

where the implication in Eq. (E4) follows from the block structure in Eq. (12). This result follows immediately from the completely general claim that the wave operator itself must be Hermitian

$$\mathcal{O}^{\mu_X}_{\nu_Y}{}^* \equiv \mathcal{O}_{\nu_Y}{}^{\mu_X}, \quad (\text{E5})$$

since, if Eq. (E5) is taken to be a convincing starting point, then one need only substitute Eq. (11) to arrive at Eq. (E4).

In summary, only the checker-Hermitian part of the coefficient matrix $\mathcal{O}_J$ contributes to the physics. Equivalently, even if $\mathcal{O}_J$ is not explicitly checker-Hermitian at its point of construction, it suffices to work only with its checker-Hermitian part.²³

2. Propagator coefficient matrix

By this point, our conclusion in Eq. (E4) and the mathematical results of Appendixes C and D allow us to derive an explicit and highly compact formula the propagator coefficient matrix. Let a_J be labels for the null vectors of the checker-Hermitian wave operator coefficient matrix block $\mathcal{O}_J$, so that $\{\mathbf{v}_{a_J}\}$ and $\{\mathbf{u}_{a_J}\}$ are orthonormal sets of right and left null vectors respectively [note that the latter can be deduced from the former via Eq. (C4)]. For almost all models of physical relevance, it is the case that these vectors are purely functions of k , being independent of the Lagrangian coupling coefficients.²⁴ From Eq. (D4) it then follows that

$$\begin{aligned} \mathcal{O}_J^+ &\equiv \mathbf{V}_J \cdot \boldsymbol{\Omega}_J^{-1} \cdot \mathbf{U}_J, \\ \mathbf{V}_J &\equiv 1 - \sum_{a_J} \mathbf{v}_{a_J} \cdot \mathbf{v}_{a_J}^\dagger, \quad \boldsymbol{\Omega}_J \equiv \mathcal{O}_J + \sum_{a_J} \mathbf{u}_{a_J} \cdot \mathbf{v}_{a_J}^\dagger, \quad \mathbf{U}_J \equiv 1 - \sum_{a_J} \mathbf{u}_{a_J} \cdot \mathbf{u}_{a_J}^\dagger, \end{aligned} \quad (\text{E6})$$

where $\boldsymbol{\Omega}_J$ is also checker-Hermitian.

APPENDIX F: GAUGE-FIXING

1. Generating functional

In this appendix, we reconcile the pseudoinverse implementation in Appendix E with the more standard process of gauge fixing. Starting again from Eq. (1), the generating functional for the source j^{μ_X} is

$$\begin{aligned} Z[j] &= \int \mathcal{D}\zeta \exp \left(i \int d^4x \sum_X \zeta_{\mu_X} \left[\sum_Y \mathcal{O}^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} - j^{\mu_X} \right] \right) \\ &\simeq \int \mathcal{D}\zeta \exp \left(\frac{i}{(2\pi)^4} \int d^4k \sum_X \left[\zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} - \frac{1}{2} \zeta_{\mu_X}^* j^{\mu_X} - \frac{1}{2} j^{*\mu_X} \zeta^{\mu_X} \right] \right). \end{aligned} \quad (\text{F1})$$

As a quantum theory, Eq. (F1) is evidently Gaussian and may thus be solved trivially by integrating out the fields ζ^{ν_Y} . This may be accomplished via the (generally nonlocal) field redefinition

$$\zeta^{\mu_X} \mapsto \zeta^{\mu_X} + \frac{1}{2} \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}^+_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y}, \quad (\text{F2})$$

²³And indeed, the this operation is always equivalent to modifying the theory by the addition of boundary terms.

²⁴Note that this is not always the case: there are instances of higher-spin Fronsda-type models which have parametric gauge symmetries, in which the null vectors are smoothly parametrized by the Lagrangian coupling coefficients.

so that, when Eq. (F2) is substituted into Eqs. (F1) and (E6) is applied, the result is

$$\begin{aligned}
 Z[j] \simeq & \int \mathcal{D}\zeta \exp \left(\frac{i}{(2\pi)^4} \int d^4k \sum_X \left[\zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} \right. \right. \\
 & - \frac{1}{2} \zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} \sum_{a_J} [\mathbf{u}_{a_J} \cdot \mathbf{u}_{a_J}^\dagger]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} \\
 & - \frac{1}{2} j_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} \sum_{a_J} [\mathbf{v}_{a_J} \cdot \mathbf{v}_{a_J}^\dagger]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} \\
 & \left. \left. - \frac{1}{4} j_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}^+_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} \right] \right). \quad (\text{F3})
 \end{aligned}$$

The four sets of terms inside the exponent of Eq. (F3) are decoupled from each other: The first term is sensitive to all *physical* variations of the fields. The second and third cross-terms are sensitive only to the variations of the fields which lie in the kernel of the wave operator, i.e., to pure *gauge* variations and the *constrained* parts of the sources; and the fourth term is sensitive only to the remaining *physical* or *unconstrained* parts of the sources. When Eq. (F3) is used to compute observables, the first term factors out entirely. The second and third cross-terms do not factor, but the fields there serve as Lagrange multipliers to constrain the source currents by enforcing the conservation laws which are conjugate to the gauge symmetries of the

theory. The fourth term depends exclusively on those parts of the source currents which are not constrained, and it is this term which defines the saturated propagator in Eq. (13).

2. Gauge-fixing

We have just seen how the pseudoinversion of the wave operator leads to the saturated propagator, and to the suppression of the constrained portions of the source currents on the shell. Working again from Eq. (F1), it is possible to implement a gauge-fixing procedure instead by writing

$$\begin{aligned}
 Z[j] \simeq & \int \mathcal{D}\zeta \exp \left(\frac{i}{(2\pi)^4} \int d^4k \sum_X \left[\zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} \right. \right. \\
 & + \zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} \sum_{a_J} [h_{a_J} \mathbf{u}_{a_J} \cdot \mathbf{v}_{a_J}^\dagger]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} \\
 & - \frac{1}{2} \zeta_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{U}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} \\
 & \left. \left. - \frac{1}{2} j_{\mu_X}^* \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{V}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} \zeta^{\nu_Y} \right] \right), \quad (\text{F4})
 \end{aligned}$$

where the h_{a_J} are arbitrary, real, nonzero coefficients which are chosen to implement the gauge fixing condition. The implicit gauge choice associated with the Moore-Penrose pseudoinverse in Eq. (E6) is that all these gauge-fixing parameters are set to unity. A shift analogous to that given in Eq. (F2) can then be used to extract from Eq. (F4) the saturated propagator, as given in the final term in the exponent of Eq. (F3).

