

# Explicit construction of the energy-momentum tensor in the large $N$ limit

Carlo Pagani<sup>\*</sup>

*Institute für Physik (WA THEP) Johannes Gutenberg Universität Staudingerweg  
7, 55099 Mainz, Germany*

Hidegori Sonoda<sup>†</sup>

*Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA*



(Received 24 August 2025; accepted 6 January 2026; published 23 January 2026)

We construct the energy-momentum tensor of the  $O(N)$  linear sigma model explicitly in the large  $N$  limit using the exact renormalization group (ERG) formalism. The energy-momentum tensor is obtained as a cutoff dependent functional of  $N$  scalar field variables. Our guiding principles behind the construction are twofold: first the energy-momentum tensor must satisfy the Ward identity for translation and rotation invariance, and second the energy-momentum tensor must satisfy a variant of the exact renormalization group equation. In the limit that the momentum cutoff goes to zero, our energy-momentum tensor gives the one-particle irreducible (1PI) effective action with the insertion of a single energy-momentum tensor operator. We verify that the energy-momentum tensor constructed satisfies the expected trace formula, and that the trace vanishes at the Wilson-Fisher critical point.

DOI: [10.1103/sfkr-vjkj](https://doi.org/10.1103/sfkr-vjkj)

## I. INTRODUCTION

The energy-momentum (EM) tensor encodes the conservation of energy, momentum, and angular momentum. As such, it is a key quantity in quantum field theory. As proven by Belinfante and Rosenfeld [1,2], the EM tensor can be made symmetric in the canonical formalism. The symmetry becomes manifest if one defines the EM tensor as a variation of the matter action with respect to an external background metric; the so defined EM tensor plays a fundamental role for the equation of motion of gravitational theories. The EM tensor also plays a key role in the construction of conformal field theories [3]. The issues related to renormalizability have been discussed in [4] within perturbation theory.

In this paper we construct explicitly the EM tensor of the  $O(N)$  linear sigma model in the large  $N$  limit within the framework of the exact renormalization group (ERG), also known as the functional renormalization group (fRG). In this framework we work with a Wilson action that is a functional of field variables and depends on a momentum

cutoff. Any local composite operator, regarded as an infinitesimal change of the Wilson action, is also realized as a functional dependent on the same cutoff. A variant of the Wilson action, namely the one-particle irreducible (1PI) part of the Wilson action, is particularly convenient since it becomes the effective action in the limit of the vanishing cutoff. Correspondingly, the cutoff dependent composite operator reduces to the effective action with a single insertion of the composite operator.

By definition the EM tensor must satisfy the Ward identities associated with translational and rotational invariance [5]. One of the advantages of working within the ERG framework is that we only deal with finite quantities at all steps thanks to the presence of a finite ultraviolet (UV) cutoff. The relevant Ward identities are first introduced in terms of Wilson actions and eventually translated to the associated one-particle-irreducible (1PI) actions, or effective average actions. (We refer the reader to [6–10] for a review of the ERG and the associated functionals.)

We focus on the large  $N$  limit of the  $O(N)$  linear sigma model in  $D$  dimensions where  $2 < D < 4$ . This allows us to explore a genuinely nonperturbative, albeit tractable, example. Our construction of the large  $N$  limit is particularly suitable to describe the theory near criticality, i.e., the Wilson-Fisher fixed point. We work in the flat Euclidean space. See Ref. [11] for an approach, in the context of ERG, to calculate the EM tensor in spaces endowed with a metric of a different signature. Thanks to our explicit construction we can also check conformal invariance at criticality by calculating the trace of the EM tensor, whose vanishing

<sup>\*</sup>Contact author: [carlo.pagani@guest.ung.si](mailto:carlo.pagani@guest.ung.si)

<sup>†</sup>Contact author: [h-sonoda@pobox.com](mailto:h-sonoda@pobox.com); Visiting Research Associate till 29 September 2026

*Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP<sup>3</sup>.*

amounts to conformal invariance [12]. This statement holds true also within the ERG framework [5, 13–17]. Thus, our findings confirm that the ERG is a powerful framework to discuss the realization of symmetries, despite the explicit presence of a UV cutoff, see [13–15, 18–20] for further discussions concerning the ERG and conformal invariance.

The paper is organized as follows. Section II is dedicated to a somewhat lengthy review of the ERG formalism, the EM tensor via the relevant Ward identities, and the large  $N$  limit in the ERG formalism. We hope this section makes the paper self-contained. The EM tensor is constructed explicitly in Sec. III by imposing the required Ward identities. As we shall see, this determines the EM tensor up to two coefficients, which are eventually fixed by requiring that the EM tensor satisfy the required ERG differential equation. Within the ERG framework, composite operators are introduced as infinitesimal changes of the Wilson action, and their cutoff dependence is given by the ERG differential equation. We explain all this in Sec. IV. Finally, in Sec. V, we complete the construction of the EM tensor by fixing the two constants and compute the trace. This is followed by a short Sec. VI where we take the cutoff to zero to obtain the effective action with the insertion of a single EM tensor. In Sec. VII we summarize our findings and discuss possible developments. Some technicalities are clarified in two Appendices.

We use the shorthand notation

$$\int_x = \int d^D x, \quad \delta(x) = \delta^{(D)}(x)$$

for the integrals in coordinate space, and

$$\int_p = \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) = (2\pi)^D \delta^{(D)}(p)$$

for those in the momentum space. We denote the Fourier transform of a function  $f(x)$  in the coordinate space by  $\tilde{f}(p)$  in the momentum space.

## II. REVIEW OF THE ENERGY-MOMENTUM TENSOR AND LARGE $N$

In this section we review the energy-momentum tensor and the large  $N$  limit within the ERG formalism. The energy-momentum tensor of a scalar field theory has been discussed in details within the context of a Wilson action and ERG in [5]. In order to make this paper self-contained as much as possible, we introduce a little more pedagogical approach. As for the large  $N$  limit within the ERG formalism, we would like to follow a particular method given in [21].

### A. Quick review of ERG

Let  $S_{\text{bare}}[\phi]$  be the bare action of a renormalized theory of the scalar field  $\phi$  in  $D$ -dimensional Euclidean space. We define a functional  $W_\Lambda[J]$  by the functional integral

$$e^{W_\Lambda[J]} \equiv \int [d\phi] \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right], \quad (1)$$

where  $\mathcal{R}_\Lambda(x)$  is a positive cutoff function that has a range of  $\frac{1}{\Lambda}$ , and normalized by

$$\int_x \mathcal{R}_\Lambda(x) = \Lambda^2. \quad (2)$$

We take  $\mathcal{R}_\Lambda(x)$  to be rotation invariant, dependent only on  $x^2$ . The Fourier transform of  $\mathcal{R}_\Lambda(x)$ , defined by

$$R_\Lambda(p) \equiv \int_x e^{-ipx} \mathcal{R}_\Lambda(x), \quad (3)$$

can be considered as a momentum dependent squared mass, suppressing the fluctuations of the fields with momenta less than  $\Lambda$ . We assume that  $R_\Lambda(p)$  has the  $\Lambda$  dependence given by

$$R_\Lambda(p) = \Lambda^2 R(p/\Lambda). \quad (4)$$

For example,  $R(p/\Lambda) = e^{-\frac{p^2}{\Lambda^2}}$  gives

$$\mathcal{R}_\Lambda(x) = \int_p e^{ipx} R_\Lambda(p) = \Lambda^2 e^{-\frac{1}{4}\Lambda^2 x^2} \left( \frac{\Lambda^2}{4\pi} \right)^{\frac{D}{2}}. \quad (5)$$

Hence, as far as  $W_\Lambda[J]$  is concerned, we can interpret  $\Lambda$  as an infrared (IR) cutoff. Since

$$\lim_{\Lambda \rightarrow 0+} \mathcal{R}_\Lambda(x) = 0, \quad (6)$$

we find

$$\lim_{\Lambda \rightarrow 0+} W_\Lambda[J] = \mathcal{W}[J], \quad (7)$$

where  $\mathcal{W}[J]$  is the generating functional of the connected correlation functions, defined by

$$e^{\mathcal{W}[J]} \equiv \int [d\phi] \exp \left( S_{\text{bare}}[\phi] + \int_x J(x) \phi(x) \right). \quad (8)$$

We then introduce the one-particle-irreducible (1PI) Wilson action (or effective average action)  $\Gamma_\Lambda[\Phi]$  as the Legendre transform of  $W_\Lambda[J]$

$$\begin{aligned} \Gamma_\Lambda[\Phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \Phi(x) \Phi(y) &= J(x) = \int_y \mathcal{R}_\Lambda(x-y) \Phi(y) - \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(x)}, \quad (15) \\ \equiv W_\Lambda[J] - \int_x J(x) \Phi(x), & \quad (9) \end{aligned}$$

we obtain

where

$$\Phi(x) \equiv \frac{\delta W_\Lambda[J]}{\delta J(x)}. \quad (10)$$

In the limit  $\Lambda \rightarrow 0+$ ,  $\mathcal{R}_\Lambda$  vanishes, and we obtain the effective action

$$\lim_{\Lambda \rightarrow 0+} \Gamma_\Lambda[\Phi] = \Gamma_{\text{eff}}[\Phi] \quad (11)$$

as the Legendre transform of  $\mathcal{W}[J]$ .

The cutoff dependence of  $W_\Lambda[J]$  is given by the ERG differential equation

$$\begin{aligned} -\Lambda \partial_\Lambda W_\Lambda[J] &= \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \\ &\times \left( \frac{\delta W_\Lambda[J]}{\delta J(x)} \frac{\delta W_\Lambda[J]}{\delta J(y)} + \frac{\delta^2 W_\Lambda[J]}{\delta J(x) \delta J(y)} \right). \quad (12) \end{aligned}$$

That of  $\Gamma_\Lambda[\Phi]$  is given by

$$-\Lambda \partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \mathcal{G}_\Lambda(x,y)[\Phi], \quad (13)$$

where

$$\mathcal{G}_\Lambda(x,y)[\Phi] \equiv \frac{\delta^2 W_\Lambda[J]}{\delta J(x) \delta J(y)} = \frac{\delta \Phi(x)}{\delta J(y)} = \frac{\delta \Phi(y)}{\delta J(x)} \quad (14)$$

is regarded as a functional of  $\Phi$ . Since the inverse Legendre transformation gives

$$\frac{\delta J(x)}{\delta \Phi(y)} = \mathcal{R}_\Lambda(x-y) - \frac{\delta^2 \Gamma_\Lambda[\Phi]}{\delta \Phi(x) \delta \Phi(y)}. \quad (16)$$

Hence,  $\mathcal{G}_\Lambda = \frac{\delta \Phi}{\delta J}$  is obtained as the inverse of  $\frac{\delta J}{\delta \Phi}$

$$\int_y \mathcal{G}_\Lambda(x,y)[\Phi] \left( \mathcal{R}_\Lambda(y-z) - \frac{\delta^2 \Gamma_\Lambda[\Phi]}{\delta \Phi(y) \delta \Phi(z)} \right) = \delta(x-z). \quad (17)$$

## B. Quick review of the energy-momentum tensor and its Ward identity

We assume that the bare action is invariant under translations in space. Under the infinitesimal change of field variables

$$\phi(x) \longrightarrow \phi(x) + \epsilon_\mu(x) \partial_\mu \phi(x), \quad (18)$$

the action changes as

$$\delta S_{\text{bare}}[\phi] = \int_x \epsilon_\mu(x) \partial_\mu \phi(x) \frac{\delta S_{\text{bare}}[\phi]}{\delta \phi(x)}. \quad (19)$$

Since this should vanish for constant  $\epsilon_\mu$ , we should be able to write this as

$$\delta S_{\text{bare}}[\phi] = \int_x \epsilon_\mu(x) \partial_\nu \Theta_{\nu\mu}^{\text{bare}}(x). \quad (20)$$

Since the functional integral (1) does not change under (18), we obtain

$$\begin{aligned} &\int [d\phi] \int_x \epsilon_\mu(x) \left( \partial_\nu \Theta_{\nu\mu}^{\text{bare}}(x) - \int_y \mathcal{R}_\Lambda(x-y) \partial_\mu \phi(x) \phi(y) + J(x) \partial_\mu \phi(x) \right) \\ &\times \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right] = 0. \quad (21) \end{aligned}$$

As a functional of  $J$ , we define the energy-momentum (EM) tensor  $\Theta_{\mu\nu}(x)$  by

$$\Theta_{\mu\nu}(x) e^{W_\Lambda[J]} \equiv \int [d\phi] \Theta_{\mu\nu}^{\text{bare}}(x) \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right]. \quad (22)$$

We can rewrite (21) as the Ward identity satisfied by the EM tensor

$$\partial_\nu \Theta_{\nu\mu}(x) e^{W_\Lambda[J]} = \left( \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\mu} \frac{\delta^2}{\delta J(x) \delta J(y)} - J(x) \frac{\delta}{\delta J(x)} \right) e^{W_\Lambda[J]}. \quad (23)$$

This gives

$$\partial_\nu \Theta_{\nu\mu}(x) = \int_y \mathcal{R}_\Lambda(x-y) \left\{ \frac{\partial}{\partial x_\mu} \frac{\delta^2 W_\Lambda[J]}{\delta J(x) \delta J(y)} + \frac{\delta W_\Lambda[J]}{\delta J(y)} \partial_\mu \frac{\delta W_\Lambda[J]}{\delta J(x)} \right\} - J(x) \partial_\mu \frac{\delta W_\Lambda[J]}{\delta J(x)}. \quad (24)$$

Regarding  $\Theta_{\mu\nu}(x)$  as a functional of  $\Phi$  instead of  $J$ , we can rewrite this further as

$$\partial_\nu \Theta_{\nu\mu}(x) = \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\mu} \mathcal{G}_\Lambda(x, y)[\Phi] + \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(x)} \partial_\mu \Phi(x), \quad (25)$$

where we have used (15).

