

Thermal static potential and pseudoscalar quarkonium spectral functions from (2 + 1)-flavor lattice QCD

Sajid Ali^{1,2}, Dibyendu Bala¹, Olaf Kaczmarek¹, and Pavan¹

(HotQCD Collaboration)

¹*Fakultät für Physik, Universität Bielefeld, D-33615 Bielefeld, Germany*

²*Government College University Lahore, Department of Physics, Lahore 54000, Pakistan*



(Received 13 June 2025; accepted 27 August 2025; published 22 September 2025)

Quarkonia, which are bound states of a heavy quark and antiquark, play a key role in probing the quark-gluon plasma (QGP). The dynamics of quarkonia in the QGP are encoded in their finite-temperature spectral functions. In this work, we estimate the quarkonium spectral functions in the pseudoscalar channel using (2 + 1)-flavor lattice QCD with a pion mass of 320 MeV, at temperatures of 220 MeV($1.2T_{pc}$), 251 MeV($1.4T_{pc}$), and 293 MeV($1.6T_{pc}$). Reconstructing the spectral function from the Euclidean lattice correlator is a well-known ill-posed problem, requiring additional physics-motivated input. We address this by smoothly matching contributions from different frequency regions of the spectral function, using appropriate physics valid for each region. The spectral function around $\omega \sim 2M_q$ is obtained using a nonperturbative complex potential, while for $\omega \gg 2M_q$ it is modeled using results from vacuum perturbation theory. Since the pseudoscalar channel does not receive a transport contribution near $\omega \sim 0$, we find that the combination of these two regions already provides a good description of the relativistic lattice pseudoscalar correlator. We observe a substantial thermal width in the $\eta_c(1S)$ state, indicating that pseudoscalar charmonium (η_c) is nearing dissolution at the studied temperatures. In comparison, the η_b ground state exhibits little change and remains well defined.

DOI: [10.1103/s8gw-n43f](https://doi.org/10.1103/s8gw-n43f)

I. INTRODUCTION

Lattice QCD calculations indicate that hadronic matter at high temperatures transforms into the quark-gluon plasma (QGP). This phase is studied in heavy-ion collision experiments at RHIC and LHC. Quarkonia (bound states of a quark and an antiquark) are important probes to study this phase [1]. In heavy-ion collisions, quarkonia are generated in the early phase of the collision through hard processes. As they move through the plasma, some dissociate due to thermal interactions with plasma constituents, while others persist and decay later in the hadronic phase after freeze-out. The heavy quark antiquark vector mesons (Υ , J/Ψ) are particularly significant due to their decay into dileptons. Experimental data reveal a suppression of Υ yields in Pb-Pb collisions compared to

expectations from proton-proton collisions, indicating significant effect of the QGP on quarkonium states [2].

Understanding the effect of the QGP on quarkonia theoretically requires the real-time evolution of quarkonium states within the QGP, together with the hydrodynamic evolution of the QGP itself [3,4]. Significant progress has been made in describing the real-time evolution in the weak coupling regime using potential non-relativistic QCD (pNRQCD) together with the open quantum system approach. In this framework, one evolves the density matrix of the $Q\bar{Q}$ pair in the singlet and octet representations using Lindblad equations [5–8]. Recently, the quarkonia evolution in the open quantum system approach has been extended to a non-Markovian process to quantify the memory effect [9].

On the other hand, one can also study the fate of quarkonium bound states and their real-time properties at a given temperature by studying the quarkonium spectral functions across different quantum number channels. The hydrodynamic evolution of the QGP assumes an (approximate) state of local thermodynamic equilibrium. Therefore, understanding the spectral properties of quarkonia at a given temperature provides important insight into

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

heavy-ion phenomenology. The spectral function is defined by

$$\rho_{\Gamma}(\omega, \vec{k}) = \int dt d^3\vec{x} \exp[i(\vec{k}\cdot\vec{x} - \omega t)] \langle [J_{\Gamma}(\vec{x}, t), J_{\Gamma}(\vec{0}, 0)] \rangle_T. \quad (1)$$

Here Γ specify the quantum channel of the point current, $J_{\Gamma} = \bar{\psi}\Gamma\psi$ and $\langle \cdots \rangle_T$ denotes the thermal average. Physically, the spectral function is related to the imaginary part of the retarded response function $\mathcal{G}_R(\omega, \vec{k})$, which represents the ability of the system to absorb energy due to a perturbation with frequency ω and momentum $\vec{k}$. Spectral function is related to the Euclidean correlation function by

$$G_{\Gamma}^E(\tau, \vec{k}) = \int_0^{\infty} \frac{d\omega}{\pi} \rho_{\Gamma}(\omega, \vec{k}) \frac{\cosh[\omega(\tau - \frac{1}{2T})]}{\sinh[\frac{\omega}{2T}]}, \quad (2)$$

where $G^E(\tau, \vec{k}) = \int d^3\vec{x} \exp(i\vec{k}\cdot\vec{x}) \langle J_{\Gamma}(\vec{x}, \tau) J_{\Gamma}(\vec{0}, 0) \rangle$ and which can be determined in first principle lattice QCD calculations. However, inverting the above equation for spectral function is a well-known ill-posed problem, as lattice QCD data involve a finite number of data points along with statistical errors. This results in a large number of possible spectral functions that agree with the lattice data within errors and further physics input is required for a reliable spectral function reconstruction. There are various studies on quarkonium spectral functions using lattice correlator. Bayesian methods, like the maximum entropy method (MEM), have been used in [10–12] to investigate charmonium in plasma using a quenched gauge field background. Similar techniques have been used to study correlators in NRQCD in background gauge configuration with dynamical highly improved staggered quark (HISQ) fermion and Wilson-Clover fermion [13,14]. While these studies provide insights into the properties of the quarkonia in plasma, they are accompanied by significant systematic uncertainties, which are inherent to Bayesian analysis. There are also other studies of quarkonia from lattice QCD, where various models of the spectral function have been assumed to fit lattice QCD data [15–17]. For the study of quarkonia from the T-matrix approach, see [18–20].

Using nonrelativistic approximations for quarkonium bound states, various potential models have also been used, where the Schrödinger equation is solved to obtain the spectral function. In [21], both the free energy [22] and internal energy of the $Q\bar{Q}$ system were used to study the spectral functions. However, the precise definition of the potential, which governs the real-time dynamics of the heavy $Q\bar{Q}$ system, has been clarified in [23]. This was achieved through a rigorous $1/M_q$ expansion of the point-split version of the correlator, resulting in a Schrödinger equation with a thermal potential. Perturbatively, this calculation revealed for the first time that the potential at

finite temperature is complex. The real part corresponds to the Debye-screened potential and the imaginary part is due to Landau damping in the $r \gg \frac{1}{T}$ region, which arises from the inelastic scattering of medium quarks and gluons with the heavy quark. Later, within the pNRQCD framework, it was shown that in the region $r \ll \frac{1}{T}$, the imaginary part of the potential receives additional contributions from singlet-to-octet transitions due to the absorption of a gluon from the medium (known as gluodissociation) [24].

This perturbative potential has been used for the spectral function calculation in various channels near the threshold [25,26]. This illustrates that the imaginary part plays a significant role in the dissociation of quarkonia in the QGP, along with the color screening of the real part. These perturbative calculations are important inputs for spectral reconstruction of the lattice correlator [27,28]. The spectral function in the entire frequency range was constructed by combining the result from the threshold region, based on the thermal potential, with spectral functions from other regions. Using these model spectral functions, the corresponding perturbative correlators have been computed using Eq. (2) and have then been compared to the lattice correlators. The lattice data was fitted by introducing some free parameters in the perturbative model spectral function.

In this paper, we used a similar approach for the spectral reconstruction, with one key difference: in the threshold region; we use a nonperturbative thermal complex potential to obtain the quarkonium spectral functions. Some of the results presented in this paper have already appeared in preliminary form in the proceedings [29]. This is particularly relevant for temperatures just a few times higher than the pseudocritical temperature T_{pc} , where nonperturbative effects in the potential may play a significant role. There are also existing spectral function calculations based on nonperturbative potential [30]. In this work we combine spectral functions from different regions with a more precise extraction of the thermal potential using data from large N_{τ} lattices. We focus on the pseudoscalar (PS) channel spectral function as in this channel no transport contribution is present in the infrared region. As a result, the large-distance part of the correlator is dominated by the spectral contribution from the bound state region. This enables us to compare the correlator obtained from the spectral function with the one directly calculated from point-to-point lattice QCD correlators.

However, calculating the nonperturbative potential itself turns out to be an inverse problem, where the correlator is related to the Wilson loop. Over the decade, significant efforts have been devoted to compute the thermal static potential nonperturbatively. These approaches range from Bayesian techniques like the MEM [31] and Bayesian reconstruction [32] to models of spectral function that are fitted to the lattice data [33–36]. Some models even suggest that there is no screening of the real part of the potential [34,35]. However, given the ill-conditioned nature of this

inverse problem, this could very well be due to the lack of physics information used during the spectral reconstruction. In fact, as we will see, and as also discussed in [33], incorporating physics-motivated information into the spectral function results in a color-screened thermal potential.

The organization of the paper is as follows: In the next section, we discuss the formalism for calculating the spectral function using the thermal static potential. In Sec. III, we provide details of our lattice simulations. Section IV covers the detailed analysis of the thermal potential extraction. In Sec. V, we determine the quark pole mass and the overall additive constant of the thermal potential, and we also provide a parametrization of the thermal potentials. In Sec. VI, we use the thermal potential to extract the spectral function in the threshold region and also match it with the UV perturbative spectral function to obtain the spectral function over the entire region. In Sec. VII, we perform a consistency check to show that the spectral function is compatible with the corresponding lattice correlator at finite temperatures, and we also discuss the physical implications of our results. The paper ends with a conclusion and outlook.

II. FORMALISM

We aim to reconstruct the spectral function using lattice correlators where the $Q\bar{Q}$ state is in the pseudoscalar (η_c, η_b) channel. This leads us to consider the following lattice correlators:

$$G_{\text{PS}}^E(\tau) = M_B^2 \int d^3\vec{x} \langle \bar{\psi}(\vec{x}, \tau) \gamma_5 \psi(\vec{x}, \tau) \bar{\psi}(\vec{0}, 0) \gamma_5 \psi(\vec{0}, 0) \rangle_c, \quad (3)$$

where ψ represents the heavy quark field on the lattice. The pseudoscalar correlator is multiplied by the square of the bare quark mass, M_B^2 , to ensure that the correlator is ultraviolet (UV) finite after renormalization [27].

It is important to note that, on the lattice, this correlator has finite renormalization factors, and a lattice spacing-dependent renormalization factor must be multiplied to make the correlators finite in the continuum limit. However, in this paper, we focus on a single lattice spacing and estimate the spectral function up to some multiplicative constant. Therefore, we do not need to consider such renormalization for our purposes.

We reconstruct the spectral function using physics-motivated input according to [27]. In the region where $\omega \gg 2M_q$, perturbative calculations become reliable. In this regime, temperature effects are also suppressed because the kinetic energy of the produced quark-antiquark pair is much larger than the thermal energy. Perturbative calculations show that at leading order (LO), temperature effects are exponentially suppressed, and at next-to-leading

order (NLO), they decrease according to a power law [37]. Thus, in this region, we estimate the spectral function using vacuum perturbation theory.

Around the threshold region $\omega \sim 2M_q$, naive perturbative calculations break down. In this region, the spectral function follows from the solution of a Schrödinger equation. This is equivalent to obtaining the spectral function in NRQCD at the order $1/M_q$ expansion. The thermal potential arises from resumming the infinite set of ladder diagrams near $\omega \sim 2M_q$ [26]. This requires solving the Schrödinger equation:

$$\left(2M_q - \frac{\nabla^2}{M_q} + V(r)\right) C_{>}^{\text{PS}}(r, t) = i \frac{\partial C_{>}^{\text{PS}}(r, t)}{\partial t}, \quad (4)$$

with the initial condition

$$C_{>}^{\text{PS}}(r, 0) = -2M_q^2 N_c \delta^3(\vec{r}).$$

Here $C_{>}^{\text{PS}}(r, t)$ is real time point-split correlator in the pseudoscalar channel.

Here, $V(r)$ is the potential, given by the long-time limit of the Wilson loop $W_M(r, t)$ as

$$V(r) = i \lim_{t \rightarrow \infty} \frac{\partial \log W_M(r, t)}{\partial t}. \quad (5)$$

The spectral function for the pseudoscalar channel is then related to $C_{>}^{\text{PS}}(r, t)$ through a Fourier transformation:

$$\rho_{\text{PS}}(\omega) = \frac{1}{2} \lim_{r \rightarrow 0} \int_{-\infty}^{\infty} C_{>}^{\text{PS}}(r, t) \exp(i\omega t) dt. \quad (6)$$

As mentioned earlier, in this paper, we will compute the potential in Eq. (5) nonperturbatively on the lattice. We will return to the formulation of this potential calculation on lattice at the end of this section.

Now in the regime $\omega \ll 2M_q$ the spectral contributions depend on the quantum number channel of the current. For example, it is well known that in the vector channel, one would expect a diffusion contribution in the $\omega \sim 0$ region, as the vector current is conserved. The diffusion coefficient has been studied extensively on the lattice [38–45]. On the other hand, for the pseudoscalar channel, one would not expect a transport peak near the $\omega \sim 0$ region due to the absence of a conservation law for the pseudoscalar channel. It has been shown up to next-to-leading order (NLO) that the spectral function indeed does not contain any transport contribution [27]. Therefore, we do not consider any transport contribution in the pseudoscalar channel.

To obtain the spectral function in the region $\omega \sim 2M_q$, we need to compute the potential defined in Eq. (5), which is related to the real-time Wilson loop. However, since lattice calculations are performed in Euclidean time, an analytic continuation is necessary to extract the real-time

Wilson loop. This continuation is achieved through the spectral function of the Wilson loop, as discussed in [31],

$$W_E(r, \tau) = \int_{-\infty}^{\infty} d\omega \rho_W(\omega, r) \exp(-\omega\tau) \quad (7)$$

$$W_M(r, t) = \int_{-\infty}^{\infty} d\omega \rho_W(\omega, r) \exp(-i\omega t). \quad (8)$$

In this paper following [32], we will calculate the potential from the Coulomb gauge fixed Wilson line correlator.

III. LATTICE DETAILS

We calculate the Euclidean correlator of the pseudoscalar $G_{\text{PS}}^E(\tau)$ defined in Eq. (3) for the charm and bottom sectors. To calculate these correlators, we used clover-improved Wilson fermions. The dynamical gauge configurations were generated using a $(2+1)$ -flavor highly improved staggered quark (HISQ) action for the fermion sector [46], and a tree-level improved Lüscher-Weisz gauge action [47], using the rational hybrid Monte Carlo algorithm. The pion mass of these configurations is $m_\pi = 320$ MeV, which corresponds to a pseudocritical temperature of $T_{pc} \sim 180$ MeV. The scale is set by $f_K = \frac{156.1}{\sqrt{2}}$ MeV [44], and corresponds to a lattice spacing of ~ 0.028 fm. These configurations have also been used in [45, 48, 49].