APPENDIX G: NO-GHOST CRITERION

1. Restoring Hermiticity

In this appendix we obtain our central result in Eq. (16), which is the most delicate change to the algorithm induced by parity violation. From the definition of the saturated propagator Eq. (13)—which in turn is derived from Eq. (F3)—the no-ghost criterion can be expressed in terms of the coefficient matrix as

$$\text{Res}_{k^2 \mapsto M_{s_J}^2} \left(\sum_{X,Y} \sum_{P,P'} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} [\mathcal{O}^+]_{i_{J^P}^X, j_{J^{P'}}^Y} j_{\mu_X}^* \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} \right) \geq 0 \quad \forall J, s_J. \quad (\text{G1})$$

The source currents j^{μ_X} are arbitrary, and the SPOs remain finite in the rest frame of the massive s_J -particle. We can therefore write Eq. (G1) in a more compact form as

$$\text{tr}(\mathcal{O}_{s_J}^+ \cdot \mathbf{J}_{s_J}) \geq 0 \quad \forall J, \quad \forall s_J, \quad \mathcal{O}_{s_J}^+ \equiv \text{Res}_{k^2 \mapsto M_{s_J}^2} (\mathcal{O}_J^+), \quad \mathbf{J}_{s_J} \equiv \text{Lim}_{k^2 \mapsto M_{s_J}^2} (\mathbf{J}_J), \quad (\text{G2})$$

where the source matrix is defined as $[\mathbf{J}_J]_{i_{J^P}^X, j_{J^{P'}}^Y} \equiv j_{\mu_X}^* \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y}$. In the 2×2 block form provided by the parity indices (see Appendix C), the relevant checker-Hermitian matrices are notated firstly in Eq. (13)—where $\mathcal{O}_{J^\mp}^+ \equiv -(\mathcal{O}_{J^\mp}^+)^{\dagger}$ by the condition $\mathcal{O}_J^+ \equiv (\mathcal{O}_J^+)^{\ddagger}$ —and second as

$$\mathbf{J}_J \equiv \begin{bmatrix} j_{\mu_X}^* \mathcal{P}(i_{J^+}^X, j_{J^+}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} & j_{\mu_X}^* \mathcal{P}(i_{J^+}^X, j_{J^-}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} \\ j_{\mu_X}^* \mathcal{P}(i_{J^-}^X, j_{J^+}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} & j_{\mu_X}^* \mathcal{P}(i_{J^-}^X, j_{J^-}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y} \end{bmatrix}. \quad (\text{G3})$$

At this point it is convenient to trade the orthonormality of the SPOs in exchange for the Hermiticity of the propagator coefficient matrix by defining²⁵

$$\bar{\mathcal{O}}_J^+ \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \mathcal{O}_J^+, \quad \bar{\mathbf{J}}_J \equiv \mathbf{J}_J \cdot 100 - 1, \quad (\text{G4})$$

so that $\bar{\mathcal{O}}_J^+ \equiv (\bar{\mathcal{O}}_J^+)^{\dagger}$ and without loss of generality the no-ghost condition in Eq. (G2) becomes

$$\text{tr}(\bar{\mathcal{O}}_{s_J}^+ \cdot \bar{\mathbf{J}}_{s_J}) \geq 0 \quad \forall J, \quad \forall s_J, \quad \bar{\mathcal{O}}_{s_J}^+ \equiv \text{Res}_{k^2 \mapsto M_{s_J}^2} (\bar{\mathcal{O}}_J^+), \quad \bar{\mathbf{J}}_{s_J} \equiv \text{Lim}_{k^2 \mapsto M_{s_J}^2} (\bar{\mathbf{J}}_J). \quad (\text{G5})$$

If $\bar{\mathcal{O}}_J^+$ has a simple pole as $k^2 \mapsto M_{s_J}^2$ then by Hermiticity $\bar{\mathcal{O}}_{s_J}^+$ must have one real nonzero eigenvalue λ_{s_J} which depends exclusively on the Lagrangian coupling coefficients.²⁶ The corresponding (normalized) eigenvector can be thought of as the direct sum of vectors²⁷ belonging to parity-even and parity-odd subspaces, so that

$$\begin{aligned} \bar{\mathcal{O}}_{s_J}^+ &\equiv \lambda_{s_J} \mathbf{v}_{s_J} \cdot \mathbf{v}_{s_J}^{\dagger} \equiv \lambda_{s_J} \begin{bmatrix} \mathbf{v}_{s_{J^+}} \cdot \mathbf{v}_{s_{J^+}}^{\dagger} & \mathbf{v}_{s_{J^+}} \cdot \mathbf{v}_{s_{J^-}}^{\dagger} \\ \mathbf{v}_{s_{J^-}} \cdot \mathbf{v}_{s_{J^+}}^{\dagger} & \mathbf{v}_{s_{J^-}} \cdot \mathbf{v}_{s_{J^-}}^{\dagger} \end{bmatrix}, \\ \mathbf{v}_{s_J} &\equiv \begin{bmatrix} \mathbf{v}_{s_{J^+}} \\ \mathbf{v}_{s_{J^-}} \end{bmatrix}, \quad \mathbf{v}_{s_J}^{\dagger} \cdot \mathbf{v}_{s_J} \equiv 1. \end{aligned} \quad (\text{G6})$$

Equation (G6) indicates that there will be two kinds of scenarios. If λ_{s_J} is nonzero only within one diagonal block, then the massive s_J -particle may be associated with the corresponding parity of that block. Otherwise, parity is not a quantum number of the s_J -particle state.

2. Parity-indefinite particles

We first consider the general case where $\mathbf{v}_{s_{J^+}}$ and $\mathbf{v}_{s_{J^-}}$ are simultaneously nonvanishing. The no-ghost criterion in Eq. (G5) takes the component form

$$\lambda_{s_J} \sum_{X,Y} \sum_{i_{J^P}^X, j_{J^{P'}}^Y} P' [\mathbf{v}_{s_J}^{\dagger}]_{i_{J^P}^X} \text{Lim}_{k^2 \mapsto M_{s_J}^2} (j_{\mu_X}^* \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu_X}_{\nu_Y} j^{\nu_Y}) [\mathbf{v}_{s_J}]_{j_{J^{P'}}^Y} \geq 0 \quad \forall J, \quad \forall s_J. \quad (\text{G7})$$

²⁵Note that this merely constitutes a change of basis for the coefficient matrix: as emphasized in Appendix E the underlying operators are always Hermitian.

²⁶See also similar arguments in [112].

²⁷We do not yet assume that $\mathbf{v}_{s_{J^+}}$ or $\mathbf{v}_{s_{J^-}}$ are individually eigenvectors.