So far we have only discussed the consequences of the translation invariance of the theory. We assume that the theory be invariant also under rotations. An infinitesimal rotation corresponds to

$$\epsilon_\mu(x) = \epsilon_{\mu\nu} x_\nu, \quad (26)$$

where  $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$  is an arbitrary constant antisymmetric tensor. Substituting this  $\epsilon_\mu(x)$  into (20), we obtain

$$\int_x (\Theta_{\mu\nu}^{\text{bare}}(x) - \Theta_{\nu\mu}^{\text{bare}}(x)) = 0. \quad (27)$$

Using a well known construction due to Belinfante [1] and Rosenfeld [2] (the details are found in [5] for example), we can redefine  $\Theta_{\mu\nu}(x)$ , while keeping the Ward identity (25) intact, so that the symmetry holds locally at any  $x$ :

$$\Theta_{\mu\nu}(x) = \Theta_{\nu\mu}(x). \quad (28)$$

Hence, as a consequence of the invariance of the theory under both translations and rotations, we can conclude the existence of a symmetric EM tensor  $\Theta_{\mu\nu}(x)$  that satisfies the Ward identity (25).

Note that  $\Theta_{\mu\nu}(x)$  is a functional of  $\Phi$ , and it should be more properly written as  $\Theta_{\mu\nu}(x)[\Phi]$ . In the limit  $\Lambda \rightarrow 0+$ , it becomes the effective action with a single insertion of the energy-momentum tensor operator. To be more specific, we obtain

$$\begin{aligned} & \lim_{\Lambda \rightarrow 0+} \Theta_{\mu\nu}(x)[\Phi] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1 \dots x_n} \Phi(x_1) \dots \Phi(x_n) \langle \Theta_{\mu\nu}(x) \phi(x_1) \dots \phi(x_n) \rangle^{\text{1PI}}, \end{aligned} \quad (29)$$

where the suffix 1PI denotes the 1PI part of the correlation function

$$\begin{aligned} & \langle \Theta_{\mu\nu}(x) \phi(x_1) \dots \phi(x_n) \rangle \\ & \equiv \int [d\phi] \Theta_{\mu\nu}^{\text{bare}}(x) \phi(x_1) \dots \phi(x_n) e^{S_{\text{bare}}[\phi]}. \end{aligned} \quad (30)$$

### C. Quick review of the large $N$ limit

In the  $O(N)$  linear sigma model we have  $N$  scalar fields  $\Phi^I (I = 1, \dots, N)$ , and the Ward identity (25) for the EM tensor is given by

$$\partial_\mu \Theta_{\mu\nu}(x) = \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \frac{\delta \Phi^I(x)}{\delta J^I(y)} + \partial_\nu \Phi^I(x) \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi^I(x)}, \quad (31)$$

where the repeated index  $I$  is summed from 1 to  $N$ . In the large  $N$  limit [22–29], the 1PI Wilson action is simplified as

$$\Gamma_\Lambda[\Phi] = -\frac{1}{2} \int_x \partial_\mu \Phi^I(x) \partial_\mu \Phi^I(x) + N \Gamma_{I\Lambda}[\rho], \quad (32)$$

where

$$\rho(x) \equiv \frac{1}{2N} \Phi^I(x) \Phi^I(x) \quad (33)$$

is the  $O(N)$  invariant squared scalar field of the mass dimension  $D - 2$ . (For the large  $N$  approximation within the ERG formalism, see [30–33]. We are following [21] for this part of review.) Defining

$$\sigma(x) \equiv \frac{\delta \Gamma_{I\Lambda}[\rho]}{\delta \rho(x)}, \quad (34)$$

we obtain

$$\frac{\delta \Phi^I(x)}{\delta J^I(y)} = \frac{\delta \Phi^I(y)}{\delta J^I(x)} \simeq \delta^{IJ} \mathcal{G}_\Lambda(x, y)[\sigma] + \dots, \quad (35)$$

where  $\mathcal{G}_\Lambda(x, y)[\sigma]$  is given by

$$\left(-\frac{\partial^2}{\partial y_\mu \partial y_\mu} - \sigma(y)\right) \mathcal{G}_\Lambda(x, y)[\sigma] + \int_z \mathcal{G}_\Lambda(x, z)[\sigma] \mathcal{R}_\Lambda(z - y) = \delta(x - y). \quad (36)$$

The suppressed terms in (35) do not give an order  $N$  contribution to the trace over  $I = J$ . Note that  $\sigma(x)$  has the mass dimension 2. We now define the high momentum propagator by

$$h_\Lambda(x) = \int_p e^{ipx} \tilde{h}_\Lambda(p) \equiv \int_p e^{ipx} \frac{1}{p^2 + R_\Lambda(p)} \quad (37)$$

that satisfies

$$-\partial^2 h_\Lambda(x - y) + \int_z h_\Lambda(x - z) \mathcal{R}_\Lambda(z - y) = \delta(x - y). \quad (38)$$

Using this, we can solve for  $\mathcal{G}_\Lambda(x, y)[\sigma]$  as a geometric series

$$\mathcal{G}_\Lambda(x, y)[\sigma] = h_\Lambda(x - y) + \int_{z_1} h_\Lambda(x - z_1) \sigma(z_1) h_\Lambda(z_1 - y) + \int_{z_1, z_2} h_\Lambda(x - z_1) \sigma(z_1) h_\Lambda(z_1 - z_2) \sigma(z_2) h_\Lambda(z_2 - y) + \dots \quad (39)$$

In large  $N$  the ERG differential equation (13) reduces to that of  $\Gamma_{I\Lambda}[\rho]$ , given by

$$-\Lambda \partial_\Lambda \Gamma_{I\Lambda}[\rho] = \frac{1}{2} \int_{x, y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x - y) \cdot \mathcal{G}_\Lambda(x, y)[\sigma]. \quad (40)$$

Using (35) and

$$\frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi^I(x)} = \partial^2 \Phi^I(x) + N \frac{\delta \Gamma_{I\Lambda}[\rho]}{\delta \Phi^I(x)} = \partial^2 \Phi^I(x) + \sigma(x) \Phi^I(x), \quad (41)$$

we can rewrite (31) as

$$\begin{aligned} \partial_\mu \Theta_{\mu\nu}(x) &= \int_y \mathcal{R}_\Lambda(x - y) N \frac{\partial}{\partial x_\nu} \mathcal{G}_\Lambda(x, y)[\sigma] + \partial_\nu \Phi^I(x) (\partial^2 \Phi^I(x) + \sigma(x) \Phi^I(x)) \\ &= \partial_\nu \Phi^I(x) \cdot \partial^2 \Phi^I(x) + N \left\{ \int_y \mathcal{R}_\Lambda(x - y) \frac{\partial}{\partial x_\nu} \mathcal{G}_\Lambda(x, y)[\sigma] + \partial_\nu \rho(x) \cdot \sigma(x) \right\}. \end{aligned} \quad (42)$$

The goal of this paper is to construct  $\Theta_{\mu\nu}(x)$  as a functional of  $\Phi^I$  that satisfies this identity.

We close this section by giving explicitly the interaction part  $\Gamma_{I\Lambda}[\rho]$  of the 1PI Wilson action. To be more accurate we give its Legendre transform explicitly. We define a functional of  $\sigma$  by

$$F_\Lambda[\sigma] \equiv \Gamma_{I\Lambda}[\rho] - \int_x \sigma(x) \rho(x). \quad (43)$$

Because of (34), this is a Legendre transformation. The ERG differential equation (40) gives that of  $F_\Lambda[\sigma]$  as

$$-\Lambda \partial_\Lambda F_\Lambda[\sigma] = \frac{1}{2} \int_{x, y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x - y) \cdot \mathcal{G}_\Lambda(x, y)[\sigma]. \quad (44)$$

The most general solution is

$$F_\Lambda[\sigma] = I_\Lambda[\sigma] + \bar{F}[\sigma], \quad (45)$$

where  $\bar{F}[\sigma]$  is an arbitrary functional independent of  $\Lambda$ .  $I_\Lambda[\sigma]$  is a particular solution to (44) given by

$$I_\Lambda[\sigma] = \int_x (c_\Lambda + c_{1\Lambda}\sigma(x)) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n} \int_{x_1, \dots, x_n} \sigma(x_1) \cdots \sigma(x_n) I_{n\Lambda}(x_1, \dots, x_n), \quad (46)$$

where

$$c_\Lambda \equiv \frac{1}{2} \int_q \ln(q^2 \tilde{h}_\Lambda(q)), \quad (47a)$$

$$c_{1\Lambda} \equiv \frac{1}{2} \int_q \left( \tilde{h}_\Lambda(q) - \frac{1}{q^2} \right), \quad (47b)$$

$$I_{n\Lambda}(x_1, \dots, x_n) \equiv h_\Lambda(x_1 - x_2) h_\Lambda(x_2 - x_3) \cdots h_\Lambda(x_n - x_1) \times (n \geq 2) \quad (47c)$$

Note that both  $c_\Lambda$  and  $c_{1\Lambda}$  are well defined, free from UV divergences for  $2 < D < 4$ ; they satisfy

$$\begin{aligned} -\Lambda \partial_\Lambda c_\Lambda &= \frac{1}{2} \int_q \Lambda \partial_\Lambda R_\Lambda(q) \cdot \tilde{h}_\Lambda(q) \\ &= \frac{1}{2} \int_x \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x) \cdot h_\Lambda(x), \end{aligned} \quad (48a)$$

$$\begin{aligned} -\Lambda \partial_\Lambda c_{1\Lambda} &= \frac{1}{2} \int_q \Lambda \partial_\Lambda R_\Lambda(q) \cdot \tilde{h}_\Lambda(q)^2 \\ &= \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \cdot h_\Lambda(x) h_\Lambda(y). \end{aligned} \quad (48b)$$

In the momentum space we can write

$$\begin{aligned} &\int_{x_1, \dots, x_n} \sigma(x_1) \cdots \sigma(x_n) I_{n\Lambda}(x_1, \dots, x_n) \\ &= \int_{p_1, \dots, p_n} \delta(p_1 + \cdots + p_n) \tilde{\sigma}(p_1) \cdots \tilde{\sigma}(p_n) \\ &\quad \times \int_q \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q + p_1) \tilde{h}_\Lambda(q + p_1 + p_2) \\ &\quad \cdots \tilde{h}_\Lambda(q + p_1 + \cdots + p_{n-1}), \quad (n \geq 2) \end{aligned} \quad (48c)$$

which makes the UV finiteness manifest.

In this paper we take

$$\bar{F}[\sigma] = \int_x \left( f_1 \sigma(x) + f_2 \frac{1}{2} \sigma(x)^2 \right), \quad (49)$$

where  $f_1$  has the mass dimension  $D - 2$ , and  $f_2 \geq 0$  has the mass dimension  $D - 4 < 0$ . Since the Legendre transform of  $\bar{F}[\sigma]$  is

$$\begin{aligned} \bar{\Gamma}[\rho] &\equiv \bar{F}[\sigma] + \int_x \sigma(x) \rho(x) \\ &= \int_x \left( -\frac{f_1^2}{2f_2} - \frac{f_1}{f_2} \rho(x) - \frac{1}{2f_2} \rho(x)^2 \right), \end{aligned} \quad (50)$$

we see that  $\frac{f_1}{f_2}$  is a squared mass parameter, and  $\frac{1}{f_2}$  is a quartic coupling. Both  $\frac{f_1}{f_2}$  and  $\frac{1}{f_2}$  are relevant parameters for the Gaussian theory  $\Gamma_{I\Lambda} = 0$ . For the critical theory at  $f_1 = f_2 = 0$ , however, only  $f_1$  is relevant, and  $f_2$  is irrelevant. At  $f_1 = 0$  the theory is massless irrespective of  $f_2 \geq 0$ , and it approaches the theory  $f_2 = 0$  (the Wilson-Fisher fixed point) at long distances and the theory  $f_2 = +\infty$  (the Gaussian fixed point or the massless free theory) at short distances.

### III. EXPLICIT CONSTRUCTION OF THE ENERGY-MOMENTUM TENSOR

Let us recapitulate our goal: we would like to construct a symmetric tensor  $\Theta_{\mu\nu}(x)$ , a functional of  $\Phi^I$ , that satisfies the Ward identity (42). In this section we construct  $\Theta_{\mu\nu}(x)$  inductively, and in Sec. V we complete the construction using the notion of  $\Lambda$ -dependent composite operators reviewed in Sec. IV.

We start with the energy-momentum tensor of the massless Gaussian theory (see [5] within the context of ERG)

$$\Theta_{\mu\nu}^G(x) \equiv \partial_\mu \Phi^I(x) \partial_\nu \Phi^I(x) - \frac{1}{2} \delta_{\mu\nu} \partial_\alpha \Phi^I(x) \partial_\alpha \Phi^I(x). \quad (51)$$

This satisfies

$$\partial_\mu \Theta_{\mu\nu}^G(x) = \partial^2 \Phi^I(x) \partial_\nu \Phi^I(x). \quad (52)$$

We then obtain, from (42),

$$\begin{aligned} &\partial_\mu (\Theta_{\mu\nu}(x) - \Theta_{\mu\nu}^G(x)) \\ &= N \left( -\sigma(x) \partial_\nu \frac{\delta F_\Lambda[\sigma]}{\delta \sigma(x)} + \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \mathcal{G}_\Lambda(x,y)[\sigma] \right), \end{aligned} \quad (53)$$

where we have used the inverse Legendre transformation

$$\rho(x) = -\frac{\delta F_\Lambda[\sigma]}{\delta \sigma(x)}. \quad (54)$$

It is convenient to construct  $\Theta_{\mu\nu}(x) - \Theta_{\mu\nu}^G(x)$  as a functional of  $\sigma$ . We first consider the dependence of the EM tensor on  $f_1, f_2$ .  $\mathcal{G}_\Lambda(x,y)[\sigma]$  is independent of  $f_1, f_2$ ; only  $\bar{F}[\sigma]$  in  $F_\Lambda[\sigma]$  depends on  $f_1, f_2$ . Since

$$-\partial_\nu \frac{\delta \bar{F}[\sigma]}{\delta \sigma(x)} \cdot \sigma(x) = -\partial_\nu (f_1 + f_2 \sigma(x)) \cdot \sigma(x) = -\partial_\mu \left( \delta_{\mu\nu} \frac{1}{2} f_2 \sigma(x)^2 \right), \quad (55)$$

we get the  $f_2$  dependent contribution

$$-N \delta_{\mu\nu} \frac{1}{2} f_2 \sigma(x)^2 \quad (56)$$

to  $\Theta_{\mu\nu}(x) - \Theta_{\mu\nu}^G(x)$ .

