Electric field fluctuations and renormalization group flows in a self-interacting scalar field theory

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We consider a self-interacting charged scalar field represented by the complex $\lambda\phi^4$ model, embedded in a background electric field exhibiting classical stochastic fluctuations. We studied the effects of the classical stochastic noise on the physical parameters of the scalar field theory, for both weak and ultrastrong electric-field regimes. The stochastic background electric field is included in the Schwinger propagator through the covariant derivative, and the generating functional of the theory is found by means of the replica trick in order to compute the statistical average over electric fluctuations. As a result of the averaging process, an effective interaction between charged currents emerges, with a coupling constant proportional to the magnitude of the autocorrelation function of the electric-field fluctuations. We obtained the dressed propagators, the interaction vertices, and the renormalization group equations of the theory, along with the corresponding streamplots in the manifold of interaction couplings.

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I. INTRODUCTION

Complex scalar fields are ubiquitous in effective quantum field theories, ranging from the analysis of critical phenomena in condensed matter physics and statistical mechanics, to the description of charged boson mesons such as pions [1]. In this second context, phenomenological scenarios in heavy-ion collisions (HIC) [2,3] and in astrophysics, such as in neutron stars [4], involve the presence of background electromagnetic fields [5] that interact with these charged bosons. HIC at relativistic momenta [2,6] constitute our current experimental approach to test the properties of the quark gluon plasma (QGP). This state of matter is predicted by QCD to emerge above a critical energy density, that was presumably attained a few microseconds after the big bang. The "fireball" generated by the nuclear collision attains a very high density, and hence the particles produced in the final state carry information about initial conditions (particularly the impact parameter), and about the properties of the high-density phase [2,3,7]. As the lightest mesons, pions constitute the pseudo-Goldstone bosons of $SU(2)$ that express the approximate, spontaneous chiral symmetry breaking in QCD. With an isospin of one, and three differently charged states π^0, π^+, and π^-, pions are abundantly produced in HICs even at the few GeV range [8,9], thus providing valuable signals from all stages of the collision. In noncentral HIC, electromagnetic fields arise due to the spectator nucleons outside the overlapping region. Particularly, in the case of asymmetric collisions such as $\mathrm{Cu}+\mathrm{Au}$, a stronger dipolelike electric field emerges due to the asymmetric distribution of the nuclear charge of the spectators [10]. Although this electric field is present on time-scales of at most $\sim$1 fm/c [11,12], its magnitude can reach average values up to $\langle qE\rangle \sim 10m_\pi^2$ for $\mathrm{Au}+\mathrm{Au}$ collisions at RHIC [13]. Moreover, the electromagnetic fields generated at the collision region [14] are expected to interact with the quark-gluon plasma, and influence the dynamics of the thousands of particles [15] produced after hadronization. For instance, charged pions produced in $Au+Au$ collisions at $\sqrt{s_{NN}}=2.4$ GeV [9] exhibit nonisotropic angular distributions which disagree with state of the art transport models [9]. These effects can in principle be attributed to the Coulomb interaction of the pions with the overall charge distribution of the expanding fireball [9]. Indeed, measurements of charge-dependent

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directed flow for charged mesons at RHIC [14], in $Au + Au$ collisions at $\sqrt{s_{NN}} = 200$ GeV and at $\sqrt{s_{NN}} = 27$ GeV, as measured by the STAR detector [14], provides direct experimental evidence for the existence of those intense electromagnetic fields.

Historically, most analytical approaches on this matter are based on the seminal work of Schwinger, which provides an integral representation of scalar and fermion propagators in terms of the so-called Schwinger proper time [16], in the presence of a constant and uniform electromagnetic field. In this context, phenomenological field theories, such as the linear sigma model coupled to quarks at finite temperature [17], have been applied to develop effective potential models describing magnetic and inverse magnetic catalysis [18]. Similarly, self-interacting scalar field theories have been developed to describe electric and inverse electric catalysis [19–21], and also in the context of the Nambu-Jona-Lasinio model [22]. These models are constructed under the approximation of constant electric or magnetic fields, respectively, and show qualitative agreement with lattice QCD calculations [23–26]. However, in real HICs the electromagnetic fields are dynamical entities evolving with the system hydrodynamics [15], and the presence of stochastic fluctuations due to the random initial conditions of the nucleons in event by event statistics has been recognized as a relevant effect on the measured signals [27]. More recently, in order to model these effects, some of us proposed a theoretical formalism to study the effects of classical stochastic fluctuations over an otherwise constant and uniform magnetic field background on the fermion propagator [28]. Interestingly, the physical picture emerging from these theories revealed that under a classical background magnetic field with stochastic fluctuations, the fermions propagate in an effective dispersive medium with an anisotropic refraction index for the group velocity [28]. Moreover, the physical consequences of this theory were explored, particularly concerning the effects on the Fermion mass [29], the generation of a screening effective mass on the photon propagator [30], and more recently the production of photons [31] under such a fluctuating magnetic medium.

In the present paper, we shall apply a similar analysis for the propagator of a charged scalar field immersed in a classical electric-field background with stochastic fluctuations. As a toy model for charged pions in this fluctuating medium, we consider a self-interacting $\lambda\phi^4$ theory, that has been used in the past to analyze the main features of $\pi - \pi$ scattering processes [1]. Moreover, the coupling of this model to a constant background electric field, in the effective potential formalism for chiral symmetry breaking, has been explored in the context of catalysis and inverse electric catalysis [19], and in pion condensation mechanisms in the presence of uniform magnetic and electric fields in a rotating framework [32]. In contrast, in this work we shall focus on the renormalization effects induced by the combination of electric field $\mathbf{E}$ and its stochastic fluctuations, in order to address how the effective couplings in the model, particularly the λ responsible for the self-interaction, renormalizes in the presence of noise. We are not explicitly analyzing here how this affects the mechanisms for catalysis and inverse electric catalysis [19], since this requires an effective potential formalism that will be communicated in separate article, but we anticipate its effects at the level of the evolution of the couplings subjected to stochastic fluctuations in the background electric field. For this purpose, we shall start by introducing the formalism to include the classical fluctuations of the background field into the Lagrangian, and the subsequent statistical averages by means of the replica trick. Later, we shall consider two different limits for the reference propagator, corresponding to: (a) The very weak electric-field limit $|q\mathbf{E}|/m^2 \ll 1$ (for q and m the particle charge and mass), and the very strong field limit $|q\mathbf{E}|/m^2 \gg 1$, respectively. In both cases, we shall study the evolution of the self-interaction coupling as a function of the noise autocorrelation and the characteristic logarithmic momentum scale $t = \ln(p^2/m^2)$, by obtaining beta functions of the theory, and then solving the corresponding Callan-Symanzik equation for the renormalization group flows.

This article is organized as follows. In Sec. II, we introduce the modified $\lambda\phi^4$ Lagrangian with stochastic fluctuations. In Sec. III, we define the scalar boson propagator in the Schwinger proper-time representation for both limiting cases, which are then used in Sec. IV to find the self-energy at one loop. Then, in Sec. V we define the physical parameters in the framework of renormalized perturbation theory, and find the corresponding self-energy counterterms. In Sec. VI, we study the spectral density of the background and dressed propagators, in order to study the effects of the interaction on the physical mass. In Sec. VII, we find the one-loop corrections to both interaction vertices, along with their corresponding counterterms. Finally, in Sec. VIII we study the renormalization group equations of the theory, in order to evaluate the evolution of both couplings with the energy scale. All detailed mathematical derivations are presented in Appendixes A–E.

II. THE MODEL

We shall consider a system of self-interacting charged bosons whose dynamics is driven by the presence of a classical static electric field with stochastic spacetime fluctuations. Therefore, the corresponding background classical gauge field is composed by an average $A^\mu_{\mathrm{BG}}(x)$ and a stochastic fluctuation $\delta A^\mu_{\mathrm{BG}}(x)$ component as follows:

$$A^\mu_{\mathrm{BG}}(x) + \delta A^\mu_{\mathrm{BG}}(x). \tag{1}$$

Here, we assume that the stochastic fluctuations satisfy the following "white-noise" statistical properties:

$$\langle \delta A^{\mu}_{\mathrm{BG}}(x) \rangle = 0, \tag{2}$$

$$\langle \delta A^{\mu}_{\mathrm{BG}}(x) \delta A^{\nu}_{\mathrm{BG}}(y) \rangle = \Delta \delta^{\mu\nu} \delta^4(x-y). \tag{3}$$

The subscript BG stands for "background," representing the external electromagnetic field imposed by experimental conditions. After Eq. (3), the magnitude of the autocorrelation function for the fluctuations is defined as Δ which, as we will see later, will play the role of an effective coupling parameter between charged currents. The statistical properties of the stochastic fluctuations correspond to a Gaussian functional distribution of the form [28]

$$dP[\delta A^{\mu}_{\mathrm{BG}}(x)] = \mathcal{N} e^{-\int d^4x \frac{(\delta A^{\mu}_{\mathrm{BG}}(x))^2}{2\Delta}} \mathcal{D}[\delta A^{\mu}_{\mathrm{BG}}(x)]. \tag{4}$$

For this purpose, we consider the standard $\lambda\phi^4$ theory for complex fields, where the fluctuating electric field is introduced, in the usual way, as a gauge connection in the covariant derivative,

$$\partial^{\mu} + iq(A^{\mu}_{\mathrm{BG}}(x) + \delta A^{\mu}_{\mathrm{BG}}(x)) \equiv D^{\mu} + iq\delta A^{\mu}_{\mathrm{BG}}(x). \tag{5}$$

Here q is the particle charge, D^{μ} is the covariant derivative with the connection $A^{\mu}_{\mathrm{BG}}(x)$ corresponding to the average background gauge field, and $\delta A^{\mu}_{\mathrm{BG}}(x)$ is the random fluctuation with respect to the mean value.