The parameters of the Wilson clover fermions also need to be fixed. The c_{SW} coefficient has been fixed to the tadpole-improved value $1/u_0^3$, where u_0 is the fourth root of the average plaquette. The hopping parameters κ have been tuned such that, at $T = 110$ MeV (zero temperature), the spin-averaged mass $\frac{1}{4}(M_{\text{PS}} + 3M_V)$ becomes identical to the PDG values [50]. This gives the following κ and masses:

- (i) For the charmonium sector, with $\kappa_c = 0.13164$, we obtain $M_{\eta_c} = 2.96 \pm 0.011$ GeV.
- (ii) For the bottomonium sector, with $\kappa_b = 0.11684$, we obtain $M_{\eta_b} = 9.40 \pm 0.01$ GeV.

For these quark masses we have calculated the pseudo-scalar correlators, Eq. (3), on the lattice at temperatures of 220, 251, and 293 MeV in the plasma phase. The lattice spacing is fixed at 0.028 fm, and the temperature is varied by adjusting the total temporal extent N_τ . All of the lattice details are summarized in Table I.

We also need to compute the Wilson loop, defined in Eq. (8), for the calculation of the potential. It has been shown perturbatively that the spectral function of the Wilson loop contains a significant UV component, in addition to a peak corresponding to the thermal potential [51]. This large UV part complicates extraction of thermal peak from the Wilson loop. On the other hand, perturbative calculations also indicate that both the Wilson loop and the Wilson line correlator yield the same thermal potential;

TABLE I. The details of the lattices studied in this paper. The term Confs represents the number of configurations used for the computation of the η_c and η_b correlators.

N_f	β_0	a (fm)	T (MeV)	$N_\sigma^3 \times N_\tau$	Confs
2 + 1	8.249	0.028053(31)	110	$64^3 \times 64$	1010
			220	$96^3 \times 32$	1700
			251	$96^3 \times 28$	1468
			293	$96^3 \times 24$	856

TABLE II. Lattices used for Wilson line correlators. For each T , numbers are for zero-flow/others.

T (MeV)	τ_F/a^2	Confs
110	0.0, 0.125, 0.2, 0.4	1599
220	0.0, 0.125, 0.2, 0.4, 0.6, 0.8	1100/2297
251	0.0, 0.125, 0.2, 0.4, 0.6, 0.8	2400/2000
293	0.0, 0.125, 0.2, 0.4, 0.6, 0.8	856/1975

however, the UV part is significantly suppressed in the Wilson line correlator [51]. As a result, identifying the peak in the spectral function becomes easier in the case of the Wilson line correlator. Consequently, it has become common practice to extract the thermal potential from the Wilson line correlator on the lattice. Following this approach, we calculate the Wilson line correlator fixed in Coulomb gauge.

However, these are purely gluonic observables and we used a much larger-statistics simulation compared to those used for measuring the heavy quark correlators. We also used gradient flow to enhance the signal at large separations. Specifically, we first perform the gradient flow using Wilson action, and then fix the configuration in Coulomb gauge to an accuracy of 10^{-6} . We have also tested that increasing the accuracy to 10^{-8} does not have an impact on the potential. The details of the configurations used and the flow times are given in Table II. The unflowed measurements are taken from Ref. [35], and therefore the number of measurements differs. However, as we will see, the zero-flow measurements are used only in the short-distance part of the real potential, which will essentially be replaced by the perturbative potential.

IV. STATIC THERMAL POTENTIAL

In this section, we discuss the calculation of the static thermal potential nonperturbatively from lattice QCD. The potential, as defined in Eq. (5), is determined by the long-time behavior of the real-time Wilson loop. At zero temperature, the transfer matrix argument shows that the Euclidean Wilson loop is a sum of exponentially decaying terms, each associated with a different energy state [52]. At large Euclidean time, the lowest energy state dominates,

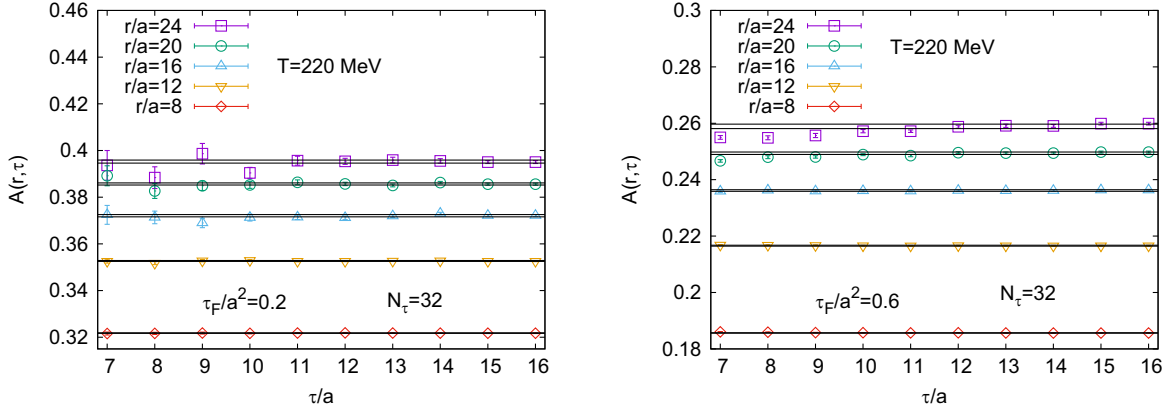


FIG. 1. $A(r, \tau)$ is plotted as a function of τ for flow time $\tau_F/a^2 = 0.2$ (left) and $\tau_F/a^2 = 0.6$ (right). The plateau behavior near $\tau \sim N_\tau/2$ demonstrates the validity of the ansatz in Eq. (9) in this region. The horizontal line represents the value of the real part of the potential obtained from the fit.

resulting in a pure exponential decay. After analytic continuation to real time, this gives the static potential in Eq. (5). This is equivalent to saying the spectral function for the Wilson line correlator in Eq. (8), has a well separated lowest lying Dirac delta peak.

At finite temperature, the extraction of the potential becomes nontrivial as there is no pure exponential decay. This is because, at finite temperature, the sharp Dirac delta peak acquires a thermal width due to interactions with the plasma, making the extraction of the thermal potential from the Euclidean correlator more difficult.

In this paper, following [33,53], we will use the hard-thermal-loop (HTL) motivated structure of the correlator and, hence, the spectral function for the extraction of the potential. Here we will briefly describe the idea behind that.

The constraint that we would like to put on the structure of Euclidean Wilson line correlator is that, after the analytic continuation $\tau \rightarrow it$ the limit in Eq. (5) must exist, as the static potential is defined in this limit. Since the HTL perturbative calculation of the Euclidean Wilson line correlator in [23,51] shows that this limit exists, we observe the τ dependence of the HTL perturbative correlator, which leads to the limit, Eq. (5), after the analytic continuation $\tau \rightarrow it$. This calculation has been extended to finite flow time in Appendix A. A close observation of this expression shows that the logarithm of the Wilson line correlator can be decomposed into a linear and a periodic part in τ . The linear part of the correlator gives the real part of the potential, as expected, since only the real part of the potential can lead to a pure exponential decay. On the other hand, the periodic part gives rise to the imaginary part and causes the correlator to deviate from exponential decay at large τ . Motivated by this observation, we parametrize the Wilson correlator at large τ as follows:

$$\log W_E(r, \tau) = -V_{re}(r)\tau + P(r, \tau). \quad (9)$$

This form can be directly verified from lattice data by considering the following quantity:

$$A(r, \tau) = -\frac{1}{2} \frac{\partial}{\partial \tau} \log \left(\frac{W_E(r, \tau)}{W_E(r, \beta - \tau)} \right). \quad (10)$$

If Eq. (9) holds, then this quantity should show a plateau-like behavior at the asymptotically large τ region, since the periodic part cancels out in this quantity. We show this quantity for two flow times in Fig. 1 at a temperature $T = 220$ MeV, which indeed exhibits a large plateau region near $\frac{\beta}{2}$. This provides a validity of the structure in Eq. (9) over a large region near $\tau \sim \frac{\beta}{2}$ directly from lattice data.

For $P(r, \tau)$, we assume a spectral representation with a function $\sigma(r, \omega)$ as follows¹:

$$P(r, \tau) = \int_{-\infty}^{\infty} \sigma(r, \omega) (\exp(\omega\tau) + \exp(\omega(\beta - \tau))) d\omega. \quad (11)$$

This form is motivated by HTL calculations, where such a representation is possible. In this HTL case, $\sigma(r, \omega)$ is related to the electric component of the gluonic spectral function.

Using this, we obtain the following form of Wilson line correlator:

$$\log W_E(r, \tau) = -V_{re}(r)\tau + \int_{-\infty}^{\infty} \sigma(r, \omega) (\exp(\omega\tau) + \exp(\omega(\beta - \tau))) d\omega. \quad (12)$$

¹Nonperturbatively, it may happen that we do not have such a spectral representation $\sigma(\omega)$, in which case understanding the analytic structure relevant to the existence of the potential becomes more challenging, and the perturbative limit may also be less apparent.

From this form we can obtain

$$\lim_{t \rightarrow \infty} i \frac{\partial \log W_E(r, \tau \rightarrow it)}{\partial t} = V_{re}(r) - \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} d\omega \omega \sigma(r, \omega) [\exp(i\omega t) - \exp(\omega(\beta - it))]. \quad (13)$$

Now using the fact that

$$\lim_{t \rightarrow \infty} \frac{\exp(i\omega t) - \exp(\omega(\beta - it))}{\omega} = 2\pi i \delta(\omega), \quad (14)$$

one can easily see that a finite result is only possible when $\sigma(r, \omega) \sim \frac{1}{\omega^2}$. In fact this $1/\omega^2$ behavior can also be seen in HTL expression, where one of the factors comes from the

Bose distribution function $n_b(\omega)$. This motivates us to write

$$\sigma(r, \omega) = n_b(\omega) \left(\frac{\beta V_{im}(r)}{2\pi\omega} + \sum_{l=0}^{\infty} c_{2l+1} \omega^{2l+1} \right). \quad (15)$$

Only odd terms are present because the combination $n_b(\omega)(\exp(\omega\tau) + \exp(\omega(\beta - \tau)))$ is odd in ω . Using this, one gets the following compact form for the Wilson line correlator:

$$W(r, \tau) = A(r) \exp \left(-V_{re}(r)\tau - \frac{\beta V_{im}(r)}{\pi} \log \left(\sin \left(\frac{\pi\tau}{\beta} \right) \right) \right) + \sum_{l=0}^{\infty} \frac{2c_{2l+1}}{\beta^{2l+2}} (2l-1)! \left(\zeta \left(2l+2, \frac{\tau}{\beta} \right) + \zeta \left(2l+2, 1 - \frac{\tau}{\beta} \right) \right). \quad (16)$$

Here, $\zeta(p, x) = \sum_n \frac{1}{(n+x)^p}$ is generalized zeta function. This parametrization is our ansatz for the calculation of the thermal potential from the lattice data. This parametrization indeed gives a well-defined potential in the long time limit, as can be seen from Eq. (5) after the analytic continuation,

$$V(r) = i \lim_{t \rightarrow \infty} \frac{\partial \log W(r, \tau \rightarrow it)}{\partial t} = V_{re}(r) - iV_{im}(r). \quad (17)$$

The other terms $c_1, c_3, \dots$ do not survive in the long time limit and, hence, do not contribute to the static potential.

We use the parametrization in Eq. (16) to fit the Wilson line correlator at zero as well as at finite flow time.

A leading order perturbative analysis of the gradient flow dependence of the Wilson line correlator, as shown in Appendix A, also suggests the structure of Eq. (12). The fit results of the correlator with respect to $A(r)$, $V_{re}(r)$, and $V_{im}(r)$ are shown in the left panel of Fig. 2 for $N_\tau = 32$ for a few distances at a flow time of $\tau_F/a^2 = 0.2$ in the range $\tau/a = 6-26$. The right panel shows the relative deviation from the prediction, illustrating predictability. The statistical error on the potential has been determined by fitting our ansatz to each bootstrap sample and taking the standard deviation over them. To find the stability of the fit, we also vary the fit range symmetrically around the midpoint, $N_\tau/2$. The results for $V_{re}(r)$ and $V_{im}(r)$, shown in Fig. 3 plotted as a function of the fitting

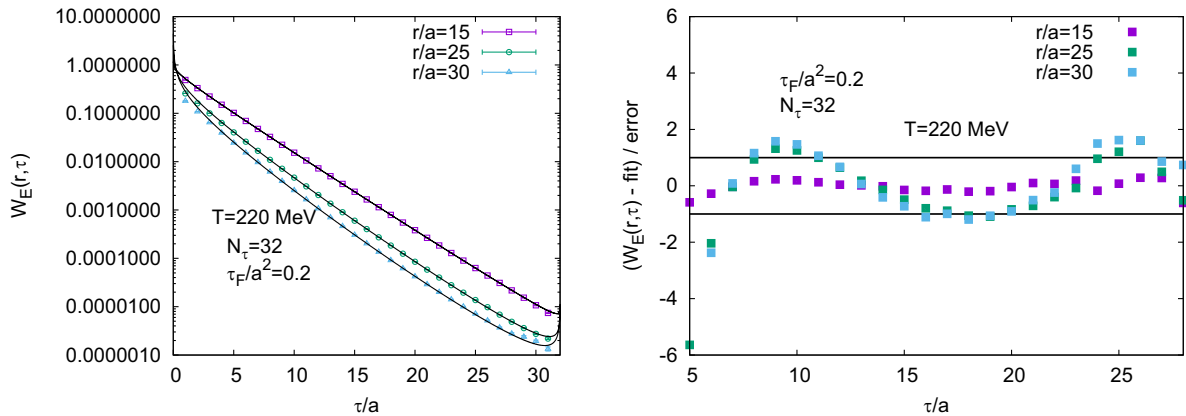


FIG. 2. Left: the Wilson line correlator from the lattice at $T = 220$ MeV ($N_\tau = 32$) fitted with Eq. (16), excluding the contributions from c_1 and c_3 , for distances $r/a = 15, 25$, and 30 . Right: the corresponding relative error from the fit is shown.

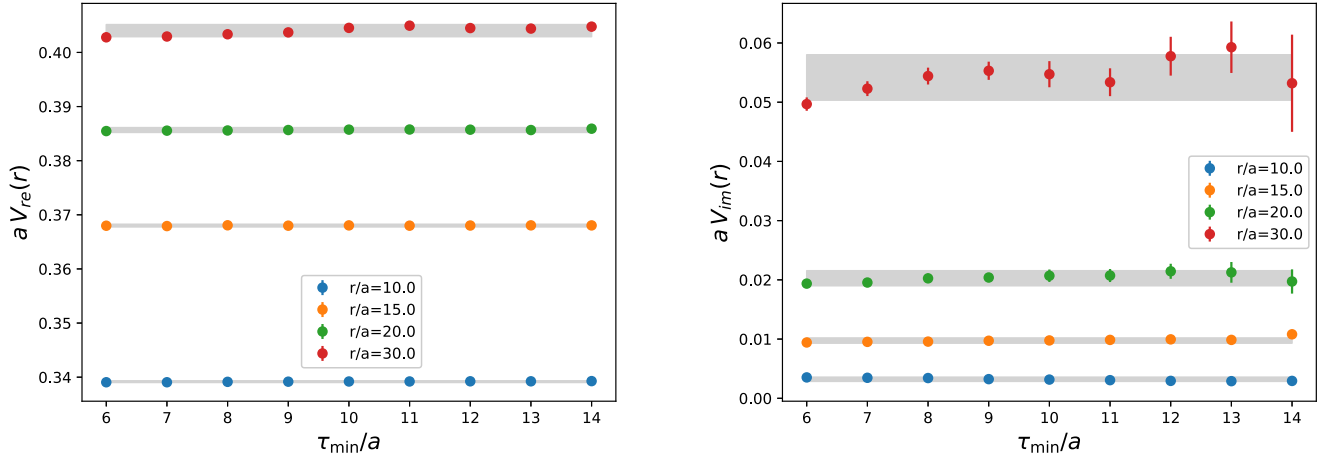


FIG. 3. The real part [$V_{re}(r)$] and the imaginary part [$V_{im}(r)$] are plotted as functions of the fit range lower limit τ_{min} . The range has been varied symmetrically with respect to $N_\tau/2$, i.e., from τ_{min} to $N_\tau - \tau_{min}$. Left: stability of the real part for $\tau_F/a^2 = 0.2$. Right: stability of the imaginary part for $\tau_F/a^2 = 0.2$. The shaded band represents the value obtained by combining V_{re} and V_{im} from different fit ranges.