Let us denote the remaining contribution as

$$\Theta'_{\mu\nu}(x) \equiv \Theta_{\mu\nu}(x) - \Theta_{\mu\nu}^G(x) + N \delta_{\mu\nu} \frac{f_2}{2} \sigma(x)^2. \quad (57)$$

This is determined from

$$\begin{aligned} \frac{1}{N} \partial_\mu \Theta'_{\mu\nu}(x) &= \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \mathcal{G}_\Lambda(x,y)[\sigma] - \sigma(x) \partial_\nu \frac{\delta I_\Lambda[\sigma]}{\delta \sigma(x)} = \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \left\{ h_\Lambda(x-y) + \int_z h_\Lambda(x-z) \sigma(z) h_\Lambda(z-y) \right. \\ &\quad \left. + \int_{z_1, z_2} h_\Lambda(x-z_1) \sigma(z_1) h_\Lambda(z_1-z_2) \sigma(z_2) h_\Lambda(z_2-y) + \dots \right\} \\ &\quad - \sum_{n=2}^{\infty} \frac{1}{2n} \sigma(x) \frac{\partial}{\partial x_\nu} \int_{x_1, \dots, x_n} \sum_{i=1}^n \delta(x-x_i) \sigma(x_1) \cdots \widehat{\sigma(x_i)} \cdots \sigma(x_n) I_{n\Lambda}(x_1, \dots, x_n) \\ &= \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \int_z h_\Lambda(x-z) \sigma(z) h_\Lambda(z-y) \\ &\quad + \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \int_{z_1, z_2} h_\Lambda(x-z_1) \sigma(z_1) h_\Lambda(z_1-z_2) \sigma(z_2) h_\Lambda(z_2-y) - \frac{1}{2} \sigma(x) \frac{\partial}{\partial x_\nu} \int_{x_1} I_{2\Lambda}(x, x_1) \sigma(x_1) \\ &\quad + \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \int_{z_1, z_2, z_3} h_\Lambda(x-z_1) \sigma(z_1) h_\Lambda(z_1-z_2) \sigma(z_2) h_\Lambda(z_2-z_3) \\ &\quad \times \sigma(z_3) h_\Lambda(z_3-y) - \frac{1}{2} \sigma(x) \frac{\partial}{\partial x_\nu} \int_{x_1, x_2} I_{3\Lambda}(x, x_1, x_2) \sigma(x_1) \sigma(x_2) + \text{terms of higher orders in } \sigma' \text{'s}. \end{aligned} \quad (58)$$

We determine the terms of order  $\sigma$ ,  $\sigma^2$ , and  $\sigma^3$ . Then it is easy to guess the higher order terms.

### A. $\sigma$ term

We find

$$(\sigma \text{ term of (58)}) = \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} \int_z h_\Lambda(x-z) \sigma(z) h_\Lambda(z-y). \quad (59)$$

We define a scalar function  $A_\Lambda(x)$  by

$$\partial_\nu A_\Lambda(x) \equiv \partial_\nu h_\Lambda(x) \int_y \mathcal{R}_\Lambda(x-y) h_\Lambda(y) \quad (60a)$$

and

$$\int_x A_\Lambda(x) = \text{finite}. \quad (60b)$$

We then obtain

$$\begin{aligned}
\frac{\partial}{\partial x_\nu} \int_y A_\Lambda(x-y) \sigma(y) &= \int_y \partial_\nu h_\Lambda(x-y) \int_z \mathcal{R}_\Lambda(x-y-z) h_\Lambda(z) \sigma(y) \\
&= \int_z \mathcal{R}_\Lambda(x-z) \frac{\partial}{\partial x_\nu} \int_y h_\Lambda(x-y) \sigma(y) h_\Lambda(y-z) = (\sigma \text{ term}).
\end{aligned} \tag{61}$$

Thus, the contribution to  $\frac{1}{N} \Theta'_{\mu\nu}(x)$  is

$$\frac{1}{N} \Theta'_{\mu\nu}(x) \equiv \delta_{\mu\nu} \int_y A_\Lambda(x-y) \sigma(y). \tag{62}$$

### B. $\sigma^2$ term

We find

$$\begin{aligned}
(\sigma^2 \text{ term of (58)}) &= \int_{z_1, z_2} \sigma(z_1) \sigma(z_2) \int_y \mathcal{R}_\Lambda(x-y) \frac{\partial}{\partial x_\nu} (h_\Lambda(x-z_1) h_\Lambda(z_1-z_2) h_\Lambda(z_2-y)) \\
&\quad - \frac{1}{2} \sigma(x) \frac{\partial}{\partial x_\nu} \int_z h_\Lambda(x-z) \sigma(z) h_\Lambda(z-x).
\end{aligned} \tag{63}$$

Using

$$\int_y \mathcal{R}_\Lambda(x-y) h_\Lambda(y) = \delta(x) + \partial^2 h_\Lambda(x), \tag{64}$$

we obtain

$$\begin{aligned}
(\sigma^2 \text{ term}) &= \int_{z_1, z_2} \sigma(z_1) \sigma(z_2) \int_y \frac{\partial}{\partial x_\nu} h_\Lambda(x-z_1) \cdot h_\Lambda(z_1-z_2) \mathcal{R}_\Lambda(x-y) h_\Lambda(z_2-y) - \frac{1}{2} \sigma(x) \frac{\partial}{\partial x_\nu} \int_z h_\Lambda(x-z) \sigma(z) h_\Lambda(z-x) \\
&= \int_{z_1, z_2} \sigma(z_1) \sigma(z_2) \partial_\nu h_\Lambda(x-z_1) \cdot h_\Lambda(z_1-z_2) (\delta(x-z_2) + \partial^2 h_\Lambda(x-z_2)) - \sigma(x) \int_z \partial_\nu h_\Lambda(x-z) \cdot h_\Lambda(z-x) \sigma(z) \\
&= \frac{1}{2} \int_{z_1, z_2} \sigma(z_1) \sigma(z_2) h_\Lambda(z_1-z_2) \{ \partial_\nu h_\Lambda(x-z_1) \cdot \partial^2 h_\Lambda(x-z_2) + \partial^2 h_\Lambda(x-z_1) \cdot \partial_\nu h_\Lambda(x-z_2) \}.
\end{aligned} \tag{65}$$

Thus, the contribution to  $\frac{1}{N} \Theta'_{\mu\nu}(x)$  is

$$\begin{aligned}
\frac{1}{N} \Theta'_{\mu\nu}(x) &\equiv \frac{1}{2} \int_{z_1, z_2} \sigma(z_1) h_\Lambda(z_1-z_2) \sigma(z_2) \{ -\delta_{\mu\nu} \partial_\alpha h_\Lambda(x-z_1) \cdot \partial_\alpha h_\Lambda(x-z_2) \\
&\quad + \partial_\mu h_\Lambda(x-z_1) \cdot \partial_\nu h_\Lambda(x-z_2) + \partial_\nu h_\Lambda(x-z_1) \cdot \partial_\mu h_\Lambda(x-z_2) \}.
\end{aligned} \tag{66}$$

To see its UV finiteness, it is easier to consider this in the momentum space

$$\begin{aligned}
\int_x e^{-ipx} \frac{1}{N} \Theta'_{\mu\nu}(x) &= \int_{p_1, p_2} \tilde{\sigma}(p_1) \tilde{\sigma}(p_2) \delta(p_1 + p_2 - p) \\
&\quad \times \frac{1}{2} \int_q \{ -\delta_{\mu\nu} q(q+p) + q_\mu(q+p)_\nu + q_\nu(q+p)_\mu \} \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p_1) \tilde{h}_\Lambda(q+p),
\end{aligned} \tag{67}$$

where  $\tilde{\sigma}(p)$  is the Fourier transform of  $\sigma(x)$ . The integral over  $q$  is UV finite for  $2 < D < 4$  because  $\tilde{h}_\Lambda(q)$  behaves as  $1/q^2$  for large  $q$ .

Going back to the order  $\sigma$  contribution, we might think that

$$\frac{1}{N} \Theta'_{\mu\nu}(x) \equiv \frac{1}{2} \int_z \sigma(z) \{ -\delta_{\mu\nu} \partial_\alpha h_\Lambda(x-z) \partial_\alpha h_\Lambda(x-z) + 2 \partial_\mu h_\Lambda(x-z) \partial_\nu h_\Lambda(x-z) \} \tag{68}$$

would do. The integral over  $z$  is UV divergent, however. In the momentum space we find

$$\int_x e^{-ipx} \frac{1}{N} \Theta_{\mu\nu}^{(1)}(x) = \tilde{\sigma}(p) \frac{1}{2} \int_q \{-\delta_{\mu\nu} q(q+p) + q_\mu(q+p)_\nu + q_\nu(q+p)_\mu\} h_\Lambda(q) h_\Lambda(q+p), \quad (69)$$

which is UV divergent for  $D > 2$ .

### C. $\sigma^3$ term

An analogous calculation gives

$$\begin{aligned} \frac{1}{N} \Theta_{\mu\nu}^{(3)}(x) &= \frac{1}{2} \int_{z_1, z_2, z_3} \sigma(z_1) h_\Lambda(z_1 - z_2) \sigma(z_2) h_\Lambda(z_2 - z_3) \sigma(z_3) \{-\delta_{\mu\nu} \partial_\alpha h_\Lambda(x - z_1) \cdot \partial_\alpha h_\Lambda(x - z_3) \\ &\quad + \partial_\mu h_\Lambda(x - z_1) \cdot \partial_\nu h_\Lambda(x - z_3) + \partial_\nu h_\Lambda(x - z_1) \cdot \partial_\mu h_\Lambda(x - z_3)\} \end{aligned} \quad (70)$$

as the order  $\sigma^3$  contribution.

From the above calculations it is easy to guess the higher order terms. We obtain, to all orders of  $\sigma$ ,

$$\begin{aligned} \frac{1}{N} \Theta_{\mu\nu}'(x) &= \delta_{\mu\nu} \int_y A_\Lambda(x - y) \sigma(y) + \frac{1}{2} \sum_{n=2}^{\infty} \int_{z_1, \dots, z_n} \sigma(z_1) h_\Lambda(z_1 - z_2) \sigma(z_2) \cdots h_\Lambda(z_{n-1} - z_n) \sigma(z_n) \\ &\quad \times \{-\delta_{\mu\nu} \partial_\alpha h_\Lambda(x - z_1) \cdot \partial_\alpha h_\Lambda(x - z_n) + \partial_\mu h_\Lambda(x - z_1) \cdot \partial_\nu h_\Lambda(x - z_n) + \partial_\nu h_\Lambda(x - z_1) \cdot \partial_\mu h_\Lambda(x - z_n)\} \\ &= \delta_{\mu\nu} \int_y A_\Lambda(x - y) \sigma(y) + \frac{1}{2} \lim_{y \rightarrow x} \left( -\delta_{\mu\nu} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\alpha} + \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} + \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial y_\mu} \right) \\ &\quad \times \left\{ \mathcal{G}_\Lambda(x, y) [\sigma] - h_\Lambda(x - y) - \int_z h_\Lambda(x - z) \sigma(z) h_\Lambda(z - y) \right\}. \end{aligned} \quad (71)$$

In conclusion we obtain, from (57) and the above,

$$\begin{aligned} \Theta_{\mu\nu}(x) &= \partial_\mu \Phi^I(x) \partial_\nu \Phi^I(x) - \frac{1}{2} \delta_{\mu\nu} \partial_\alpha \Phi^I(x) \partial_\alpha \Phi^I(x) - N \delta_{\mu\nu} \frac{f_2}{2} \sigma(x)^2 \\ &\quad + N \left[ \delta_{\mu\nu} \left( \alpha_\Lambda + \int_y A_\Lambda(x - y) \sigma(y) \right) + (-\delta_{\mu\nu} \partial^2 + \partial_\mu \partial_\nu) \int_y \beta_\Lambda(x - y) \sigma(y) \right. \\ &\quad \left. + \frac{1}{2} \lim_{y \rightarrow x} \left( -\delta_{\mu\nu} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\alpha} + \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} + \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial y_\mu} \right) \left\{ \mathcal{G}_\Lambda(x, y) [\sigma] - h_\Lambda(x - y) - \int_z h_\Lambda(x - z) \sigma(z) h_\Lambda(z - y) \right\} \right], \end{aligned} \quad (72)$$

where the function  $A_\Lambda(x)$  is defined by (60). Please note that we have introduced two extra terms, a constant proportional to  $\delta_{\mu\nu}$  and another proportional to  $\delta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu$ . Neither contributes to  $\partial_\mu \Theta_{\mu\nu}(x)$ , and we cannot determine their coefficients  $\alpha_\Lambda$  and  $\beta_\Lambda(x - y)$  at this point.

