

Concept study of a storage ring-based gravitational wave observatory: Terrestrial gravity noise from Rayleigh waves

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We extend the concept study of a storage ring-based millihertz gravitational wave (GW) detector of our previous work by estimating the terrestrial gravity noise (TGN) for a singly charged di-uranium ion circulating in a model storage ring with the circumference of the Large Hadron Collider (LHC). In our estimate, we consider solely seismic activity originating from Rayleigh waves, which are one of the most dominant source of TGN. Our first result is an analytical calculation of the time delay caused by the gravitational pull of a mass fluctuation in the vicinity of such a storage ring that a particle builds up compared to the unperturbed circulation time. Furthermore we simulate a Rayleigh wave field in a homogeneous half space scaled with the Peterson low noise model and compute the gravitational forces acting on the particle stored in the model ring. We relate our analytical considerations to the numerical results for the time delay by solving the geodesic equations for a storage ring and a conservative background potential and analytically derive the function of the time delay from the result. By performing a Fourier transform we compute the characteristic noise strain numerically and analytically. Based on the results we conclude that the terrestrial gravity noise in a storage ring-based gravitational wave detector is weak compared to the GW signature. However, we expect that seismic perturbations of the storage ring structure leading to a shaking of the experimental apparatus can cause a very large noise contribution. This noise, however, is mitigable in principle—in contrast to the TGN.

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I. INTRODUCTION

Current-generation gravitational wave (GW) detectors are increasingly limited by the low-frequency noise sources, mostly dominated by terrestrial gravity noise (TGN) [1,2]. This noise source originates from the fluctuations of the gravitational potential in the vicinity of the GW detector, e.g., from ocean waves or due to atmospheric turbulence [3,4] or seismic activity [5]. In these cases, the density of the soil or bodies of water changes as a function of time, which leads to a frequency-dependent redistribution of matter—the source of a force that pulls in corresponding direction and strength on the probes used for the detection of GWs. Due to its presence everywhere on Earth, this type of noise is of primary concern, when low-frequency GW detection on Earth is envisioned,

e.g., for operational interferometer-based detectors, such as LIGO, Virgo, and KAGRA [6–8], or for more recently emerged concepts, such as torsion bar GW antennae [9–11] or the storage ring-based gravitational wave observatory (SRGO) [12,13]. A suitable choice of detector sites, active noise cancellation, noise subtraction, and detector design suitable for single-Hz frequency GW detection, such as planned for Einstein Telescope [14,15], can only reduce the impact of TGN on the detector signal, but will not entirely circumvent the coupling of the noise source to the detector probe [3]. Ultimately, optimized detectors operating on Earth will likely be limited to measurements close and above 10 Hz [16,17]. However, a new detector design, pioneered with a first concept study of the SRGO [12,18], could potentially solve the problem of TGN mitigation, by avoiding the dominating gravitational coupling on the test masses almost entirely. The key difference of the SRGO compared to an interferometer-based detector is that the test masses are not suspended in fixed locations, but circulate in free fall along closed trajectories within the inhomogeneous spacetime due to a GW and the conservative background gravitational potential of the Earth. Spacetime strain is then measured as the change of the accumulated time-of-flight

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signal of a particle stored in the ring, compared to the expected time-of-flight signal of a hypothetical reference particle that is not subject to the same forces. While the particles circulate in the storage ring with frequencies of kilohertz, the part of the gravitational potential interesting to Earth-based GW astronomy changes with tens or hundreds of millihertz. Appearing as basically static to the particle for the short time span of one or even a few circulations within the ring at a time, the spatial inhomogeneity of the gravitational potential leads to a change of the particle velocity. However, a particle traveling on closed loops in a conservative potential will be deaccelerated and accelerated in a way, such that it returns to its initial velocity upon the end of each circulation. Consequently, the time delay that builds up in the conservative gravitational potential is just a sum of the time delay accumulated in each round individually, where during a fraction of time the particle does not travel with the same velocity as the reference particle. In contrast to that, a gravitational wave applies work on these circulating test masses [19–21] and coherently deaccelerates or accelerates it over minutes up to hours [12,18] without restoring them to the initial velocity after completion of a closed trajectory. Hence, the time delay accumulated over successive circulations originates in this case not only from the inhomogeneity of the spacetime, but also from the velocity change due to previous motion. Thus, even for much weaker interaction strength of the GW compared to the gravitational potential, the measurable time delay can be much larger. In this work we present the implications of this fundamental difference between the signature of a gravitational wave [18] and background gravitational potential in the time-of-flight signal. Furthermore, we derive the expected characteristic noise strain for a simulated seismic Rayleigh wave field expected in a model storage ring with the circumference of the LHC. We predict that the Rayleigh wave field changes the duration of a circulation on the order of 10^{-9} fs per second, which is orders of magnitude smaller than the 10^{-1} fs per second of a gravitational wave. Compared to other heavy objects, such as planes ($m = 500 \times 10^3$ kg, elevation ~ 13 km) which we estimate to cause a time delay for our model, which we describe in Sec. II, on the order of 10^{-13} fs per second and a heavy cloud ($m = 500 \times 10^6$ kg, elevation ~ 4.5 km) which could lead to a time delay up to 10^{-9} fs per second, we expect the Rayleigh wave field to be one of the most dominant noise sources and, thus, we consider our first estimation of TGN in storage ring-based detectors as representative for many different TGN sources.