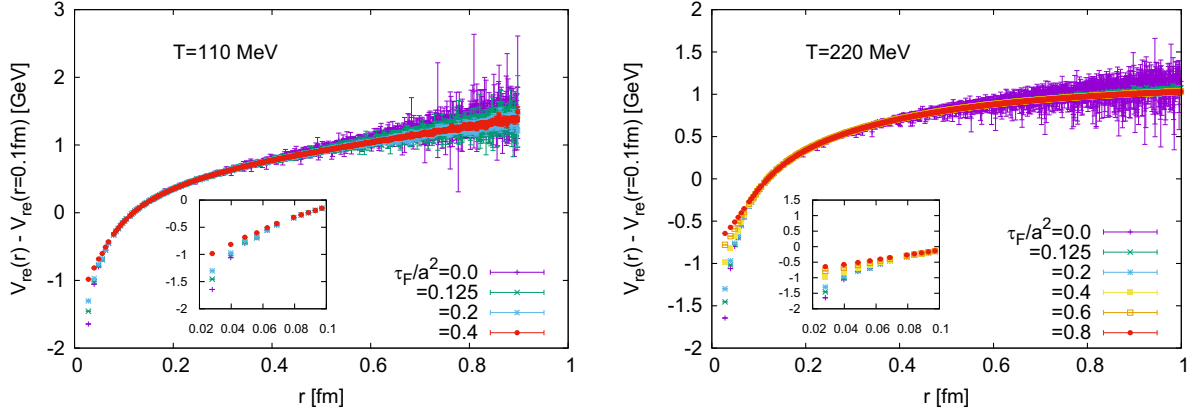


FIG. 4. The flow-time dependence of the real part of the potential at $T = 110$ MeV (left) and $T = 220$ MeV (right).

range, show a very mild dependence on the fitting range, illustrating the stability of the fit. The statistical average over the fitting range, shown by the shaded band in the figure, is taken to be the final value of the potential from a given flow depth. The real part of the potential for various flow depths at temperatures of 110 and 220 MeV is shown in Fig. 4. The temperature 110 MeV is much smaller than the crossover temperature and the potential at this temperature is purely real. Therefore the potential for this temperature has been extracted using a standard exponential fit to the Wilson line correlator at large τ . As the real part of the potential contains an additive renormalization, which depends on the flow depth, we have plotted $V(r) - V(r = 0.1 \text{ fm})$ in Fig. 4. As one can see, the flow reduces the noise of the potential at large distances. At the same time, the flow deforms the short distance part of the potential. This behavior can also be observed in perturbative calculation of the flowed potential in Fig. 18 of Appendix A. Therefore, we calculate the potential at

several flow times, including zero flow depth. We would like to have a potential where the large-distance part is taken from the larger flow-time data, while the short-distance part is taken from the zero flow-time data to avoid flow artifacts at short distances. In the intermediate-distance region, we smoothly interpolate between these two regions using intermediate flow-time potential data. To do this, we retain the $\tau_F = 0$ potential up to $r_0 = 0.1 \text{ fm}$, below which there is a significant flow-time effect. Beyond this point, higher-flow-time potentials are used after adjusting additive constants. The procedure is as follows: we first interpolate the potential and then minimize the difference between the derivatives of consecutive flow times in the neighborhood of the previous matching points. Specifically, if we find a matching point at r_n between flow times τ_F^n and τ_F^{n-1} , the next matching point between τ_F^n and τ_F^{n+1} is chosen within an interval $r_n + \delta r$. We do this recursively to obtain the full potential. In this work, we take $\delta r \sim 0.1 \text{ fm}$. This is illustrated in the

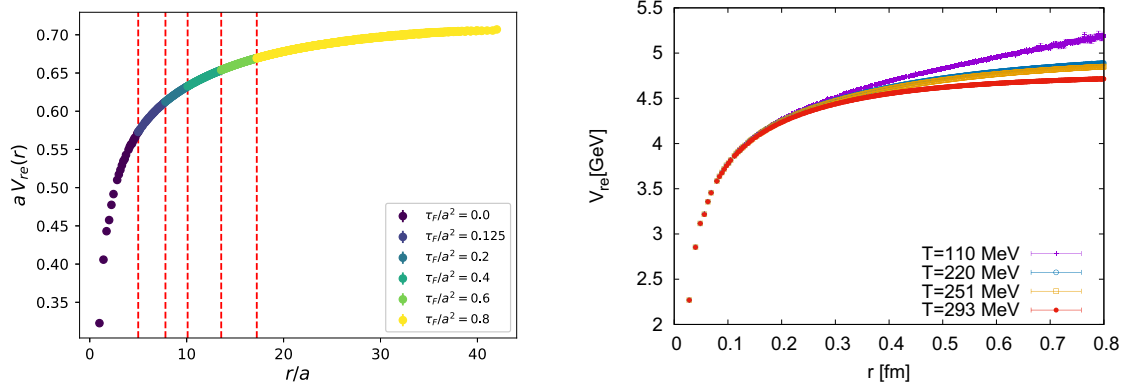


FIG. 5. Left: the real part of the thermal potential is smoothly matched across different flow times at a temperature of 220 MeV. The dashed vertical lines indicate the matching points of the real part of the potential between consecutive flow times. Right: real part of thermal potential at different temperature.

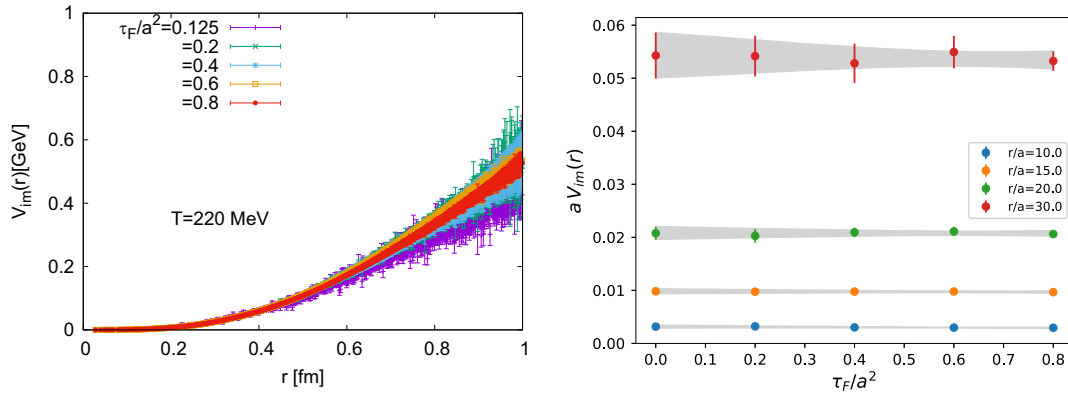


FIG. 6. Left: imaginary part at different flow times for $T = 220$ MeV. Right: zero flow-time extrapolation at selected distances.

left panel of Fig. 5, where the vertical lines represent the matching points.²

On the right panel of Fig. 5, we plot the real part of the potential obtained in this way for various temperatures. As we can see, below T_{pc} the potential is confining, but as the temperature rises above T_{pc} , the potential shows a much flatter behavior because of the medium modification due to the presence of color screening in the QGP phase, and the strength of screening increases with temperature. This result is in agreement with earlier studies in [30,32,33]. However, it contradicts the study in [35], where the real part of the potential is independent of temperature up to ~ 350 MeV. The contradiction serves as an indication of the inversion problem's ill-posed nature in Eq. (8) and the physics input required for meaningful spectral reconstruction. The short distance part of the potential in Fig. 5 is from the zero flow time correlator, however this region also gets affected by the lattice artifacts and in the next section we will replace this region by a three-loop perturbative potential.

²We use cubic spline interpolation. We have verified that alternative interpolation schemes and variations of δr have no visible impact on the resulting potential.

The imaginary part for different flow time has been shown in Fig. 6 for temperature 220 MeV. Though the perturbative prediction in Appendix A shows that the imaginary part gets affected by the gradient flow, we do not find much flow time dependence in the lattice data within error. Nevertheless, we have performed a linear extrapolation to zero flow time as shown in Fig. 6. The zero flow time extrapolated imaginary part for different temperature is shown on the right panel of Fig. 7. The imaginary part is increasing with both distances and temperature. The imaginary part is expected to saturate at large distances because, at large distances, the quark-antiquark pair is expected to be scattered by thermal gluons independently. However, in the distance scale we studied on the lattice, we do not see any such saturation.

In this section, we have shown the results using the first three parameters in Eq. (16), namely $A(r)$, $V_{re}(r)$, and $V_{im}(r)$, which already describe a large range of the Wilson line correlator. We have also performed the fit, including c_1 and c_3 individually. We find that, although there is no change in the real part of the potential, the imaginary part of the potential shows slight changes at large distances, and the error bars increase, as shown in the right panel of Fig. 7.

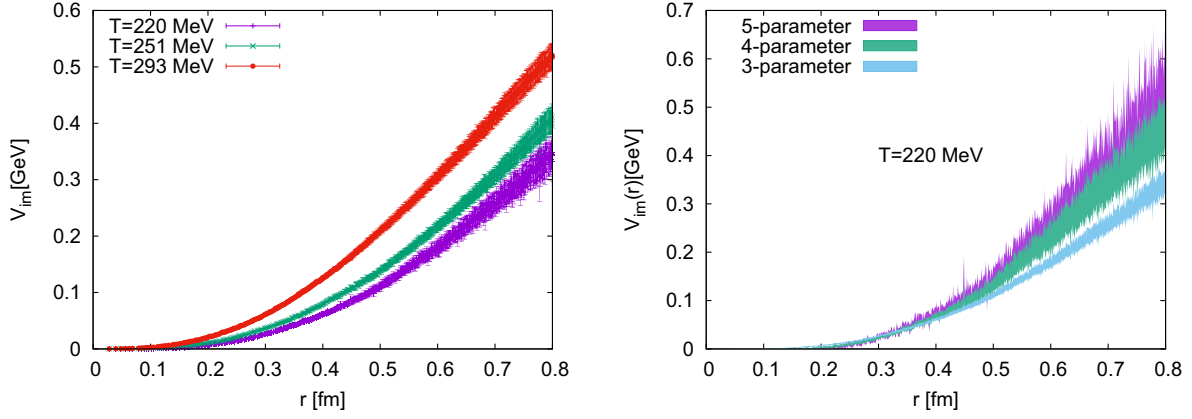


FIG. 7. Left: zero flow-time extrapolated imaginary part at different temperatures. Right: dependence of the zero flow-time extrapolated imaginary part on the fit parameters.

We have taken this spread in the imaginary part as a systematic uncertainty in determining the imaginary part, and it has been included in the determination of the quarkonia spectral function in Sec. VI.

V. POLE-MASS EXTRACTION AND PARAMETRIZATION OF THERMAL POTENTIAL

We need to obtain the quark mass appearing in the Schrödinger equation for spectral function in Eq. (4), which we will call “pole mass” in our context. We will use the

zero-temperature potential to extract the pole mass of heavy quarks. However, the potential calculated from lattice is defined up to an additive renormalization constant. On the other hand, the short distance part of the potential obtained from lattice has artifacts from finite lattice spacing. The short distance ($r \ll \frac{1}{\Lambda_{\text{QCD}}}$) part of the potential, however, can be calculated reliably from perturbation theory. Since the potential is temperature independent in this region, one can use vacuum perturbation theory. The perturbative potential up to three-loop is given by [54,55]

$$V(r) = \int \frac{d^3 \vec{q}}{(2\pi)^3} \exp(i\vec{q} \cdot \vec{r}) \frac{\alpha_V(q^2)}{q^2}$$

where

$$\alpha_V(q^2) = -4\pi C_F \alpha_s(|\vec{q}|) \left\{ 1 + \frac{\alpha_s(|\vec{q}|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi} \right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi} \right)^3 a_3 + \mathcal{O}(\alpha_s^4) \right\}. \quad (18)$$

Here, a_1 , a_2 , and a_3 depend on number of flavor N_f and number of color N_c . For the running coupling $\alpha_s = g^2/4\pi$, a four-loop beta function [56] has been used.

However, the perturbative potential suffers from a well-known renormalon problem originating from the integration region $|\vec{q}| = \Lambda_{\text{QCD}}$, where the perturbative QCD coupling constant has a pole [55]. Therefore, the perturbative potential soon becomes unreliable, and smooth matching with the lattice potential becomes difficult. As a result, we subtract the renormalon contribution by avoiding integration in Eq. (18) over the entire region $0 < |\vec{q}| < \infty$. Instead, we perform the integration only up to some cutoff $\mu_r < |\vec{q}| < \infty$, where $\mu_r \gg \Lambda_{\text{QCD}}$. This will create an additive renormalization of $\mathcal{O}(\Lambda_{\text{QCD}})$ originating from renormalon pole and weak r^2 dependence, which can be neglected at short distance [55]. Nevertheless, renormalon subtraction will make the perturbative potential well

behaved up to a larger distance and a smooth matching with lattice QCD potential is possible. We checked that the short distance part of the potential has a weak dependence on the value of μ_r up to an additive constant and we fix the value $\mu_r = 1 \text{ GeV}$.

To match the zero temperature lattice potential with this renormalon-subtracted (RS) potential, we first fit the lattice potential using the Cornell form:

$$V(r) = -\frac{\alpha}{r} + \sigma r + c \quad (19)$$

within the range $[r_{\min}, r_{\max}]$. We find that setting $r_{\min} = 3a$ provides a good description of the lattice data up to $r_{\max} = 25a$. From this fit we obtained a string tension $\sqrt{\sigma} = 0.45 \text{ GeV}$ and $\alpha = 0.42$. After the parametrization, we performed a smooth matching with the RS potential

at a distance of $r_m \sim 0.1$ fm, by adjusting the additive renormalization of the lattice potential, similar to [57]. However, this matching still carries an additive uncertainty of $\mathcal{O}(\Lambda_{\text{QCD}})$ from the renormalon pole in the RS potential, which needs to be fixed.

To fix this additive constant, we use the $\eta_c(1S)$ and $\eta_b(1S)$ masses extracted from lattice correlators, as described in Sec. III. The pole mass of the bottom quark is fixed to $m_b = 4.78$ GeV, as quoted by the PDG [50]. Using this mass, we solve the Schrödinger equation and adjust the additive constant of the potential such that the ground state mass matches the $\eta_b(1S)$ mass obtained from the lattice. Using the same additive constant, we then tune the charm quark pole mass by solving the Schrödinger equation so that the ground state mass matches the $\eta_c(1S)$ mass obtained from the zero-temperature lattice correlator. This leads to a pole charm quark mass 1.35 ± 0.01 MeV. The matched zero-temperature potential, after fixing the additive constant, is shown in the left panel of Fig. 8. These mass estimates, of course, suffer from an uncertainty of order $\mathcal{O}(\Lambda_{\text{QCD}})$, but this uncertainty can always be absorbed into the renormalon ambiguity of the potential, since the combination $2M + V(r)$ is expected to be free from renormalon ambiguity [55].