Recalling once again that $j^{\mu x}$ is arbitrary, we can equivalently parametrize it by arbitrary $\tilde{j}^{\mu x}$ such that

$$j^{\mu x} \equiv \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X j_{J^{P'}}^Y} \mathcal{P}(i_{J^P}^X, j_{J^{P'}}^Y)^{\mu x}{}_{\nu_Y} \tilde{j}^{\nu_Y} [\mathbf{V}_{s_J}(k_{J^{P'}}^Z)]_{i_{J^P}^X j_{J^{P'}}^Y}^{\nu_Y}, \quad (\text{G8a})$$

$$j_{\mu x}^* \equiv \sum_Y \sum_{J,P,P'} \sum_{i_{J^P}^X j_{J^{P'}}^Y} [\mathbf{V}_{s_J}^\dagger(k_{J^{P'}}^Z)]_{i_{J^P}^X j_{J^{P'}}^Y} \tilde{j}_{\nu_Y}^* \mathcal{P}(i_{J^P}^Y, j_{J^{P'}}^Y)^{\nu_Y}{}_{\mu x}, \quad (\text{G8b})$$

where $\mathbf{V}_{s_J}^\dagger(k_{J^{P'}}^Z)$ is any complete (i.e., full-rank) row-matrix of orthonormal basis vectors, of which we choose the vector at position label $k_{J^{P'}}^Z$ to be $\mathbf{v}_{s_J}^\dagger$. Note that Eq. (G8b) follows from Eq. (G8a) due to the checker-Hermitian property in (10b). When Eqs. (G8a) and (G8b) are substituted into Eq. (G7) we obtain

$$\lambda_{s_J} \text{Lim}_{k^2 \rightarrow M_{s_J}^2} (P'' \tilde{j}_{\mu Z}^* \mathcal{P}(k_{J^{P''}}^Z, k_{J^{P'}}^Z)^{\mu Z}{}_{\nu Z} \tilde{j}^{\nu Z}) \geq 0 \quad \forall J, \quad \forall s_J. \quad (\text{G9})$$

Due to the positivity property of the SPOs in Eq. (A3d), it follows that Eq. (G9) implies the simple result $\lambda_{s_J} > 0$ for all the states s_J across all J . The most economical way to determine the eigenvalue is by taking the invariant trace of the residue matrix, so that Eq. (G9) reduces simply to Eq. (16). As already stated, the mixed-parity scenario may be easily detected by simply inspecting the block-structure of the residue matrix $\mathbf{O}_{s_J}^+$. Once this is done, the formula in Eq. (16) may be implemented without needing to compute the eigenvector $\mathbf{v}_{s_J}$ or performing any other operations.

3. Parity-definite particles

The only other contingency that can arise is one where $\mathbf{v}_{s_{J^P}}$ is nonvanishing, but $\mathbf{v}_{s_{J^{P'}}}$ is vanishing for $P' \neq P$. In this case, the considerations that led to Eq. (16) still hold. As before, P can be determined easily by inspection of the block structure, at which point is it more sensible to denote the various masses using the label s_{J^P} rather than s_J . The no-ghost criterion in Eq. (16) then becomes

$$\text{Res}_{k^2 \rightarrow M_{s_{J^P}}^2} (\text{Ptr} \mathbf{O}_{s_{J^P}}^+) > 0 \quad \forall J, P, \quad \forall s_{J^P}. \quad (\text{G10})$$

Of course, Eq. (G10) is also the no-ghost criterion in cases where $\mathbf{O}_J^+$ is already block-diagonal before its pole residues are computed, such as happens without any parity violation: it was obtained already in [39,112].

APPENDIX H: ANALYTIC CALIBRATION

1. Parity-violating massive two-form

The action for a two-form with a parity-odd mass term is given in Eq. (21). This can be brought into the following first-order form

$$S = \int d^4x \left[-\epsilon^{\mu\nu\rho\sigma} \partial_\rho \mathcal{B}_{\mu\nu} X_\sigma + \frac{3}{2\alpha} X_\mu X^\mu + \gamma \epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\mu\nu} \mathcal{B}_{\rho\sigma} \right], \quad (\text{H1})$$

where X_μ is an auxiliary four-vector field; its equation of motion is

$$X_\sigma = -\frac{\alpha}{3} \epsilon_{\mu\nu\rho\sigma} \partial^\rho \mathcal{B}^{\mu\nu}, \quad (\text{H2})$$

and when Eq. (H2) is plugged into Eq. (H1), we find Eq. (21). On the other hand, the equations of motion for the two-form give

$$\mathcal{B}_{\mu\nu} = -\frac{1}{\gamma} F_{\mu\nu}, \quad (\text{H3})$$

with $F_{\mu\nu} \equiv \partial_\mu X_\nu - \partial_\nu X_\mu$. Therefore, Eq. (H2) yields

$$X_\mu = 0, \quad (\text{H4})$$

so there are no propagating d.o.f. Equivalently, taking Eq. (H1) on-shell gives

$$S = \int d^4x \left[-\frac{2}{\gamma} F^{\mu\nu} \tilde{F}_{\mu\nu} + \frac{3}{2\alpha} X_\mu X^\mu \right], \quad (\text{H5})$$

with $\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$. Notice that the vector appears without a kinetic term—remember, $F^{\mu\nu} \tilde{F}_{\mu\nu}$ is a total derivative—and so its equations of motion are as in Eq. (H4).

2. Parity-indefinite massive two-form

There is no difficulty in working out analytically the dynamics of a massive two-form with both parity-even and parity-odd mass terms. In terms of X_μ , Eq. (22) becomes

$$S = \int d^4x \left[-\epsilon^{\mu\nu\rho\sigma} \partial_\rho \mathcal{B}_{\mu\nu} X_\sigma + \frac{3}{2\alpha} X_\mu X^\mu + \beta \mathcal{B}_{\mu\nu} \mathcal{B}^{\mu\nu} + \gamma \epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\mu\nu} \mathcal{B}_{\rho\sigma} \right]. \quad (\text{H6})$$

The equations of motion read

$$\beta \mathcal{B}_{\mu\nu} + \gamma \epsilon_{\mu\nu\rho\sigma} \mathcal{B}^{\rho\sigma} + 2\tilde{F}_{\mu\nu} = 0, \quad (\text{H7})$$

from which we find

$$\mathcal{B}_{\mu\nu} = -\frac{1}{\beta^2 + 4\gamma^2} \left(\gamma F_{\mu\nu} + \frac{\beta}{2} \tilde{F}_{\mu\nu} \right), \quad (\text{H8})$$

and Eq. (H6) becomes (after dropping full derivatives)

$$S = \int d^4x \left[\frac{\beta}{4(\beta^2 + 4\gamma^2)} F_{\mu\nu} F^{\mu\nu} + \frac{3}{2a} X_\mu X^\mu \right]. \quad (\text{H9})$$

In full accordance with the PSALTer result, we see that the theory propagates a healthy massive spin-one field with square mass $-3(\beta^2 + 4\gamma^2)/\alpha\beta$, provided that $\alpha > 0$ and $\beta < 0$.