Our work is structured as follows; in Sec. II we analyze the time delay that a particle builds up as it moves along a closed loop under the gravitational pull of a heavy object located somewhere in the vicinity. The obtained expression for the round trip time delay is a central result we use in later sections. In Sec. III we present our simulation of a

homogeneous half space filled with a Rayleigh wave field, that has a power spectral density (PSD) scaled with the Peterson low noise model. We describe how we use this field to compute the density variation leading to a fluctuating gravitational force at the different parts of the storage ring. In the next step, which we describe in Sec. IV, we analytically derive an expression for the arrival time due to an external gravitational force by relating it to the solution of the geodesic equations for a particle circulating in a conservative background potential. We compare the result to the numerical simulation of tracking a particle within the model storage ring. In Sec. V we use analytical considerations and the result for the time delay from Sec. II to derive an analytical expression for the noise power spectral density of a Rayleigh wave field scaled with the Peterson low noise model. We compare it to the results of the numerical tracking simulations. Lastly, we summarize our findings in Sec. VI.

II. TIME DELAY FROM MASS FLUCTUATION

In our setup an object at position $\mathbf{r}$ is assumed stationary for at least one full round of a particle circulating in a storage ring, with a linear velocity $\dot{r}_0 = \beta c$, where $\beta = 0.32$ in our case and c is the speed of light. This amounts to a revolution time of $278 \mu\text{s}$ and corresponds to sampling TGN with a rate of ≈ 3.6 kHz. In practice, however, for the frequency band 2.5×10^{-3} Hz– 0.1 Hz, which is interesting to our detection concept of low-frequency GW detection, this sampling rate is unreasonably high and, therefore, in our numerical computation the object at $\mathbf{r}$ will be assumed to be stationary for $k = 3000$ rounds, which corresponds to the time interval $\Delta t = 0.834$ s. As will be discussed in Sec. III, the lower frequency part of the spectrum is dominating the displacement amplitude of the ground and, thus, our TGN estimate. Hence, we do not simulate Rayleigh waves with frequencies higher than 0.1 Hz and consider $\frac{1}{\Delta t} \approx 1.2$ Hz as a suitable sampling frequency for the part of the spectrum that is interesting for our analysis. The Newtonian force $\mathbf{F}(\mathbf{r}_0, \mathbf{r})$ between the stored particle with mass m at position $\mathbf{r}_0$ and an object at position $\mathbf{r}$ and with mass M is given by

$$\mathbf{F}(\mathbf{r}_0, \mathbf{r}) = G \frac{mM}{|\mathbf{r}_0 - \mathbf{r}|^3} (\mathbf{r}_0 - \mathbf{r}). \quad (1)$$

For the geometry as shown in Fig. 1 where the storage ring is assumed to be located at a height H above and parallel to the (x, y) -plane, the longitudinal force on the circulating particle is given by $F_{\parallel} = \mathbf{F}(\mathbf{r}_0, \mathbf{r}) \cdot \frac{\dot{\mathbf{r}}_0}{\dot{r}_0}$, where $\mathbf{r}_0 = (R \cos \omega_{\text{Ring}} t, R \sin \omega_{\text{Ring}} t, H)$, with $\omega_{\text{Ring}} = \frac{\dot{r}_0}{R}$, which is the unperturbed angular velocity of the stored ion. The integration to obtain the time delay due to this force is difficult to perform in Cartesian coordinates, hence spherical coordinates for the position

$\mathbf{r} = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ with azimuth and polar angle ϕ and θ of the mass M are used.