For the finite temperature potential in Fig. 5 we see that, up to the matching distance ($r_m = 0.1$ fm), thermal effects in the potential can be neglected in the temperature range we studied. In this region, we replace the thermal potential with the perturbative potential; therefore we subtract the same additive constant that has been used at zero temperature. This is crucial to always replace the temperature-independent part of the thermal potential with the zero-temperature potential and use the same additive constant from the zero-temperature matching.

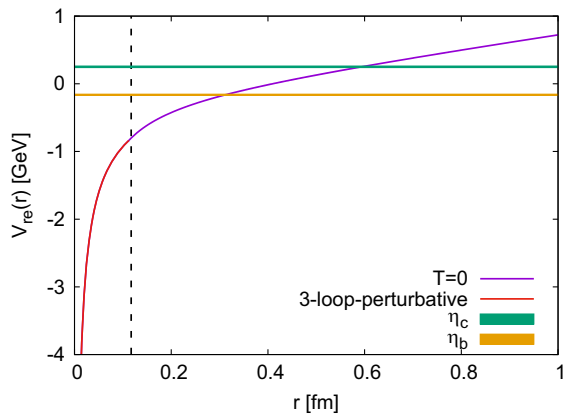


FIG. 8. The lattice zero-temperature potential smoothly matches the renormalon-subtracted vacuum potential at short distances. The vertical dashed line represents the matching point between the renormalon-subtracted vacuum potential and the zero-temperature lattice potential. The overall additive constant is fixed to reproduce the ground state masses of η_b and η_c .

TABLE III. Parameters at different temperatures for real part parametrization.

T (MeV)	$\alpha(T)$	$\sqrt{\sigma(T)}$ (GeV)	$m_r(T)$ (GeV)	$C(T)$ (GeV)
220	0.326	0.710	0.521	4.047
251	0.314	0.747	0.586	3.991
293	0.306	0.838	0.811	3.881

This ensures that the relative separation between the zero and finite temperature potentials remains the same.

In the spectral function calculation, one has to solve the Schrödinger equation with the thermal potential, for which we need a parametrization of the thermal potential as well that describes the lattice data in Fig. 5. We parametrize the finite temperature potential with the Karsch-Mehr-Satz form [58],

$$V_{re}(r, T) = -\frac{\alpha(T)}{r} \exp(-m_r r) + \frac{\sigma}{m_r} (1 - \exp(-m_r r)) + C. \quad (20)$$

We fit the thermal lattice potential with this form with respect to the parameters α, σ, m_r, C . We found a reasonably good description of our lattice data with this form in the range $r_{\min} = 3a$ to $r_{\max} = 35a$. The fitted parameters are given in Table III. In the Table we have shown the value obtained from the fits. We do not quote the error bar, as it is negligible compared to the uncertainty introduced in the spectral function due to mass tuning.

The matched thermal potential along with the zero temperature potential is shown in Fig. 9. In the same plot, we plotted the perturbative thermal potential from [23] given by

$$V_T^re(r) = -\frac{g(T)^2}{4\pi} C_F \left[m_d + \frac{\exp(-m_d r)}{r} \right]. \quad (21)$$

We used the two-loop formula for the thermal coupling and Debye mass from [59]. We find that color screening is weaker in the nonperturbative potential than in the perturbative one.

We close the discussion of this section by also providing a parametrization of the imaginary part shown in Fig. 7, which will also be needed in the next section for the spectral function calculation. Motivated from [60], we parametrize the imaginary part by

$$V_{im}(r) = a_1 \phi_2(m_i r) + a_2 \phi_3(m_i r) + a_3 \phi_4(m_i r), \quad (22)$$

where

$$\phi_n(x) = 2 \int_0^\infty \frac{z dz}{(z^2 + 1)^n} \left[1 - \frac{\sin(zx)}{zx} \right]. \quad (23)$$

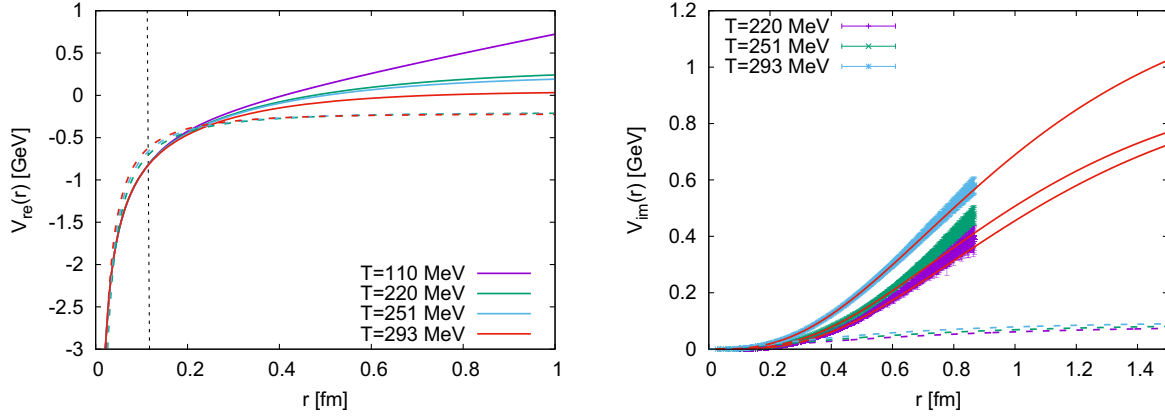


FIG. 9. Left: the thermal potential matches the renormalon-subtracted vacuum potential at short distances, using the same additive constant as the zero-temperature potential. The vertical dashed line represents the matching point between the renormalon-subtracted vacuum potential and the real part of lattice potential. The perturbative real part is shown with dashed lines. Right: parametrization of the imaginary part. The perturbative imaginary part is shown with dashed lines.

Here a_1 , a_2 , a_3 , and m_i are the fit parameters. This parametrization is consistent with the expectation that, at large distances, the imaginary part should saturate to a finite value.

The parametrization is illustrated in Fig. 9 for the case of a number of parameters of three in Eq. (16). We have also performed fitting with the above form, using the imaginary part obtained by changing the number of parameters to 4 and 5 in Eq. (16). The error in the imaginary part becomes larger at large distances. As a result, we perform the parametrization for various upper limits of the fit range. In Appendix B, we study the impact of this fit range on the spectral function, which turns out to be very small. Further details of the systematic uncertainty estimation of the spectral function from the imaginary part are given in Appendix B.

In Fig. 9 we also show the HTL perturbative imaginary part given by

$$V_T^{\text{im}}(r) = \frac{g^2}{4\pi} C_F T \int_0^\infty \frac{z dz}{(z^2 + 1)^2} \left[1 - \frac{\sin(zm_d r)}{zm_d r} \right]. \quad (24)$$

We observe that the nonperturbative imaginary part is significantly larger than the perturbative one, resulting in a higher dissociation rate of quarkonia in the plasma.

We emphasize that the forms given in Eqs. (20) and (22) are simple parametrizations used solely for solving the Schrödinger equation. Any other parametrization consistent with the lattice data should also be allowed. Therefore, these forms are purely phenomenological in nature.

VI. SPECTRAL FUNCTION

In this section, we will discuss the computation of the spectral function for the correlator given in Eq. (3). Let us begin with the calculation of the spectral function in the region $\omega \gg 2M_q$. In this region temperature effects are power suppressed and we use zero temperature perturbation

theory. As discussed in [27], the vacuum perturbative spectral function normalized by the pole quark mass, $\frac{\rho_{\text{ps}}}{M_q^2 \omega^2}$, shows poor convergence even at large ω , whereas when it is normalized by the $\overline{\text{MS}}$ mass, $\frac{\rho_{\text{ps}}}{m(\bar{\mu})^2 \omega^2}$ shows relatively good convergence at large ω .

We estimate the charm and bottom quark $\overline{\text{MS}}$ mass from the corresponding pole quark mass, obtained from the zero-temperature potential in Sec. V. We use the self-consistency condition from [27], namely at large ω one would expect $\frac{\rho_{\text{ps}}}{M_q^2 \omega^2} = \frac{\rho_{\text{ps}}}{m(\bar{\mu})^2 \omega^2} \big|_{\mu=\omega, \omega=xM_q}$, where we take $x = 4$. The charm and bottom quark $\overline{\text{MS}}$ mass obtained in this way is given by 1.06 ± 0.01 and 5.3 GeV at the scale $\mu_{\text{ref}} = 2$ GeV. Again the errorbar in the charm mass is due to the uncertainty in tuning of zero temperature $\eta_c(1S)$ and $\eta_b(1S)$ mass.

Near the threshold region $\omega \sim 2M_q$, thermal effects become important, and the spectral function needs to be calculated using the thermal potential from Fig. 9, following the formalism outlined in Sec. II. We solve the Schrödinger equation in Eq. (4) using the algorithm given in [26] to obtain the spectral function in this region. Much below the threshold, $\omega \ll 2M_q$, the Schrödinger equation overestimates the spectral function. The NLO calculation given in [27] shows that in the $\omega < 2M_q$ region, the spectral function undergoes exponential suppression. This kind of suppression is not included in the Schrödinger estimate of the spectral function and requires further modeling in this region. Motivated by this, and following [27], we multiply the imaginary part in the $\omega < 2M_q$ region by $\exp(\frac{2M_q - \omega}{T})$ to model this suppression. The spectral function in both the asymptotic and thermal regions has been matched as follows:

$$\rho_{\text{matched}} = A_0 \rho_T(\omega) \theta(\omega_0 - \omega) + \rho_{\text{vac}}(\omega) \theta(\omega - \omega_0). \quad (25)$$

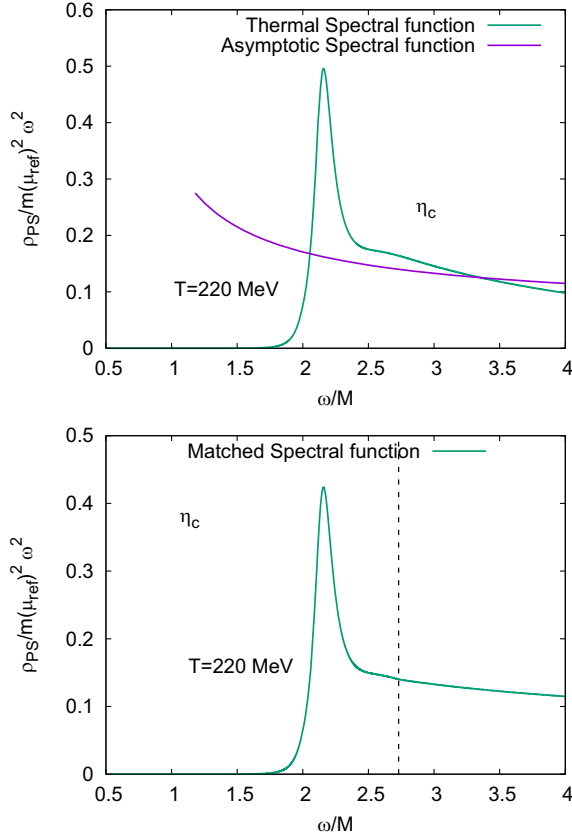


FIG. 10. Top: the spectral function obtained from the thermal potential in the threshold region at $T = 220$ MeV temperature (green curve). The blue curve represents the asymptotic vacuum perturbative spectral function in the PS channel. Bottom: the matched spectral function in the entire region. The vertical line represents the matching point between the perturbative and thermal spectral functions.

The parameters A_0 and ω_0 were determined by equating $A_0 \rho_T(\omega_0)$ with $\rho_{vac}(\omega_0)$ at ω_0 , along with a smooth matching at ω_0 . We multiply the thermal spectral function by a factor A_0 because it is obtained from NRQCD by solving a Schrödinger equation. As such, it requires a

matching coefficient to relate it to full QCD. This coefficient is poorly determined in perturbation theory, so we treat A_0 as a free parameter and fix it by matching to the perturbative spectral function at ω_0 [27]. The matching is performed in a region where the thermal spectral function has no temperature dependence, as this region is being replaced with the vacuum spectral function. This keeps the relative effect of temperature on the spectral function near the threshold unchanged after the matching. We obtain these matching coefficients by minimizing the difference of the first derivative of the thermal spectral function and the perturbative spectral function, along with implementing continuity of the entire spectral function. We find that $\omega \sim 2.7M_q$ for charm and $\omega \sim 2.2M_q$ for bottom and $A_0 \sim 1$ for both cases. The spectral function matching procedure is illustrated in Fig. 10. In the top panel of Fig. 10, the spectral functions from the thermal potential and from the perturbative vacuum regions are shown separately, while in the bottom panel, they are smoothly matched. For the pseudoscalar channel, we do not need to consider the region for $\omega \sim 0$, as there is no transport contribution in this channel.

The spectral functions for the η_c and η_b at various temperatures are shown in Fig. 11, with the left and right panels displaying them, respectively. At a temperature of 110 MeV, we observe a bound state peak corresponding to the peak position of the vacuum bound state mass. As the temperature rises above the crossover temperature, we observe significant thermal modification of the spectral functions. We see in both, the η_c and η_b channel, the excited state peaks melt at the temperatures we studied. We would like to emphasize that the absence of any excited state peak is not a consequence of replacing the $\omega \gg 2M_q$ region with the vacuum spectral function. Even prior to this replacement, no excited state peak is observed in the thermal spectral function. The sharp ground state peak, however, broadens due to thermal interactions and its peak position shifts towards smaller masses depending on the quark mass. This is consistent with the spectral function of the

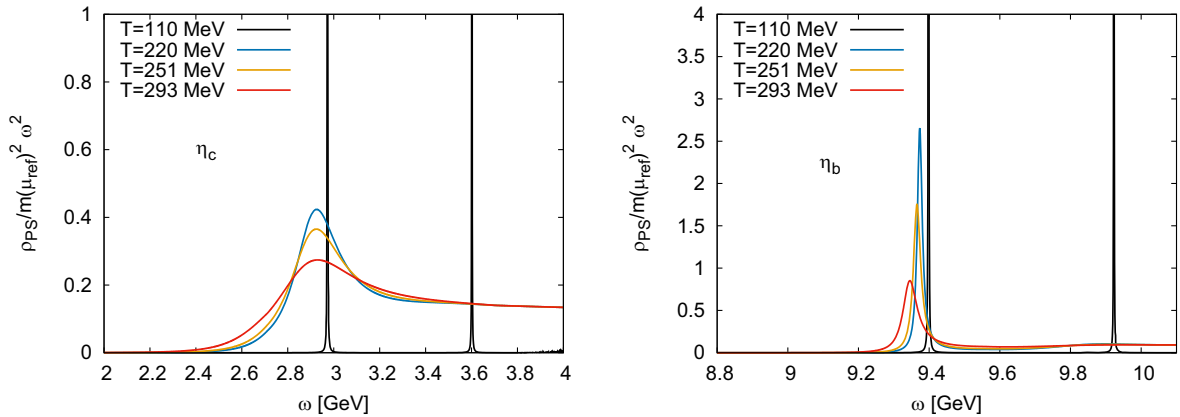


FIG. 11. The spectral function at various temperatures for the η_c state is shown on the left, and for the η_b state on the right.

point source from the T -matrix approach [18,19], where no excited state peaks were observed in the temperature range we studied. However, in the study [19], when an extended source was used for bottomonium, excited state peaks did appear—but with very large thermal widths. As noted in that work, the existence of such peaks becomes questionable due to the large width associated with the extended operator. Similarly, when model spectral functions such as a Gaussian or cut-Lorentzian were used to analyze NRQCD bottomonium correlators with extended operators, a large width was also observed for excited state [15–17].