In the next section we review and discuss cutoff dependent composite operators in large  $N$ . We will be able to determine both  $\alpha_\Lambda$  and  $\beta_\Lambda(x - y)$  by demanding that  $\Theta_{\mu\nu}(x)$  have the correct cutoff dependence.<sup>1</sup>

## IV. CUTOFF DEPENDENT COMPOSITE OPERATORS IN LARGE $N$

To introduce cutoff dependent composite operators,<sup>2</sup> we go back to the cutoff dependent generating functional  $W_\Lambda[J]$  defined by

$$e^{W_\Lambda[J]} = \int [d\phi] \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x - y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right]. \quad (73)$$

<sup>1</sup>As we will explain in Sec. V,  $\beta_\Lambda(x - y)$  is unique up to the addition of a constant multiple of  $f_2 \delta(x - y)$ .

<sup>2</sup>These were first introduced in [34]. Here we adopt a pedagogical approach.

We consider a single field  $\phi$  since the generalization to  $N$  fields is straightforward. An infinitesimal change of the bare action  $S_{\text{bare}}[\phi]$  by  $\mathcal{O}_{\text{bare}}[\phi]$  induces an infinitesimal change  $\mathcal{O}_\Lambda[J]$  of  $W_\Lambda[J]$  by

$$\mathcal{O}_\Lambda[J]e^{W_\Lambda[J]} = \int [d\phi] \mathcal{O}_{\text{bare}}[\phi] \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right]. \quad (74)$$

Regarding  $\mathcal{O}_\Lambda[J]$  as a functional of  $\Phi(x) \equiv \frac{\delta W_\Lambda[J]}{\delta J(x)}$ , we obtain a cutoff dependent composite operator  $\mathcal{O}_\Lambda[\Phi]$  (using the same symbol). The cutoff dependence of  $\mathcal{O}_\Lambda[\Phi]$  is obtained as

$$-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{O}_\Lambda[\Phi] = \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \int_{x',y'} \mathcal{G}_\Lambda(x,x')[\Phi] \frac{\delta}{\delta \Phi(x')} \mathcal{O}_\Lambda[\Phi] \frac{\tilde{\delta}}{\delta \Phi(y')} \mathcal{G}_\Lambda(y,y')[\Phi], \quad (75)$$

where  $\mathcal{G}_\Lambda(x,y)[\Phi]$  is defined by (17).<sup>3</sup> The simplest example is  $\mathcal{O}_\Lambda[\Phi] = \Phi(x)$ , which corresponds to  $\mathcal{O}_{\text{bare}}[\phi] = \phi(x)$ . The EM tensor  $\Theta_{\mu\nu}(x)$  is another example of a cutoff dependent composite operator for the choice  $\mathcal{O}_{\text{bare}}[\phi] = \Theta_{\mu\nu}^{\text{bare}}(x)$ .

Note that  $\mathcal{O}_\Lambda[\Phi]$  is a functional of  $\Phi$ . In the limit  $\Lambda \rightarrow 0+$ , we obtain the 1PI correlation functions

$$\lim_{\Lambda \rightarrow 0+} \mathcal{O}_\Lambda[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1, \dots, x_n} \Phi(x_1) \cdots \Phi(x_n) \langle \mathcal{O}\phi(x_1) \cdots \phi(x_n) \rangle^{\text{1PI}}, \quad (76)$$

where the suffix 1PI denotes the 1PI part of the correlation function

$$\langle \mathcal{O}\phi(x_1) \cdots \phi(x_n) \rangle \equiv \int [d\phi] \mathcal{O}_{\text{bare}}[\phi] \phi(x_1) \cdots \phi(x_n) e^{S_{\text{bare}}[\phi]}. \quad (77)$$

In large  $N$  we find

$$\mathcal{G}_\Lambda^{IJ}(x,y)[\Phi] \simeq \delta^{IJ} \mathcal{G}_\Lambda(x,y)[\sigma] \quad (78)$$

which, regarded as a functional of  $\sigma$ , is determined by (36). Hence, in large  $N$ , the cutoff dependence of a composite operator is given by

$$-\Lambda \partial_\Lambda \mathcal{O}_\Lambda[\Phi] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \int_{x',y'} \mathcal{G}_\Lambda(x,x')[\sigma] \mathcal{G}_\Lambda(y,y')[\sigma] \frac{\delta^2 \mathcal{O}_\Lambda[\Phi]}{\delta \Phi^I(x') \delta \Phi^I(y')}. \quad (79)$$

If  $\mathcal{O}_\Lambda[\Phi]$  depends only on  $\rho$ , this ERG equation can be simplified further. Since

$$\frac{\delta \mathcal{O}_\Lambda[\rho]}{\delta \Phi^J(x)} = \frac{\delta \mathcal{O}_\Lambda[\rho]}{\delta \rho(x)} \frac{1}{N} \Phi^J(x), \quad (80)$$

we obtain

$$\frac{\delta^2 \mathcal{O}_\Lambda[\rho]}{\delta \Phi^J(x) \delta \Phi^K(y)} = \frac{1}{N} \delta^{JK} \delta(x-y) \frac{\delta \mathcal{O}_\Lambda[\rho]}{\delta \rho(x)} + \frac{1}{N^2} \Phi^J(x) \Phi^K(y) \frac{\delta^2 \mathcal{O}_\Lambda[\rho]}{\delta \rho(x) \delta \rho(y)} \simeq \frac{1}{N} \delta^{JK} \delta(x-y) \frac{\delta \mathcal{O}_\Lambda[\rho]}{\delta \rho(x)}. \quad (81)$$

Hence, in large  $N$ , the ERG equation is simplified as

$$-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{O}_\Lambda[\rho] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \int_z \mathcal{G}_\Lambda(x,z)[\sigma] \frac{\delta \mathcal{O}_\Lambda[\rho]}{\delta \rho(z)} \mathcal{G}_\Lambda(z,y)[\sigma]. \quad (82)$$

Note that the right-hand side involves only the first order differential with respect to  $\rho$ . Composite operators in the  $O(N)$  model have been studied before in Refs. [17,35–37].

<sup>3</sup>We derive this in Appendix A.

### A. $\sigma$ as a cutoff dependent composite operator

Let us show that  $\sigma(x)$  and its products are cutoff dependent composite operators satisfying (82). By definition (34),  $\sigma(x)$  is a functional of  $\rho$ . In the large  $N$  approximation (13) reduces to

$$-\Lambda \partial_\Lambda \Gamma_{I\Lambda}[\rho] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \mathcal{G}_\Lambda(x,y)[\sigma]. \quad (83)$$

Differentiating this with respect to  $\rho(x)$ , we obtain

$$-\Lambda \partial_\Lambda \sigma(x) \Big|_{\rho \text{ fixed}} = \frac{1}{2} \int_{y,z} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(y-z) \cdot \frac{\delta}{\delta \rho(x)} \mathcal{G}_\Lambda(y,z)[\sigma], \quad (84)$$

where  $\sigma$  is regarded as a functional of  $\rho$  on the left-hand side. Differentiating (36) with respect to  $\rho(x)$ , we obtain

$$\int_z \frac{\delta \mathcal{G}_\Lambda(y,z)[\sigma]}{\delta \rho(x)} \{ -(\partial_z^2 + \sigma(z)) \delta(z-w) + \mathcal{R}_\Lambda(z-w) \} - \mathcal{G}_\Lambda(y,w) \frac{\delta^2 \Gamma_{I\Lambda}}{\delta \rho(w) \delta \rho(x)} = 0. \quad (85)$$

Using (36), we can solve this as

$$\frac{\delta \mathcal{G}_\Lambda(y,z)[\sigma]}{\delta \rho(x)} = \int_w \mathcal{G}_\Lambda(y,w)[\sigma] \frac{\delta^2 \Gamma_{I\Lambda}[\rho]}{\delta \rho(w) \delta \rho(x)} \mathcal{G}_\Lambda(w,z)[\sigma]. \quad (86)$$

Hence, we obtain

$$-\Lambda \partial_\Lambda \sigma(x) \Big|_{\rho \text{ fixed}} = \frac{1}{2} \int_{y,z} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(y-z) \times \int_w \mathcal{G}_\Lambda(y,w)[\sigma] \frac{\delta \sigma(x)}{\delta \rho(w)} \mathcal{G}_\Lambda(w,z)[\sigma]. \quad (87)$$

Thus, as a functional of  $\rho$ ,  $\sigma(x)$  is a cutoff dependent composite operator. Moreover, any product of  $\sigma$ 's

$$\mathcal{O}_\Lambda[\sigma] \equiv \int_{x_1, \dots, x_n} C_n(x_1, \dots, x_n) \sigma(x_1) \cdots \sigma(x_n) \quad (88)$$

is also a composite operator as long as the coefficient  $C_n$  has no  $\Lambda$  dependence, because (82) involves only the first order differential with respect to  $\rho$ . For example, both

$$\frac{\partial}{\partial f_1} F_\Lambda[\sigma] = \int_x \sigma(x) \quad (89)$$

and

$$\frac{\partial}{\partial f_2} F_\Lambda[\sigma] = \frac{1}{2} \int_x \sigma(x)^2 \quad (90)$$

are cutoff dependent composite operators.

### B. A class of cutoff dependent composite operators

Let

$$\mathcal{O}_\Lambda(x)[\Phi] \equiv \frac{1}{2} \int_{x_1, x_2} C(x-x_1, x-x_2) \times \left( \frac{1}{N} \Phi^I(x_1) \Phi^I(x_2) + \mathcal{G}_\Lambda(x_1, x_2)[\sigma] \right). \quad (91)$$

As the coefficient, we take

$$C(x-x_1, x-x_2) = \int_{p_1, p_2} e^{ip_1(x-x_1) + ip_2(x-x_2)} \tilde{C}(p_1, p_2), \quad (92)$$

where  $\tilde{C}$  is a symmetric polynomial of  $p_1, p_2$  independent of  $\Lambda$ . In other words  $C$  is a sum of derivatives of  $\delta(x-x_1)\delta(x-x_2)$  with respect to  $x_1, x_2$ .

We wish to show that (91) is a  $\Lambda$ -dependent composite operator satisfying (79). As a preparation, we compute

$$\begin{aligned} -\Lambda \partial_\Lambda \mathcal{G}_\Lambda(x,y)[\sigma] \Big|_{\rho \text{ fixed}} &= -\Lambda \partial_\Lambda \mathcal{G}_\Lambda(x,y)[\sigma] \Big|_{\sigma \text{ fixed}} \\ &+ \int_z (-\Lambda \partial_\Lambda \sigma(z))_{\rho \text{ fixed}} \frac{\delta \mathcal{G}_\Lambda(x,y)[\sigma]}{\delta \sigma(z)} \\ &= -\Lambda \partial_\Lambda \mathcal{G}_{\Lambda; -p,q}(x,y)[\sigma] \Big|_{\sigma \text{ fixed}} \\ &+ \frac{1}{2} \int_{u,v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u-v) \\ &\times \int_z \mathcal{G}_\Lambda(u,z) \frac{\delta \mathcal{G}_\Lambda(x,y)}{\delta \rho(z)} \mathcal{G}_\Lambda(z,v), \end{aligned} \quad (93)$$

where we have used (87). Differentiating (36) with respect to  $\Lambda$ , we obtain

$$\begin{aligned} &\int_y (-\Lambda \partial_\Lambda \mathcal{G}_\Lambda(x,y)[\sigma]) \{ (-\partial_y^2 - \sigma(y)) \delta(y-z) + \mathcal{R}_\Lambda(y-z) \} \\ &= \int_y \mathcal{G}_\Lambda(x,y)[\sigma] \Lambda \partial_\Lambda \mathcal{R}_\Lambda(y-z). \end{aligned} \quad (94)$$

This gives

$$\begin{aligned} &-\Lambda \partial_\Lambda \mathcal{G}_\Lambda(x,y)[\sigma] \Big|_{\sigma \text{ fixed}} \\ &= \int_{u,v} \mathcal{G}_\Lambda(x,u) \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u-v) \mathcal{G}_\Lambda(v,y). \end{aligned} \quad (95)$$

Hence,

$$-\Lambda \partial_\Lambda \mathcal{G}_\Lambda(x, y)[\sigma] \Big|_{\rho \text{ fixed}} = \int_{u, v} \mathcal{G}_\Lambda(x, u)[\sigma] \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u - v) \mathcal{G}_\Lambda(v, y)[\sigma] + \frac{1}{2} \int_{u, v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u - v) \int_z \mathcal{G}_\Lambda(u, z) \frac{\delta \mathcal{G}_\Lambda(x, y)}{\delta \rho(z)} \mathcal{G}_\Lambda(z, v). \quad (96)$$

We therefore obtain

$$\begin{aligned} -\Lambda \partial_\Lambda \mathcal{O}_\Lambda(x)[\Phi] &= \frac{1}{2} \int_{x_1, x_2} C(x - x_1, x - x_2)(-) \Lambda \partial_\Lambda \mathcal{G}_\Lambda(x_1, x_2) \Big|_{\rho \text{ fixed}} \\ &= \frac{1}{2} \int_{x_1, x_2} C(x - x_1, x - x_2) \int_{u, v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u - v) \left\{ \mathcal{G}_\Lambda(x_1, u) \mathcal{G}_\Lambda(v, x_2) + \frac{1}{2} \int_z \mathcal{G}_\Lambda(u, z) \frac{\delta \mathcal{G}_\Lambda(x_1, x_2)}{\delta \rho(z)} \mathcal{G}_\Lambda(z, v) \right\}. \end{aligned} \quad (97)$$

We now compare this with

$$\begin{aligned} \frac{1}{2} \int_{u, v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u - v) \int_{z, w} \mathcal{G}_\Lambda(u, z) \mathcal{G}_\Lambda(v, w) \frac{\delta^2 \mathcal{O}_\Lambda(x)}{\delta \Phi^I(z) \delta \Phi^I(w)} &\simeq \frac{1}{2} \int_{u, v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u - v) \int_{z, w} \mathcal{G}_\Lambda(u, z) \mathcal{G}_\Lambda(v, w) \int_{x_1, x_2} C(x - x_1, x - x_2) \\ &\quad \times \left( \delta(z - x_1) \delta(w - x_2) + \frac{1}{2} \frac{\delta^2}{\delta \Phi^I(z) \delta \Phi^I(w)} \mathcal{G}_\Lambda(x_1, x_2)[\sigma] \right) \\ &\simeq \frac{1}{2} \int_{u, v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u - v) \int_{x_1, x_2} C(x - x_1, x - x_2) \\ &\quad \times \left\{ \mathcal{G}_\Lambda(u, x_1) \mathcal{G}_\Lambda(v, x_2) + \frac{1}{2} \int_z \mathcal{G}_\Lambda(u, z) \mathcal{G}_\Lambda(v, z) \frac{\delta \mathcal{G}_\Lambda(x_1, x_2)}{\delta \rho(z)} \right\}, \end{aligned} \quad (98)$$

where we have used

$$\frac{\delta^2}{\delta \Phi^I(z) \delta \Phi^I(w)} \mathcal{G}_\Lambda(x_1, x_2)[\sigma] \simeq \delta(z - w) \frac{\delta \mathcal{G}_\Lambda(x_1, x_2)}{\delta \rho(w)}, \quad (99)$$

valid in large  $N$ . Hence,  $\mathcal{O}_\Lambda(x)[\Phi]$  has the correct  $\Lambda$  dependence (79).