3. One-by-two CSKR theory

Instead of the usual Higgs mechanism, there exists yet another way to induce a mass for a vector field, “topologically” [113,114]. This requires that it couple to a massless two-form, the latter eventually playing the role of the Stückelberg field. Let us make this maximally explicit, by considering Eq. (23). As we did in the pure two-form case, we rewrite the model by using a vector X_μ as

$$S = \int d^4x \left[\alpha \partial_{[\mu} \mathcal{A}_{\nu]} \partial^{[\mu} \mathcal{A}^{\nu]} - \epsilon^{\mu\nu\rho\sigma} \partial_\rho \mathcal{B}_{\mu\nu} X_\sigma + \frac{3}{2\beta} X_\mu X^\mu + \gamma \epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\mu\nu} \partial_{[\rho} \mathcal{A}_{\sigma]} \right]. \quad (\text{H10})$$

The equations of motion for $\mathcal{A}_\mu$, X_μ , and $\mathcal{B}_{\mu\nu}$ give

$$2\alpha \partial^\nu \partial_{[\mu} \mathcal{A}_{\nu]} - \gamma \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \mathcal{B}^{\nu\rho} = 0, \quad (\text{H11a})$$

$$X_\mu = -\frac{\beta}{3} \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \mathcal{B}^{\nu\rho}, \quad (\text{H11b})$$

$$\partial_{[\mu} X_{\nu]} = \gamma \partial_{[\mu} \mathcal{A}_{\nu]}, \quad (\text{H11c})$$

respectively; note that Eq. (H11c) dictates that

$$X_\mu = \gamma (\mathcal{A}_\mu - \partial_\mu \chi), \quad (\text{H12})$$

with χ a scalar. Combining appropriately Eqs. (H11a), (H11b), and (H12), we find

$$2\alpha \partial^\nu \partial_{[\mu} \mathcal{A}_{\nu]} + \frac{3\gamma^2}{\beta} (\mathcal{A}_\mu - \partial_\mu \chi) = 0, \quad (\text{H13})$$

which is the equation of motion for a massive spin-one field, with square mass $-3\gamma^2/\alpha\beta$, and the consistency conditions on the coefficients are $\alpha < 0$ and $\beta > 0$.

4. Zero-by-three CSKR

For the model of Eq. (24), the equations of motion for the three-form and the scalar are

$$\partial^\sigma \left(\beta \partial_{[\mu} \mathcal{C}_{\nu\rho\sigma]} + \frac{\gamma}{2} \epsilon_{\mu\nu\rho\sigma} \phi \right) = 0, \quad (\text{H14a})$$

$$\alpha \partial^\alpha \partial_\alpha \phi + \frac{\gamma}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\sigma \mathcal{C}_{\mu\nu\rho} = 0, \quad (\text{H14b})$$

respectively. We see that

$$\partial_{[\mu} \mathcal{C}_{\nu\rho\sigma]} = -\frac{\gamma}{2\beta} \epsilon_{\mu\nu\rho\sigma} \phi, \quad (\text{H15})$$

and

$$\alpha \partial^\alpha \partial_\alpha \phi - \frac{6\gamma^2}{\beta} \phi = 0, \quad (\text{H16})$$

meaning that the theory propagates a spin-zero particle with square mass $-6\gamma^2/\alpha\beta$ which is healthy as long as $\alpha > 0$ and $\beta < 0$. Once again, the explicit computation is in full agreement with the PSALTer result.

5. Parity-indefinite Einstein-Cartan gravity

When it comes to studying such models by hand, it is useful to transition from the gauge picture with variables $e^\alpha_{\hat{\mu}}$ and $\omega^{\alpha\beta}_{\hat{\mu}}$, to the affine one with variables the metric $g_{\hat{\mu}\hat{\nu}}$ and (affine) connection $\Gamma^{\hat{\mu}}_{\hat{\nu}\hat{\rho}}$; the latter are related to the former as

$$g_{\hat{\mu}\hat{\nu}} = e^\alpha_{\hat{\mu}} e^\beta_{\hat{\nu}} \eta_{\alpha\beta}, \quad \Gamma^{\hat{\mu}}_{\hat{\nu}\hat{\rho}} = e^\alpha_{\hat{\nu}} (\partial_{\hat{\rho}} e^\alpha_{\hat{\mu}} + \omega^{\alpha}_{\hat{\mu}\hat{\rho}} e^\beta_{\hat{\nu}}). \quad (\text{H17})$$

From the above it can be shown that the affine torsion and curvature tensors read

$$\begin{aligned} \mathcal{T}^{\hat{\mu}}_{\hat{\nu}\hat{\rho}} &= \Gamma^{\hat{\mu}}_{\hat{\nu}\hat{\rho}} - \Gamma^{\hat{\mu}}_{\hat{\rho}\hat{\nu}}, \quad \mathcal{R}^{\hat{\rho}}_{\hat{\sigma}\hat{\mu}\hat{\nu}} \\ &= \partial_{\hat{\mu}} \Gamma^{\hat{\rho}}_{\hat{\nu}\hat{\sigma}} - \partial_{\hat{\nu}} \Gamma^{\hat{\rho}}_{\hat{\mu}\hat{\sigma}} + \Gamma^{\hat{\rho}}_{\hat{\mu}\hat{\lambda}} \Gamma^{\hat{\lambda}}_{\hat{\nu}\hat{\sigma}} - \Gamma^{\hat{\rho}}_{\hat{\nu}\hat{\lambda}} \Gamma^{\hat{\lambda}}_{\hat{\mu}\hat{\sigma}}. \end{aligned} \quad (\text{H18})$$

For what follows, we also introduce the vector $v_{\hat{\mu}}$, pseudovector $a_{\hat{\mu}}$ and reduced torsion tensor $\tau_{\hat{\mu}\hat{\nu}\hat{\rho}}$ defined as [18,19]

$$\begin{aligned} v^{\hat{\mu}} &= \mathcal{T}^{\hat{\nu}}_{\hat{\mu}\hat{\nu}}, \quad a^{\hat{\mu}} = E^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \mathcal{T}_{\hat{\nu}\hat{\rho}\hat{\sigma}}, \\ \tau_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \frac{2}{3} \mathcal{T}_{\hat{\mu}\hat{\nu}\hat{\rho}} + \frac{1}{3} (g_{\hat{\mu}\hat{\nu}} v_{\hat{\rho}} - g_{\hat{\rho}\hat{\mu}} v_{\hat{\nu}}) - \frac{1}{3} (\mathcal{T}_{\hat{\nu}\hat{\rho}\hat{\mu}} - \mathcal{T}_{\hat{\rho}\hat{\nu}\hat{\mu}}), \end{aligned} \quad (\text{H19})$$

with $\tau^{\hat{\nu}}_{\hat{\nu}\hat{\mu}} = \tau^{\hat{\nu}}_{\hat{\mu}\hat{\nu}} = E^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \tau_{\hat{\nu}\hat{\rho}\hat{\sigma}} = 0$, and $E^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} / \sqrt{g}$, and $g = -\det(g_{\hat{\mu}\hat{\nu}})$. The scalar $\mathcal{R}$ and pseudoscalar $\tilde{\mathcal{R}}$ curvatures in the affine basis read

$$\mathcal{R} = g^{\hat{\sigma}\hat{\nu}} \delta^{\hat{\mu}}_{\hat{\rho}} \mathcal{R}^{\hat{\rho}}_{\hat{\sigma}\hat{\mu}\hat{\nu}}, \quad \tilde{\mathcal{R}} = E^{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} \mathcal{R}_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}. \quad (\text{H20})$$