The Lagrangian density for the model will be given by the sum of two contributions,

$$\mathcal{L} = \mathcal{L}_{BG} + \mathcal{L}_{\delta}, \tag{6}$$

where the first one represents the standard $\lambda\phi^4$ theory coupled to a static electric field

$$\mathcal{L}_{BG} = |D_{\mu}\phi(x)|^2 - m^2\phi^{\dagger}(x)\phi(x) - \frac{\lambda}{4}[\phi^{\dagger}(x)\phi(x)]^2. \tag{7}$$

Here, ϕ is a complex scalar field, the operator D_{μ} represents the covariant derivative in the absence of fluctuations, and $\lambda > 0$ is a coupling constant. The second contribution represents the interaction between the scalar field and the background fluctuations

$$\mathcal{L}_{\delta} = -iq\delta A^{\mu}_{\mathrm{BG}}(x) j_{\mu}(x), \tag{8}$$

where the currents will be given by

$$j_{\mu}(x) = \phi^{\dagger}(x) D_{\mu}\phi(x) - (D_{\mu}\phi(x))^{\dagger}\phi(x). \tag{9}$$

To determine the evolution of the physical parameters of the system, we must find the generating functional in the presence of the background electromagnetic field with stochastic fluctuations. The generating functional of the theory, in the absence of sources, and for a given realization of the background field and its fluctuations is given by the expression

$$Z[A_{\mathrm{BG}}] = \int \mathcal{D}[\phi^{\dagger}, \phi] e^{i\int d^4x (\mathcal{L}_{BG} + \mathcal{L}_{\delta})}. \tag{10}$$

In contrast with the quantum scalar field $\phi(x)$, the background fluctuations $\delta A^{\mu}_{BG}(x)$ are classical stochastic variables distributed according to Eq. (4). Since all the physical information is encoded in the averaged connected $2n$-point quantum correlation functions,

$$\overline{\langle \hat{T}\phi(x_1)\ldots\phi(x_n)\phi^{\dagger}(x_{n+1})\ldots\phi^{\dagger}(x_{2n})\rangle} = \frac{(-i)^{2n}\delta^{2n}}{\delta J(x_{2n})\ldots\delta J^{\dagger}(x_1)} \overline{\ln\left(Z[A_{BG}; J^{\dagger}, J]\right)}|_{J=0}, \tag{11}$$

we must find the statistical average of $\ln Z$ over the background noise

$$\overline{\ln(Z[A_{BG}])} = \int dP[\delta A_{BG}] \ln\left(Z[A_{BG} + \delta A_{BG}]\right). \tag{12}$$

Computing $\overline{\ln Z}$ poses technical difficulties, which can be circumvented by the application of the replica trick [33], following the mathematical identity:

$$\overline{\ln\left(Z[A_{BG}]\right)} = \lim_{n\to 0} \frac{\overline{Z[A_{BG}]} - 1}{n}. \tag{13}$$

Here n represents the total number of replicas of the bosonic field. The method implies the introduction of n replica fields $\phi_a(x)$, for $1 \le a \le n$. This way, the average generating functional can be explicitly computed, leading to the expression

$$\overline{Z^n} = \int \prod_{a=1}^{n} \mathcal{D}[\phi_a^{\dagger}, \phi_a] dP[\delta A^{\mu}_{\mathrm{BG}}(x)] e^{i\int d^4x \sum_{a=1=}^{n} (\mathcal{L}_{BG} + \mathcal{L}_{\delta})} = \int \prod_{a=1}^{n} \mathcal{D}[\phi_a^{\dagger}, \phi_a] e^{i\int d^4x \mathcal{L}_{\mathrm{eff}}[\phi_a^{\dagger}, \phi_a]}, \tag{14}$$

where in the last step, we performed Gaussian functional integration over the probability distribution defined by Eq. (4). The resulting effective Lagrangian is thus given by the sum of two contributions,

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{BG}} + \mathcal{L}_{\mathrm{int}}, \tag{15}$$

where

$$\mathcal{L}_{\mathrm{BG}} = \sum_a |D_{\mu}\phi_a(x)|^2 - m^2\phi_a^{\dagger}(x)\phi_a(x), \tag{16}$$

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{4}\sum_a [\phi_a^\dagger(x)\phi_a(x)]^2 - iq^2\Delta\sum_{a,b} j_{\mu,a}(x)j^{\mu,b}(x), \quad (17)$$

represent the background and interaction Lagrangian, respectively. Notice that, for $d \to 4$ the spacetime dimensions, the corresponding dimension of the currents defined by Eq. (9) is m^{d-1}, and hence the coupling Δ has dimensions of

$$[\Delta] = m^{2-d}. \quad (18)$$

On the other hand, the parameter λ has dimensions of

$$[\lambda] = m^{4-d}. \quad (19)$$

Hence, for consistency, introduce a factor of m^2 in the definition of the effective noise coupling, thus defining the dimensionless coupling (in $d = 4$),

$$\tilde{\Delta} = q^2 m^2 \Delta. \quad (20)$$

In summary, after averaging over the electromagnetic fluctuations, we are left with an effective theory describing a Klein-Gordon Lagrangian coupled to a constant background electromagnetic field, with a self-interaction $\lambda\phi^4$, and an effective interaction between charged currents, the latter proportional to the magnitude of the noise autocorrelation function Δ.

III. BOSON PROPAGATOR IN THE PRESENCE OF A UNIFORM ELECTRIC FIELD

To approach the phenomenological scenario of the in-plane electric field present in asymmetric heavy-ion collisions, we shall neglect the magnetic field, and consider a static and uniform electric field in the $\hat{z}$-direction as follows:

$$\boldsymbol{E} = E\hat{z}, \quad (21)$$

$$\boldsymbol{B} = \boldsymbol{0}. \quad (22)$$

These fields are incorporated into the definition of the electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Using $-\partial_3 A^0 = E$ (with other components equal to zero), the explicit form of the tensor is

$$\hat{F} = \begin{pmatrix} 0 & 0 & 0 & iE \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -iE & 0 & 0 & 0 \end{pmatrix}, \quad (23)$$

which is written in Euclidean metric $A_0 \to iA_0$ for technical purposes. For a uniform background electric field, the Schwinger's proper-time representation for the free scalar propagator is [16,34]

$$D_E(x,y) = \Phi(x,y)\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} D_E(p), \quad (24)$$

where $\Phi(x,y)$ is the Schwinger phase, defined on a straight line path $\xi^\mu(t)$, $t \in [0,1]$ connecting the points $x^\mu = \xi^\mu(1)$ and $y^\mu = \xi^\mu(0)$, respectively,

$$\Phi(x,y) = e^{-iq\int_y^x d\xi_\mu A^\mu_{\rm BG}(\xi)}. \quad (25)$$

The translational invariant part of the propagator is expressed in momentum space $D_E(p)$, and for the case of a uniform electric field, it takes the form [19,20]

$$D_E(p) = \int_0^\infty ds \frac{e^{-s[\frac{\tanh(qEs)}{qEs}p_\parallel^2 + p_\perp^2 + m^2]}}{\cosh(qEs)}, \quad (26)$$

where s is the Schwinger proper time parameter, and the subscript E stands for "electric." Here, the momentum

$$p^2 = p_\parallel^2 + p_\perp^2, \quad (27)$$

is in Euclidean metric, with parallel and transverse components defined by

$$p_\parallel^2 = p_4^2 + p_3^2, \quad (28)$$

$$p_\perp^2 = p_1^2 + p_2^2. \quad (29)$$

A. Propagator in the weak electric-field regime

Restricting ourselves to the weak electric-field regime ($|qE| \ll m^2$), the background propagator is given by the series expansion (see Appendix A for details)

$$D_E(p) = \frac{1}{p^2+m^2} + (qE)^2\left[\frac{-1}{(p^2+m^2)^3} + \frac{2p_\parallel^2}{(p^2+m^2)^4}\right]. \quad (30)$$

As both contributions to the interacting Lagrangian from Eq. (17) are added separately, regular procedures of perturbation theory can be carried out for both interactions without further complications. In particular, we shall obtain the self-energy and the vertex corrections at first and second order in the couplings λ and $\tilde{\Delta}$, respectively.

B. Propagator in the strong electric-field regime

A more general representation of the scalar propagator for a finite electric field is given by the infinite series [19]

$$D_E(p) = 2\sum_{l=0}^{\infty}(-1)^l \frac{e^{-\frac{p_\parallel^2}{qE}} L_l(2p_\parallel^2/(qE))}{p_\perp^2 + (2l+1)qE + m^2}. \quad (31)$$

The gap between the lowest level $l = 0$, and the first excited level $l = 1$ grows with the magnitude of the electric field. Therefore, in the very strong field limit $|qE|/m^2 \gg 1$, the propagator will be dominated by the $l = 0$ mode, and hence

$$D_E(p) \approx 2\frac{e^{-\frac{p_\parallel^2}{qE}}}{p_\perp^2 + m_E^2}, \quad (32)$$

where

$$m_E^2 = qE + m^2 \quad (33)$$

is defined as an effective electric mass.

In what follows, we shall consider the renormalization of the theory, in the presence of the electromagnetic noise, for both regimes in the magnitude of the average background electric field, i.e., (a) Very weak field $|qE|/m^2 \ll 1$, and (b) Very strong field $|qE|/m^2 \gg 1$, respectively.

IV. SELF-ENERGY AT FIRST ORDER

Expanding the averaged generating functional from Eq. (14) up to first order in the interaction Lagrangian, the two-point, time-ordered correlation function averaged over fluctuations, as defined in Eq. (11), is given by the expression

$$\overline{\langle \hat{T}\phi_\alpha(x)\phi_\beta^\dagger(y)\rangle} = \int \prod_{a=1}^{n} \mathcal{D}[\phi_a^\dagger(z), \phi_a(z)] e^{i\int d^4z \mathcal{L}_{\rm BG}} \times \left(1 + i\int d^4z \mathcal{L}_{\rm int}\right)\phi_\alpha(x)\phi_\beta^\dagger(y). \quad (34)$$

Here, the propagator for the reference system, i.e., a noninteracting scalar theory in a constant background electric field, is given by the Schwinger propagator discussed in the previous section,

$$\langle \hat{T}\phi_\alpha(x)\phi_\beta^\dagger(y)\rangle_{\rm BG} = \delta_{\alpha\beta} i D_E(x-y) = i\delta_{\alpha\beta}\Phi(x,y)\int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} D_E(p), \quad (35)$$

which is $\propto \delta_{\alpha\beta}$, and hence explicitly diagonal in the replica component space.

As $\mathcal{L}_{\rm int}$ contains the contributions from both types of interaction, proportional to the couplings λ and $\tilde{\Delta}$, respectively, the averaged dressed propagator

$$\overline{\langle \hat{T}\phi_\alpha(x)\phi_\beta^\dagger(y)\rangle} = i\delta_{\alpha\beta}\Phi(x,y)\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} D_{\rm int}(p) \quad (36)$$

satisfies a Dyson equation. Using the property of the Schwinger phase $\Phi(x,z)\Phi(z,y) = \Phi(x,y)$, the Dyson equation in momentum space is given by

$$iD_{\rm int}(p) = iD_E(p) + iD_E(p)[-i\Sigma(p)]iD_{\rm int}(p). \quad (37)$$

Here, the self-energy is given by the sum of two contributions: the first one corresponds to the λ-interaction, while the second one is associated to the effective interaction ($\tilde{\Delta}$-interaction) induced after averaging over background electric fluctuations,

$$\Sigma(p,E) = \Sigma_\lambda(p,E) + \Sigma_\Delta(p,E). \quad (38)$$

Notice that the self-energy has an explicit dependence on the background electric field $E = |\mathbf{E}|$, which will become relevant when we renormalize the theory.