There is one subtlety, however. The integral

$$\frac{1}{2} \int_{x_1, x_2} C(x - x_1, x - x_2) \mathcal{G}_\Lambda(x_1, x_2)[\sigma] \quad (100)$$

may not converge if  $C$  involves too many derivatives. Suppose  $C(x - x_1, x - x_2)$  involves two derivatives so that  $\tilde{C}(p_1, p_2)$  is a quadratic polynomial. Expanding in powers of  $\sigma$ , we obtain

$$\begin{aligned} &\int_x e^{-ipx} \frac{1}{2} \int_{x_1, x_2} C(x - x_1, x - x_2) \mathcal{G}_\Lambda(x_1, x_2)[\sigma] \\ &= \frac{1}{2} \int_{p_1, p_2} \delta(p_1 + p_2 - p) \tilde{C}(p_1, p_2) \times \left[ \tilde{h}_\Lambda(p_1) \delta(p) + \tilde{h}_\Lambda(p_1) \right. \\ &\quad \times \left. \left\{ \tilde{\sigma}(p) + \int_{q_1, q_2} \delta(q_1 + q_2 - p) \tilde{\sigma}(q_1) \tilde{h}_\Lambda(-p_1 + q_1) \tilde{\sigma}(q_2) + \dots \right\} \tilde{h}_\Lambda(p_2) \right]. \end{aligned} \quad (101)$$

The first two terms

$$a_\Lambda = \frac{1}{2} \int_{p_1, p_2} \delta(p_1 + p_2) \tilde{C}(p_1, p_2) \tilde{h}_\Lambda(p_1) = \frac{1}{2} \int_q \tilde{C}(-q, q) \tilde{h}_\Lambda(q), \quad (102a)$$

$$\tilde{b}_\Lambda(p) = \frac{1}{2} \int_{p_1, p_2} \delta(p_1 + p_2 - p) \tilde{C}(p_1, p_2) \tilde{h}_\Lambda(p_1) \tilde{h}_\Lambda(p_2) = \frac{1}{2} \int_q \tilde{C}(-q, q + p) \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q + p) \quad (102b)$$

may be UV divergent for  $2 < D < 4$ . But we can still define them as the solution to the differential equations

$$-\Lambda \partial_\Lambda a_\Lambda = \frac{1}{2} \int_q \tilde{C}(-q, q) \tilde{f}_\Lambda(q), \quad (103a)$$

$$-\Lambda \partial_\Lambda \tilde{b}_\Lambda(p) = \frac{1}{2} \int_q \tilde{C}(-q, q + p) (\tilde{f}_\Lambda(q) \tilde{h}_\Lambda(q + p) + \tilde{h}_\Lambda(q) \tilde{f}_\Lambda(q + p)), \quad (103b)$$

where

$$\tilde{f}_\Lambda(q) \equiv -\Lambda \partial_\Lambda \tilde{h}_\Lambda(q) = \Lambda \partial_\Lambda R_\Lambda(q) \cdot \tilde{h}_\Lambda(q)^2. \quad (104)$$

The right-hand sides of (103) are absolutely convergent because  $\tilde{f}_\Lambda(q)$  decays fast for large  $q$ .

### C. An example

We consider the simplest example from the class of composite operators discussed in Sec. B above. We choose

$$C(x_1, x_2) = \delta(x - x_1) \delta(x - x_2) \quad (105)$$

to obtain

$$\begin{aligned} [\rho(x)]_\Lambda &\equiv \frac{1}{2N} \Phi'(x) \Phi'(x) + a_\Lambda + \frac{1}{2} \int_z h_\Lambda(x - z) \sigma(z) h_\Lambda(z - x) \\ &\quad + \frac{1}{2} \lim_{x' \rightarrow x} \left\{ \mathcal{G}_\Lambda(x, x') - h_\Lambda(x - x') - \int_z h_\Lambda(x - z) \sigma(z) h_\Lambda(z - x') \right\} \\ &= \frac{1}{2N} \Phi'(x) \Phi'(x) + a_\Lambda + \frac{1}{2} \lim_{x' \rightarrow x} \{ \mathcal{G}_\Lambda(x, x') - h_\Lambda(x - x') \}, \end{aligned} \quad (106)$$

where  $a_\Lambda$  is determined by

$$-\Lambda \partial_\Lambda a_\Lambda = \frac{1}{2} \int_q \tilde{f}_\Lambda(q), \quad (107)$$

and

$$\int_z h_\Lambda(x - z) \sigma(z) h_\Lambda(z - x) \quad (108)$$

is finite. Solving (107) we obtain

$$a_\Lambda = \frac{1}{2-D} \frac{1}{2} \int_q \tilde{f}_\Lambda(q) = \frac{1}{2} \int_q \left( \tilde{h}_\Lambda(q) - \frac{1}{q^2} \right) = c_{1\Lambda}, \quad (109)$$

where  $c_{1\Lambda}$  is defined in (47b).

We now show that  $[\rho(x)]_\Lambda$  can in fact be written in terms of  $\sigma$  only, and that it vanishes at  $f_1 = f_2 = 0$ . We recall that the inverse Legendre transformation from  $F_\Lambda[\sigma]$  to  $\Gamma_{I\Lambda}[\rho]$  gives

$$\rho(x) = -\frac{\delta F_\Lambda[\sigma]}{\delta \sigma(x)} = -f_1 - f_2 \sigma(x) - \frac{\delta I_\Lambda[\sigma]}{\delta \sigma(x)}, \quad (110)$$

where  $I_\Lambda[\sigma]$  is defined by (46). Hence, we obtain

$$\begin{aligned} -f_1 - f_2 \sigma(x) &= \rho(x) + \frac{\delta I_\Lambda[\sigma]}{\delta \sigma(x)} = \rho(x) + c_{1\Lambda} \\ &\quad + \frac{1}{2} \sum_{n=1}^{\infty} \int_{x_1, \dots, x_n} \sigma(x_1) \cdots \sigma(x_n) I_{n\Lambda}(x, x_1, \dots, x_n) \\ &= \rho(x) + c_{1\Lambda} + \frac{1}{2} \lim_{x' \rightarrow x} \{ \mathcal{G}_\Lambda(x, x') - h_\Lambda(x - x') \} \\ &= [\rho(x)]_\Lambda. \end{aligned} \quad (111)$$

Generalizing this result, we find that any  $\Lambda$ -dependent composite operator written in terms of  $\rho$  can be expressed in terms of  $\sigma$ . Those involving a term like  $\partial_\mu \Phi^I \partial_\nu \Phi^I$  are not of this type.

### D. Equation of motion operators

There is a special class of cutoff dependent composite operators, called equation of motion operators. [9,34,38] They correspond to bare fields of the type

$$\mathcal{E}_{\text{bare}}(x) = -e^{-S_{\text{bare}}[\phi]} \frac{\delta}{\delta\phi(x)} (\mathcal{O}_{\text{bare}}[\phi] e^{S_{\text{bare}}[\phi]}). \quad (112)$$

This has the correlation function

$$\begin{aligned} \langle \mathcal{E}(x) \phi(x_1) \cdots \phi(x_n) \rangle &\equiv \int [d\phi] \mathcal{E}_{\text{bare}}(x) \phi(x_1) \cdots \phi(x_n) e^{S_{\text{bare}}[\phi]} \\ &= \int [d\phi] \mathcal{O}_{\text{bare}}[\phi] \frac{\delta}{\delta\phi(x)} \{ \phi(x_1) \cdots \phi(x_n) \} e^{S_{\text{bare}}[\phi]} \\ &= \sum_{i=1}^n \delta(x - x_i) \langle \mathcal{O} \phi(x_1) \cdots \widehat{\phi(x_i)} \cdots \phi(x_n) \rangle, \end{aligned} \quad (113)$$

where  $\phi(x_i)$  is replaced by  $\mathcal{O}$ . These operators were originally introduced by Wegner [39] and called redundant operators since they only introduce change of field variables but keep the theory unchanged.

Corresponding to  $\mathcal{E}_{\text{bare}}(x)$ , we obtain the cutoff dependent composite operator  $\mathcal{E}_\Lambda(x)$

$$\begin{aligned} \mathcal{E}_\Lambda(x) e^{W_\Lambda[J]} &\equiv \int [d\phi] \mathcal{E}_{\text{bare}}(x) \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right] \\ &= \int [d\phi] \mathcal{O}_{\text{bare}}[\phi] \left( - \int_y \mathcal{R}_\Lambda(x-y) \phi(y) + J(x) \right) \exp \left[ S_{\text{bare}}[\phi] - \frac{1}{2} \int_{x,y} \mathcal{R}_\Lambda(x-y) \phi(x) \phi(y) + \int_x J(x) \phi(x) \right] \\ &= \left\{ J(x) \mathcal{O}_\Lambda[J] - \int_y \mathcal{R}_\Lambda(x-y) \left( \frac{\delta \mathcal{O}_\Lambda[J]}{\delta J(y)} + \mathcal{O}_\Lambda[J] \frac{\delta W_\Lambda[J]}{\delta J(y)} \right) \right\} e^{W_\Lambda[J]}, \end{aligned} \quad (114)$$

where  $\mathcal{O}_\Lambda$  is the cutoff dependent composite operator given by (74). Regarding  $\mathcal{E}_\Lambda(x)$  as a functional of  $\Phi(x) \equiv \frac{\delta W_\Lambda[J]}{\delta J(x)}$ , we obtain

$$\mathcal{E}_\Lambda(x)[\Phi] = -\mathcal{O}_\Lambda[\Phi] \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(x)} - \int_{y,z} \mathcal{R}_\Lambda(x-y) \mathcal{G}_\Lambda(y,z)[\Phi] \frac{\delta \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(z)}. \quad (115)$$

The choice  $\mathcal{O}_{\text{bare}}[\phi] = \phi(x)$  gives the simplest example of  $\mathcal{E}_\Lambda[\Phi]$

$$\mathcal{N}_\Lambda(x) \equiv -\Phi(x) \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(x)} - \int_y \mathcal{R}_\Lambda(x-y) \mathcal{G}_\Lambda(x,y)[\Phi]. \quad (116)$$

We call this a number operator because it counts the number of scalar fields in the correlation function (113).

For the linear sigma model, the number operator is generalized to

$$\mathcal{N}_\Lambda(x) \equiv -\Phi^I(x) \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi^I(x)} - \int_y \mathcal{R}_\Lambda(x-y) \mathcal{G}_\Lambda^H(x,y)[\Phi]. \quad (117)$$

In large  $N$ , we obtain

$$\begin{aligned} \mathcal{N}_\Lambda(x) &= -\Phi^I(x) \partial^2 \Phi^I(x) - 2N\rho(x)\sigma(x) \\ &\quad - N \int_y \mathcal{R}_\Lambda(x-y) \mathcal{G}_\Lambda(x,y)[\sigma]. \end{aligned} \quad (118)$$

By construction this should have the correct cutoff dependence given by (79). In the remaining part we show that  $\mathcal{N}_\Lambda(x)$  consists of terms proportional to  $\sigma(x)$ ,  $\sigma(x)^2$ , and one of the cutoff dependent composite operators discussed in Sec. IV B.