We then proceed to compute the longitudinal separation Δl_{round} and thus, by division with βc , the time delay after one round Δt_{round} as a double integration of the acceleration obtained from $F_{\parallel}$ defined above. After the variable transformation $\tau := \omega_{\text{Ring}} t$ we find

$$\begin{aligned} \Delta t_{\text{round}} &= -\frac{\Delta l_{\text{round}}}{\beta c} \\ &= -\frac{GM}{\beta c} \int_0^{2\pi} \frac{d\tau}{\omega_{\text{Ring}}} \int_0^{\tau} \frac{d\tau'}{\omega_{\text{Ring}}} \frac{r \sin \theta \sin(\tau' - \phi)}{\sqrt{R^2 + r^2 \sin^2 \theta - 2rR \cos(\phi - \tau') \sin \theta + (r \cos \theta - H)^2}}, \end{aligned} \quad (2)$$

where now $\phi = \arctan(y/x)$, $\theta = \arccos z/r$ and $r = \sqrt{x^2 + y^2 + z^2}$ and we use $\beta = \frac{v}{c}$. Details of performing the integration are provided in the Appendix A. The solution to this integral is given by

$$\Delta t_{\text{round}} = -\frac{RGM}{(\beta c)^3} \left[\frac{2\pi}{\sqrt{H^2 + r^2 + R^2 - 2Hr \cos \theta - 2rR \cos \phi \sin \theta}} - \frac{4K\left(-\frac{4Rr \sin \theta}{H^2 + r^2 + R^2 - 2r(H \cos \theta + R \sin \theta)}\right)}{\sqrt{H^2 + r^2 + R^2 - 2rH \cos \theta - 2rR \sin \theta}} \right], \quad (3)$$

where $K(k^2)$ is the complete elliptic integral of the first kind.

Equation (3) computes the time delay of a particle due to the mass M after one full circulation relative to a design particle that is not subject to the force of such a mass. It is important to note that the result is valid for a ring with fixed radius. This is an idealized ring, where the radius of curvature of the charged particles path in the magnetic fields stays constant despite the momentum difference compared to the design particle. In the following $\Delta t_{\text{round}}(t)$ is written with explicit time dependence to denote that the three-dimensional coordinates of M at $\mathbf{r}$ are time dependent. We need to multiply Eq. (3) by $1/T_0 = \frac{\omega_{\text{Ring}}}{2\pi}$, the revolution frequency, to obtain the temporal derivative of the accumulated time delay per second

$$\frac{d\Delta T^{\text{fixed}}(t)}{dt} = \frac{\omega_{\text{Ring}}}{2\pi} \Delta t_{\text{round}}(t). \quad (4)$$

Equation (4) denotes the rate at which the accumulated time delay in the ring with fixed radius changes. In order to obtain the result that would be seen in a measurement, that is in a real storage ring, the rate must be integrated and multiplied with a factor to account for the relativistic mass change and ring geometry [18], leading to

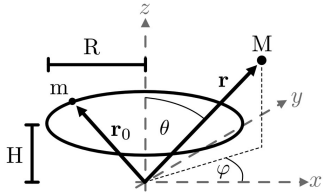


FIG. 1. The coordinate system for an object with mass m traveling on a circle with radius R and another object at position $\mathbf{r} = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ with mass M .

$$\Delta T^{\text{Ring}}(t) = -\eta \gamma^2 \int_0^t \frac{\omega_{\text{Ring}}}{2\pi} \Delta t_{\text{round}}(t') dt', \quad (5)$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and η is the slip factor of the storage ring.

Under the assumption that the position of M stays constant for the duration $\Delta t = t_{i+1} - t_i$ which the particle in the ring requires to pass $k = \frac{\Delta t}{T_0}$ full circulations, this result can be simplified to a sum, more suitable for numerical calculations, leading to

$$\Delta T^{\text{Ring}}(t) = -\eta \gamma^2 \sum_{i=1}^{N_k} \Delta t_{\text{round}}(t_i) k, \quad (6)$$

where N_k is the total number of time steps during the time interval $[0, t]$.