A. First-order in λ contribution to the self-energy

We now show that the λ-contribution to the self-energy is obtained in a straightforward way. After Eq. (34), the first order in λ correction to the averaged propagator is given by the expression

$$\overline{\langle \hat{T}\phi_\alpha(x)\phi_\beta^\dagger(y)\rangle}_\lambda = -\frac{i\lambda}{4}\int d^4z \langle \hat{T}\phi_a^\dagger(z)\phi_a(z)\phi_b^\dagger(z)\phi_b(z)\phi_\alpha(x)\phi_\beta^\dagger(y)\rangle\rangle_{\rm BG}, \quad (39)$$

where the expectation value of the field product is defined for a reference system given by the Lagrangian with a constant background electric field $\mathcal{L}_{\rm BG}$, as defined in Eq. (17). Then, applying Wick's theorem, the contractions of the fields are factored as usual, where each contraction is expressed in terms of the reference system Schwinger propagator Eq. (35).

Following this procedure for each contraction, and integrating over z, we find that the self-energy contribution arising from the λ-interaction is given by

$$\Sigma_\lambda(p,E) = -\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4} D_E(k), \quad (40)$$

which is independent of the external momentum and corresponds to the typical one loop correction for $\lambda\phi^4$ theory shown in Fig. 1. The difference in the present case lies in the fact that the propagator is now dressed by the effect of the background electric field, as defined in Eq. (30).

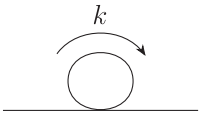


FIG. 1. Feynman diagram for the first order in λ-correction to the propagator.

B. First-order in Δ contribution to the self-energy

For the effective interaction induced by the statistical average over electromagnetic fluctuations, the procedure is very similar to the λ case already discussed, but we must also consider the effect of the covariant derivative $D_\mu(x) = \partial_\mu + iqA^\mu_{\mathrm{BG}}(x)$ on the currents defined by Eq. (9). The action of this operator over the propagator is as follows:

$$D_\mu(x) iD_E(x-y) = \Phi(x,y) \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} p \cdot iD_E(p). \tag{41}$$

Taking this property into account, and following a similar procedure as in the previous case, the self-energy contribution due to averaging over background electromagnetic fluctuations is depicted as a Feynman diagram in Fig. 2, and given by the analytical expression

$$\Sigma_\Delta(p,E) = -\frac{2i\tilde{\Delta}}{m^2} \int \frac{d^4k}{(2\pi)^4} D_E(k)(p-k)^2. \tag{42}$$

By joining the two contributions at first order in λ and $\tilde{\Delta}$, respectively, we notice that the total self-energy is a complex quantity, with its real and imaginary parts given by

$$\mathrm{Re}[\Sigma] = \Sigma_\lambda(p,E), \tag{43}$$

$$\mathrm{Im}[\Sigma] = \Sigma_\Delta(p,E). \tag{44}$$

V. SELF-ENERGY COUNTERTERMS AND RENORMALIZED PARAMETERS

The self-energy contributions defined by Eqs. (40) and (42), respectively, are computed in detail in Appendix B. As a result, we find that the electric-field independent contributions to the self-energy will result in logarithmically divergent integrals, for which we use dimensional regularization (see Appendix B for details). Then, we must proceed in the framework of renormalized perturbation theory in order to renormalize the mass and couplings in the Lagrangian.

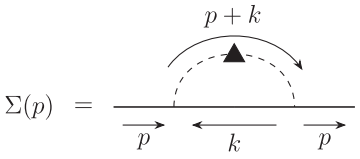


FIG. 2. Self-energy diagram at first order in $\tilde{\Delta}$.

We first start by rescaling the bare field, introducing the renormalization factor Z as

$$\phi_a^{(0)}(x) = \sqrt{Z}\phi_a(x), \tag{45}$$

where $\phi_a(x)$ stands for the renormalized field components. Then, the rescaled field is introduced in the Lagrangian, which is rewritten in terms of the bare (m_0, λ_0, $\tilde{\Delta}_0$) and physical parameters (m^2, $\tilde{\Delta}$, λ), as shown in Appendix C. In particular, we define the following counterterms for the mass, field strength, and couplings, respectively,

$$\delta_Z = Z - 1, \tag{46}$$

$$\delta_{m^2} = m_0^2 Z - m^2, \tag{47}$$

$$\delta_\lambda = \lambda_0 Z^2 - \lambda, \tag{48}$$

$$\delta_{\tilde{\Delta}} = \tilde{\Delta}_0 Z^2 - \tilde{\Delta}. \tag{49}$$

Taking these counterterms into account, the renormalized self-energy is given, up to one-loop, by the expression in Euclidean metric,

$$-i\Sigma^R(p,E) = -i\Sigma(p,E) - i(p^2\delta_Z + \delta_{m^2}). \tag{50}$$

We define the renormalization conditions as follows:

$$\lim_{p^2\to m^2} \Sigma^R(p,E) = \lim_{p^2\to m^2} \Sigma(p,E) - (m^2\delta_Z + \delta_{m^2}) = 0 \tag{51}$$

and

$$\lim_{p^2\to m^2} \frac{\partial}{\partial p^2}\Sigma^R(p,E) = \lim_{p^2\to m^2} \frac{\partial}{\partial p^2}\Sigma(p,E) - \delta_Z = 0. \tag{52}$$

A. Weak-field limit $|qE|/m^2 \ll 1$

As shown in Appendix B, the total self-energy at first-order in the couplings λ and $\tilde{\Delta}$ is given by

$$\Sigma(p,E) = \frac{1}{(4\pi)^2}\left[\frac{\lambda}{2} + \frac{2i\tilde{\Delta}}{m^2}(p^2-m^2)\right] \times \left\{ m^2\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 1\right] + \frac{(qE)^2}{6m^2}\right\}, \tag{53}$$

where $\epsilon = 4 - d$ is the dimensional regularization parameter, and μ is the renormalization energy scale.

Solving the system formed by Eqs. (51) and (52) explicitly, we find the counterterms for the mass and the field strength, respectively, in the weak field regime as follows:

$$\delta_Z = -\frac{2i\tilde{\Delta}}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 1\right], \tag{54}$$

$$\delta_{m^2} = \frac{m^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right) + 1\right]\left[\frac{\lambda}{2} + 2i\tilde{\Delta}\right]. \tag{55}$$

We notice that both counterterms are complex due to the electromagnetic noise contribution $i\tilde{\Delta}$, and according to the definition of the renormalized parameters from Eqs. (C5)–(C8) this will give rise to complex renormalized couplings.

B. Very strong-field regime $|qE|/m^2 \gg 1$

As shown in detail in the Appendix B, the self-energy in the very strong field limit is given by the expression

$$\Sigma(p,E) = \frac{1}{(4\pi)^2}\frac{(qE)}{m^2}[4i\tilde{\Delta}(p^2 - m^2) - m^2\lambda] \times \left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m_E^2}\right)\right], \tag{56}$$

with m_E^2 as defined in Eq. (33). In this case, we also define the renormalized self-energy according to Eq. (50). Then, after applying the renormalization conditions defined in Eqs. (51) and (52), we obtain explicit expressions for the counterterms in the very strong-field limit $|qE|/m^2 \gg 1$

$$\delta_Z = -\frac{4i\tilde{\Delta}}{(4\pi)^2}\frac{(qE)}{m^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m_E^2}\right)\right], \tag{57}$$

$$\delta_{m^2} = \frac{(qE)}{(4\pi)^2}[\lambda + 4i\tilde{\Delta}]\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m_E^2}\right)\right]. \tag{58}$$

VI. SPECTRAL DENSITY ANALYSIS

To analyze the effects of the background and noisy electric fields over the physical mass, we must compute the spectral densities of the theory. The dressed propagator is defined by the Dyson equation (37), where the self-energy will have real and imaginary contributions as defined in Eq. (44). The spectral density will be defined by the means of the Källén-Lehmann representation of the two point function as

$$\rho_{\rm int}(p) = -\frac{1}{\pi}\mathrm{Im}[D_{\rm int}(p)]. \tag{59}$$

This definition is valid in Minkowski space, and hence we revert the propagators and self-energy contributions to Minkowski metric by performing a Wick rotation on the 4-momentum as $p_4 \to -ip_0$. Notice that this changes the definition of the parallel component of the momentum as

$$p_\parallel^2 \to -p_0^2 + p_3^2. \tag{60}$$

We will also compare the effects of the interaction with the reference system, for which we will define the spectral density

$$\rho_E(p) = -\frac{1}{\pi}\mathrm{Im}[D_E(p)]. \tag{61}$$

Here, $D_E(p)$ corresponds to the reference system propagator, either for the weak, Eq. (30), or the strong, Eq. (32), electric field cases, respectively.

A. Weak electric-field regime $|qE|/m^2 \ll 1$

In this case, the renormalized self-energy is given by

$$\Sigma^R(p,E) = \frac{1}{(4\pi)^2}\frac{(qE)^2}{6m^2}\left[\frac{\lambda}{2} + \frac{2i\tilde{\Delta}}{m^2}(p^2 - m^2)\right]. \tag{62}$$

The corresponding spectral density for the dressed propagator is shown in Fig. 3, as a function of the momentum scale p_0, considering for the sake of the example the numerical parameters $\lambda = 2,\ 06$, $\tilde{\Delta} = 5 \times 10^{-7}$, the pion mass $m_\pi = 140$ MeV, and an average background electric field $|qE| = 10^{-6}$ MeV^2. The spectral density plot shows the emergence of quasi-particle states, with well-defined mass and lifetime, the later being inversely proportional to $\tilde{\Delta}$.

The real contribution to the self-energy slightly shifts the resonance to the right, as shown by the inset plot in Fig. 3, meaning that the pole mass is increased by the effect of the self-interaction. On the other hand, the effect of the imaginary part of the self-energy, given by the fluctuations, is observed by the spectral broadening of the resonance, thus taking the form of a Lorentzian distribution. This resonance implies the emergence of quasiparticle states, where the width of the Lorentzian is inversely proportional

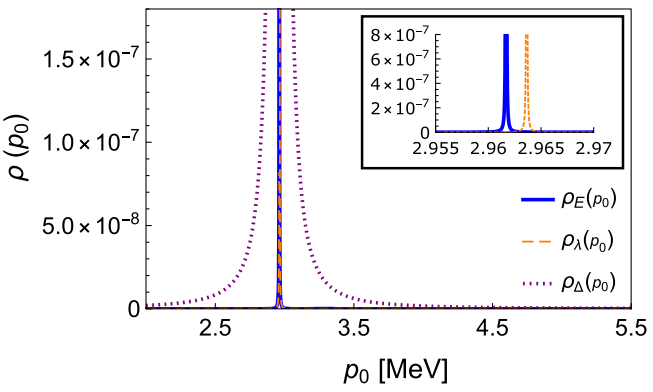


FIG. 3. Spectral density for the bare and dressed propagators for fixed values of: mass ($m_\pi = 140$ MeV), couplings ($\lambda = 2.06$, $\tilde{\Delta} = 5 \times 10^{-7}$), and electric field ($|qE| = 10^{-6}$ MeV^2).

to the lifetime of these states, i.e., inversely proportional to the noise autocorrelation $\tilde{\Delta}$.