Using

$$\rho(x) = -\frac{\delta F_\Lambda[\sigma]}{\delta \sigma(x)} = -f_1 - f_2 \sigma(x) - \frac{\delta I_\Lambda[\sigma]}{\delta \sigma(x)}, \quad (119)$$

we obtain

$$\mathcal{N}_\Lambda(x) = -\Phi^I(x)\partial^2\Phi^I(x) + 2N(f_1\sigma(x) + f_2\sigma(x)^2) + 2N\sigma(x)\frac{\delta I_\Lambda}{\delta\sigma(x)} - N\int_y \mathcal{R}_\Lambda(x-y)\mathcal{G}_\Lambda(x,y)[\sigma]. \quad (120)$$

Since

$$\begin{aligned} & -2\sigma(x)\frac{\delta I_\Lambda[\sigma]}{\delta\sigma(x)} + \int_y \mathcal{R}_\Lambda(x-y)\mathcal{G}_\Lambda(x,y)[\sigma] \\ &= \int_y \mathcal{R}_\Lambda(y)h_\Lambda(y) + \int_{y,z} \mathcal{R}_\Lambda(x-y)h_\Lambda(x-z)\sigma(z)h_\Lambda(z-y) - 2c_{1\Lambda}\sigma(x) \\ & \quad + \lim_{y \rightarrow x} \frac{1}{2}(\partial_x^2 + \partial_y^2) \left( \mathcal{G}_\Lambda(x,y)[\sigma] - h_\Lambda(x-y) - \int_z h_\Lambda(x-z)\sigma(z)h_\Lambda(z-y) \right), \end{aligned} \quad (121)$$

where (38) is used, we obtain

$$\begin{aligned} \mathcal{N}_\Lambda(x) &= 2N(f_1\sigma(x) + f_2\sigma(x)^2) - \Phi^I(x)\partial^2\Phi^I(x) + N \left[ a_\Lambda + \int_y b_\Lambda(x-y)\sigma(y) \right. \\ & \quad \left. - \lim_{y \rightarrow x} \frac{1}{2}(\partial_x^2 + \partial_y^2) \left( \mathcal{G}_\Lambda(x,y)[\sigma] - h_\Lambda(x-y) - \int_z h_\Lambda(x-z)\sigma(z)h_\Lambda(z-y) \right) \right], \end{aligned} \quad (122)$$

where

$$a_\Lambda \equiv - \int_y \mathcal{R}_\Lambda(y)h_\Lambda(y) = - \int_q R_\Lambda(q)\tilde{h}_\Lambda(q), \quad (123a)$$

$$b_\Lambda(x-y) \equiv 2c_{1\Lambda}\delta(x-y) - h_\Lambda(x-y) \int_z \mathcal{R}_\Lambda(x-z)h_\Lambda(z-y). \quad (123b)$$

The part starting with  $-\Phi^I\partial^2\Phi^I$  must belong to the class of composite operators in Sec. IV B, corresponding to  $\tilde{C}(p_1, p_2) = p_1^2 + p_2^2$ . Hence,  $a_\Lambda$  and

$$\tilde{b}_\Lambda(p) \equiv \int_x e^{-ipx}b_\Lambda(x) = 2c_{1\Lambda} - \int_q R_\Lambda(q)\tilde{h}_\Lambda(q)\tilde{h}_\Lambda(q+p) \quad (124)$$

must satisfy

$$-\Lambda\partial_\Lambda a_\Lambda = \int_q q^2 \tilde{f}_\Lambda(q), \quad (125a)$$

$$-\Lambda\partial_\Lambda \tilde{b}_\Lambda(p) = \int_q q^2 (\tilde{f}_\Lambda(q)\tilde{h}_\Lambda(q+p) + \tilde{f}_\Lambda(q+p)\tilde{h}_\Lambda(q)), \quad (125b)$$

respectively.

Let us check (125a) first. Differentiating (123a), we find

$$-\Lambda\partial_\Lambda a_\Lambda = \int_q (\Lambda\partial_\Lambda R_\Lambda(q) \cdot \tilde{h}_\Lambda(q) - R_\Lambda(q)\tilde{f}_\Lambda(q)) = \int_q \tilde{f}_\Lambda(q) \left( \frac{1}{\tilde{h}_\Lambda(q)} - R_\Lambda(q) \right) = \int_q q^2 \tilde{f}_\Lambda(q), \quad (126)$$

verifying (125a). We next differentiate (124) to find

$$\begin{aligned}
-\Lambda \partial_\Lambda \tilde{b}_\Lambda(p) &= \int_q [\Lambda \partial_\Lambda R_\Lambda(q) \cdot \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p) \\
&\quad - R_\Lambda(q) (\tilde{f}_\Lambda(q) \tilde{h}_\Lambda(q+p) + \tilde{h}_\Lambda(q) \tilde{f}_\Lambda(q+p)) + \tilde{f}_\Lambda(q)] \\
&= \int_q [\tilde{f}_\Lambda(q) q^2 \tilde{h}_\Lambda(q+p) + \tilde{f}_\Lambda(q+p) (-1 + q^2 \tilde{h}_\Lambda(q)) + \tilde{f}_\Lambda(q+p)] \\
&= \int_q q^2 (\tilde{f}_\Lambda(q) \tilde{h}_\Lambda(q+p) + \tilde{h}_\Lambda(q) \tilde{f}_\Lambda(q+p)), \tag{127}
\end{aligned}$$

thus verifying (125b).

## V. COMPLETION OF THE ENERGY-MOMENTUM TENSOR

In Sec. III we have constructed the energy-momentum tensor  $\Theta_{\mu\nu}(x)$  as given by (72). We are still missing two coefficients  $\alpha_\Lambda$  and  $\beta_\Lambda(x-y)$ . We wish to determine these by demanding that the energy-momentum tensor have the correct cutoff dependence (79) explained in Sec. IV.

Before starting we must ask if it is enough to know the  $\Lambda$  dependence of  $\alpha_\Lambda$  and  $\beta_\Lambda(x-y)$  to determine them completely. We start with  $\alpha_\Lambda$ . It is a constant with the mass dimension  $D$ , the same as the EM tensor  $\Theta_{\mu\nu}(x)$ . Its  $\Lambda$  independent part must be determined by the two parameters  $f_1, f_2$  of the theory. Since the mass dimension of  $f_1$  is  $D-2$  and that of  $f_2$  is  $D-4$ , the only possibility is  $\frac{f_1^2}{f_2}$ ,

which diverges at  $f_2 = 0$ . Hence, we expect that the  $\Lambda$  dependence determines  $\alpha_\Lambda$  uniquely. We next consider  $\beta_\Lambda(x-y)$ , which has the mass dimension  $2D-4$ . The only  $\Lambda$  independent possibility is  $f_2 \delta(x-y)$ , which would give a term proportional to

$$f_2 (-\delta_{\mu\nu} \partial^2 + \partial_\mu \partial_\nu) \sigma(x) \tag{128}$$

to  $\Theta_{\mu\nu}(x)$ . We thus conclude that the  $\Lambda$  dependence determines  $\beta_\Lambda(x-y)$  only up to an additive term proportional to (128). Thus, the Ward identity (42) and the symmetry (28) leaves  $\Theta_{\mu\nu}(x)$  ambiguous up to a constant multiple of (128).

We now determine the  $\Lambda$  dependence of both  $\alpha_\Lambda$  and  $\beta_\Lambda(x-y)$  by demanding that the trace  $\Theta(x) \equiv \Theta_{\mu\mu}(x)$  have the correct cutoff dependence. Equation (72) gives the trace as

$$\begin{aligned}
\frac{1}{N} \Theta(x) &= -D \frac{1}{2} f_2 \sigma(x)^2 - (D-2) \frac{1}{2N} \partial_\mu \Phi^I(x) \partial_\mu \Phi^I(x) + \alpha_{\Theta\Lambda} + \int_y b_{\Theta\Lambda}(x-y) \sigma(y) \\
&\quad + \frac{1}{2} (D-2) \lim_{y \rightarrow x} \frac{\partial^2}{\partial x_\mu \partial y_\mu} \left( \mathcal{G}_\Lambda(x, y) [\sigma] - h_\Lambda(x-y) - \int_z h_\Lambda(x-z) \sigma(z) h_\Lambda(z-y) \right), \tag{129}
\end{aligned}$$

where we have defined

$$a_{\Theta,\Lambda} \equiv D \alpha_\Lambda, \tag{130a}$$

$$b_{\Theta,\Lambda}(x-y) \equiv D A_\Lambda(x-y) - (D-1) \partial^2 \beta_\Lambda(x-y). \tag{130b}$$

The second part of the trace, starting with  $\partial_\mu \Phi^I \partial_\mu \Phi^I$ , should belong to the class of composite operators in Sec. IV B. Since

$$\lim_{y \rightarrow x} \frac{\partial^2}{\partial x_\mu \partial y_\mu} (\mathcal{G}_\Lambda(x, y) - \dots) = \frac{1}{2} \left[ \partial_x^2 \lim_{y \rightarrow x} (\mathcal{G}_\Lambda(x, y) - \dots) - \lim_{y \rightarrow x} (\partial_x^2 + \partial_y^2) (\mathcal{G}_\Lambda(x, y) - \dots) \right], \tag{131}$$

we obtain

$$\frac{1}{N} \Theta(x) = -D \frac{f_2}{2} \sigma(x)^2 - k f_2 (D-1) \partial^2 \sigma(x) - \frac{D-2}{2} \partial^2 [\rho(x)]_\Lambda - \frac{D-2}{2} \left( \frac{1}{N} \mathcal{N}_\Lambda(x) - 2 f_1 \sigma(x) - 2 f_2 \sigma(x)^2 \right), \tag{132}$$

where we have used (106) and (122). The term proportional to a dimensionless constant  $k$  comes from the additive ambiguity of  $\beta_\Lambda(x-y)$  by  $k f_2 \delta(x-y)$ .

Using (109) and (123), we obtain

$$a_{\Theta,\Lambda} = \frac{D-2}{2} \int_y \mathcal{R}_\Lambda(y) h_\Lambda(y), \quad (133a)$$

$$b_{\Theta,\Lambda}(x) = \frac{D-2}{2} \left[ -2c_{1\Lambda} \delta(x) + h_\Lambda(x) \int_y \mathcal{R}_\Lambda(x-y) h_\Lambda(y) - \partial^2 \frac{1}{2} h_\Lambda(x)^2 \right] - kf_2(D-1) \partial^2 \delta(x). \quad (133b)$$

We then obtain

$$\alpha_\Lambda = \frac{1}{D} a_{\Theta,\Lambda} = \frac{D-2}{2D} \int_y \mathcal{R}_\Lambda(y) h_\Lambda(y), \quad (134a)$$

$$\begin{aligned} -\partial^2 \beta_\Lambda(x) &= \frac{1}{D-1} (b_{\Theta,\Lambda}(x) - DA_\Lambda(x)) = \frac{D-2}{2(D-1)} \left[ -2c_{1\Lambda} \delta(x) + h_\Lambda(x) \int_y \mathcal{R}_\Lambda(x-y) h_\Lambda(y) - \frac{2D}{D-2} A_\Lambda(x) \right] \\ &\quad - \partial^2 \left\{ \frac{D-2}{2(D-1)} \frac{1}{2} h_\Lambda(x)^2 + kf_2 \delta(x) \right\}, \end{aligned} \quad (134b)$$

where  $c_{1\Lambda}$  is defined by (47b), and  $A_\Lambda(x)$  by (60). In Appendix B we solve (134b) for  $\beta_\Lambda(x)$  in the momentum space. This completes the construction of the energy-momentum tensor given by (72).

We now discuss the trace of the energy-momentum tensor. From (132), using (111), we obtain

$$\begin{aligned} \frac{1}{N} \Theta(x) &= -Df_2 \frac{1}{2} \sigma(x)^2 - kf_2(D-1) \partial^2 \sigma(x) - \frac{D-2}{2} \partial^2 (-f_1 - f_2 \sigma(x)) - \frac{D-2}{2} \left( \frac{1}{N} \mathcal{N}_\Lambda(x) - 2f_1 \sigma(x) - 2f_2 \int_q \sigma(x)^2 \right) \\ &= -\frac{D-2}{2} \frac{1}{N} \mathcal{N}_\Lambda(x) + (D-2)f_1 \sigma(x) + (D-4)f_2 \frac{1}{2} \sigma(x)^2 + \left( \frac{D-2}{2} - k(D-1) \right) f_2 \partial^2 \sigma(x). \end{aligned} \quad (135)$$

Integrating this over space, we obtain

$$\int_x \left( \Theta(x) + \frac{D-2}{2} \mathcal{N}_\Lambda(x) \right) = \left[ (D-2)f_1 \frac{\partial}{\partial f_1} + (D-4)f_2 \frac{\partial}{\partial f_2} \right] N\Gamma_{I\Lambda}[\rho], \quad (136)$$

where  $D-2$  and  $D-4$  are the scale dimensions of the parameters  $f_1, f_2$ , respectively. We have thus verified the trace formula that relates the trace of the energy momentum tensor to the scale transformation. [4]

As has been shown in [12], conformal invariance amounts to the vanishing of the trace  $\Theta(x)$  up to the second order derivative of a local operator. Within the context of ERG, we must find

$$\Theta(x) + \left( \frac{D-2}{2} + \gamma \right) \mathcal{N}_\Lambda(x) = 0 \quad (137)$$

up to the second order derivative of a composite operator, where  $\gamma$  is the anomalous dimension of the scalar field. This was first shown in [14], and subsequently discussed in [13,5]. In large  $N$  the anomalous dimension vanishes  $\gamma = 0$ , and at  $f_1 = f_2 = 0$ , (135) gives

$$\Theta(x) + \frac{D-2}{2} \mathcal{N}_\Lambda(x) = 0. \quad (138)$$

Hence, the theory at  $f_1 = f_2 = 0$  has conformal invariance.

Equation (135) for the trace of the EM tensor is appropriate in the neighborhood of the critical point  $f_1 = f_2 = 0$ . In the neighborhood of the massless Gaussian theory  $f_1 = 0, f_2 = +\infty$ , it is more convenient to rewrite (136) using the scalar field  $\rho$ . The squared mass parameter  $m_\Lambda^2$ , which can be negative, is given by

$$f_1 - f_2 m_\Lambda^2 + \frac{1}{2} \int_p \left( \frac{1}{p^2 + m_\Lambda^2 + R_\Lambda(p)} - \frac{1}{p^2} \right) = 0. \quad (139)$$

Since

$$\Gamma_{I\Lambda}[\rho] \xrightarrow{f_2 \rightarrow +\infty} \int_x \left( -m_\Lambda^2 \rho(x) - \frac{1}{2f_2} \rho(x)^2 \right) \quad (140)$$

up to an additive constant if we ignore terms suppressed by  $\frac{1}{f_2^2}$  and higher, we find

$$\sigma(x) = \frac{\delta\Gamma_{I\Lambda}[\rho]}{\delta\rho(x)} \xrightarrow{f_2 \rightarrow +\infty} -m_\Lambda^2 - \lambda\rho(x), \quad (141)$$

where  $\lambda \equiv \frac{1}{f_2^2}$ . Hence, (136) gives

$$\begin{aligned} \Theta_\Lambda(x) + \frac{D-2}{2} \mathcal{N}_\Lambda(x) &\xrightarrow{f_2 \rightarrow +\infty} N \left[ -(D-2)f_1(m_\Lambda^2 + \lambda\rho(x)) \right. \\ &\quad \left. + (D-4)\frac{1}{2\lambda}(m_\Lambda^2 + \lambda\rho(x))(m_\Lambda^2 + \lambda\rho(x)) + \left(-\frac{D-2}{2} + k(D-1)\right)\partial^2\rho(x) \right] \\ &= N \left[ -D\frac{1}{2\lambda}m_\Lambda^4 - 2m_\Lambda^2\rho(x) - (4-D)\frac{\lambda}{2}\rho(x)^2 + \left(-\frac{D-2}{2} + k(D-1)\right)\partial^2\rho(x) \right], \end{aligned} \quad (142)$$

which is a more familiar expression for the trace. We note that the scale dimensions of  $\frac{1}{\lambda}m_\Lambda^4, m_\Lambda^2, \lambda$  are respectively given by  $D, 2, 4-D$ .