The most general parity-indefinite action that comprises all invariants which are at most quadratic in torsion and the

scalar and pseudoscalar curvatures was given in the main text, see Eq. (29). In terms of Eq. (H19), it reads

$$S = \int d^4x \sqrt{g} \left[c_1 \mathcal{R} + c_2 \tilde{\mathcal{R}} + c_3 \mathcal{R}^2 + c_4 \mathcal{R} \tilde{\mathcal{R}} + c_5 \tilde{\mathcal{R}}^2 + \frac{C_{vv}}{3} v_\mu v^\mu - \frac{C_{aa}}{24} a_\mu a^\mu + \frac{C_{\tau\tau}}{2} \tau_{\dot{\mu}\dot{\nu}\dot{\rho}} \tau^{\dot{\mu}\dot{\nu}\dot{\rho}} + \frac{2C_{va}}{3} a_\mu v^\mu + \frac{\tilde{C}_{\tau\tau}}{2} E^{\dot{\mu}\dot{\nu}\dot{\rho}\dot{\sigma}} \tau_{\dot{\lambda}\dot{\mu}\dot{\nu}} \tau^{\dot{\lambda}\dot{\rho}\dot{\sigma}} \right], \quad (\text{H21})$$

$$C_{vv} = 2c_6 - c_7 + 3c_8, \quad C_{aa} = 4(c_6 + c_7), \quad C_{va} = 2c_9 - c_{10}, \quad (\text{H22})$$

and $C_{\tau\tau}$ and $\tilde{C}_{\tau\tau}$ depend on c_6 , c_7 and c_9 , c_{10} , respectively—the explicit relations can be easily worked out but are completely irrelevant for the following. We can get rid of the quadratic-in-curvature terms by introducing two auxiliary fields χ and ϕ

where

$$S = \int d^4x \sqrt{g} \left[(c_1 + \chi) \mathcal{R} + (c_2 + q\chi + \phi) \tilde{\mathcal{R}} - \frac{\chi^2}{4c_3} - \frac{c_3 \phi^2}{4c_3 c_5 - c_4^2} + \frac{C_{vv}}{3} v_\mu v^\mu - \frac{C_{aa}}{24} a_\mu a^\mu - \frac{C_{\tau\tau}}{2} \tau_{\dot{\mu}\dot{\nu}\dot{\rho}} \tau^{\dot{\mu}\dot{\nu}\dot{\rho}} + \frac{2C_{va}}{3} a_\mu v^\mu + \frac{\tilde{C}_{\tau\tau}}{2} E^{\dot{\mu}\dot{\nu}\dot{\rho}\dot{\sigma}} \tau_{\dot{\lambda}\dot{\mu}\dot{\nu}} \tau^{\dot{\lambda}\dot{\rho}\dot{\sigma}} \right], \quad (\text{H23})$$

with

$$q = \frac{c_4}{2c_3}. \quad (\text{H24})$$

The next step consists in plugging into the above the standard decomposition of the scalar $\mathcal{R}$ and pseudoscalar $\tilde{\mathcal{R}}$ curvatures in Riemannian (denoted with a “ $\circ$ ” on top) and post-Riemannian contributions

$$\begin{aligned} \mathcal{R} &= \mathring{R} + 2\mathring{\nabla}_\mu v^\mu - \frac{2}{3} v_\mu v^\mu + \frac{1}{24} a_\mu a^\mu + \frac{1}{2} \tau_{\dot{\mu}\dot{\nu}\dot{\rho}} \tau^{\dot{\mu}\dot{\nu}\dot{\rho}}, \\ \tilde{\mathcal{R}} &= -\mathring{\nabla}_\mu a^\mu + \frac{2}{3} a_\mu v^\mu + \frac{1}{2} E^{\dot{\mu}\dot{\nu}\dot{\rho}\dot{\sigma}} \tau_{\dot{\lambda}\dot{\mu}\dot{\nu}} \tau^{\dot{\lambda}\dot{\rho}\dot{\sigma}}, \end{aligned} \quad (\text{H25})$$

to obtain

$$\begin{aligned} S &= \int d^4x \sqrt{g} \left[(c_1 + \chi) \mathring{R} - \frac{\chi^2}{4c_3} - \frac{c_3 \phi^2}{4c_3 c_5 - c_4^2} - 2v^\mu \partial_\mu \chi + a^\mu (q \partial_\mu \chi + \partial_\mu \phi) - \frac{2}{3} \left(\frac{\Upsilon_1}{2} + \chi \right) v_\mu v^\mu \right. \\ &\quad + \frac{1}{24} (\Upsilon_4 + \chi) a_\mu a^\mu + \frac{c_{\tau\tau} + \chi}{2} \tau_{\dot{\mu}\dot{\nu}\dot{\rho}} \tau^{\dot{\mu}\dot{\nu}\dot{\rho}} \\ &\quad \left. - \frac{2}{3} (\Upsilon_2 - q\chi - \phi) a_\mu v^\mu + \frac{\tilde{c}_{\tau\tau} + q\chi + \phi}{2} E^{\dot{\mu}\dot{\nu}\dot{\rho}\dot{\sigma}} \tau_{\dot{\lambda}\dot{\mu}\dot{\nu}} \tau^{\dot{\lambda}\dot{\rho}\dot{\sigma}} \right], \end{aligned} \quad (\text{H26})$$

where

$$\Upsilon_1 = 2c_1 - 2c_6 + c_7 + 3c_8, \quad \Upsilon_2 = c_{10} - 2c_9 - c_2, \quad \Upsilon_4 = c_1 - 4(c_6 + c_7), \quad (\text{H27})$$

have already appeared in the main text (see Fig. 5), and we also introduced

$$c_{\tau\tau} = C_{\tau\tau} + c_1, \quad \tilde{c}_{\tau\tau} = \tilde{C}_{\tau\tau} + c_2. \quad (\text{H28})$$

One notices that torsion appears algebraically in the action Eq. (H26) and can thus be integrated out via the corresponding equations of motion. Variation of the above with respect to τ dictates that the reduced tensor vanish on-shell

$$\tau_{\dot{\mu}\dot{\nu}\dot{\rho}} = 0, \quad (\text{H29})$$

while for the vector and pseudovector, we find

$$v_{\dot{\mu}} = -3 \frac{(\Upsilon_4 - 4q\Upsilon_2 + (1 + 4q^2)\chi + 4q\phi)\partial_{\dot{\mu}}\chi - 4(\Upsilon_2 - q\chi - \phi)\partial_{\dot{\mu}}\phi}{D}, \quad (\text{H30a})$$

$$a_{\dot{\mu}} = -12 \frac{(q\Upsilon_1 + 2\Upsilon_2 - 2\phi)\partial_{\dot{\mu}}\chi + (\Upsilon_1 + 2\chi)\partial_{\dot{\mu}}\phi}{D}, \quad (\text{H30b})$$

and to keep the expressions short we introduced

$$D = \Upsilon_1\Upsilon_4 + 8\Upsilon_2^2 + (\Upsilon_1 - 16q\Upsilon_2 + 2\Upsilon_4)\chi + 2(1 + 4q^2)\chi^2 - 16\phi(\Upsilon_2 - q\chi) + 8\phi^2. \quad (\text{H31})$$