B. Very strong electric-field regime $|qE|/m^2 \gg 1$

In the very strong field regime $|qE|/m^2 \gg 1$, the self-energy given by Eq. (56) is UV-logarithmically divergent with no finite contributions. Therefore, after adding the counterterms δ_Z and δ_{m^2} to the renormalized Lagrangian, the corresponding renormalized self-energy vanishes. Therefore, the spectral density of the dressed propagator will not receive contributions from neither interaction at this order.

The only finite spectral density function will be given by $\rho_E(p)$. Notice that the form of the propagator in Eq. (32) cannot be described using the Källén-Lehmann representation. Nonetheless, if we evaluate the spectral density at $p_\parallel^2 = 0$, then the propagator takes a more conventional form, which in Euclidean metric results

$$D_E(p_\perp) \xrightarrow{p_\parallel^2=0} \frac{1}{p_\perp^2 + m_E^2}, \tag{63}$$

and then, the spectral density can be represented using Eq. (59). By the form of the propagator, we can see that the pole mass will be only modified by the value of the electric field, as defined in Eq. (33). As the magnitude of the field increases, the spectral density will approach that of a free-particle, which is consistent with a theory that exhibits asymptotic freedom, an issue that will be discussed further in Sec. VIII B.

VII. VERTEX CORRECTIONS UP TO SECOND ORDER

As we expand the four-point function up to second order in the interaction Lagrangian, we find three contributions; one for each separate coupling λ and $\tilde{\Delta}$, respectively, and a mixing term proportional to both of them. As we are interested in a comparison of both interaction effects, for the sake of simplicity we shall ignore the mixing term in our present analysis.

The vertex amplitude for both couplings is given by the similar diagrams as for the usual $\lambda\phi^4$ theory, as shown in Fig. 4. In particular, the correction to the vertex is given by the sum of all s, t, and u channels, such that we define the vertex amplitude for the coupling $g = (\lambda, \tilde{\Delta})$ as

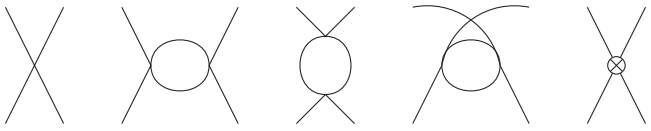


FIG. 4. Diagrams contributing to the four-point vertex amplitude.

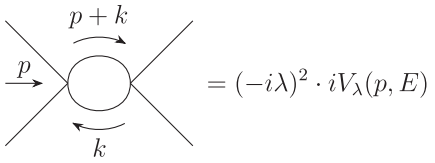


FIG. 5. s-channel contribution to the self-interaction vertex correction at second order in λ.

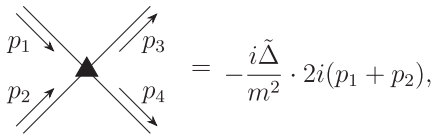


FIG. 6. Feynman rule for the $\tilde{\Delta}$-vertex at tree-level.

$$i\mathcal{M}_g(p_1 p_2 \to p_3 p_4) = -ig + (-ig)^2[iV_g(4m^2, E) + 2iV_g(0, E)] + i\delta_g. \tag{64}$$

Here, $V_g(p, E)$ represents the vertex function, to be defined later for each coupling explicitly, and δ_g is the corresponding counterterm at the given order.

The s-channel diagram contributing to the vertex correction at second order in λ is shown in Fig. 5.

Regarding the $\tilde{\Delta}$ vertex, we must adapt the general structure of the Feynman diagrams in Fig. 2 to obtain the corresponding vertex rule as depicted in the Feynman diagram in Fig. 6.

Here, momentum conservation condition $p_1 + p_2 = p_3 + p_4$ is satisfied. Then, at each vertex we have a factor of $(2ip)$, where p is the incoming momentum. Using the Feynman rule for the interaction, the s-channel of the vertex correction is depicted in the Feynman diagram in Fig. 7.

while the t and u channels are depicted in the Feynman diagrams in Fig. 8.

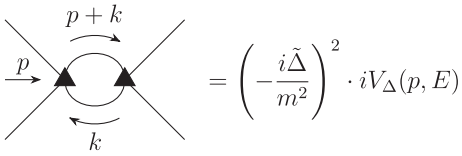


FIG. 7. s-channel diagram for the $\tilde{\Delta}$-vertex at second order.

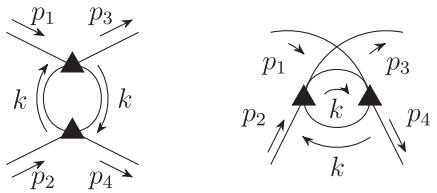


FIG. 8. t- and u-channel diagrams for the one-loop correction to the $\tilde{\Delta}$-vertex.

Finally, the vertex functions for both contributions are given by

$$iV_\lambda(p,E) = -\frac{i}{2}\cdot iV(p,E), \tag{65}$$

$$iV_\Delta(p,E) = -4ip^2\cdot iV(p,E), \tag{66}$$

where all expressions are in Euclidean metric. In particular, for the s-channel diagram, we define the generalized vertex function as

$$iV(p,E) = \int \frac{d^4k}{(2\pi)^4} iD_E(k) iD_E(p+k), \tag{67}$$

whose explicit form will depend on the specialized approximation for the propagator used for each electric-field regime.

A. Weak-field regime $|qE|/m^2 \ll 1$

As we are interested in obtaining the beta functions for the renormalized parameters in the theory, we only need to compute the divergent terms of the vertex function, which are independent of the background electric field. In particular, this contribution is given by the integral

$$iV(p,0) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{[(p+k)^2+m^2](k^2+m^2)}. \tag{68}$$

As shown in detail in Appendix D, this integral is computed using the standard technique of Feynman parameters and dimensional regularization, resulting in

$$iV(p,0) = \frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + \text{finite terms}, \tag{69}$$

where $\epsilon = 4-d$ and the finite contribution corresponds to terms regular in p.

B. Very strong-field regime $|qE|/m^2 \gg 1$

The procedure to compute the vertex function is shown in Appendix D, resulting in

$$iV(p,E) = \frac{qE}{2(4\pi p_\perp)^2}\frac{e^{-\frac{p_\parallel^2}{2qE}}}{(x_2-x_1)}\ln\left(\frac{1-x_2}{1-x_1}\cdot\frac{x_1}{x_2}\right), \tag{70}$$

where

$$x_{1,2} = \frac{1}{2} \pm \sqrt{1+\frac{4m_E^2}{p_\perp^2}}. \tag{71}$$

We remark that the vertex correction in the strong field case is free of divergences, and hence it does not contribute to the renormalization of the couplings, at least up to one loop.

C. Vertex counterterms

From the previous results, we notice that the vertex amplitude is UV-logarithmically divergent in the weak electric-field regime, but it is free of divergences in the very strong field regime. As in the case of the self-energy, the divergent integrals for the vertex function are independent of the electric field. Recalling the definition of the vertex amplitude from Eq. (64), the vertex counterterms for the couplings $g = (\lambda, \tilde{\Delta})$ are defined by

$$i\delta_g = (-ig)^2[iV_g(4m^2, E=0) + 2iV_g(0, E=0)]. \tag{72}$$

Using this definition and the divergent contribution to the vertex functions, we find the vertex counterterms in the weak electric-field regime $|qE|/m^2 \ll 1$ for both couplings as follows:

$$\delta_\lambda = -\frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right], \tag{73}$$

$$\delta_{\tilde{\Delta}} = -\frac{\tilde{\Delta}^2}{\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]. \tag{74}$$

In contrast, in the very strong electric field regime $|qE|/m^2 \gg 1$ we have

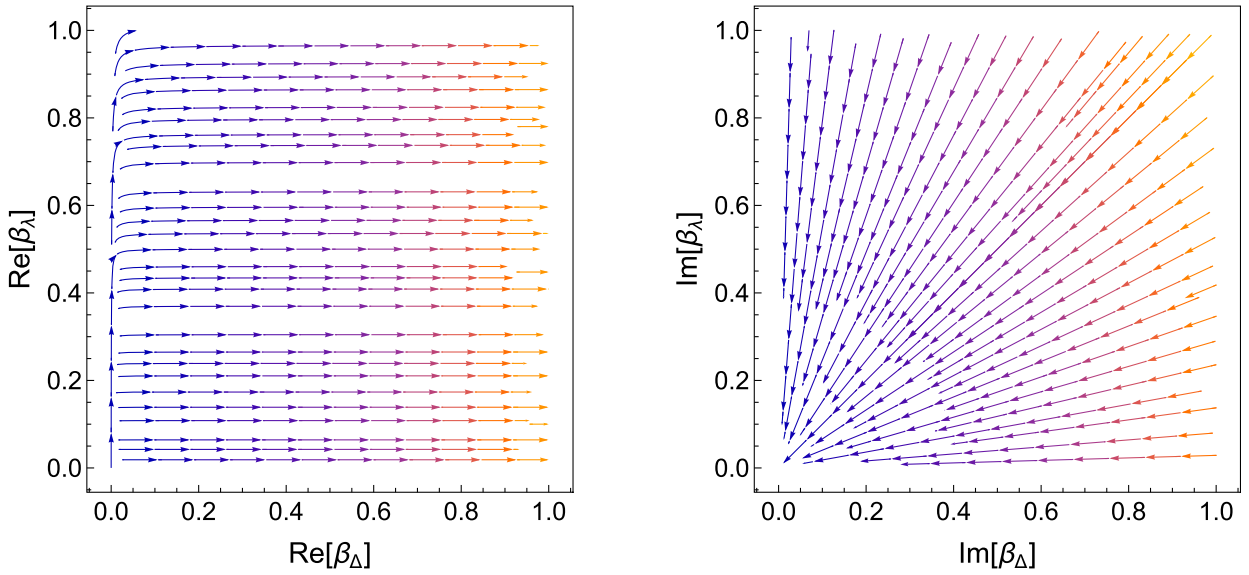
$$\delta_\lambda = \delta_{\tilde{\Delta}} = 0. \tag{75}$$

VIII. RENORMALIZATION GROUP EQUATIONS

As sown in the previous section, the vertex functions renormalize very differently depending on the electric-field intensity. In the weak electric field regime, the vertex function exhibits UV-logarithmic divergences, which are absorbed in suitable counterterms, while in the very strong electric-field regime the function is free of divergences, at least up to one-loop order. Nonetheless, the couplings will still flow with the logarithmic momentum scale, as both couplings also depend on the field-strength renormalization factor Z, after the definition in Eqs. (C7) and (C8). Therefore, for both regimes we formulate the renormalization group equations (RGE), in order to find the running couplings and their dependence on the logarithmic momentum scale. We define the beta function for each parameter $g = (\tilde{\Delta}, \lambda)$ as

$$\beta(g) = \frac{dg}{d\ln\mu}. \tag{76}$$

Then, considering the definition of the counterterms, the RGE for the couplings is



(a) Real part streamplot. (b) Imaginary part streamplot.