## VI. THE $\Lambda \rightarrow 0+$ LIMIT OF $\Theta_{\mu\nu}$

As we have explained in Sec. IV, a cutoff dependent composite operator becomes the 1PI effective action with the insertion of a single composite operator in the vanishing cutoff limit. Let us compute the limit

$$\lim_{\Lambda \rightarrow 0+} \Theta_{\mu\nu}(x).$$

We find it easier to compute in the momentum space where

$$R_\Lambda(p) = \Lambda^2 R(p/\Lambda), \quad (143a)$$

$$\tilde{h}_\Lambda(p) = \int_x e^{-ipx} h_\Lambda(x) = \frac{1}{p^2 + R_\Lambda(p)} = \frac{1}{\Lambda^2} \frac{1}{\frac{p^2}{\Lambda^2} + R(p/\Lambda)} = \frac{1}{\Lambda^2} \tilde{h}(p/\Lambda), \quad (143b)$$

$$\tilde{f}_\Lambda(p) = \int_x e^{-ipx} f_\Lambda(x) = \frac{\Lambda \partial_\Lambda R_\Lambda(p)}{(p^2 + R_\Lambda(p))^2} = \frac{1}{\Lambda^2} \frac{(2 - p \cdot \partial_p) R(p/\Lambda)}{(\frac{p^2}{\Lambda^2} + R(p/\Lambda))^2} = \frac{1}{\Lambda^2} \tilde{f}(p/\Lambda). \quad (143c)$$

From (72) we find

$$\begin{aligned} \tilde{\Theta}_{\mu\nu}(p) &\equiv \int_x e^{-ipx} \Theta_{\mu\nu}(x) = -N \delta_{\mu\nu} f_2 \frac{1}{2} \int_q \tilde{\sigma}(p+q) \tilde{\sigma}(-q) + N [\delta_{\mu\nu} \alpha_\Lambda \delta(p) + \{\delta_{\mu\nu} \tilde{A}_\Lambda(p) + (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \tilde{\beta}_\Lambda(p)\} \tilde{\sigma}(p)] \\ &\quad + \frac{1}{2} \int_q \{-\delta_{\mu\nu} q(q+p) + q_\mu(q+p)_\nu + q_\nu(q+p)_\mu\} \\ &\quad \times [\tilde{\Phi}^I(-q) \tilde{\Phi}^I(q+p) + N \{\tilde{\mathcal{G}}_{\Lambda;q,-q-p}[\sigma] - \tilde{h}_\Lambda(q) \delta(p) - \tilde{h}_\Lambda(q) \tilde{\sigma}(p) \tilde{h}_\Lambda(q+p)\}], \end{aligned} \quad (144)$$

where

$$\begin{aligned}
\tilde{\mathcal{G}}_{\Lambda;q,-q-p}[\sigma] &\equiv \int_{x,y} e^{iqx-i(q+p)y} \mathcal{G}_{\Lambda}(x,y)[\sigma] \\
&= \tilde{h}_{\Lambda}(q)\delta(p) + \tilde{h}_{\Lambda}(q) \left[ \tilde{\sigma}(p) + \int_{p_1,p_2} \delta(p_1+p_2-p) \tilde{\sigma}(p_1) \tilde{h}_{\Lambda}(q+p_1) \tilde{\sigma}(p_2) \right. \\
&\quad \left. + \int_{p_1,p_2,p_3} \delta(p_1+p_2+p_3-p) \tilde{\sigma}(p_1) \tilde{h}_{\Lambda}(q+p_1) \tilde{\sigma}(p_2) \tilde{h}_{\Lambda}(q+p_1+p_2) \tilde{\sigma}(p_3) + \dots \right] \tilde{h}_{\Lambda}(q+p). \quad (145)
\end{aligned}$$

$\alpha_{\Lambda}$ , given by (134a), vanishes in the limit

$$\alpha_{\Lambda} = \frac{D-2}{2D} \int_q R_{\Lambda}(q) \tilde{h}_{\Lambda}(q) = \frac{D-2}{2D} \Lambda^D \int_q R(q) \tilde{h}(q) \xrightarrow{\Lambda \rightarrow 0^+} 0. \quad (146)$$

Similarly, (60) gives

$$\begin{aligned}
p_{\nu} \tilde{A}_{\Lambda}(p) &\equiv \int_x e^{-ipx} \frac{1}{i} \partial_{\nu} A_{\Lambda}(x) = \int_q (p+q)_{\nu} R_{\Lambda}(q) \tilde{h}_{\Lambda}(q) \tilde{h}_{\Lambda}(q+p) \\
&= \Lambda^{D-1} \int_q \left( q + \frac{p}{\Lambda} \right)_{\nu} R(q) \tilde{h}(q) \tilde{h} \left( q + \frac{p}{\Lambda} \right) \xrightarrow{\Lambda \rightarrow 0^+} p_{\nu} \frac{\Lambda^D}{p^2} \int_q R(q) \tilde{h}(q) \longrightarrow 0. \quad (147)
\end{aligned}$$

(134b) gives

$$\begin{aligned}
p^2 \tilde{\beta}_{\Lambda}(p) &\equiv \int_x e^{-ipx} (-\partial^2) \beta_{\Lambda}(x) = \frac{D-2}{2(D-1)} \left[ -2c_{1\Lambda} + \int_q R_{\Lambda}(q) \tilde{h}_{\Lambda}(q) \tilde{h}_{\Lambda}(q+p) - \frac{2D}{D-2} \tilde{A}_{\Lambda}(p) \right] \\
&\quad + p^2 \left\{ \frac{D-2}{4(D-1)} \int_q \tilde{h}_{\Lambda}(q) \tilde{h}_{\Lambda}(q+p) + kf_2 \right\}, \quad (148)
\end{aligned}$$

where (47b) gives

$$2c_{1\Lambda} = \int_q \left( \tilde{h}_{\Lambda}(q) - \frac{1}{q^2} \right) = \Lambda^{D-2} \int_q \left( \tilde{h}(q) - \frac{1}{q^2} \right) \xrightarrow{\Lambda \rightarrow 0^+} 0, \quad (149)$$

and

$$\int_q R_{\Lambda}(q) \tilde{h}_{\Lambda}(q) \tilde{h}_{\Lambda}(q+p) = \Lambda^{D-2} \int_q R(q) \tilde{h}(q) \tilde{h} \left( q + \frac{p}{\Lambda} \right) \xrightarrow{\Lambda \rightarrow 0^+} \Lambda^D \frac{1}{p^2} \int_q R(q) \tilde{h}(q) \longrightarrow 0. \quad (150)$$

Hence, we obtain

$$\lim_{\Lambda \rightarrow 0^+} \tilde{\beta}_{\Lambda}(p) = \frac{D-2}{4(D-1)} \int_q \frac{1}{q^2(q+p)^2} + kf_2. \quad (151)$$

We thus obtain the effective action with the insertion of a single EM tensor as

$$\begin{aligned}
\lim_{\Lambda \rightarrow 0^+} \tilde{\Theta}_{\mu\nu}(p) &= -N \delta_{\mu\nu} f_2 \frac{1}{2} \int_q \tilde{\sigma}(-q) \tilde{\sigma}(q+p) + N(p^2 \delta_{\mu\nu} - p_{\mu} p_{\nu}) \left( \frac{D-2}{4(D-1)} \int_q \frac{1}{q^2(q+p)^2} + kf_2 \right) \tilde{\sigma}(p) \\
&\quad + \frac{1}{2} N \int_q \{ -\delta_{\mu\nu} q(q+p) + q_{\mu}(q+p)_{\nu} + q_{\nu}(q+p)_{\mu} \} \left[ \frac{1}{N} \tilde{\Phi}^I(-q) \tilde{\Phi}^I(q+p) \right. \\
&\quad + \frac{1}{q^2} \left\{ \int_{p_1,p_2} \delta(p_1+p_2-p) \tilde{\sigma}(p_1) \frac{1}{(q+p_1)^2} \tilde{\sigma}(p_2) \right. \\
&\quad \left. \left. + \int_{p_1,p_2,p_3} \delta(p_1+p_2+p_3-p) \tilde{\sigma}(p_1) \frac{1}{(q+p_1)^2} \tilde{\sigma}(p_2) \frac{1}{(q+p_1+p_2)^2} \tilde{\sigma}(p_3) + \dots \right\} \frac{1}{(q+p)^2} \right], \quad (152)
\end{aligned}$$

where  $\tilde{\sigma}(p)$  is given by

$$\tilde{\sigma}(p) \equiv \frac{\delta \Gamma_{\text{eff},I}[\rho]}{\delta \tilde{\rho}(-p)} \quad (153)$$

and

$$\Gamma_{\text{eff},I}[\rho] \equiv \lim_{\Lambda \rightarrow 0+} \Gamma_{I\Lambda}[\rho]. \quad (154)$$

## VII. CONCLUSIONS

In this paper we have applied the exact renormalization group formalism to construct the energy-momentum (EM) tensor in the large  $N$  limit of the  $O(N)$  linear sigma model in  $D$  dimensions, where  $2 < D < 4$ . The EM tensor is a functional of scalar fields, and its cutoff dependence is given by the exact renormalization group equation. By taking the momentum cutoff to zero, we obtain the effective action with a single insertion of the energy-momentum tensor.

Our result for the cutoff dependent EM tensor is given by (72) of Sec. III, where the coefficient  $A_\Lambda(x)$  is given by (60), and  $\alpha_\Lambda$  and  $\beta_\Lambda(x)$  are given respectively by (134a), (134b). The field  $\sigma(x)$  is defined by (34). The EM tensor has been constructed to satisfy the Ward identity (42) for the translation and rotation invariance. In Sec. VI we take the zero cutoff limit to obtain (152).

In general, given a Wilson action, we should be able to write down the energy-momentum tensor explicitly. We hope we have presented such an example for an interacting theory even though we have needed the help of large  $N$  approximations. We have verified that our energy-momentum tensor satisfies the expected trace formula. We have also confirmed that the critical  $O(N)$  model is conformally

invariant in the large  $N$  limit (see [20] for a different approach valid for any finite  $N$ ).

Now that the EM tensor is constructed explicitly within the ERG formalism, we should be able to compute also the short distance singularities in the products of the tensor. This is a task left for the future; we refer the reader to [40,41] for some explicit examples of calculations of short distance singularities of operator products in the ERG framework. Another issue left for a future work is to find the relation between the EM tensor defined in this paper and the EM tensor defined by coupling the system to an external metric and possibly an additional background metric. In this case we expect that the Ward identity associated with the shift of the background metric also plays a role; it would be interesting to study this along the lines of [42]. This approach could have some merit also for the study of the role of these Ward identities within the asymptotic safety scenario for quantum gravity [43,44].

## DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

## APPENDIX A: DERIVATION OF (75)

A cutoff dependent composite operator  $\mathcal{O}_\Lambda[J]$  is defined by (74). Equation (75) gives the cutoff dependence of  $\mathcal{O}_\Lambda[\Phi] = \mathcal{O}_\Lambda[J]$  regarded as a functional of  $\Phi = \frac{\delta W_\Lambda[J]}{\delta J}$  instead of  $J$ . It is the purpose of this appendix to derive (75).