This broadening is attributed to the imaginary part of the potential, which represents the thermal decay width due to the inelastic scattering of heavy quarks by plasma constituents. The real part of the potential leads to a screening effect, shifting the peak position towards lower values of ω . The imaginary part on the other hand tends to push the peak towards higher ω . The interplay between these two effects determines the final position of the peak.

In the left panel of Fig. 11, we observe that the $\eta_c(1S)$ state has a much larger thermal decay width compared to the thermal decay width of the $\eta_b(1S)$ state. This difference is expected, given that the charm quark mass is significantly smaller than the bottom quark mass, leading to stronger thermal effects in the charm sector. As the temperature increases, the broadening of these states becomes more pronounced. At approximately 293 MeV, the ground state in the charm sector shows a significant increase in broadening, while in the bottom sector, a well-defined ground state peak still persists at this temperature.

The thermal potential and quark masses used to obtain the spectral function have both systematic and statistical uncertainties. However, the systematic uncertainty is larger compared to the statistical uncertainty. We find that most of the dominant source of error arises from the uncertainty in

determining the quark masses from the zero-temperature bound state mass and from the imaginary part of the potential, which depends on the number of parameters used. As a result, we study the spectral function by varying the quark mass and the imaginary part within the error bar as discussed in Appendix B. In Fig. 11, the spectral function is computed for the central value of the quark masses and the imaginary part obtained from a three-parameter fit. To assess the effect of systematic uncertainty, we study the width of the peak and the mass position as a function of temperature.

To get an estimate of the width, we fit the spectral function around the peak using the skewed Breit-Wigner form [30]:

$$\rho(\omega \approx M) = A \frac{\Gamma/2}{\Gamma^2/4 + (\omega - M)^2} + \delta \frac{(\omega - M)\Gamma}{(\omega - M)^2 + \Gamma^2/4} + \dots \quad (26)$$

This form is motivated by the fact that thermal interactions shift the pole of the retarded propagator from the real axis into the complex plane, resulting in a spectral function (the imaginary part of the retarded correlator) with a Breit-Wigner structure near the peak.

The fit for the spectral function is illustrated in Fig. 12, where dashed curves represent the fit function. The fitted width, $\Gamma(1S)$, and the thermal mass, M , obtained from the peak position of the spectral function are shown in Fig. 13. The error bars in the width and mass arise from the uncertainty in the estimates of the zero-temperature mass and the imaginary part, as discussed in Appendix B. As can be seen from the figure, the thermal decay width increases with temperature. The results suggest a width for the $\eta_c(1S)$ in the range $\Gamma \sim 300$ –600 MeV, while the $\eta_b(1S)$ exhibits a significantly smaller decay width, approximately 10–15 times smaller, in the range $\Gamma \sim 20$ –60 MeV.

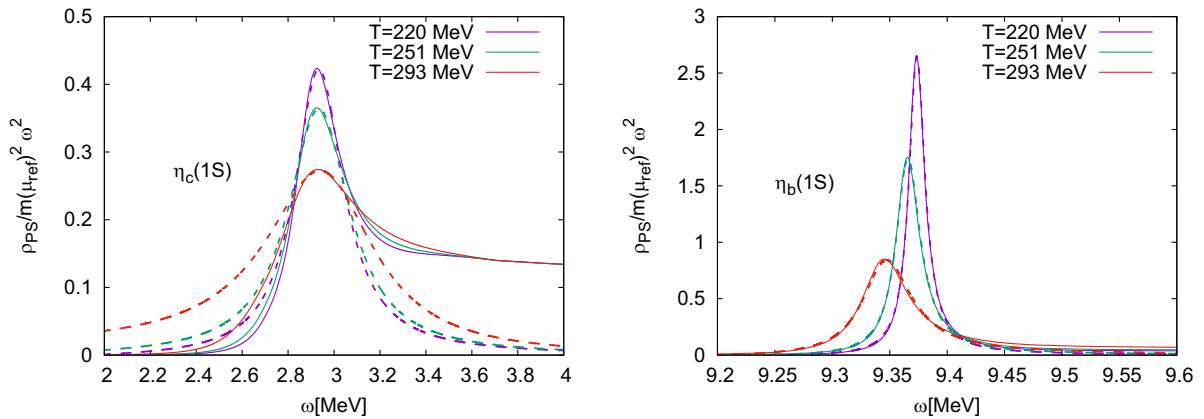


FIG. 12. The spectral function has been fitted near the peak using Eq. (26). The left panel shows the η_c case, while the right panel corresponds to η_b . The solid curve represents the spectral function from Fig. 11, while the dashed lines correspond to the fit.

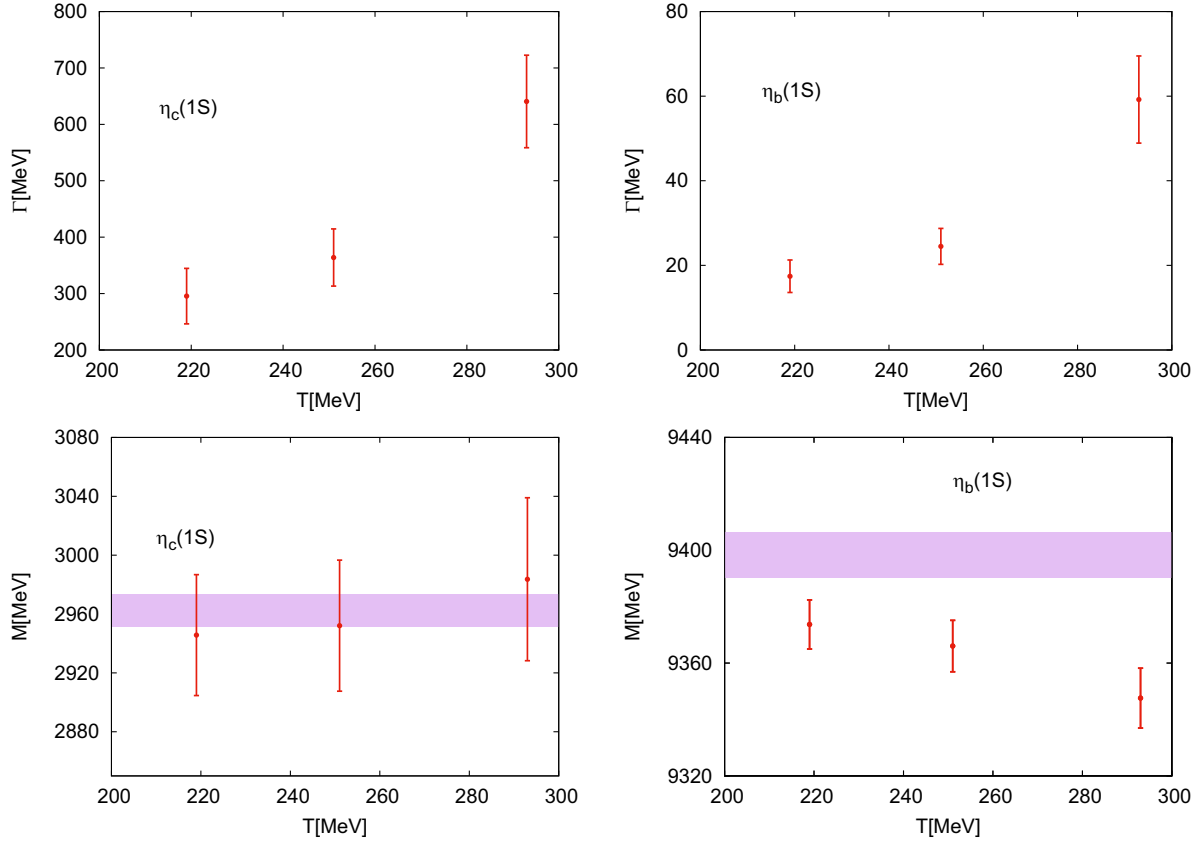


FIG. 13. Thermal mass shifts and decay widths of the quarkonium $1S$ states are extracted by fitting the spectral function near the peak, as described in Eq. (26). Top panels: decay width as a function of temperature for the charm sector (left) and bottom sector (right). Bottom panels: thermal mass as a function of temperature for the charm sector (left) and bottom sector (right). The purple band corresponds to the zero-temperature mass along with its error band.

The thermal mass of the $\eta_c(1S)$ state is consistent with the zero-temperature mass within error bars, while the thermal mass of the $\eta_b(1S)$ state shows a departure from the zero-temperature mass, decreasing with increasing temperature. As mentioned earlier, this is due to the fact that the real part of the potential always tends to shift the peak position toward smaller values, whereas the imaginary part pushes the peak position toward larger values. This suggests that the impact of the imaginary part on the charm state is much larger than that of the real part in the temperature range we considered, and that $\eta_c(1S)$ is already close to its melting temperature at our highest temperature, 293 MeV. In contrast, the impact of the real part on the bottom state is much larger compared to the imaginary part, resulting in a downward shift with increasing temperature.

VII. COMPARISON WITH LATTICE

As the pseudoscalar spectral function does not contain any additional structure in the $\omega \sim 0$ region, we can use the spectral function from Fig. 11 to describe the long-distance part of the lattice correlator in Eq. (3).

On the lattice, we directly calculate $G^{\text{PS}}(\tau T)/M_B^2$, while the spectral function we calculated is $\rho^{\text{PS}}(\omega)/m(\mu_{\text{ref}})^2$. A multiplicative renormalization factor is needed for the lattice correlator to transform it to the $\overline{\text{MS}}$ scheme at the mass scale $\mu = 1/a$, and subsequently multiply by the factor $(\frac{m(\mu=1/a)}{m(\mu_{\text{ref}})})^2$. However, this renormalization factor is currently not known for $(2+1)$ -flavor QCD. Therefore, for comparison we calculate the following quantity from the lattice QCD data:

$$aM_{\text{eff}}(\tau_i) = \log \left(\frac{G_{\text{PS}}^E(\tau_i)}{G_{\text{PS}}^E(\tau_{i+1})} \right). \quad (27)$$

In this ratio, the renormalization constant cancels out, allowing for a direct comparison between the lattice data and the correlation function obtained from the spectral function in Fig. 11. This comparison is shown in Fig. 14, where we also include the error of the spectral function in the correlator. From this, we observe that the correlator obtained from the spectral function in Fig. 11 is consistent with lattice QCD data at large τ values within error bars across all temperatures.

To quantify the level of agreement, we calculate the following χ^2 , defined as

$$\chi^2 = \sum_{\tau_i > \tau_{\min}} \sum_{\tau_j > \tau_{\min}} \left(M_{\text{eff}}^{\text{data}}(\tau_i) - a M_{\text{eff}}^{\text{prediction}}(\tau_i) \right) [C^{-1}]_{ij} \left(M_{\text{eff}}^{\text{data}}(\tau_j) - a M_{\text{eff}}^{\text{prediction}}(\tau_j) \right), \quad (28)$$

where C is the covariance matrix:

$$C_{ij} = C_{ij}^{\text{data}} + \delta_{ij} \left(\sigma_i^{\text{prediction}} \right)^2, \quad (29)$$

where C^{data} represents the covariance matrix on the lattice effective masses, and $\sigma^{\text{prediction}}$ is the systematic uncertainty in the effective mass from the spectral function. For the η_c and η_b channels, we find that $\tau_{\min} = 0.22$ and $\tau_{\min} = 0.28$ fm give a χ^2/ndf range of 0.1–1.22. The value of $\tau_{\min}$ is expected to be larger for the η_b state than for the η_c

state as the bottom quark mass is much larger than the charm quark mass, leading to larger lattice artifacts at larger distances for the η_b case.

Figure 14 also indicates that it should be possible to fit the lattice correlator at all temperatures using the spectral function with a single multiplicative constant. The overall conversion factor that was mentioned before can be absorbed into the overall temperature-independent fitting coefficient. Since the prediction also contains errors, we minimize the χ^2 function, defined as

$$\chi^2(A) = \sum_T \sum_{\tau_i, \tau_j} (\text{corr}^{\text{data}}(\tau_i) - A \text{corr}^{\text{prediction}}(\tau_i)) [C^{-1}]_{ij} (\text{corr}^{\text{data}}(\tau_j) - A \text{corr}^{\text{prediction}}(\tau_j)), \quad (30)$$

where C is the covariance matrix:

$$C_{ij} = C_{ij}^{\text{data}} + \delta_{ij} \left(A \sigma_i^{\text{prediction}} \right)^2. \quad (31)$$

Again, C^{data} represents the covariance matrix for the correlator, and $\sigma^{\text{prediction}}$ represents the uncertainty in the prediction of the correlator.

This yields $A \sim 11$ for the η_c correlator and $A \sim 7$ for the η_b correlator. In Fig. 15, we have plotted $G_{\text{PS}}/G_{\text{PS}}^{\text{free}}$ from both, the direct lattice correlators and the correlators obtained from the spectral function. Here, we divide the

correlator by $G_{\text{PS}}^{\text{free}}$, the correlator in the free theory, to improve visualization.

From Figs. 14 and 15, we observe that our spectral function, obtained from the Schrödinger equation using a screened thermal potential, provides a valid description of the η_c and η_b correlators obtained directly from the lattice. We therefore conclude that the non-perturbatively determined complex potential, with a real part that describes color screening of the QGP, is consistent with all the lattice measurements of pseudoscalar charmonium and bottomonium correlators we studied.

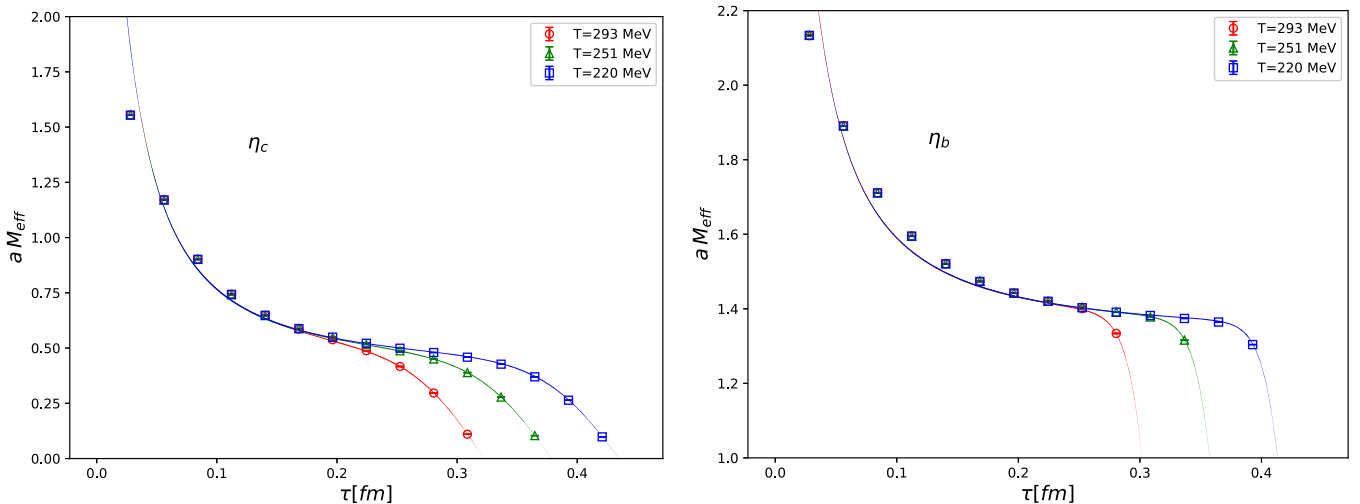


FIG. 14. The effective mass, M_{eff} , is compared with lattice data and the corresponding spectral functions in Sec. VI: left panel for η_c and right panel for η_b .