FIG. 9. Stream plot of the beta functions in the very weak electric-field regime $|qE|/m^2 \ll 1$.

$$\beta_g = \frac{d}{d\ln\mu}(g_0 Z^2 - \delta_g) = \frac{d}{d\ln\mu}[2g_0\delta_Z - \delta_g]. \tag{77}$$

In this section, we will study the RGE flow of the system of β functions, along with the solution of the system in both the weak and strong electric-field regimes, respectively.

A. Weak electric-field regime

Using the counterterms for the vertex from Eqs. (73) and (74), and the counterterm for the field strength factor from Eq. (54) we find the beta functions for the renormalized couplings $\lambda = \lambda(t)$ and $\tilde{\Delta} = \tilde{\Delta}(t)$ to be given by

$$\beta_{\tilde{\Delta}} = \frac{2}{\pi^2}\tilde{\Delta}^2\left(1 - \frac{i}{4}\right), \tag{78}$$

$$\beta_\lambda = \frac{3}{16\pi^2}\lambda^2 - \frac{i}{2\pi^2}\lambda\tilde{\Delta}, \tag{79}$$

where we defined the logarithmic momentum scale

$$t = \frac{1}{2}\ln\left(\frac{p^2}{\mu^2}\right). \tag{80}$$

Before finding the solution, we can study the competition between both couplings by the means of the stream plots shown in Figs. 9(a) and 9(b). This panel of figures represents a vector field with components $(\beta_{\tilde{\Delta}}, \beta_\lambda)$, where the arrows indicate the direction of the RGE flow as a function of $t = \frac{1}{2}\ln(p^2/\mu^2)$, for the real [Fig. 9(a)] and imaginary [Fig. 9(b)] parts, respectively. Notice that the scale of $\tilde{\Delta}$ has been amplified for illustration purposes. As seen in Fig. 9(a), the real parts of both couplings grow with the logarithmic momentum scale t, with λ growing faster than $\tilde{\Delta}$. The noise contribution works as a damping factor for the self-interaction, with $\tilde{\Delta}$ remaining almost constant as the energy scale increases.

The imaginary part of the stream plot, displayed in Fig. 9(b) shows the existence of a UV-stable fixed point at $\mathrm{Im}\,\tilde{\Delta} = \mathrm{Im}\,\lambda = 0$, thus implying that the imaginary part of both couplings becomes negligible at large energy scales.

Regarding the solution of the renormalization group equations, Eqs. (78) and (79) constitute a set of three coupled nonlinear differential equations that can be solved analytically, as shown in detail in Appendix E. For the running effective coupling related to the electric noise, we obtain

$$\tilde{\Delta}(t) = \frac{\tilde{\Delta}_0}{1 - \frac{2}{\pi^2}\left(1 - \frac{i}{4}\right)\tilde{\Delta}_0 t}. \tag{81}$$

The real part of the denominator defines a first Landau pole in the logarithmic momentum scale, such that the solution is applicable for $t < \pi^2/(2\tilde{\Delta}_0)$.

On the other hand, the self-interaction running coupling $\lambda(t)$ evolves as a function of the logarithmic momentum scale as follows:

$$\lambda(t) = \frac{\lambda_0(\tilde{\Delta}(t)/\tilde{\Delta}_0)^{\frac{i}{i-4}}}{1 - \frac{3\lambda_0}{16\pi^2}\frac{\pi^2}{2\tilde{\Delta}_0}[1 - (\tilde{\Delta}(t)/\tilde{\Delta}_0)^{\frac{4}{i-4}}]}. \tag{82}$$

It is important to remark that, in the noiseless limit $\tilde{\Delta}_0 \to 0$, the expression for the running self-interacting coupling reduces to the well-known result

$$\lim_{\tilde{\Delta}_0 \to 0} \lambda(t) = \frac{\lambda_0}{1 - \frac{3\lambda_0}{16\pi^2} t}, \tag{83}$$

which is valid for the logarithmic momentum scale well below the Landau pole, i.e., $t < 16\pi^2/(3\lambda_0)$. Taking into account both Landau poles, we have that the analysis is valid for

$$t < \min\left\{\frac{\pi^2}{(2\tilde{\Delta}_0)}, \frac{3\lambda_0}{(16\pi^2)}\right\}. \tag{84}$$

B. Very strong electric-field regime

Considerig the counterterm for the field strength factor in Eq. (58), the beta functions for the renormalized couplings are given by

$$\beta_{\tilde{\Delta}} = -i\frac{qE}{\pi^2 m^2}\tilde{\Delta}^2, \tag{85}$$

$$\beta_{\lambda} = -i\frac{qE}{\pi^2 m^2}\lambda\tilde{\Delta}. \tag{86}$$

The dynamics of both equations are somewhat trivial, and the RGE flow behaves the same as the imaginary part of the RGE flow in the very weak electric-field case, represented by Fig. 9(b).

Equations (85) and (86) can be easily solved by separation of variables, as shown in Appendix E 2. Then, the evolution of both couplings as a function of the logarithmic momentum scale is given by the expressions

$$\tilde{\Delta}(t) = \frac{\tilde{\Delta}_0}{1 + i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t}, \tag{87}$$

$$\begin{aligned}\lambda(t) &= \frac{\lambda_0}{\tilde{\Delta}_0}\tilde{\Delta}(t) \\ &= \frac{\lambda_0}{1 + \left(\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t\right)^2} - i\frac{\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0\lambda_0 t}{1 + \left(\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t\right)^2}.\end{aligned} \tag{88}$$

We remark that in the noiseless limit, the bare self-interaction coupling is recovered, as expected

$$\lim_{\tilde{\Delta}_0 \to 0} \lambda(t) = \lambda_0. \tag{89}$$

Furthermore, both expressions lack the presence of a Landau pole, as opposed to the weak electric-field case. Then, both couplings decay with the energy scale t, which is consistent with the emergence of asymptotic freedom at very large values of the electric field. It is instructive to appreciate this effect in terms of the real and imaginary components of the running coupling $\lambda(t)$ in this regime, after the explicit analytical result in Eq. (88). While the imaginary part of $\lambda(t)$ contributes to an spectral quasiparticle broadening, analogous to a Breit-Wigner resonance, its real part contributes to the physical self-interaction effects in the usual sense. Remarkably, as can be appreciated in Eq. (88), the combination of the strong average electric field in the regime $qE/m^2 \gg 1$, *and* its stochastic fluctuation $\tilde{\Delta}_0$, via the product of both effects, weakens the running coupling for the effective self-interaction.

IX. CONCLUSIONS

In this work, we have studied the effects of a classical electric-field background with stochastic fluctuations over a self-interacting theory of complex scalar fields. Although a toy model, it represents a technically tractable effective field theory for pions under nontrivial background conditions, such as very weak ($|qE|/m^2 \ll 1$), or very strong ($|qE|/m^2 \gg 1$) average electric fields, respectively, subjected to stochastic fluctuations. This configuration is relevant in a variety of physical scenarios, particularly in heavy-ion collisions, where an in-plane electric field emerges due to nuclear charge anisotropies, specially when asymmetric nuclei are involved.

The background electric field was introduced in the Lagrangian density via minimal coupling, as a connection in the covariant derivative. The space-time anisotropies in the background field are modeled as white noise, meaning the stochastic fluctuations satisfy a Gaussian functional distribution. Starting from the usual, complex $\lambda\phi^4$ Lagrangian, where the covariant derivative was modified to introduce the electric field with stochastic fluctuations $A^{\mu}_{\mathrm{BG}} + \delta A^{\mu}_{\mathrm{BG}}$, we followed the procedure established by some of us in previous works [28–31]; to calculate the statistical average of the generating functional, given the many different realizations of the Lagrangian for each possible configuration of the fluctuating background field. For this purpose we used the replica trick, and then integrated out the fluctuations. As a result, a new effective interaction between charged currents arises in the averaged Lagrangian density. This interaction is proportional to the magnitude of the autocorrelation function in the electromagnetic fluctuations and to the charge of the field $q^2\Delta$.

Regarding the self-energy calculations, in the weak electric-field limit, the divergent amplitudes remain the same as for the bare $\lambda\phi^4$ theory, which are reduced to the vacuum diagram. This is a consequence of the perturbative treatment of the weak electric-field propagator, which in the limit $|qE| \to 0$ recovers the expression for the free scalar propagator in vacuum. The spectral density function in this weak electric-field regime shows that the self-interaction

increases the value of the pole mass, whereas the fluctuations produce quasiparticle states of finite lifetime inversely proportional to Δ. On the other hand, our results for the strong electric-field regime show that the self-energy is directly proportional to the electric field in this limit, and free of logarithmic UV divergences. As a consequence, after we add the corresponding counterterms to the Lagrangian, the renormalized self-energy vanishes in the very strong electric-field regime. Hence, in the perturbative approach, the physical mass is not modified by the presence of electromagnetic noise.