III. SIMULATION OF RAYLEIGH WAVE FIELD

We consider a very simple model already employed by Saulson [22], the homogeneous half space, filled with a Rayleigh wave field, to quantify the effect of TGN on the particle in the storage ring. The numerical simulation is adapted from [23] and will be modeled similarly by the superposition of a number of plane waves, with amplitudes scaled by the Peterson low noise model [24]. Rayleigh waves exhibit a characteristic surface displacement and we include only this type in our wave fields due to their dominant contribution to seismic motion at the surface [23]. Far away from any sources of seismic activity, we can consider the superposition of many different Rayleigh waves propagating in different directions parallel to the surface, to approximate a homogeneous field [23]. While a more sophisticated wave field will likely modify our estimates,

we expect our simplifications and assumptions to provide a useful approximation to the effect in first order.

Our model consists of plain waves in a three-dimensional grid of points, with the storage ring in the (x, y) -plane and its center being displaced from the ground by the height $H = 1$ m. The grid spans $3\lambda_R$ in all four horizontal and $2\lambda_R$ in downward vertical direction from the origin, where each block with edge length λ_R is subdivided further into 16 parts to sample the wave forms with enough accuracy.¹ We do not perform a simulation of ground motion by including scattering events, but, as we mentioned above, assume that the waves are located far away from the source. Thus, we can simply add each plane wave to our wave field and consider the result to be an approximation of a homogeneous seismic field. We start by considering the root $c_R/c_S = \chi < 1$ of the dispersion relation [25]

$$\chi^6 - 8\chi^4 + 8\left(\frac{2-\nu}{1-\nu}\right)\chi^2 - \frac{8}{1-\nu} = 0, \quad (7)$$

where we use for the P-wave velocity $c_P = 3000$ m/s and for the S-wave velocity $c_S = 1732$ m/s, together with a Poisson ratio of $\nu = 0.25$, to fix the Rayleigh wave speed c_R . For convenience we introduce the dimensionless quantities $p = \sqrt{1 - c_R^2/c_S^2}$ and $q = \sqrt{1 - c_R^2/c_P^2}$ [23] and, since we are interested in a proof of principle only, we leave $p \approx 0.39$ and $q \approx 0.85$ fixed for the numerical simulation. We construct the plane Rayleigh waves with *horizontal* and *vertical* components using the conventions in [23]

$$\begin{aligned} \xi_{\text{hor}}(\mathbf{r}, t) &= Ak_R \cdot \left(e^{-qk_R z} - \frac{2qp}{1+p^2} e^{-pk_R z} \right) \\ &\quad \times \sin(\mathbf{k}_R \mathbf{r} - \omega t + \phi) \\ \xi_{\text{ver}}(\mathbf{r}, t) &= -qk_R A \cdot \left(e^{-qk_R z} - \frac{2}{1+p^2} e^{-pk_R z} \right) \\ &\quad \times \cos(\mathbf{k}_R \mathbf{r} - \omega t + \phi). \end{aligned} \quad (8)$$

We use as dispersion relation [23]

$$c_R(f) = (150 + 1000e^{-f/1.5}) \text{ m/s}, \quad (9)$$

where $\lambda_R = c_R(f)/f$, to obtain a somewhat realistic Rayleigh wave speed, which, due to our choice of frequency interval $2.5 \text{ mHz} < f < 100 \text{ mHz}$, is in the range of $c_R \approx 1085 \text{ m/s} - 1150 \text{ m/s}$. This, however, is an empirical fit for the Virgo site [23] for higher frequencies and we

¹We performed a numerical convergence study, which revealed that a larger grid with e.g. length of $5\lambda_R$ or $7\lambda_R$ in each direction or using a higher subdivision into 32 parts changes the effective gravitational force from the wave field in our target frequency interval on the order of 1%. For our analysis we limit ourselves therefore to a smaller grid in order to reduce the numerical complexity of our calculation.

note that the above approximation for the dimensionless parameters and velocity of the Rayleigh wave is a simplification. Given, however, that the wave velocity is not strongly varying in our target frequency interval and that we are primarily interested in modeling the power spectral density of the vertical ground displacement with the aim to obtain an order-of-magnitude estimate only, we do not necessarily require a more precise description of the wave field. Furthermore, exact measurements beyond the Peterson low-noise model are difficult to obtain and highly location specific and, thus, our simplified wave field model with not be able to reflect the noise level at any specific site, but is intended to give a rather general indication and we can, hence, accept this level of oversimplification.