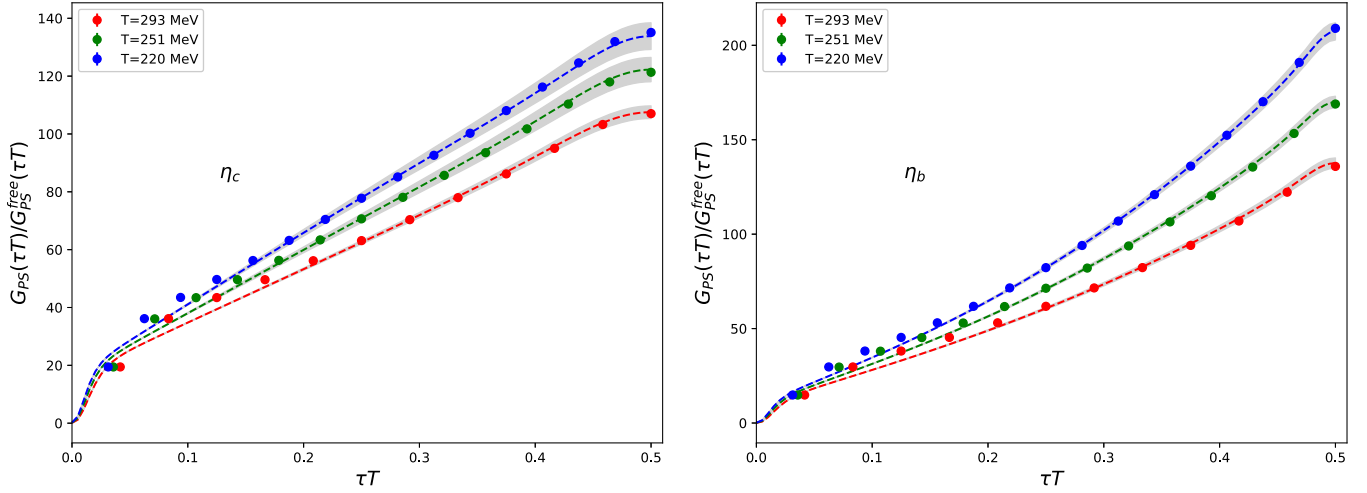


FIG. 15. Prediction of the pseudoscalar correlator from the spectral functions in Sec. VI, compared with the lattice correlator using relativistic quarks: left panel for η_c , right panel for η_b .

VIII. CONCLUSION

In this paper we study quarkonium spectral functions in the pseudoscalar channel using a nonperturbatively determined thermal lattice potential and compared this to a calculation of the full relativistic pseudoscalar charmonium and bottomonium point-point correlators in lattice QCD. For the quark propagator we used clover-improved Wilson fermions, and for the background gauge field, we used $(2+1)$ -flavor HISQ configurations with a pion mass of 320 MeV. For the spectral reconstruction process of this correlator, we combined the spectral function in different regions using physics-based information. The method used to calculate the thermal potential is based on a physically motivated ansatz that satisfies the basic definition given in Eq. (5). This approach has been used in several previous lattice studies [33,53] and has consistently resulted in a screened real part of the thermal potential, which our findings in this paper also confirm. While there are also studies in Refs. [34,35] that do not show such screening, they adopt different definitions or frameworks. In these works, a different method has been used to extract the potential, and the connection to the potential defined as a limit in Eq. (5) is not made explicit. This reflects the inherent ambiguity and the ill-posed nature of analytic continuation from Euclidean data, emphasizing the role of physics motivated inputs in such analyses. In this work, we further show that the spectral function of pseudoscalar quarkonia calculated from the thermal potential is consistent with direct lattice QCD measurements of the corresponding correlators.

As pointed out, different regions of the spectral function can be computed using different methods. The perturbative vacuum spectral function was used for the region $\omega \gg 2M_q$, whereas for $\omega \sim 2M_q$, a nonperturbatively determined complex thermal potential was used in the Schrödinger equation to extract the spectral function.

For the pseudoscalar channel, the combination of the spectral functions from these two regions provides a sufficient description.

The thermal complex potential was calculated from nonperturbatively determined Wilson line correlators, which itself is an analytic continuation problem. To extract the potential, an ansatz for the parametrization of the Wilson line correlators was used, motivated by the general τ dependence of the HTL spectral function, as shown in Eq. (16). This guarantees the existence of the limit in Eq. (5). As demonstrated in Fig. 2, the parametrization effectively describes the lattice QCD correlator within a reasonable range in the intermediate region. The $\tau \sim 0$ and $\tau \sim \beta$ regions are not expected to be described by this correlator, as the thermal potential originates from the long-time behavior of the Wilson line correlator.

At long distances, the potential becomes quite noisy, so gradient flow at different depths was used to enhance the signal at large separations, as seen in Fig. 4. However, it was observed that the flow distorts the short-distance behavior of the real part of the potential. A smooth matching of the potentials at different flow depths was performed to obtain a reliable estimate for the real part. The results indicate the presence of color screening in the QGP phase, as shown in Fig. 5. We found that the imaginary part of the potential does not show significant flow dependence within error bars, as shown in Fig. 6. Nevertheless, a zero flow-time extrapolation using a linear fit was carried out to obtain the final imaginary part. The imaginary part increases with both distance and temperature, as illustrated in Fig. 7. Note that our distances are too small to observe the saturation of the imaginary part expected to occur at larger distances. For solving the Schrödinger equation in Eq. (4), the real and imaginary parts of the potential were parametrized as in Eqs. (20) and (22). The real part of the potential at short distance, found to be temperature

independent, was replaced with the renormalon-subtracted perturbative vacuum potential, as demonstrated in Fig. 9.

The spectral function was calculated using the potentials in Sec. VI. The quark pole mass was determined using the zero-temperature potential. By solving the Schrödinger equation, the spectral function in the threshold region was obtained and subsequently matched with the perturbative spectral function in the ultraviolet regime, as shown in Fig. 10. The resulting pseudoscalar quarkonia spectral functions at various temperatures are shown in Fig. 11, representing one of the main results of this paper. It was observed that charmonium states are significantly more sensitive to temperature compared to bottomonium states. For charm quarks, thermal effects are predominantly influenced by the imaginary part of the potential, while for bottom quarks, the real part plays a more critical role in the temperature range considered.

In Sec. VII, it is shown that these spectral functions can predict the lattice correlator calculated using the relativistic quark propagator. This agreement between the predicted and measured lattice correlators based on the thermal screened potential supports the conclusion that the screening of the QCD plasma is consistent with all lattice QCD data measured. There are various ways to extend this work. We have studied the pseudoscalar channel; this analysis can be extended to the vector channel, where, in addition to the bound state region, one needs to model the transport part of the spectral function. Here, we use the thermal potential in the static limit; at high temperatures, the mass correction to the thermal potential becomes important. Therefore, relativistic corrections to the static potential also need to be calculated. We would also like to extend the formalism of this paper to small nonzero baryonic density, which will be relevant for low-energy RHIC and FAIR experiments.

ACKNOWLEDGMENTS

The generation of gauge configurations and the measurement of Wilson line correlators were carried out using SIMULATEQCD [61]. These configurations have also been used in other studies by the HotQCD collaboration [35,45,49]. Some of the Wilson line correlator data used in this paper was taken from [35], and we thank the authors

for providing it. We also thank Luis Altenkort and Hai-Tao Shu for their efforts in integrating the mesonic correlator into the QUDA [62] code, which was used for the measurement of charm and bottom-point correlation functions. We would like to thank Saumen Datta for various discussions and reading the manuscript. The authors acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the CRC-TR 211 “Strong-interaction matter under extreme conditions”—Project No. 315477589—TRR 211. For the computational work we used the Bielefeld GPU cluster and the LUMI-G supercomputer. We acknowledge the EuroHPC Joint Undertaking for awarding this project access to the EuroHPC supercomputer LUMI-G, hosted by CSC (Finland) and the LUMI consortium through a EuroHPC Extreme Scale Access call.

DATA AVAILABILITY

The data that support the findings of this article are openly available [63].

APPENDIX A: LEADING ORDER PERTURBATIVE CALCULATION

In this Appendix, we will calculate the gradient-flowed Wilson line correlator at leading order to understand the effect of flow on the thermal potential. The Wilson line correlator, in terms of Wilson lines, is defined in Coulomb gauge as

$$W(r_L, r_R, \tau; \tau_F) = \frac{1}{N_c} \langle \text{Tr}[U_L(r_L, \tau_F) U_R^\dagger(r_R, \tau_F)] \rangle_T,$$

where $\langle O \rangle_T$ denotes the thermal average.

Here, U_L and U_R is the Wilson line at finite flow time at position r_L (position of quark) and r_R (position of antiquark) given by

$$U_{L/R}(r_{L/R}, \tau_F) = \exp \left(ig \int_0^\infty A_0(r_{L/R}, \tau; \tau_F) \right).$$

Expanding up to $O(g^2)$,

$$\begin{aligned} W(\vec{r}_L, \vec{r}_R, \tau, \tau_F) = & 1 - \frac{g^2}{4N_c} \int_0^\tau \int_0^\tau \langle A_0^a(\vec{r}_L, \tau_{L1}; \tau_F) A_0^a(\vec{r}_L, \tau_{L2}, \tau_F) \rangle_T d\tau_{L1} d\tau_{L2} \\ & - \frac{g^2}{4N_c} \int_0^\tau \int_0^\tau \langle A_0^a(\vec{r}_R, \tau_{R1}, \tau_F) A_0^a(\vec{r}_R, \tau_{R2}, \tau_F) \rangle_T d\tau_{R1} d\tau_{R2} \\ & + \frac{g^2}{2N_c} \int_0^\tau \int_0^\tau \langle A_0^a(\vec{r}_L, \tau_L, \tau_F) A_0^a(\vec{r}_R, \tau_R; \tau_F) \rangle_T d\tau_L d\tau_R, \end{aligned} \quad (\text{A1})$$

where τ_L and τ_R denote the time coordinate of the flowed gauge field at r_L and r_R , respectively.

We denote the position space propagator as

$$D^{ab}(r_1 - r_2, \tau_1 - \tau_2; \tau_F) = \langle A_0^a(r_1, \tau_1; \tau_F) A_0^b(r_2, \tau_2; \tau_F) \rangle.$$

At finite flow time τ_F , the propagator in the leading order is given by

$$D_{00}^{ab}(\vec{r}, \tau; \tau_F) = T \int \frac{d^3 \vec{p}}{(2\pi)^3} \sum_n \exp(i\omega_n \tau - i\vec{p} \cdot \vec{r}) \exp[-2\tau_F(\omega_n^2 + p^2)] D_{00}^{ab}(\omega_n, \vec{p}),$$

where $\omega_n = 2\pi nT$ is the Matsubara frequency. $D^{ab}(\omega_n, \vec{p})$ is the standard HTL momentum space propagator in the Coulomb gauge given by

$$D_{00}^{ab}(\omega_n, p) = \frac{\delta^{ab}}{\omega_n^2 + p^2 + \Pi_E(\omega_n, p)} \frac{\omega_n^2 + p^2}{p^2}.$$

Substituting this into Eq. (A1) and performing the integration over $\tau_{L/R}$, we obtain

$$\begin{aligned} W(r = |\vec{r}_L - \vec{r}_R|, \tau; \tau_F) &= 1 - \frac{g^2}{2N_c} \sum_n \int_p \exp[-2\tau_F(\omega_n^2 + p^2)] \frac{2 - \exp(i\omega_n \tau) - \exp(-i\omega_n \tau)}{\omega_n^2} D_{00}^{aa}(\omega_n, \vec{p}) \\ &+ \frac{g^2}{2N_c} \sum_n \int_p \exp[-2\tau_F(\omega_n^2 + p^2)] \exp(-i\vec{p} \cdot \vec{r}) \frac{2 - \exp(i\omega_n \tau) - \exp(-i\omega_n \tau)}{\omega_n^2} D_{00}^{aa}(\omega_n, \vec{p}). \end{aligned}$$

Simplifying, we get

$$W(r, \tau, \tau_F) = 1 - g^2 \int \frac{d^3 \vec{p}}{(2\pi)^3} (1 - \exp(i\vec{p} \cdot \vec{r})) \exp(-2\tau_F p^2) I(\tau, \tau_F, p), \quad (\text{A2})$$

where

$$I(\tau, \tau_F, p) = \frac{1}{2N_c} \sum_n \exp(-2\tau_F \omega_n^2) \frac{2 - \exp(i\omega_n \tau) - \exp(-i\omega_n \tau)}{\omega_n^2} D^{aa}(\omega_n, \vec{p}). \quad (\text{A3})$$

Separating out the $\omega_n = 0$ mode and using the propagator's spectral representation,

$$\frac{1}{\omega_n^2 + p^2 + \Pi_E(\omega_n, p)} = \int_{-\infty}^{\infty} \frac{dp_0}{\pi} \frac{\rho_E(p_0, \vec{p})}{p_0 - i\omega_n},$$

$\Pi_E(\omega_n, \vec{p})$ is the longitudinal gluon self-energy. We obtain

$$I(\tau, \tau_F, p) = g^2 C_F \left[\frac{\tau^2 T}{p^2 + \Pi_E(0, \vec{p})} + \int \frac{dp_0}{\pi} F(\tau, \tau_F, p_0, p) \rho_E(p_0, \vec{p}) \right],$$

where $F(\tau, \tau_F, p_0, p)$ contains the following Matsubara sum:

$$F(\tau, \tau_F, p_0, p) = p_0 T \sum_{n \neq 0} (2 - \exp(i\omega_n \tau) - \exp(-i\omega_n \tau)) \exp(-2\omega_n^2 \tau_F) \left(\frac{1}{\omega_n^2} + \frac{1}{\vec{p}^2} \right) \frac{1}{p_0^2 + \omega_n^2}.$$

To perform the Matsubara sum, we apply the Laplace transformation to the above equation with respect to τ_F :

$$F(s, p_0, p) = \mathcal{L}[F(\tau_F, p_0, p)] = \int_0^{\infty} F(\tau_F, p_0, p) e^{-s\tau_F} d\tau_F.$$

This leads to

$$F(s, p_0, p) = p_0 T \sum_{n \neq 0} (2 - e^{i\omega_n \tau} - e^{-i\omega_n \tau}) \frac{1}{s + 2\omega_n^2} \left(\frac{1}{\omega_n^2} + \frac{1}{\vec{p}^2} \right) \frac{1}{p_0^2 + \omega_n^2}.$$

In the Laplace domain, the Matsubara sum can be easily evaluated using partial fractions. Performing the Matsubara sum, followed by the inverse Laplace transform and some algebraic simplifications, leads to the following expression:

$$I(\tau, \tau_F, p) = C_F \left[\frac{\tau}{p^2 + \Pi_E(0, \vec{p})} + \int \frac{dp_0}{\pi} \mathcal{L}^{-1} \left\{ p_0 \frac{R(\tau, \sqrt{s/2})}{(s - 2p_0^2)} \left(\frac{2}{s} - \frac{1}{p^2} \right) + \frac{p_0 R(\tau, p_0)}{(s - 2p_0^2)} \left(\frac{1}{p^2} - \frac{1}{p_0^2} \right) \right\} \rho_E(p_0, \vec{p}) \right]. \quad (\text{A4})$$