The vertex correction at one loop allow us to find the vertex counterterms, and consequently, the renormalization group equations for both couplings. The RGE flow shows that both parameters, $\lambda(t)$ and $\tilde{\Delta}(t)$, grow with the logarithmic energy scale t in the weak electric-field regime, with the noise working as a damping factor for the self-interaction strength. In contrast, in the very strong electric-field regime, the beta functions show the emergence of asymptotic freedom, thus implying that the scalar fields behave as free particles at high energy values under these conditions. The later effect is expected from a simple heuristic argument; for two identical charged particles, a very strong background electric field will apply a local force to separate them, thus weakening the self-interacting effects and leading to an effective quasifree particle description. The physical consequences of the evolution of the self-interacting coupling λ in the presence of stochastic noise, in the context of the catalysis and inverse electric catalysis, are under current development in the effective potential formalism, and will be communicated in a separate publication. However, from the results obtained in this work, we can already anticipate some qualitative effects. As reported in [19], the spontaneous chiral symmetry breaking in the massive (with bare mass m) scalar field $\phi(x) = \sigma(x) + i\chi(x)$, when the vacuum expectation value (VEV) of $\sigma(x)$ is v, leads to finite quasiparticle masses $m_\sigma^2 = \frac{3}{4}\lambda v^2 - m^2$, and $m_\chi^2 = \frac{1}{4}\lambda v^2 - m^2$, respectively. As both masses are proportional to the coupling λ, the presence of an imaginary component in this running coupling due to electric fluctuations [see Eqs. (82) and (88)] will generate a spectral broadening, similar to a Breit-Wigner resonance, in both quasiparticles σ and χ. Moreover, in the zero temperature limit, and at the tree level, the corresponding effective potential reduces to [19] $V_{T=0}^{\text{tree}} = -\frac{1}{2}m^2v^2 + \frac{1}{16}\lambda^2 v^4$, with the VEV v acting as an order parameter, thus leading to a nontrivial solution $v^2 = 4m^2/\lambda^2$ in this limit [19], which is *inversely proportional* to λ. A similar effect can be concluded from the effective scalar theory presented in [21], that includes one-loop corrections. As we showed in the present work, in the presence of a very strong background electric field $q|\mathbf{E}|/m^2 \gg 1$, with fluctuations $\tilde{\Delta}$, the real part of the running coupling [see Eq. (88)] $\mathrm{Re}\lambda(t) = \lambda_0(1 + (\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t)^2)^{-1}$ decreases with the product of both effects. Therefore, based on the previous argument, we also expect a consistent reduction in the order parameter v, and on the masses m_σ and m_χ, in the presence of stochastic fluctuations in the strong electric-field limit. Even though this behavior may in principle suggest the full restoration of the chiral symmetry (i.e., $v \to 0$) in the limit of very strong field and noise combined $qE\tilde{\Delta}_0/m^2 \to \infty$, we should not precipitate ourselves towards this seemingly obvious, but rather superficial conclusion yet. We shall instead save it for a deeper analysis, in the context of a finite temperature effective potential model that also captures the effects of stochastic noise over the critical temperature, currently under development and to be communicated elsewhere.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THE PROPAGATOR FOR A VERY WEAK ELECTRIC FIELD

From the exact Schwinger proper-time parametrization for the scalar propagator,

$$D_E(p) = \int_0^\infty ds \frac{e^{-s[\frac{\tanh(qEs)}{qEs}p_\parallel^2 + p_\perp^2 + m^2]}}{\cosh(qEs)}, \tag{A1}$$

it is straightforward to derive an explicit power series representation in terns of the electric field $\varepsilon = qE$, as follows:

$$D_E(p,\varepsilon) = D_E(p,0) + \varepsilon\frac{\partial}{\partial\varepsilon}D_E(p,\varepsilon=0) + \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial\varepsilon^2}D_E(p,\varepsilon=0) + O(\varepsilon^3). \tag{A2}$$

From the trivial identities $\lim_{\varepsilon\to 0}\cosh(s\varepsilon) = 1$, and $\lim_{\varepsilon\to 0}\tanh(s\varepsilon)/(s\varepsilon) = 1$, we obtain the first term

$$\begin{aligned} D_E(p,0) &= \lim_{\varepsilon\to 0} D_E(p,\varepsilon) \\ &= \int_0^\infty ds\, e^{-s[p^2+m^2]} \\ &= \frac{1}{p^2+m^2}. \end{aligned} \tag{A3}$$

For the next term, we notice that

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}D_E(p,\varepsilon)=\lim_{\varepsilon\to 0}\int_0^\infty ds\, e^{-s\left[\frac{\tanh(s\varepsilon)}{s\varepsilon}p_\parallel^2+p_\perp^2+m^2\right]}\frac{s}{\cosh(s\varepsilon)}\left[-\tanh(s\varepsilon)+\frac{p_\parallel^2}{\varepsilon}\left(\frac{\tanh(s\varepsilon)}{s\varepsilon}-1\right)\right]=0. \tag{A4}$$

So the first-order contribution in the electric field vanishes exactly. Finally, the second-order term is given by

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\frac{\partial^2}{\partial\varepsilon^2}D_E(p,\varepsilon)=\int_0^\infty ds e^{-s[p^2+m^2]}\left(\frac{2}{3}p_\parallel^2 s^3-s^2\right), \tag{A5}$$

and applying the elementary identity, for $n\in\mathbb{N}_0$,

$$\int_0^\infty ds e^{-s\alpha}s^n=\frac{\Gamma(n+1)}{\alpha^{n+1}}=\frac{n!}{\alpha^{n+1}}, \tag{A6}$$

we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{2}\frac{\partial^2}{\partial\varepsilon^2}D_E(p,\varepsilon)&=\frac{2}{3}\frac{3!p_\parallel^2}{(p^2+m^2)^4}-\frac{2!}{(p^2+m^2)^3}\\&=2\left[\frac{2p_\parallel^2}{(p^2+m^2)^4}-\frac{1}{(p^2+m^2)^3}\right].\end{aligned} \tag{A7}$$

Therefore, substituting these partial results into the Taylor expansion in Eq. (A1), we obtain the final form for the weak field scalar propagator

$$\begin{aligned}D_E(p)=&\frac{1}{p^2+m^2}+(qE)^2\left[\frac{2p_\parallel^2}{(p^2+m^2)^4}-\frac{1}{(p^2+m^2)^3}\right]\\&+O(qE)^4.\end{aligned} \tag{A8}$$

APPENDIX B: INTEGRALS INVOLVED IN THE SELF-ENERGY CALCULATIONS

Recall that we defined the dimensionless coupling

$$\tilde{\Delta}=q^2m^2\Delta. \tag{B1}$$

As discussed in the main text, the total self energy involves two additive contributions proportional to each coupling

$$\begin{aligned}\Sigma(p,E)=&-\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}D_E(k)-2i\frac{\tilde{\Delta}}{m^2}\\&\times\int\frac{d^4k}{(2\pi)^4}D_E(k)(p-k)^2.\end{aligned} \tag{B2}$$

Since we considered two different limits for the scalar propagator, we shall compute the integrals in both limits separately.

1. Case (A): Very weak field limit $|qE|/m^2\ll 1$

The self-energy contribution arising from the $\lambda\phi^4$ interaction is given by

$$\begin{aligned}\Sigma_\lambda(E)=&-\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\left\{\frac{1}{k^2+m^2}\right.\\&\left.+(qE)^2\left[\frac{-1}{(k^2+m^2)^3}+\frac{2k_\parallel^2}{(k^2+m^2)^4}\right]\right\}.\end{aligned} \tag{B3}$$

The first integral is logarithmically divergent, and can be evaluated using dimensional regularization for $\epsilon=4-d\to 0^+$

$$\begin{aligned}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}&=\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\\&=\frac{1}{(4\pi)^2}\Gamma(1-\epsilon/2)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2}\\&=-\frac{m^2}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+1\right],\end{aligned} \tag{B4}$$

where $\gamma\approx 0.5772$ is the Euler-Mascheroni constant and μ is the energy scale. The second integral is finite at $d=4$, resulting in

$$\begin{aligned}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^3}&=\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)^3}\\&=\mu^\epsilon\frac{(m^2)^{d/2-3}}{(4\pi)^{d/2}}\frac{\Gamma(3-d/2)}{\Gamma(3)}\\&=\frac{1}{2(4\pi)^2m^2}.\end{aligned} \tag{B5}$$

For the remaining integral we separate the integrating measure into parallel and perpendicular components as $d^4k=d^2k_\parallel d^2k_\perp$, thus obtaining the expression

$$\int \frac{d^4k}{(2\pi)^4}\frac{2k_\parallel^2}{(k^2+m^2)^4} = \int \frac{d^2k_\perp}{(2\pi)^2}\int \frac{d^2k_\parallel}{(2\pi)^2}\frac{2k_\parallel^2}{(k_\parallel^2+k_\perp^2+m^2)^4} = \mu^{2\epsilon}\int \frac{d^dk_\perp}{(2\pi)^d}\int \frac{d^dk_\parallel}{(2\pi)^d}\frac{2k_\parallel^2}{(k_\parallel^2+k_\perp^2+m^2)^4}, \quad \text{(B6)}$$

where $d = 2-\epsilon$ in this case. Evaluating we have

$$\mu^\epsilon \int \frac{d^dk_\parallel}{(2\pi)^d}\frac{2k_\parallel^2}{(k_\parallel^2+k_\perp^2+m^2)^4} = \frac{2}{4\pi}\frac{\Gamma(2)}{\Gamma(4)}\frac{1}{(k_\perp^2+m^2)^2}, \quad \text{(B7)}$$

and then integrating over the perpendicular components we obtain

$$\mu^\epsilon \int \frac{d^dk_\perp}{(2\pi)^d}\frac{1}{(k_\perp^2+m^2)^2} = \frac{1}{4\pi m^2}. \quad \text{(B8)}$$

Therefore, the λ contribution to the self-energy is given by the expression

$$\Sigma_\lambda(E) = \frac{\lambda m^2}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+1-\frac{(qE)^2}{6m^4}\right]. \quad \text{(B9)}$$

For the noise contribution, we have

$$\Sigma_\Delta(p,E) = -2i\frac{\tilde{\Delta}}{m^2}\int \frac{d^4k}{(2\pi)^4}(p^2-2pk+k^2)\left\{\frac{1}{k^2+m^2} + (qE)^2\left[\frac{-1}{(k^2+m^2)^3}+\frac{2k_\parallel^2}{(k^2+m^2)^4}\right]\right\}. \quad \text{(B10)}$$

The integrals proportional to the external momentum are the same as the ones previously computed at (B4), (B5), and (B11). The integrals over $2pk$ vanish, since they are odd functions over an even integration interval. We are left to evaluate the integrals proportional to the factor k^2. Using dimensional regularization, the first integral reduces to the result (for $\epsilon = 4-d$ in this case)

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^2}{k^2+m^2} = \mu^\epsilon\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2+m^2} = \frac{m^4}{(4\pi)^2}\left(2-\frac{\epsilon}{2}\right)\Gamma(\epsilon/2-2)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = \frac{m^4}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+1\right]. \quad \text{(B11)}$$

In the same fashion, the second integral reduces to the result

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2+m^2)^3} = \mu^\epsilon\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2+m^2} = \frac{1}{2(4\pi)^2}\left(2-\frac{\epsilon}{2}\right)\Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = \frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)-\frac{1}{2}\right]. \quad \text{(B12)}$$

Finally, for the last integral we must separate into parallel and perpendicular components for $d = 2-\epsilon$, resulting in

$$\int \frac{d^4k}{(2\pi)^4}\frac{2k^2k_\parallel^2}{(k^2+m^2)^4} = \int \frac{d^2k_\perp}{(2\pi)^2}k_\perp^2\int \frac{d^2k_\parallel}{(2\pi)^2}\frac{k_\parallel^2}{(k_\parallel^2+k_\perp^2+m^2)^4} + \int \frac{d^2k_\perp}{(2\pi)^2}\int \frac{d^2k_\parallel}{(2\pi)^2}\frac{k_\parallel^4}{(k_\parallel^2+k_\perp^2+m^2)^4}$$
$$= \frac{1}{6(4\pi)}\mu^\epsilon\int \frac{d^dk_\perp}{(2\pi)^d}\frac{k_\perp^2}{(k_\perp^2+m^2)^2} + \frac{1}{3(4\pi)}\mu^\epsilon\int \frac{d^dk_\perp}{(2\pi)^2}\frac{1}{k_\perp^2+m^2}$$
$$= \frac{1}{6(4\pi)^2}\left(1-\frac{\epsilon}{2}\right)\Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{m^2}\right)^{(\epsilon/2)} + \frac{1}{3(4\pi)^2}\Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{m^2}\right)^{(\epsilon/2)}$$
$$= \frac{1}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)-\frac{1}{3}\right]. \quad \text{(B13)}$$