Plugging Eq. (H29) to Eq. (H30b) into Eq. (H26), we obtain

$$S = \int d^4x \sqrt{g} \left[(c_1 + \chi)\dot{R} + \partial_{\dot{\mu}}\phi^T \cdot \mathbf{g} \cdot \partial^{\dot{\mu}}\phi - \frac{\chi^2}{4c_3} - \frac{c_3\phi^2}{4c_3c_5 - c_4^2} \right], \quad (\text{H32})$$

where $\phi^T = [\chi, \phi]$, and

$$\mathbf{g} = \frac{3}{D} \begin{bmatrix} \mathbf{g}_{\chi\chi} & \mathbf{g}_{\chi\phi} \\ \mathbf{g}_{\chi\phi} & \mathbf{g}_{\phi\phi} \end{bmatrix}, \quad (\text{H33})$$

is the metric of the kinetic manifold whose components are

$$\begin{aligned} \mathbf{g}_{\chi\chi} &= \Upsilon_4 - 2q(q\Upsilon_1 + 4\Upsilon_2) + (1 + 4q^2)\chi + 8q\phi, & \mathbf{g}_{\chi\phi} &= -2(q\Upsilon_1 + 2\Upsilon_2 - 2\phi), \\ \mathbf{g}_{\phi\phi} &= -2(\Upsilon_1 + 2\chi). \end{aligned} \quad (\text{H34})$$

We can eliminate the nonminimal coupling of χ to gravity via a Weyl rescaling of $g_{\mu\nu}$

$$g_{\mu\nu} \mapsto \Omega^{-2} g_{\mu\nu}, \quad \Omega^2 = \frac{c_1 + \chi}{c_1}. \quad (\text{H35})$$

This results into the following Einstein-frame action

$$S = \int d^4x \sqrt{\tilde{g}} [c_1 \dot{R} + \partial_{\dot{\mu}}\phi^T \cdot \tilde{\mathbf{g}} \cdot \partial^{\dot{\mu}}\phi - V], \quad (\text{H36})$$

where the Weyl-transformed field-space metric $\tilde{\mathbf{g}}$ is given by

$$\tilde{\mathbf{g}} = \frac{3}{D\Omega^2} \begin{bmatrix} \mathbf{g}_{\chi\chi} - \frac{D}{2c_1\Omega^2} & \mathbf{g}_{\chi\phi} \\ \mathbf{g}_{\chi\phi} & \mathbf{g}_{\phi\phi} \end{bmatrix}, \quad (\text{H37})$$

and the potential reads

$$V = \frac{c_1^2}{(c_1 + \chi)^2} \left(\frac{\chi^2}{4c_3} + \frac{c_3\phi^2}{4c_3c_5 - c_4^2} \right). \quad (\text{H38})$$

Since gravity is canonical, we can immediately conclude that

$$c_1 < 0, \quad (\text{H39})$$

as expected. Turning now to the scalar sector, we first consider the potential, that is extremized for

$$\chi = \phi = 0. \quad (\text{H40})$$

For its Hessian to be positive-definite when evaluated on the extremum, we find

$$(c_3 > 0) \wedge (4c_3c_5 - c_4^2 > 0), \quad (\text{H41})$$

from which it follows that $c_5 > 0$. These are exactly the no-ghost conditions we derived with the SPOs in the main text. Finally, we consider the kinetic terms of the fields.²⁸ We evaluate the field-space metric Eq. (H37) on Eq. (H40), and then require that its determinant and trace be positive. This reproduces the constraints of Eqs. (36) and (37).

²⁸Although not needed for the considerations here, note that there is no difficulty in diagonalizing the kinetic terms of the scalars and also making one of them (χ) canonical; this is achieved by introducing

$$\begin{aligned} \Phi &= \log \left[\frac{c_4\Upsilon_1 + 4c_3(\Upsilon_2 - \phi)}{2c_3(\Upsilon_1 + 2\chi)} \right], \\ X &= 2\sqrt{3c_1} \tan^{-1} \left[\sqrt{\frac{\Upsilon_1 + 2\chi}{2c_1 - \Upsilon_1}} \right]. \end{aligned} \quad (\text{H42})$$

APPENDIX I: SPECTRAL CALIBRATION

1. General EC/Poincaré gravity

In this appendix we perform the most sophisticated possible calibration for the PSALT_{er} implementation. In this procedure we compare the software output with the results of [32], where the most general parity-violating theory up to quadratic order in curvature and torsion (see also [28,30]) was already studied in the SPO formalism. In [32], however,

different Lagrangian coupling coefficients were used relative to those introduced in Sec. III B. Specifically, the $\tilde{\mathcal{R}}$ and the $\mathcal{R}^2$ operators were not included, the first being related to $\epsilon^{\mu\nu\rho\sigma} \mathcal{T}_{\mu\nu\lambda} \mathcal{T}_{\rho\sigma}{}^\lambda$ (see also Footnote 16), and the latter to the squares of the Ricci and Riemann tensors via the Gauss-Bonnet identity. To facilitate the comparison, we now utilize the parametrization of [32]. Accordingly, the action in Eq. (29) is extended and reparametrized as

$$\begin{aligned}
 S = \int d^4x e & \left[-\lambda \mathcal{R} + \frac{1}{6} (2r_1 + r_2) \mathcal{R}_{\alpha\beta\gamma\delta} \mathcal{R}^{\alpha\beta\gamma\delta} + \frac{2}{3} (r_1 - r_2) \mathcal{R}_{\alpha\beta\gamma\delta} \mathcal{R}^{\alpha\gamma\beta\delta} \right. \\
 & + \frac{1}{6} (2r_1 + r_2 - 6r_3) \mathcal{R}_{\alpha\beta\gamma\delta} \mathcal{R}^{\gamma\delta\alpha\beta} + (r_4 + r_5) \mathcal{R}_{\alpha\beta} \mathcal{R}^{\alpha\beta} + (r_4 - r_5) \mathcal{R}_{\alpha\beta} \mathcal{R}^{\beta\alpha} \\
 & + \frac{1}{6} (r_6 - r_8) \mathcal{R} \tilde{\mathcal{R}} - \frac{1}{8} (r_7 + r_8) \epsilon^{\alpha\beta\mu\nu} \mathcal{R}_{\alpha\beta\rho\sigma} \mathcal{R}_{\mu\nu}{}^{\rho\sigma} + \frac{1}{4} (r_7 - r_8) \epsilon^{\alpha\beta\mu\nu} \mathcal{R}_{\alpha\beta\rho\sigma} \mathcal{R}^{\rho\sigma}{}_{\mu\nu} \\
 & + \frac{1}{12} (4t_1 + t_2 + 3\lambda) \mathcal{T}_{\alpha\beta\gamma} \mathcal{T}^{\alpha\beta\gamma} - \frac{1}{3} (t_1 - 2t_3 + 3\lambda) \mathcal{T}_\alpha \mathcal{T}^\alpha \\
 & - \frac{1}{6} (2t_1 - t_2 + 3\lambda) \mathcal{T}_{\alpha\beta\gamma} \mathcal{T}^{\beta\gamma\alpha} - \frac{1}{12} (t_4 + 4t_5) \epsilon^{\alpha\beta\mu\nu} \mathcal{T}_{\rho\alpha\beta} \mathcal{T}^\rho{}_{\mu\nu} \\
 & \left. + \frac{1}{3} (t_4 - 2t_5) \epsilon^{\alpha\beta\mu\nu} \mathcal{T}_{\alpha\beta\rho} \mathcal{T}_{\mu\nu}{}^\rho \right], \tag{I1}
 \end{aligned}$$

where $\lambda, r_1, \dots, r_8$ and $t_1, \dots, t_5$ are constants $k\Lambda$, kR_1 , through to kT_5 .