First, the cutoff dependence of  $\mathcal{O}_\Lambda[J]$  as a functional of  $J$  is obtained, from (74), as

$$-\Lambda \partial_\Lambda \mathcal{O}_\Lambda[J] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \left\{ \frac{\delta^2 \mathcal{O}_\Lambda[J]}{\delta J(x) \delta J(y)} + 2 \frac{\delta \mathcal{O}_\Lambda[J]}{\delta J(x)} \frac{\delta W_\Lambda[J]}{\delta J(y)} \right\}. \quad (A1)$$

We then obtain

$$\begin{aligned} -\Lambda \partial_\Lambda \mathcal{O}_\Lambda[\Phi] &= -\Lambda \partial_\Lambda \mathcal{O}_\Lambda[J] + \int_x \frac{\delta \mathcal{O}_\Lambda[J]}{\delta J(x)} (-\Lambda \partial_\Lambda) J(x) |_{\Phi \text{ fixed}} \\ &= -\Lambda \partial_\Lambda \mathcal{O}_\Lambda[J] + \int_x \frac{\delta \mathcal{O}_\Lambda[J]}{\delta J(x)} (-\Lambda \partial_\Lambda) \left( \int_y \mathcal{R}_\Lambda(x-y) \Phi(y) - \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(x)} \right), \end{aligned} \quad (A2)$$

where (15) is used. Using

$$\frac{\delta \mathcal{O}_\Lambda[J]}{\delta J(x)} = \int_y \frac{\delta \Phi(y)}{\delta J(x)} \frac{\delta \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(y)} = \int_y \mathcal{G}_\Lambda(x,y) [\Phi] \frac{\delta \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(y)}, \quad (A3)$$

$$\frac{\delta^2 \mathcal{O}_\Lambda[J]}{\delta J(x) \delta J(y)} = \int_v \mathcal{G}_\Lambda(y,v) [\Phi] \frac{\delta}{\delta \Phi(v)} \int_u \mathcal{G}_\Lambda(x,u) [\Phi] \frac{\delta \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(u)} = \int_{u,v} \mathcal{G}_\Lambda(y,v) [\Phi] \left( \frac{\delta \mathcal{G}_\Lambda(x,u)}{\delta \Phi(v)} \frac{\delta \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(u)} + \mathcal{G}_\Lambda(x,u) [\Phi] \frac{\delta^2 \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(u) \delta \Phi(v)} \right), \quad (A4)$$

$$-\Lambda \partial_\Lambda \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(x)} = \frac{\delta}{\delta \Phi(x)} \frac{1}{2} \int_{u,v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u-v) \mathcal{G}_\Lambda(u,v)[\Phi] = \frac{1}{2} \int_{u,v} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(u-v) \frac{\delta G_\Lambda(u,v)[\Phi]}{\delta \Phi(x)}, \quad (\text{A5})$$

we obtain

$$-\Lambda \partial_\Lambda \mathcal{O}_\Lambda[\Phi] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \int_{u,v} \mathcal{G}_\Lambda(x,u) \mathcal{G}_\Lambda(y,v) \frac{\delta^2 \mathcal{O}_\Lambda[\Phi]}{\delta \Phi(u) \delta \Phi(v)} \quad (\text{A6})$$

which is (75). For the linear sigma model, this generalizes to

$$-\Lambda \partial_\Lambda \mathcal{O}_\Lambda[\Phi] = \frac{1}{2} \int_{x,y} \Lambda \partial_\Lambda \mathcal{R}_\Lambda(x-y) \int_{u,v} \mathcal{G}_\Lambda^{IJ}(x,u) \mathcal{G}_\Lambda^{IK}(y,v) \frac{\delta^2 \mathcal{O}_\Lambda[\Phi]}{\delta \Phi^J(u) \delta \Phi^K(v)}. \quad (\text{A7})$$

## APPENDIX B: SOLUTION TO (134b)

In this appendix we solve (134b) in the momentum space, where the equation becomes

$$\begin{aligned} p^2(\tilde{\beta}_\Lambda(p) - kf_2) &= \frac{1}{D-1} (\tilde{b}_{\Theta,\Lambda}(p) - D\tilde{A}_\Lambda(p)) \\ &= \frac{1}{D-1} \frac{D-2}{2} \left[ \int_q \left( R_\Lambda(q) \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p) - h_\Lambda(q) + \frac{1}{q^2} \right) \right. \\ &\quad \left. - \frac{2D}{D-2} \tilde{A}_\Lambda(p) + \frac{1}{2} p^2 \int_q \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p) \right]. \end{aligned} \quad (\text{B1})$$

This is solved by

$$\begin{aligned} \tilde{\beta}_\Lambda(p) &= kf_2 + \frac{1}{D-1} \frac{D-2}{2} \frac{1}{2} \int_q \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p) \\ &\quad + \frac{1}{D-1} \frac{D-2}{2} \frac{1}{p^2} \left[ \int_q \left( R_\Lambda(q) \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p) - \tilde{h}_\Lambda(q) + \frac{1}{q^2} \right) - \frac{2D}{D-2} \tilde{A}_\Lambda(p) \right]. \end{aligned} \quad (\text{B2})$$

For this solution to make sense, we must show that the coefficient function  $\tilde{\beta}_\Lambda(p)$  defined by (134b) is finite at  $p = 0$ . We need to show

$$\frac{2D}{D-2} \tilde{A}_\Lambda(0) = \int_q \left( R_\Lambda(q) \tilde{h}_\Lambda(q)^2 - \tilde{h}_\Lambda(q) + \frac{1}{q^2} \right), \quad (\text{B3})$$

where  $\tilde{A}_\Lambda(p)$  is determined by (60) as

$$p_\nu \tilde{A}_\Lambda(p) = \int_q (p+q)_\nu R_\Lambda(q) \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p). \quad (\text{B4})$$

Since

$$\begin{aligned} p_\nu \tilde{A}_\Lambda(p) &= \int_q (q+p)_\nu R_\Lambda(q) \tilde{h}_\Lambda(q) \tilde{h}_\Lambda(q+p) \xrightarrow{p \rightarrow 0} p_\nu \int_q R_\Lambda(q) \tilde{h}_\Lambda(q)^2 + \int_q q_\nu R_\Lambda(q) \tilde{h}_\Lambda(q) p_\mu \frac{\partial}{\partial q_\mu} \tilde{h}_\Lambda(q) \\ &= p_\nu \left( \int_q R_\Lambda(q) \tilde{h}_\Lambda(q)^2 + \frac{1}{D} \int_q R_\Lambda(q) \tilde{h}_\Lambda(q) q \cdot \partial_q \tilde{h}_\Lambda(q) \right), \end{aligned} \quad (\text{B5})$$

we obtain

$$\begin{aligned}
\tilde{A}_\nu(0) &= \int_q R_\Lambda(q) \tilde{h}_\Lambda(q)^2 + \frac{1}{2D} \int_q R_\Lambda(q) q \cdot \partial_q \tilde{h}_\Lambda(q)^2 \\
&= \int_q R_\Lambda \tilde{h}_\Lambda^2 + \frac{1}{2D} \int_q (-D - q \cdot \partial_q) R_\Lambda(q) \cdot \tilde{h}_\Lambda(q)^2 \\
&= \int_q R_\Lambda \tilde{h}_\Lambda^2 + \frac{1}{2D} \int_q (2 - q \cdot \partial_q - D - 2) R_\Lambda(q) \cdot \tilde{h}_\Lambda(q)^2 \\
&= \int_q R_\Lambda \tilde{h}_\Lambda^2 + \frac{1}{2D} \int_q (\tilde{f}_\Lambda(q) - (D+2) R_\Lambda(q) \tilde{h}_\Lambda(q)^2) \\
&= \frac{D-2}{2D} \int_q R_\Lambda(q) \tilde{h}_\Lambda(q)^2 + \frac{1}{2D} \int_q \tilde{f}_\Lambda(q).
\end{aligned} \tag{B6}$$

Using

$$\frac{1}{D} \int_q \tilde{f}_\Lambda(q) = \frac{D-2}{D} \int_q \left( \frac{1}{q^2} - \tilde{h}_\Lambda(q) \right), \tag{B7}$$

we obtain the desired equality (B3).

- 
- [1] F. J. Belinfante, On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields, *Physica (Amsterdam)* **7**, 449 (1940).
  - [2] L. Rosenfeld, On the energy-momentum tensor, *Mém. Acad. R. Belg.* **18**, 1 (1940), [https://en.wikipedia.org/wiki/Belinfante%E2%80%93Rosenfeld\\_stress%E2%80%93energy\\_tensor](https://en.wikipedia.org/wiki/Belinfante%E2%80%93Rosenfeld_stress%E2%80%93energy_tensor).
  - [3] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997).
  - [4] Curtis G. Callan, Jr., Sidney R. Coleman, and Roman Jackiw, A new improved energy—momentum tensor, *Ann. Phys. (N.Y.)* **59**, 42 (1970).
  - [5] H. Sonoda, Construction of the energy-momentum tensor for Wilson actions, *Phys. Rev. D* **92**, 065016 (2015).
  - [6] Juergen Berges, Nikolaos Tetradis, and Christof Wetterich, Nonperturbative renormalization flow in quantum field theory and statistical physics, *Phys. Rep.* **363**, 223 (2002).
  - [7] Jan M. Pawłowski, Aspects of the functional renormalisation group, *Ann. Phys. (Amsterdam)* **322**, 2831 (2007).
  - [8] Bertrand Delamotte, An introduction to the nonperturbative renormalization group, *Lect. Notes Phys.* **852**, 49 (2012).
  - [9] Yuji Igarashi, Katsumi Itoh, and Hidenori Sonoda, Realization of symmetry in the ERG approach to quantum field theory, *Prog. Theor. Phys. Suppl.* **181**, 1 (2010).
  - [10] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawłowski, M. Tissier, and N. Wschebor, The nonperturbative functional renormalization group and its applications, *Phys. Rep.* **910**, 1 (2021).
  - [11] Markus Heller and Jan M. Pawłowski, Causal temporal renormalisation group flow of the energy-momentum tensor, [arXiv:2112.12652](https://arxiv.org/abs/2112.12652).
  - [12] Joseph Polchinski, Scale and conformal invariance in quantum field theory, *Nucl. Phys.* **B303**, 226 (1988).
  - [13] Oliver J. Rosten, On functional representations of the conformal algebra, *Eur. Phys. J. C* **77**, 477 (2017).
  - [14] Bertrand Delamotte, Matthieu Tissier, and Nicolás Wschebor, Scale invariance implies conformal invariance for the three-dimensional Ising model, *Phys. Rev. E* **93**, 012144 (2016).
  - [15] Hidenori Sonoda, Conformal invariance for Wilson actions, *Prog. Theor. Exp. Phys.* **2017**, 083B05 (2017).
  - [16] Bertrand Delamotte, Gonzalo De Polsi, Matthieu Tissier, and Nicolás Wschebor, Conformal invariance and composite operators: A strategy for improving the derivative expansion of the nonperturbative renormalization group, *Phys. Rev. E* **109**, 064152 (2024).
  - [17] Santiago Cabrera, Gonzalo De Polsi, and Nicolás Wschebor, Conformal invariance constraints in the  $O(N)$  models: A study within the nonperturbative renormalization group, *Phys. Rev. E* **111**, 054126 (2025).
  - [18] Oliver J. Rosten, A conformal fixed-point equation for the effective average action, *Int. J. Mod. Phys. A* **34**, 1950027 (2019).
  - [19] Oliver J. Rosten, A Wilsonian energy-momentum tensor, *Eur. Phys. J. C* **78**, 312 (2018).
  - [20] Gonzalo De Polsi, Matthieu Tissier, and Nicolás Wschebor, Conformal invariance and vector operators in the  $O(N)$  model, *J. Stat. Phys.* **177**, 1089 (2019).
  - [21] Hidenori Sonoda, Exact renormalization group in large  $N$ , [arXiv:2302.09914](https://arxiv.org/abs/2302.09914).
  - [22] T. H. Berlin and M. Kac, The spherical model of a ferromagnet, *Phys. Rev.* **86**, 821 (1952).
  - [23] H. E. Stanley, Spherical model as the limit of infinite spin dimensionality, *Phys. Rev.* **176**, 718 (1968).

- [24] Kenneth G. Wilson, Quantum field theory models in less than four-dimensions, *Phys. Rev. D* **7**, 2911 (1973).
- [25] Shang-keng Ma, Introduction to the renormalization group, *Rev. Mod. Phys.* **45**, 589 (1973).
- [26] Howard J. Schnitzer, Nonperturbative effective potential for  $\lambda\phi^4$  theory in the many field limit, *Phys. Rev. D* **10**, 1800 (1974).
- [27] Howard J. Schnitzer, The Hartree approximation in relativistic field theory, *Phys. Rev. D* **10**, 2042 (1974).
- [28] S. R. Coleman, R. Jackiw, and H. D. Politzer, Spontaneous symmetry breaking in the  $O(N)$  model for large  $N$ , *Phys. Rev. D* **10**, 2491 (1974).
- [29] Moshe Moshe and Jean Zinn-Justin, Quantum field theory in the large  $N$  limit: A review, *Phys. Rep.* **385**, 69 (2003).
- [30] Marco D’Attanasio and Tim R. Morris, Large  $N$  and the renormalization group, *Phys. Lett. B* **409**, 363 (1997).
- [31] Tim R. Morris and Michael D. Turner, Derivative expansion of the renormalization group in  $O(N)$  scalar field theory, *Nucl. Phys. B* **509**, 637 (1998).
- [32] J. P. Blaizot, Ramon Mendez Galain, and Nicolas Wschebor, A new method to solve the non perturbative renormalization group equations, *Phys. Lett. B* **632**, 571 (2006).
- [33] Daniel F. Litim and Matthew J. Trott, Asymptotic safety of scalar field theories, *Phys. Rev. D* **98**, 125006 (2018).
- [34] C. Becchi, On the construction of renormalized gauge theories using renormalization group techniques, [arXiv: hep-th/9607188](https://arxiv.org/abs/hep-th/9607188).
- [35] Félix Rose, Frédéric Léonard, and Nicolas Dupuis, Higgs amplitude mode in the vicinity of a  $(2 + 1)$ -dimensional quantum critical point: A nonperturbative renormalization-group approach, *Phys. Rev. B* **91**, 224501 (2015).
- [36] Félix Rose and Nicolas Dupuis, Nonperturbative functional renormalization-group approach to transport in the vicinity of a  $(2 + 1)$ -dimensional  $O(N)$ -symmetric quantum critical point, *Phys. Rev. B* **95**, 014513 (2017).
- [37] Félix Rose, Carlo Pagani, and Nicolas Dupuis, Operator product expansion coefficients from the nonperturbative functional renormalization group, *Phys. Rev. D* **105**, 065020 (2022).
- [38] Juergen A. Dietz and Tim R. Morris, Redundant operators in the exact renormalisation group and in the  $f(R)$  approximation to asymptotic safety, *J. High Energy Phys.* **07** (2013) 064.
- [39] F. J. Wegner, Some invariance properties of the renormalization group, *J. Phys. C* **7**, 2098 (1974).
- [40] C. Pagani and H. Sonoda, Products of composite operators in the exact renormalization group formalism, *Prog. Theor. Exp. Phys.* **2018**, 023B02 (2018).
- [41] C. Pagani and H. Sonoda, Operator product expansion coefficients in the exact renormalization group formalism, *Phys. Rev. D* **101**, 105007 (2020).
- [42] Carlo Pagani and Hidenori Sonoda, Background dependent cutoff for Wilson actions, *Phys. Rev. D* **111**, 105006 (2025).
- [43] Martin Reuter and Frank Saueressig, *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety* (Cambridge University Press, Cambridge, England, 2019).
- [44] Robert Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, 100 Years of General Relativity (World Scientific, Singapore, 2017), Vol. 3.