Here, $R(\tau, p_0)$ is a periodic function given by

$$R(\tau, p_0) = \frac{1 + \exp(\beta p_0) - \exp((\beta - \tau)p_0) - \exp(\tau p_0)}{p_0(\exp(\beta p_0) - 1)}.$$

Using the convolution theorem, we can write

$$I(\tau, \tau_F, p) = C_F \left[\frac{1}{p^2 + \Pi_E(0, \vec{p})} + \int \frac{dp_0}{\pi} \left(\mathcal{L}^{-1} \left\{ \frac{p_0}{(s - 2p_0^2)} \left(\frac{2}{s} - \frac{1}{p^2} \right) \right\} * \mathcal{L}^{-1} \left\{ R(\sqrt{s/2}, \beta, \tau) \right\} \right. \right. \\ \left. \left. + p_0 \exp(2p_0^2 \tau_F) R(p_0, \beta, \tau) \left(\frac{1}{p^2} - \frac{1}{p_0^2} \right) \right) \rho_E(p_0, \vec{p}) \right]. \quad (\text{A5})$$

We further refine this to obtain

$$I(\tau, \tau_F, p) = C_F \left[\frac{\tau}{p^2 + \Pi_E(0, \vec{p})} + \int \frac{dp_0 p_0}{\pi} \left\{ g(\tau_F) * Q(\tau_F, \beta, \tau) + \exp(2p_0^2 \tau_F) R(p_0, \beta, \tau) \left(\frac{1}{p^2} - \frac{1}{p_0^2} \right) \right\} \rho_E(p_0, \vec{p}) \right]. \quad (\text{A6})$$

Here,

$$g(\tau_F) = \mathcal{L}^{-1} \left\{ \frac{\tau}{(s - 2p_0^2)} \left(\frac{2}{s} - \frac{1}{p^2} \right) \right\} = -\frac{1}{p_0^2} - \left(\frac{1}{p^2} - \frac{1}{p_0^2} \right) \exp(2p_0^2 \tau_F), \\ Q(\tau_F, \beta, \tau) = \mathcal{L}^{-1} \{ R(\sqrt{s/2}, \beta, \tau) \}.$$

The following convolution can be further simplified:

$$g(\tau_F) * Q(\tau_F, \beta, \tau) = \int_0^{\tau_F} d\nu Q(\nu, \beta, \tau) g(\tau_F - \nu) \\ = -\left[\frac{1}{p_0^2} P(0, \tau_F, \beta, \tau) + \left(\frac{1}{p^2} - \frac{1}{p_0^2} \right) \exp(2p_0^2 \tau_F) P(2p_0^2, \tau_F, \beta, \tau) \right].$$

Here, $P(2p_0^2, \tau_F, \beta, \tau) = \int_0^{\tau_F} d\nu Q(\nu, \beta, \tau) \exp(-2p_0^2 \nu)$.

If $\tau_F \rightarrow \infty$, then $P(2p_0^2, \tau_F, \beta, \tau)$ corresponds to the Laplace transform of Q at the point $2p_0^2$. Since Q is already the inverse Laplace transform of $R(\sqrt{s/2}, \beta, \tau)$, performing the Laplace transform will lead us back to $R(\sqrt{s/2}, \beta, \tau)$ at the point $2p_0^2$. To generalize this, we write

$$P(2p_0^2, \tau_F, \beta, \tau) = \int_0^\infty \theta(\tau_F - \nu) Q(\nu, \beta, \tau) \exp(-2p_0^2 \nu) d\nu \\ = \int_0^\infty d\nu \int_{-\infty}^\infty dz \frac{1}{2\pi i} \frac{1}{z - i\epsilon} \exp(i(\tau_F - \nu)z) Q(\nu, \beta, \tau) \exp(-2p_0^2 \nu) \\ = \int_{-\infty}^\infty dz \frac{\exp(i\tau_F z)}{2\pi i(z - i\epsilon)} \int_0^\infty d\nu \exp[-(2p_0^2 + iz)\nu] Q(\nu, \beta, \tau) \\ = \int_{-\infty}^\infty dz \frac{\exp(i\tau_F z)}{2\pi i(z - i\epsilon)} R\left(\sqrt{p_0^2 + i\frac{z}{2}}, \beta, \tau\right).$$

Substituting everything back into $I(\tau, p_0, p)$ in Eq. (A6), we obtain

$$I(\tau, \tau_F, p) = C_F \left[\frac{\tau - P(0, \tau_F, \beta, \tau)}{p^2 + \Pi_E(0, \vec{p})} + \int \frac{dp_0 p_0}{\pi} \left(\frac{1}{p^2} - \frac{1}{p_0^2} \right) \exp(2p_0^2 \tau_F) \{R(p_0, \beta, \tau) - P(2p_0^2, \tau_F, \beta, \tau)\} \rho_E(p_0, \vec{p}) \right].$$

The Wilson line correlator is then

$$\begin{aligned} W(r, \tau; \tau_F) &= 1 - g^2 \int \frac{d^3 \vec{p}}{(2\pi)^3} (1 - \exp(i\vec{p} \cdot \vec{r})) \exp(-2\tau_F p^2) I(\tau, \tau_F, p) + \dots \\ &\approx \exp \left(-g^2 \int \frac{d^3 \vec{p}}{(2\pi)^3} (1 - \exp(i\vec{p} \cdot \vec{r})) \exp(-2\tau_F p^2) I(\tau, \tau_F, p) \right). \end{aligned} \quad (\text{A7})$$

From this equation, we can see that our ansatz for fitting the Wilson line correlator in Eq. (16) also holds in the case of leading-order flowed configurations. Specifically, we observed that $\log(W(r, \tau))$ can be decomposed into a combination of a linear τ term and a periodic τ term. At zero flow time, Eq. (A7) reproduces the expression given in [23].

To see the effect of the flow on this correlator, we compute the above expression numerically and calculate the effective mass, defined as

$$m_{\text{eff}}(r, \tau_i; \tau_F) = \log \left(\frac{W(r, \tau_i, \tau_F)}{W(r, \tau_i + a, \tau_F)} \right), \quad (\text{A8})$$

where a is the lattice spacing.

We use the two-loop thermal coupling $g^2(T)$ and the Debye mass $m_D^2 = \Pi_E(0, p)$ from [59]. The renormalization scale is chosen as $\mu_{\min} = 9.082T$, corresponding to the minimum sensitivity scale. The spectral function $\rho_E(p_0, p)$ is taken from [23] and is given by

$$\rho_E(p_0, p) = \pi \text{sign}(p_0) \delta(p_0^2 - p^2 - \text{Re}[\Pi_E(p_0, p)]), \quad p < p_0, \quad (\text{A9})$$

$$\rho_E(p_0, p) = -\frac{\pi m_D^2 p_0}{2|\vec{p}|(p^2 + m_D^2)^2}, \quad p > p_0. \quad (\text{A10})$$

The momentum cutoff in the integrations in Eq. (A7) is set to π/a .

In Fig. 16, we plot $m_{\text{eff}}(r, \tau) - m_{\text{eff}}(r, N_\tau/2)$ as a function of τ for various flow times at $T = 220$ MeV ($N_\tau = 32$). The subtraction with respect to midpoint $N_\tau/2$ removes the flow-time-dependent additive renormalization constant. We observe that the flow time has a significant effect in the regions $\tau \sim 0$ and $\tau \sim \beta$. As the flow time increases, larger regions are affected. However, in the intermediate region, the effect of the flow remains small. On the right panel, we show $m_{\text{eff}}(r, \tau)$ from lattice data, which qualitatively exhibits similar behavior.

To compute the potential, we first need to perform analytic continuation $\tau \rightarrow it$ and then evaluate $\lim_{t \rightarrow \infty} t \frac{\partial \log(W(r, it; \tau_F))}{\partial t}$. The analytic continuation of the P term in Eq. (A7) turns out to be complicated. However, the effects of this should be small in the determination of the potential. This can already be seen from the plot in Fig. 17, where we compare the effective mass with and without the $P(p_0, \tau_F, \beta, \tau)$ term in Eq. (A7) for various flow times. This shows that this term primarily affects the regions $\tau \sim 0$ and $\tau \sim \beta$, while its influence in the intermediate region is negligible. Furthermore, the $P(p_0, \tau_F, \beta, \tau)$ term is entirely responsible for the nonmonotonic behavior of the effective mass near $\tau \sim 0$ and $\tau \sim \beta$. Since the effect of $P(p_0, \tau_F, \beta, \tau)$ is small in the intermediate region, and the potential is extracted from this region, we neglect its contribution in the potential computation. The potential can then be easily obtained from the remaining terms, as given by

$$V_T^{\text{re}}[\tau_F, m, r] = C_F g^2 \int_{-\infty}^{\infty} \frac{d^3 \vec{p}}{(2\pi)^3} (\exp(i\vec{p} \cdot \vec{r}) - 1) \frac{\exp(-2\tau_F p^2)}{p^2 + m_D^2} \quad (\text{A11})$$

$$V_T^{\text{im}}[\tau_F, m, r] = T C_F m_D^2 g^2 \int_{-\infty}^{\infty} \frac{d^3 \vec{p}}{(2\pi)^3} (\exp(i\vec{p} \cdot \vec{r}) - 1) \frac{\exp(-2\tau_F p^2)}{|\vec{p}|(p^2 + m_D^2)^2}. \quad (\text{A12})$$

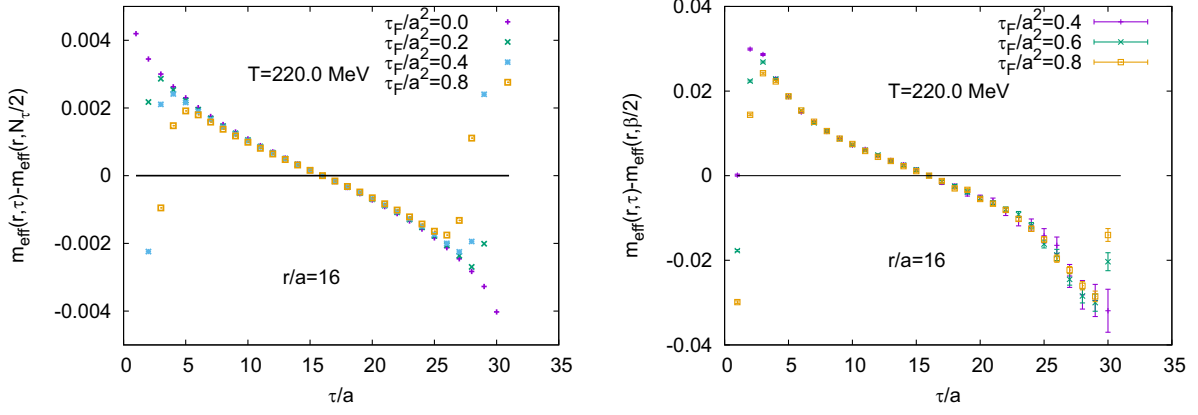


FIG. 16. The flow time dependence of the effective mass at $T = 220$ MeV and $r/a = 16$. Left: LO HTL perturbative result. Right: lattice QCD data.

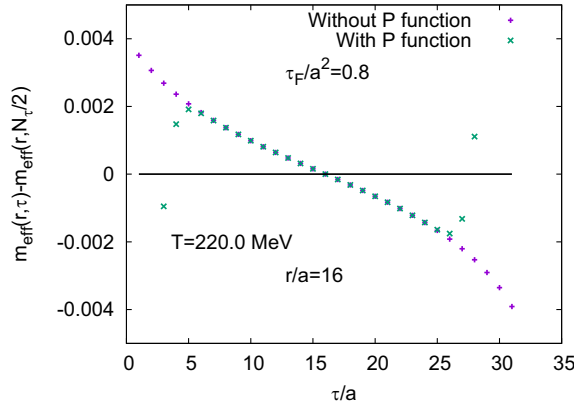


FIG. 17. Comparison of LO effective mass with and without P function.

This potential is plotted in Fig. 18 at $T = 220$ MeV for various flow times. In the left panel, we observe that the flow time dependence of the LO potential at short distances is quite similar to that of the nonperturbative potential in Fig. 4. Here, too, the effect of flow is significant at short distances compared to larger distances.

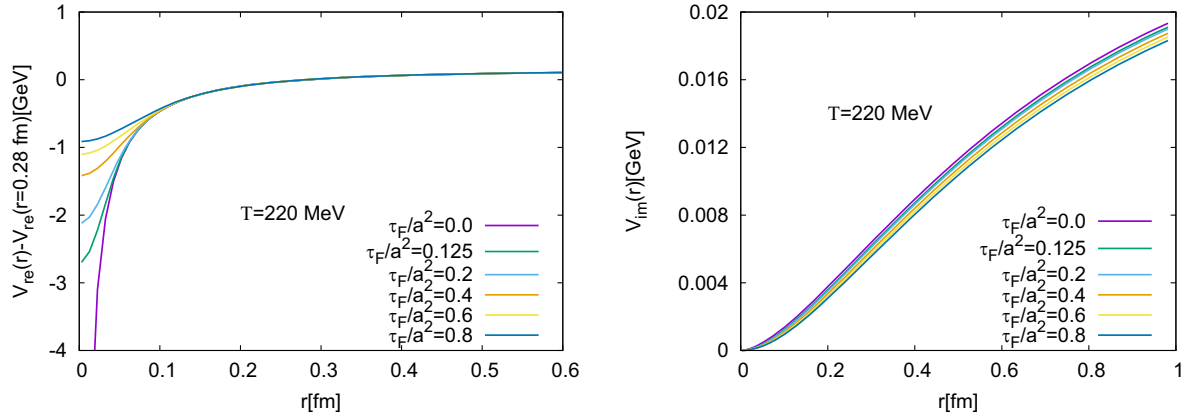


FIG. 18. The flow time dependence of the thermal potential at LO. Left: real part. Right: imaginary part.

However, the impact of flow on the LO real part is much stronger than what we observed nonperturbatively in Fig. 4.

In the right panel, we see that the imaginary part exhibits mild flow time dependence at all distances, with flow reducing its magnitude. However, this weak dependence is within the error bars of our lattice data and, therefore, cannot be resolved as shown in Fig. 6.

APPENDIX B: SYSTEMATIC UNCERTAINTY IN THE SPECTRAL FUNCTION

In this Appendix, we discuss the systematic uncertainty in the estimate of the spectral function in Fig. 11, as well as the width and mass shift in Fig. 13. As mentioned earlier, most of the error arises from the tuning of zero-temperature lattice masses, as shown in Sec. III, and the imaginary part. Any uncertainty in the determination of the zero-temperature mass will introduce uncertainty in the determination of the pole quark mass and the additive renormalization of the potential, which, in turn, will affect the final spectral function and its characterization parameters.