Then, the self-energy contribution due to the statistical average over electric field fluctuations is given by

$$\Sigma_\Delta(p,E) = \frac{2i\tilde{\Delta}}{(4\pi)^2}\left\{(p^2-m^2)\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi\mu^2}{m^2}\right)+1\right]+\frac{(qE)^2}{6m^4}\right\}. \quad \text{(B14)}$$

2. Case (B): Very strong field limit $|qE|/m^2 \gg 1$

In this case, we compute the self-energy contributions by using the very strong limit form of the propagator, Eq. (32),

$$\begin{aligned}\Sigma_\lambda(p,E) &= -\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4} D_E(k) \\ &= -\frac{\lambda}{2} 2 \int \frac{d^2k_\parallel}{(2\pi)^2} e^{-\frac{k_\parallel^2}{qE}} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + m_E^2} \\ &= -\lambda \frac{\pi qE}{(2\pi)^2} \mu^\epsilon \int \frac{d^dk_\perp}{(2\pi)^d} \frac{1}{k_\perp^2 + m_E^2}. \end{aligned} \tag{B15}$$

The first integral over $k_\perp$ is trivially convergent, while the second one has a logarithmic divergence that is handled by dimensional regularization in $d = 2 - \epsilon$, such that we have

$$\Sigma_\lambda(p,E) = -\lambda \frac{qE}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma - \ln\left(\frac{4\pi\mu^2}{m_E^2}\right)\right]. \tag{B16}$$

Similarly, for the other contribution to the self-energy, we have

$$\begin{aligned}\Sigma_\Delta(p,E) &= \frac{2i\tilde{\Delta}}{m^2}\int \frac{d^4k}{(2\pi)^4} D_E(k)(p^2 + 2k\cdot p + k^2) \\ &= \frac{2i\tilde{\Delta}}{m^2}\tilde{\Delta}\left\{\left[p^2 \int \frac{d^2k_\parallel}{(2\pi)^2} e^{-\frac{k_\parallel^2}{qE}} + \int \frac{d^2k_\parallel}{(2\pi)^2} e^{-\frac{k_\parallel^2}{qE}} k_\parallel^2\right]\mu^\epsilon \int \frac{d^dk_\perp}{(2\pi)^d}\frac{1}{k_\perp^2 + m_E^2} + \int \frac{d^2k_\parallel}{(2\pi)^2} e^{-\frac{k_\parallel^2}{qE}} \mu^\epsilon \int \frac{d^dk_\perp}{(2\pi)^d}\frac{k_\perp^2}{k_\perp^2 + m_E^2}\right\} \\ &= \frac{2i\tilde{\Delta}}{m^2}\left\{\frac{(p^2(qE) + (qE)^2)}{4\pi}\mu^\epsilon \int \frac{d^dk_\perp}{(2\pi)^d}\frac{1}{k_\perp^2 + m_E^2} + \frac{qE}{4\pi}\mu^\epsilon \int \frac{d^dk_\perp}{(2\pi)^d}\frac{k_\perp^2}{k_\perp^2 + m_E^2}\right\}. \end{aligned} \tag{B17}$$

Now, using the results from Eqs. (B4) and (B11), we obtain

$$\Sigma_\Delta(p,E) = \frac{4i\tilde{\Delta}}{(4\pi)^2}(p^2 - m^2)\frac{(qE)}{m^2}\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m_E^2}\right)\right]. \tag{B18}$$

Therefore, the total self-energy in the very strong field limit is given by

$$\Sigma(p,E) = \Sigma_\lambda(p,E) + \Sigma_\Delta(p,E) \tag{B19}$$

$$= \frac{1}{(4\pi)^2}\frac{(qE)}{m^2}[-\lambda m^2 + 4i\tilde{\Delta}(p^2 - m^2)]\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\mu^2}{m_E^2}\right)\right]. \tag{B20}$$

APPENDIX C: RENORMALIZED LAGRANGIAN DENSITY

As shown in the previous appendix, the integrals involved in the self-energy calculations are logarithmically divergent. Hence, we must introduce counterterms in the effective Lagrangian, which can be rewritten in terms of the bare parameters as

$$\mathcal{L}_{\text{eff}} = |D_\mu \phi_0^a|^2 - m_0^2(\phi_0^a)^\dagger \phi_0^a - \frac{1}{4}\lambda_0[(\phi_0^a)^\dagger \phi_0^a]^2 - \frac{i\tilde{\Delta}}{m^2} j_{\mu 0}^a j_{b0}^\mu, \tag{C1}$$

where the sum over replica indices a and b is implicit and the currents depend on the bare field as

$$j_\mu(x) = \phi_{a0}^\dagger(x) D_\mu \phi_0^a(x) - (D_\mu \phi_0^a(x))^\dagger \phi_0^a(x). \tag{C2}$$

Rescaling the bare field as

$$\phi_a^{(0)}(x) = \sqrt{Z}\phi_a(x), \tag{C3}$$

the Lagrangian density results in the expression

$$\mathcal{L}_{\text{eff}} = Z|D^\mu \phi^a|^2 - m_0^2 Z \phi_a^\dagger \phi^a - \frac{\lambda_0}{4} Z^2 (\phi_a^\dagger \phi_2)^2 - \frac{i\tilde{\Delta}_0}{m^2} j_\mu^a j_b^\mu. \tag{C4}$$

Then, by defining the following counterterms for the mass, field strength and couplings, respectively, as

$$\delta_Z = Z - 1, \tag{C5}$$

$$\delta_{m^2} = m_0^2 Z - m^2, \tag{C6}$$

$$\delta_\lambda = \lambda_0 Z^2 - \lambda, \tag{C7}$$

$$\delta_{\tilde{\Delta}} = \tilde{\Delta}_0 Z^2 - \tilde{\Delta}, \tag{C8}$$

the full renormalized Lagrangian is given by the expression

$$\mathcal{L}_{\text{eff}} = \sum_{a=1}^{n}\left[|D_\mu\phi_a|^2 - m(\phi_a)^\dagger\phi_a - \frac{1}{4}\lambda(\phi_a^\dagger\phi_a)^2\right] - i\tilde{\Delta}\sum_{a,b}^{n} j_{\mu,a}j^{\mu,b} + \sum_{a=1}^{n}\left[\delta_Z|D_\mu\phi_a|^2 - \delta_{m^2}\phi_a^\dagger\phi_a - \frac{1}{4}\delta_\lambda(\phi_a\phi_a)^2\right] + i\delta_{\tilde{\Delta}}\sum_{a,b}^{n} j_{\mu,a}j^{\mu,b}. \tag{C9}$$

APPENDIX D: INTEGRALS INVOLVED IN THE VERTEX CORRECTIONS

1. Case (A): Very weak field limit $|qE|/m^2 \ll 1$

As discussed in the main text, the s-channel contribution to the generalized vertex function defined in Eq. (67) is given by the integral

$$iV(p,E) = \int \frac{d^4k}{(2\pi)^4} iD_E(k) iD_E(p+k). \tag{D1}$$

Using the propagator for the weak-field limit, given by Eq. (30), the generalized vertex function is given by the expression

$$iV(p,E) = \int \frac{d^4k}{(2\pi)^4}\left\{\frac{1}{k^2+m^2} + (qE)^2\left[\frac{-1}{(k^2+m^2)^3} + \frac{2k_\parallel^2}{(k^2+m^2)^4}\right]\right\} \times \left\{\frac{1}{(p+k)^2+m^2} + (qE)^2\left[\frac{-1}{[(p+k)^2+m^2]^3} + \frac{2(p+k)_\parallel^2}{[(p+k)^2+m^2]^4}\right]\right\}. \tag{D2}$$

Since we aim at determining the beta functions for this theory, we only need to compute the divergent contribution, which is given by the integral

$$A_1 = \int \frac{d^4k}{(2\pi)^4}\frac{1}{[(p+k)^2+m^2](k^2+m^2)}. \tag{D3}$$

Using Feynman parametrization and dimensional regularization, we have

$$A_1 = \int_0^\infty dx \int \frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+\kappa)^2}, \tag{D4}$$

where

$$l = k + xp, \tag{D5}$$

$$\kappa = xp^2(1-x) + m^2. \tag{D6}$$

Using dimensional regularization with $\epsilon = 4-d \to 0^+$, we find the following expression:

$$A_1 = \frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln(\mu^2)\right] + \frac{1}{(4\pi)^2}\int_0^1 dx \ln\left(\frac{4\pi}{xp^2(1-x)+m^2}\right). \tag{D7}$$

Therefore, the divergent generalized vertex function is given by

$$iV(p^2,0) = \frac{4ip^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln(\mu^2)\right] + \text{finite terms}. \tag{D8}$$

The vertex functions for each type of interaction are given in terms of the generalized one according to the following expressions

$$iV_\lambda(p,E) = -\frac{i}{2}\cdot iV(p,E), \tag{D9}$$

$$iV_\Delta(p,E) = -4ip^2\cdot iV(p,E). \tag{D10}$$

The scattering amplitude involving the renormalized vertex for each coupling $g = (\lambda, \tilde{\Delta})$ is defined as

$$i\mathcal{M}_g(p_1p_2 \to p_3p_4) = -ig + (-ig)^2[iV_g(4m^2,E) + 2iV_g(0,E)] + i\delta_g. \tag{D11}$$

Hence, the counterterms for both couplings are given by the expressions

$$\delta_\lambda = -\frac{3\lambda^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln(\mu^2)\right], \tag{D12}$$

$$\delta_\Delta = -\frac{\tilde{\Delta}^2}{\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln(\mu^2)\right]. \tag{D13}$$

2. Case (B): Very strong field limit $|qE|/m^2 \gg 1$

In the very strong field limit, we use the propagator defined in Eq. (32). Then, the generalized vertex function defined by Eq. (67) is given by the expression

$$iV(p,E) = \int \frac{d^4k}{(2\pi)^4} \frac{2e^{-\frac{k_\parallel^2}{qE}}}{k_\perp^2 + qE + m^2} \times \frac{2e^{-\frac{(p_\parallel + k_\parallel)^2}{qE}}}{(p_\perp + k_\perp)^2 + qE + m^2}. \tag{D14}$$