2. Results of the calibration

The quadratic part of Eq. (I1) is inevitably a very long expression:

```

In[#]:= ParticleSpectrum[ ((kT1 + kT2) * SpinConnection[-a, -b,
  ↳ -c] * SpinConnection[a, b, c])/3 + ((kT4 - 2 * kT5) *
  ↳ epsilonG[-b, -c, -d, -i] * SpinConnection[-a, d, i] *
  ↳ SpinConnection[a, b, c])/3 + ((kT1 - 2 * kT2) *
  ↳ SpinConnection[a, b, c] * SpinConnection[-b, -a, -c])/3 - (2
  ↳ * (kT4 - 2 * kT5) * (*omitted 4264 characters for brevity*)
  ↳ b]] * CD[i][TetradPerturbation[d, -b]])/3 - (2 * (kR6 - kR8)
  ↳ * epsilonG[-c, -d, -i, -j] * CD[-b][SpinConnection[a, -a,
  ↳ b]] * CD[j][SpinConnection[c, d, i]])/3, TheoryName ->
  ↳ "GeneralParityViolatingPGT", MaxLaurentDepth -> 1,
  ↳ AspectRatio -> Portrait, ShowPropagator -> False];

```

The output is shown in Fig. 6. Apart from polynomial factors in k^2 whose couplings are numerical,²⁹ each determinant is quadratic in k^2 with couplings that depend on the coefficients in Eq. (I1). The roots of these quadratic

equations are the masses of the two nongraviton particles in each spin sector. Importantly, the mass expressions are identical to the ones presented in [32], and so are the no-tachyon conditions [34] that follow by requiring that these be (real and) positive. This is the first nontrivial sanity-check that the code passes successfully. It should be noted that there are in fact two differences, attributed to choices of

²⁹These factors are artefacts of Moore-Penrose gauge fixing, and do not imply the presence of massive poles.

$\omega_{0+}^{\#1}$	$f_{0+}^{\#1}$	$f_{0+}^{\#2}$	$\omega_{0+}^{\#1}$
$\omega_{0+}^{\#1} \uparrow$	$2Y_1 k^2 + t_3$	$i\sqrt{2} k t_3$	0
$f_{0+}^{\#1} \uparrow$	$-i\sqrt{2} k t_3$	$-2Y_2 k^2$	0
$f_{0+}^{\#2} \uparrow$	0	0	0
$\omega_{0+}^{\#1} \uparrow^{a\beta\chi}$	$-ik^2 r_6 + it_4$	$-\sqrt{2} k t_4$	0
$\omega_{1+}^{\#1}$	$\frac{1}{6}(Y_3 + 6Y_4 k^2)$	$-\frac{Y_9}{3\sqrt{2}}$	$\frac{iY_9 k}{3\sqrt{2}}$
$\omega_{1+}^{\#2}$	$-\frac{Y_9}{3\sqrt{2}}$	$\frac{Y_8}{3}$	$-\frac{1}{3}iY_8 k$
$f_{1+}^{\#1}$	$-\frac{iY_9 k}{3\sqrt{2}}$	$\frac{iY_8 k}{3}$	$-\frac{1}{3}\sqrt{2}Y_7 k$
$\omega_{1+}^{\#1} \uparrow^a$	$\frac{1}{3}(iY_6 - 3ik^2 r_7)$	$\frac{1}{3}i\sqrt{2}Y_7$	$\frac{1}{6}(Y_{10} + 6Y_{11} k^2)$
$\omega_{1+}^{\#2} \uparrow^a$	$-\frac{1}{3}i\sqrt{2}Y_7$	$-\frac{iY_9}{3}$	$\frac{Y_{12}}{3\sqrt{2}}$
$f_{1+}^{\#1} \uparrow^a$	0	0	0
$f_{1+}^{\#2} \uparrow^a$	$\frac{2Y_7 k}{3}$	$\frac{1}{3}\sqrt{2}Y_9 k$	$\frac{iY_{12} k}{3}$
$\omega_{2+}^{\#1}$	$\frac{1}{2}(2Y_{14} k^2 + t_1)$	$\frac{i k t_1}{\sqrt{2}}$	$-\frac{1}{2}ik^2 r_8 - it_5$
$f_{2+}^{\#1}$	$-\frac{i k t_1}{\sqrt{2}}$	$Y_{15} k^2$	$-\sqrt{2} k t_5$
$\omega_{2+}^{\#1} \uparrow^{a\beta\chi}$	$-ik^2 r_8 - it_5$	$\sqrt{2} k t_5$	$\frac{1}{2}(2k^2 r_1 + t_1)$