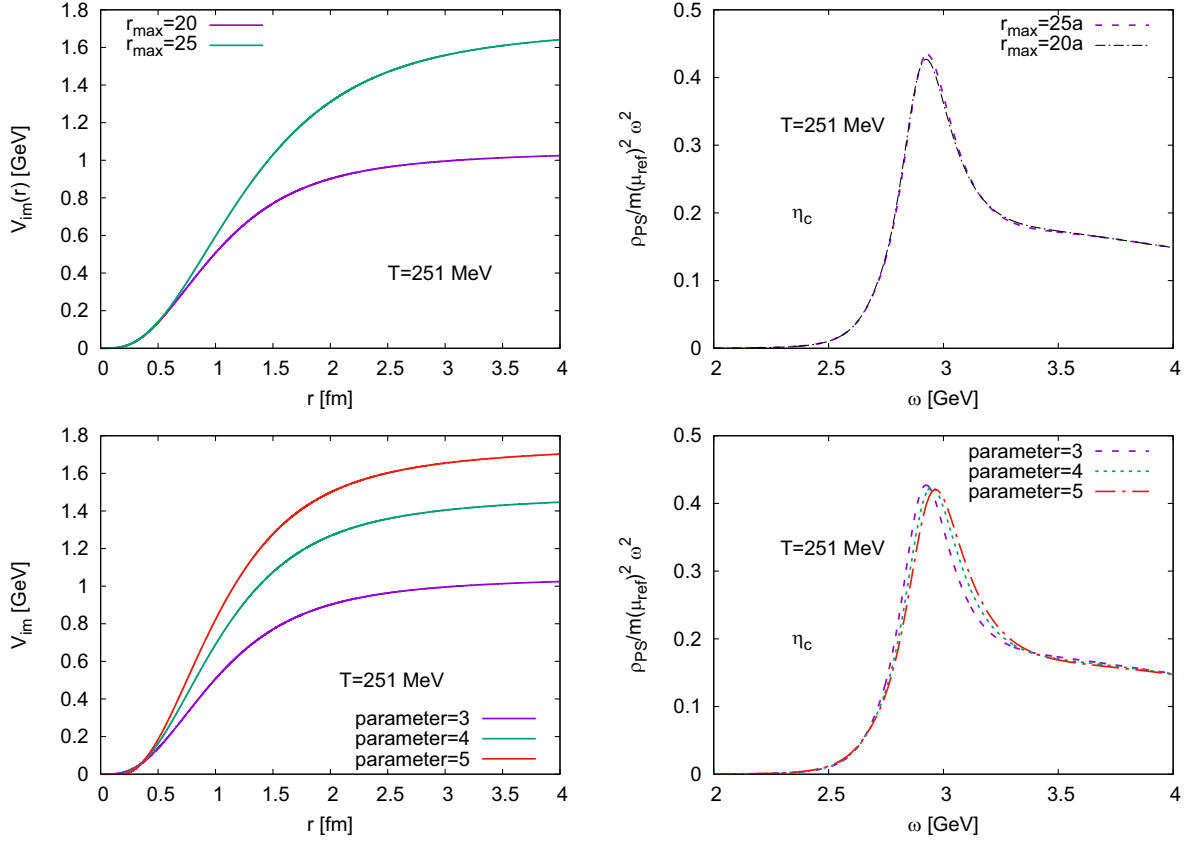


FIG. 19. Dependence of the spectral function on the imaginary part at $T = 251$ MeV for η_c . Top: the right panel shows the dependence of the spectral function on the imaginary part, fitted using the parametrization in Eq. (22) from the range $r_{\max} = 20a$ to $r_{\max} = 25a$, as depicted in the left panel. Bottom: The right panel illustrates the dependence of the spectral function on the imaginary part obtained using different fit parameters from Eq. (16), as shown in the left panel.

However, performing bootstrap error propagation for the spectral function is difficult in this case. As a result, we adopt a simpler approach. As mentioned earlier, we have fixed the bottom quark mass to 4.78 GeV and tuned the additive constant of the zero-temperature Cornell potential such that the ground state mass matches the lattice-determined $\eta_b(1S)$ mass. To account for the error, we performed the matching for three values: $M + \sigma_M$, M , and $M - \sigma_M$ for the $\eta_b(1S)$ mass. This results in three different additive constants for the potential. The error in the η_b spectral function due to the uncertainty thus arises through the uncertainty in these additive constants at each temperature.

Next, we tune the charm quark mass using this additive constant by solving the Schrödinger equation for three values: $M - \sigma_M$, M , and $M + \sigma_M$. Since there are three additive renormalizations, this results in nine charm quark mass values, contributing to the error in the η_c spectral function from zero temperature mass tuning. The average charm quark mass is found to be 1.35 ± 0.01 GeV. Now, we have to include the error of the thermal potential in the spectral function. We ignore the error on the real part because it is small compared to error introduced from other sources. On the other hand, the error in the imaginary part is

significant, and therefore, we consider the effect of this uncertainty on the imaginary part.

The main source of uncertainty in the imaginary part comes from the number of fitting parameters in the ansatz given in Eq. (16). We have also studied the systematic uncertainty by changing the fitting range and parametrization for each value of the number of parameters. The very short-distance region, around ~ 0.2 fm, cannot be fitted with this form, as in this region V_{im} becomes very small and is consistent with zero. This very small value causes numerical accuracy problems in solving the Schrödinger equation. To avoid this, we have regularized the imaginary part as $\max(10^{-4}M\alpha^2, V_{im}(r))$. We have checked that the details of the parametrization at short distances do not have a significant impact on the spectral function.

In the top left panel of Fig. 19 we have shown the imaginary part from three parameter fit for the fitting with $r_{\min} = 3a$ and $r_{\max} = 20a - 25a$. On the right panel of Fig. 19 we see that, although the large distance part of this is very different, nevertheless the effect of this difference on the spectral function is negligible.

In the bottom left panel of Fig. 19, we show the imaginary part with different fit parameter range for the

fit range with $r_{\max} = 20a$, and the corresponding spectral function on the bottom right panel is shown to have a mild dependence on the fit parameters.

We have taken these two sources (mass tuning and imaginary part) of uncertainty into account when

calculating the decay width and thermal mass from the spectral function, Fig. 13, and also in the computation of the Euclidean correlator for the comparison with the directly measured lattice correlator in Figs. 14 and 15.

-
- [1] T. Matsui and H. Satz, *Phys. Lett. B* **178**, 416 (1986).
 - [2] A. M. Sirunyan *et al.* (CMS Collaboration), *Phys. Lett. B* **790**, 270 (2019).
 - [3] B. Krouppa, A. Rothkopf, and M. Strickland, *Phys. Rev. D* **97**, 016017 (2018).
 - [4] L. Dong, Y. Guo, A. Islam, A. Rothkopf, and M. Strickland, *J. High Energy Phys.* **09** (2022) 200.
 - [5] S. Kajimoto, Y. Akamatsu, M. Asakawa, and A. Rothkopf, *Phys. Rev. D* **97**, 014003 (2018).
 - [6] N. Brambilla, M. A. Escobedo, A. Islam, M. Strickland, A. Tiwari, A. Vairo, and P. Vander Griend, *J. High Energy Phys.* **08** (2022) 303.
 - [7] N. Brambilla, M. A. Escobedo, A. Islam, M. Strickland, A. Tiwari, A. Vairo, and P. Vander Griend, *Phys. Rev. D* **108**, L011502 (2023).
 - [8] R. Sharma and A. Tiwari, *Phys. Rev. D* **101**, 074004 (2020).
 - [9] V. B. R and R. Sharma, *arXiv:2504.19348*.
 - [10] M. Asakawa and T. Hatsuda, *Phys. Rev. Lett.* **92**, 012001 (2004).
 - [11] S. Datta, F. Karsch, P. Petreczky, and I. Wetzorke, *Phys. Rev. D* **69**, 094507 (2004).
 - [12] H. T. Ding, A. Francis, O. Kaczmarek, F. Karsch, H. Satz, and W. Soeldner, *Phys. Rev. D* **86**, 014509 (2012).
 - [13] S. Kim, P. Petreczky, and A. Rothkopf, *J. High Energy Phys.* **11** (2018) 088.
 - [14] G. Aarts, C. Allton, T. Harris, S. Kim, M. P. Lombardo, S. M. Ryan, and J.-I. Skullerud, *J. High Energy Phys.* **07** (2014) 097.
 - [15] R. Larsen, S. Meinel, S. Mukherjee, and P. Petreczky, *Phys. Rev. D* **100**, 074506 (2019).
 - [16] R. Larsen, S. Meinel, S. Mukherjee, and P. Petreczky, *Phys. Lett. B* **800**, 135119 (2020).
 - [17] H. T. Ding, W. P. Huang, R. Larsen, S. Meinel, S. Mukherjee, P. Petreczky, and Z. Tang, *J. High Energy Phys.* **05** (2025) 149.
 - [18] S. Y. F. Liu and R. Rapp, *Phys. Rev. C* **97**, 034918 (2018).
 - [19] Z. Tang, S. Mukherjee, P. Petreczky, and R. Rapp, *Phys. Rev. D* **112**, 034030 (2025).
 - [20] Z. Tang, B. Wu, A. Hanlon, S. Mukherjee, P. Petreczky, and R. Rapp, *arXiv:2502.09044*.
 - [21] A. Mocsy and P. Petreczky, *Phys. Rev. D* **77**, 014501 (2008).
 - [22] O. Kaczmarek, F. Karsch, P. Petreczky, and F. Zantow, *Phys. Lett. B* **543**, 41 (2002).
 - [23] M. Laine, O. Philipsen, P. Romatschke, and M. Tassler, *J. High Energy Phys.* **03** (2007) 054.
 - [24] N. Brambilla, J. Ghiglieri, A. Vairo, and P. Petreczky, *Phys. Rev. D* **78**, 014017 (2008).
 - [25] M. Laine, *J. High Energy Phys.* **05** (2007) 028.
 - [26] Y. Burnier, M. Laine, and M. Vepsalainen, *J. High Energy Phys.* **01** (2008) 043.
 - [27] Y. Burnier, H. T. Ding, O. Kaczmarek, A. L. Kruse, M. Laine, H. Ohno, and H. Sandmeyer, *J. High Energy Phys.* **11** (2017) 206.
 - [28] H.-T. Ding, O. Kaczmarek, A.-L. Lorenz, H. Ohno, H. Sandmeyer, and H.-T. Shu, *Phys. Rev. D* **104**, 114508 (2021).
 - [29] D. Bala, S. Ali, O. Kaczmarek, and Pavan (HotQCD Collaboration), *J. Subatomic Part. Cosmol.* **3**, 100042 (2025).
 - [30] Y. Burnier, O. Kaczmarek, and A. Rothkopf, *J. High Energy Phys.* **12** (2015) 101.
 - [31] A. Rothkopf, T. Hatsuda, and S. Sasaki, *Phys. Rev. Lett.* **108**, 162001 (2012).
 - [32] Y. Burnier, O. Kaczmarek, and A. Rothkopf, *Phys. Rev. Lett.* **114**, 082001 (2015).
 - [33] D. Bala and S. Datta, *Phys. Rev. D* **101**, 034507 (2020).
 - [34] D. Bala, O. Kaczmarek, R. Larsen, S. Mukherjee, G. Parkar, P. Petreczky, A. Rothkopf, and J. H. Weber (HotQCD Collaboration), *Phys. Rev. D* **105**, 054513 (2022).
 - [35] A. Bazavov, D. Hoyer, R. N. Larsen, S. Mukherjee, P. Petreczky, A. Rothkopf, and J. H. Weber (HotQCD Collaboration), *Phys. Rev. D* **109**, 074504 (2024).
 - [36] R. N. Larsen, G. Parkar, A. Rothkopf, and J. H. Weber, *Phys. Rev. D* **110**, 114501 (2024).
 - [37] S. Caron-Huot, *Phys. Rev. D* **79**, 125009 (2009).
 - [38] D. Banerjee, S. Datta, R. Gavai, and P. Majumdar, *Phys. Rev. D* **85**, 014510 (2012).
 - [39] D. Banerjee, S. Datta, R. V. Gavai, and P. Majumdar, *Nucl. Phys. A* **1038**, 122721 (2023).
 - [40] D. Banerjee, S. Datta, and M. Laine, *J. High Energy Phys.* **08** (2022) 128.
 - [41] L. Altenkort, A. M. Eller, O. Kaczmarek, L. Mazur, G. D. Moore, and H.-T. Shu, *Phys. Rev. D* **103**, 014511 (2021).
 - [42] N. Brambilla, V. Leino, J. Mayer-Steudte, and P. Petreczky (TUMQCD Collaboration), *Phys. Rev. D* **107**, 054508 (2023).
 - [43] H. Pandey, S. Schlichting, and S. Sharma, *Phys. Rev. Lett.* **132**, 222301 (2024).
 - [44] R. Sommer, *Proc. Sci., LATTICE2013* (2014) 015 [*arXiv:1401.3270*].
 - [45] L. Altenkort, O. Kaczmarek, R. Larsen, S. Mukherjee, P. Petreczky, H.-T. Shu, and S. Stendebach (HotQCD Collaboration), *Phys. Rev. Lett.* **130**, 231902 (2023).
 - [46] E. Follana, Q. Mason, C. Davies, K. Hornbostel, G. P. Lepage, J. Shigemitsu, H. Trottier, and K. Wong (HPQCD

- and UKQCD Collaborations), *Phys. Rev. D* **75**, 054502 (2007).
- [47] M. Luscher and P. Weisz, *Commun. Math. Phys.* **98**, 433 (1985); **98**, 433(E) (1985).
- [48] L. Altenkort, D. de la Cruz, O. Kaczmarek, R. Larsen, G. D. Moore, S. Mukherjee, P. Petreczky, H.-T. Shu, and S. Stendebach (HotQCD Collaboration), *Phys. Rev. Lett.* **132**, 051902 (2024).
- [49] S. Ali, D. Bala, A. Francis, G. Jackson, O. Kaczmarek, J. Turnwald, T. Ueding, and N. Wink (HotQCD Collaboration), *Phys. Rev. D* **110**, 054518 (2024).
- [50] S. Navas *et al.* (Particle Data Group), *Phys. Rev. D* **110**, 030001 (2024).
- [51] Y. Burnier and A. Rothkopf, *Phys. Rev. D* **87**, 114019 (2013).
- [52] O. Philipsen and M. Wagner, *Phys. Rev. D* **89**, 014509 (2014).
- [53] D. Bala and S. Datta, *Phys. Rev. D* **103**, 014512 (2021).
- [54] A. V. Smirnov, V. A. Smirnov, and M. Steinhauser, *Phys. Rev. Lett.* **104**, 112002 (2010).
- [55] Y. Sumino, [arXiv:1411.7853](#).
- [56] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren, and A. Vogt, *J. High Energy Phys.* **02** (2017) 090.
- [57] A. Laschka, N. Kaiser, and W. Weise, *Phys. Rev. D* **83**, 094002 (2011).
- [58] F. Karsch, M. T. Mehr, and H. Satz, *Z. Phys. C* **37**, 617 (1988).
- [59] M. Laine and Y. Schroder, *J. High Energy Phys.* **03** (2005) 067.
- [60] Y. Guo, L. Dong, J. Pan, and M. R. Moldes, *Phys. Rev. D* **100**, 036011 (2019).
- [61] L. Mazur *et al.* (HotQCD Collaboration), *Comput. Phys. Commun.* **300**, 109164 (2024).
- [62] M. A. Clark, R. Babich, K. Barros, R. C. Brower, and C. Rebbi (QUDA Collaboration), *Comput. Phys. Commun.* **181**, 1517 (2010).
- [63] D. Bala, O. Kaczmarek, S. Ali, and Pavan, Data publication for “Thermal static potential and pseudoscalar quarkonium spectral functions from 2+1 flavor lattice QCD”, [10.4119/unibi/3006414](#).