Separating the integral in the perpendicular and parallel components, respectively, we get

$$iV(p,E) = 4\int \frac{d^2k_\parallel}{(2\pi)^2} e^{\frac{-k_\parallel^2 - (p_\parallel + k_\parallel)^2}{qE}} \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + m_E^2} \cdot \frac{1}{(p_\perp + k_\perp)^2 + m_E^2}. \tag{D15}$$

The first integral is quadratic in $k_\parallel$, and can be solved by completing squares as

$$k_\parallel^2 + (p_\parallel + k_\parallel)^2 = 2\left(k_\parallel + \frac{p_\parallel}{2}\right)^2 + \frac{p_\parallel^2}{2} = 2l_\parallel^2 + \frac{p_\parallel^2}{2}, \tag{D16}$$

where $l_\parallel = k_\parallel + \frac{p_\parallel}{2}$. By first computing the integral over the parallel component, we obtain

$$A_1 = \int \frac{d^2k_\parallel}{(2\pi)^2} e^{\frac{-k_\parallel^2 - (p_\parallel + k_\parallel)^2}{qE}} \tag{D17}$$

$$= e^{-\frac{p_\parallel^2}{2qE}} \int \frac{d^2k_\parallel}{(2\pi)^2} e^{-\frac{2l_\parallel^2}{qE}} = \frac{e^{-\frac{p_\parallel^2}{2qE}}}{(2\pi)^2} \frac{qE\pi}{2}. \tag{D18}$$

The secular integral over the perpendicular components is given by

$$A_2 = \int \frac{d^2k_\perp}{(2\pi)^2} \frac{1}{k_\perp^2 + m_E^2} \cdot \frac{1}{(p_\perp + k_\perp)^2 + m_E^2}, \tag{D19}$$

which can be solved with Feynman parameters as follows:

$$A_2 = \int_0^1 dx \int \frac{d^2l_\perp}{(2\pi)^2} \frac{1}{(l_\perp^2 + \kappa)^2}, \tag{D20}$$

where

$$l_\perp = k_\perp + xp_\perp, \tag{D21}$$

$$\kappa = x(1-x)p_\perp^2 + m_E^2. \tag{D22}$$

This integral is regular in two dimensions, resulting in the expression

$$A_2 = \frac{1}{4\pi} \int_0^1 dx \frac{1}{x(1-x)p_\perp^2 + m_E^2}. \tag{D23}$$

To solve this integral, we factorize the denominator as

$$x(1-x)p_\perp^2 + m_E^2 = -p_\perp^2 (x - x_1)(x - x_2), \tag{D24}$$

where

$$x_{1,2} = \frac{1}{2} \pm \sqrt{1 + \frac{4m_E^2}{p_\perp^2}}. \tag{D25}$$

Then, by partial-fraction decomposition, we get

$$A_2 = -\frac{1}{4\pi p_\perp^2} \int_0^1 dx \frac{1}{(x - x_1)(x - x_2)} = -\frac{1}{4\pi p_\perp^2 (x_2 - x_1)} \int_0^1 dx \left[\frac{1}{(x - x_2)} - \frac{1}{(x - x_1)}\right] = -\frac{1}{4\pi p_\perp^2 (x_2 - x_1)} \ln\left(\frac{1 - x_2}{1 - x_1} \cdot \frac{x_1}{x_2}\right). \tag{D26}$$

Finally, we thus obtain the following result for the generalized vertex function:

$$iV(p,E) = \frac{qE}{2(4\pi p_\perp)^2} \frac{e^{-\frac{p_\parallel^2}{2qE}}}{(x_2 - x_1)} \ln\left(\frac{1 - x_2}{1 - x_1} \cdot \frac{x_1}{x_2}\right). \tag{D27}$$

APPENDIX E: SOLUTION OF THE RUNNING COUPLINGS

1. Weak electric field regime

In this section, we consider the analytical solution to the system of differential equations for the couplings $g = (\lambda, \tilde{\Delta})$ in the very weak field regime $qE/m^2 \ll 1$, as follows:

$$\beta_{\tilde{\Delta}} = \frac{d\tilde{\Delta}}{dt} = \frac{2}{\pi^2} \tilde{\Delta}^2 \left(1 - \frac{i}{4}\right), \tag{E1}$$

$$\beta_\lambda = \frac{d\lambda}{dt} = \frac{3}{16\pi^2} \lambda^2 - \frac{i}{2\pi^2} \lambda\tilde{\Delta}. \tag{E2}$$

We start by solving Eq. (E1) by separation of variables, as follows:

$$\int_{\tilde{\Delta}_0}^{\tilde{\Delta}(t)} \frac{d\tilde{\Delta}}{\tilde{\Delta}^2} = \frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\int_0^t dt', \tag{E3}$$

such that we obtain

$$\frac{1}{\tilde{\Delta}_0} - \frac{1}{\tilde{\Delta}(t)} = \frac{2}{\pi^2}\left(1-\frac{i}{4}\right)t, \tag{E4}$$

and after some elementary algebra, we solve in favor of $\tilde{\Delta}(t)$

$$\tilde{\Delta}(t) = \frac{\tilde{\Delta}_0}{1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t}. \tag{E5}$$

Let us now consider Eq. (E2), first dividing by λ^2, and then reorganizing the terms as follows:

$$-\frac{1}{\lambda(t)^2}\frac{d\lambda}{dt} - \frac{i}{2\pi^2}\frac{\tilde{\Delta}(t)}{\lambda(t)} = -\frac{3}{16\pi^2}. \tag{E6}$$

Let us now introduce the auxiliary function

$$u(t) \equiv \frac{1}{\lambda(t)}, \tag{E7}$$

so that the differential Eq. (E6) becomes

$$\frac{du}{dt} - \frac{i}{2\pi^2}\tilde{\Delta}(t)u(t) = -\frac{3}{16\pi^2}. \tag{E8}$$

An integrating factor for this equation is the function

$$\eta(t) = \exp\left[-\frac{i}{2\pi^2}\int_0^t dt'\tilde{\Delta}(t')\right], \tag{E9}$$

such that

$$\frac{d}{dt}[u(t)\eta(t)] = -\frac{3}{16\pi^2}\eta(t). \tag{E10}$$

After integration, we then obtain the exact solution

$$\begin{aligned}\int_0^t \frac{d}{dt'}[u(t')\eta(t')]dt' &= u(t)\eta(t) - u(0) \\ &= -\frac{3}{16\pi^2}\int_0^t \eta(t')dt',\end{aligned} \tag{E11}$$

where we used the initial condition $\eta(t=0)=1$ that trivially follows from Eq. (E9). Solving for $u(t) = 1/\lambda(t)$, after some elementary algebra, we obtain

$$\frac{1}{\lambda(t)} = \frac{1}{\eta(t)}\left[\frac{1}{\lambda_0} - \frac{3}{16\pi^2}\int_0^t \eta(t')dt'\right], \tag{E12}$$

that solving explicitly for $\lambda(t)$ yields the expression

$$\lambda(t) = \frac{\lambda_0\eta(t)}{1-\frac{3}{16\pi^2}\lambda_0\int_0^t \eta(t')dt'}. \tag{E13}$$

All the effects of the electric noise over the effective coupling $\lambda(t)$ are contained in the function $\eta(t)$. Now, we solve explicitly for $\eta(t)$, using the exact solution Eq. (E5), to obtain

$$\begin{aligned}\eta(t) &= \exp\left[-\frac{i\tilde{\Delta}_0}{2\pi^2}\int_0^t \frac{dt'}{1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t'}\right] \\ &= \exp\left[\frac{i}{4-i}\ln\left(1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t\right)\right] \\ &= \left[1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t\right]^{\frac{i}{4-i}}.\end{aligned} \tag{E14}$$

Similarly, we trivially obtain the integral

$$\begin{aligned}\int_0^t \eta(t')dt' &= \int_0^t \left(1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t'\right)^{\frac{i}{4-i}}dt' \\ &= \frac{\pi^2}{2\tilde{\Delta}_0}\left[1-\left(1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t\right)^{\frac{4}{4-i}}\right].\end{aligned} \tag{E15}$$

We can finally solve explicitly for the running coupling $\lambda(t = 1/2\ln(p^2/m^2))$, by inserting Eqs. (E14) and (E15) into Eq. (E13), to obtain the final result

$$\lambda(t) = \frac{\lambda_0\left(1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t\right)^{\frac{i}{4-i}}}{1-\frac{3\lambda_0}{16\pi^2}\frac{\pi^2}{2\tilde{\Delta}_0}\left[1-\left(1-\frac{2}{\pi^2}\left(1-\frac{i}{4}\right)\tilde{\Delta}_0 t\right)^{\frac{4}{4-i}}\right]}. \tag{E16}$$

2. Very strong electric field regime

The beta functions for the couplings and the physical mass parameters, in the very strong electric field regime $qE/m^2 \gg 1$, are given by

$$\beta_{\tilde{\Delta}} = \frac{d\tilde{\Delta}}{dt} = -i\frac{qE}{\pi^2 m^2}\tilde{\Delta}^2, \tag{E17}$$

$$\beta_\lambda = \frac{d\lambda}{dt} = -i\frac{qE}{\pi^2 m^2}\lambda\tilde{\Delta}. \tag{E18}$$

The first two equations can be solved by direct integration. The solution to the first equation is obtained by separation of variables as follows:

$$\int_{\tilde{\Delta}_0}^{\tilde{\Delta}(t)} \frac{d\tilde{\Delta}}{(\tilde{\Delta})^2} = -i\frac{qE}{\pi^2 m^2}\int_0^t dt', \tag{E19}$$

thus resulting in the expression

$$\frac{1}{\tilde{\Delta}(t)} - \frac{1}{\tilde{\Delta}_0} = i\frac{qE}{\pi^2 m^2}t. \tag{E20}$$

Then, the running coupling $\tilde{\Delta}(t)$ is given by

$$\tilde{\Delta}(t) = \frac{\tilde{\Delta}_0}{1 + i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t}. \tag{E21}$$

With this solution, the second differential equation for $\lambda(t)$ reduces to the expression

$$\frac{d\lambda}{dt} = -i\frac{qE}{\pi^2 m^2}\lambda(t)\cdot\frac{\tilde{\Delta}_0}{1 + i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t}. \tag{E22}$$

This equation can also be solved by separation of variables, as follows:

$$\int_{\lambda_0}^{\lambda(t)} \frac{d\lambda}{\lambda} = -i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0\int_0^t \frac{dt'}{1 + i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t}. \tag{E23}$$

Direct integration yields the explicit result

$$\ln\left(\frac{\lambda(t)}{\lambda_0}\right) = \ln\left(\frac{1}{1 + i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t}\right). \tag{E24}$$

Finally, solving the later by exponentiation, we obtain

$$\lambda(t) = \frac{\lambda_0}{1 + i\frac{qE}{\pi^2 m^2}\tilde{\Delta}_0 t} = \frac{\lambda_0}{\tilde{\Delta}_0}\tilde{\Delta}(t). \tag{E25}$$

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