Abbreviations used in matrices			
$Y_1 = r_1 - r_3 + 2r_4$ && $Y_2 = \lambda - t_3$ && $Y_3 = t_1 + 4t_2$ && $Y_4 = 2r_3 + r_5$ && $Y_5 = t_1 - 2t_2$ && $Y_6 = 2t_4 - t_5$ && $Y_7 = t_4 + t_5$ && $Y_8 = t_1 + t_2$ && $Y_9 = t_4 - 2t_5$ && $Y_{10} = t_1 + 4t_3$ && $Y_{11} = r_1 + r_4 + r_5$ && $Y_{12} = t_1 - 2t_3$ && $Y_{13} = t_1 + t_3$ && $Y_{14} = 2r_1 - 2r_3 + r_4$ && $Y_{15} = \lambda + t_1$ && $\text{Det}(0) = -2(k^6(2r_2(r_1 - r_3 + 2r_4) + r_6^2)(\lambda - t_3) + k^2\lambda(t_2 t_3 + t_4^2) +$ $k^4(\lambda(2(r_1 - r_3 + 2r_4)t_2 + r_2 t_3 - 2r_6 t_4) - 2(r_1 - r_3 + 2r_4)(t_2 t_3 + t_4^2)))$ && $\text{Det}(1) = \frac{1}{36}(1 + 3k^2 + 2k^4)(4k^4((2r_3 + r_5)(r_1 + r_4 + r_5) + r_7^2)$ $((t_1 + t_2)(t_1 + t_3) + (t_4 - 2t_5)^2) + 9(t_2 t_3 + t_4^2)(t_1^2 + 4t_5^2) +$ $6k^2(t_1(r_1 t_1 t_2 + r_4 t_1 t_2 + r_5 t_1 t_2 + 2r_3 t_1 t_3 + r_5 t_1 t_3 + r_1 t_2 t_3 +$ $2r_3 t_2 t_3 + r_4 t_2 t_3 + 2r_5 t_2 t_3 - 2r_7 t_1 t_4 + (r_1 + 2r_3 + r_4 + 2r_5)t_4^2) +$ $4r_7(t_2 t_3 + t_4^2)t_5 + 4((r_1 + r_4 + r_5)t_2 + 2r_3 t_3 + r_5 t_3 - 2r_7 t_4)t_5^2))$ && $\text{Det}(2) = \frac{1}{4}(4k^6(r_1(2r_1 - 2r_3 + r_4) + r_6^2)(\lambda + t_1) + k^2\lambda(t_1^2 + 4t_5^2) +$ $2k^4(\lambda(3r_1 - 2r_3 + r_4)t_1 + 4\lambda r_8 t_5 + (2r_1 - 2r_3 + r_4)(t_1^2 + 4t_5^2)))$			
Added source term(s):	$f^{a\beta} \tau_{a\beta} + \omega^{a\beta\chi} \sigma_{a\beta\chi}$		
Source constraint(s)	# constraint(s)	Covariant form	
$\tau_{0+}^{\#2} = 0$	1	$\partial_\beta \partial_\alpha \tau^{a\beta} = 0$	
$\tau_{1+}^{\#2\alpha} - 2ik \sigma_{1+}^{\#2\alpha} = 0$	3	$\partial_\chi \partial_\beta \partial^\alpha \tau^{\beta\chi} + 2\partial_\beta \partial^\alpha \partial_\chi \sigma^{\beta\alpha\chi} = \partial_\chi \partial^\alpha \partial_\beta \tau^{a\beta}$	
$\tau_{1+}^{\#1\alpha} = 0$	3	$\partial_\chi \partial_\beta \partial^\alpha \tau^{\beta\chi} = \partial_\chi \partial^\alpha \partial_\beta \tau^{a\beta}$	
$\tau_{1+}^{\#1\alpha\beta} - ik \sigma_{1+}^{\#2\alpha\beta} = 0$	3	$\partial_\chi \partial^\alpha \tau^{\beta\chi} + \partial_\chi \partial^\beta \tau^{\alpha\chi} + \partial_\chi \partial^\chi \tau^{a\beta} + 2\partial_\beta \partial_\chi \partial^\alpha \sigma^{\chi a\beta} =$ $\partial_\chi \partial^\alpha \tau^{\chi\beta} + \partial_\chi \partial^\beta \tau^{\alpha\chi} + \partial_\chi \partial^\chi \tau^{\beta a} + 2\partial_\beta \partial_\chi \partial^\alpha \sigma^{\chi\beta a} + 2\partial_\beta \partial^\alpha \partial_\chi \sigma^{\chi a\beta}$	
Total # constraint(s):	10		
Unresolved pole(s)	# d.o.f.	Pole structure(s)	
	2	$k^4(2r_2(r_1 - r_3 + 2r_4) + r_6^2)(\lambda - t_3) + \lambda(t_2 t_3 + t_4^2) +$ $k^2(\lambda(2(r_1 - r_3 + 2r_4)t_2 + r_2 t_3 - 2r_6 t_4) - 2(r_1 - r_3 + 2r_4)(t_2 t_3 + t_4^2))$	
	6	$4k^4((2r_3 + r_5)(r_1 + r_4 + r_5) + r_7^2)((t_1 + t_2)(t_1 + t_3) + (t_4 - 2t_5)^2) +$ $9(t_2 t_3 + t_4^2)(t_1^2 + 4t_5^2) +$ $6k^2(t_1(r_1 t_1 t_2 + r_4 t_1 t_2 + r_5 t_1 t_2 + 2r_3 t_1 t_3 + r_5 t_1 t_3 + r_1 t_2 t_3 +$ $2r_3 t_2 t_3 + r_4 t_2 t_3 + 2r_5 t_2 t_3 - 2r_7 t_1 t_4 + (r_1 + 2r_3 + r_4 + 2r_5)t_4^2) +$ $4r_7(t_2 t_3 + t_4^2)t_5 + 4((r_1 + r_4 + r_5)t_2 + 2r_3 t_3 + r_5 t_3 - 2r_7 t_4)t_5^2)$	
	10	$4k^4(r_1(2r_1 - 2r_3 + r_4) + r_6^2)(\lambda + t_1) + \lambda(t_1^2 + 4t_5^2) +$ $2k^2(\lambda(3r_1 - 2r_3 + r_4)t_1 + 4\lambda r_8 t_5 + (2r_1 - 2r_3 + r_4)(t_1^2 + 4t_5^2))$	
Resolved pole(s)	# polarization(s)	Square mass	Residue
	2	0	$\frac{1+8\rho^2}{\lambda}$
Resolved unitarity condition(s):		$\lambda > 0$	

FIG. 6. Partial particle spectrum of the most general parity-violating PGT. Due to the square masses of the new species being irrational functions of the Lagrangian coupling coefficients, PSALTER does not yet attempt to evaluate the massive no-ghost criteria. The elements of the wave operator matrices are fully consistent with those in [32]. The determinants are quadratic in k^2 , and the (generally massive) poles defined by their roots are also consistent with the mass formulas in [32]. We note the appearance of ten gauge generators: precisely this number is expected due to the Poincaré gauge symmetry. All quantities are defined in Tables VI and VII.

convention, between the matrix elements in Fig. 6 and those in [32]; nevertheless, neither affects the physics and the coefficient matrices are perfectly consistent with each other. The first difference is that all the off-diagonal (parity-violating) blocks differ by a factor of i . This is because PSALTer assumes the convention whereby the parity-violating spin-projection operators are symmetric and imaginary, whereas [32] takes the same operators to be real, but skew-symmetric. As we showed in Sec. II A, the requirement for physicality is actually that these blocks have a *skew-Hermitian* structure, and so either of these conventions is valid. The second difference is that the degeneracy of the spin-one matrix in Fig. 6 is visible in the form of *two* repeated rows and *one* row of zeros (and likewise for columns). In [32], on the other hand, there are *three* repeated rows and columns (similar matrices appear in [115]). The actual difference in this case is due to the direct decomposition of the negative parity spin-one modes in Table VI. These modes are linear combinations of the

modes used in [32,115], which are obtained after first breaking the tetrad perturbation into symmetric and anti-symmetric parts. The second nontrivial cross-check for the validity of the results obtained by PSALTer is provided by deriving the no-ghost conditions—with our simplified method of Sec. II C, this can be done almost trivially by inspection of the matrices in Fig. 6. We obtain

$$J = 0: r_2 < 0, \quad 2r_2(r_1 - r_3 + 2r_4) + r_6^2 < 0; \quad (I2)$$

$$J = 1: r_1 + r_4 + r_5 < 0, \quad (r_1 + r_4 + r_5)(2r_3 + r_5) + r_7^2 < 0; \quad (I3)$$

$$J = 2: r_1 < 0, \quad r_1(2r_1 - 2r_3 + r_4) + r_8^2 < 0; \quad (I4)$$

which are identical to the findings of [32,34]. As is well known, the above constraints for the spin-one and spin-two sectors contradict each other [34,116].

-
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