We model our ground with a mean density of $\rho_0 = 2.0 \text{ g/cm}^3$ and we define the three-dimensional displacement vector of the Rayleigh wave

$$\boldsymbol{\xi}(\mathbf{r}, t) = \xi_{\text{hor}}(\mathbf{r}, t) \frac{\mathbf{k}_R}{|\mathbf{k}_R|} + \xi_{\text{ver}}(\mathbf{r}, t) \mathbf{e}_z. \quad (10)$$

We consider frequency bins of width 0.5 mHz from 2.5 mHz up to 0.1 Hz. We sample each bin using 20 statistically independent waves with identical amplitude in frequency steps of $2.5 \times 10^{-2} \text{ mHz}$, but we ensure spatial homogeneity of our wave field model by sampling the random phase shift and the direction $\frac{\mathbf{k}_R}{|\mathbf{k}_R|} \sim \{\cos \theta, \sin \theta, 0\}$, such that $\phi, \theta \in [0, 2\pi)$ using a uniform distribution.

In addition to the three-dimensional displacement amplitude at each grid point, which we require for the estimate of the acceleration due to the mass fluctuations, we compute a time series from the vertical displacement vector of the total Rayleigh wave field at the surface $\xi_{\text{ver}}(\mathbf{0}, t)$.

By performing a Fourier transformation of this time series and computing the corresponding vertical displacement power spectral density, we then choose the amplitude A in Eq. (8) for the 20 waves in each frequency bin according to the Peterson low-noise model [24], as is shown in Fig. 2. A minimum in the spectrum is observable at around $2 \times 10^{-2} \text{ Hz}$, and at lower frequencies, the noise power increases by about 3 orders of magnitude. Thus, lower frequency seismic waves dominate the displacement spectrum by far. The Rayleigh waves with these amplitudes are also responsible for the mass fluctuations in the soil and having fixed the amplitudes A for each frequency bin enables us to compute the three-dimensional acceleration at the location $\mathbf{r}_j$ of the different magnetic lattice elements of the storage ring via [23,26]

$$\delta \mathbf{a}(\mathbf{r}_j) = G\rho_0 \sum_i \frac{1}{|\mathbf{r}_{ij}/\lambda|^3} (\boldsymbol{\xi}_i - 3[\hat{\mathbf{r}}_{ij} \cdot \boldsymbol{\xi}_i] \cdot \hat{\mathbf{r}}_{ij}) \frac{V_i}{\lambda^3}, \quad (11)$$

where $\mathbf{r}_j$ is the position of the j th storage ring element and $\mathbf{r}_i$ that of the i th grid point. Their distance is given as

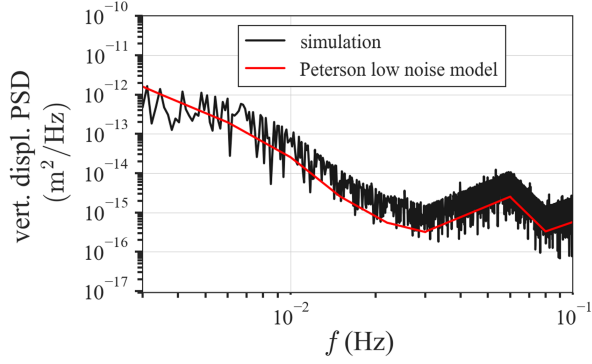


FIG. 2. The PSD of the Peterson low-noise model (red) and the z -displacement PSD (black) normal to the surface layer obtained from the Rayleigh wave field simulation. The wave field has a frequency range from 2.5 mHz up to 0.1 Hz where each bin of width 0.5 mHz contains 20 waves with identical amplitude in frequency steps of 2.5×10^{-2} mHz.

$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ and for convenience the unit vector $\hat{\mathbf{r}}_{ij} := \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|}$ is introduced, as well as the short hand notation $\xi_i := \xi(\mathbf{r}_i, t)$. The λ -dependence in Eq. (11) is written explicitly to emphasize that the resulting acceleration is independent of the wavelength and thus frequency of the Rayleigh waves [26], the t -dependence is omitted for brevity, but contained implicitly. We sample this Rayleigh field in time steps Δt during which it is kept constant in time, but spatially varying.

We evaluate Eq. (11) for the position of all magnets and drift spaces in our model. However, we exclude those grid points from the sum, which are closer than $\lambda/4$ to any element of the storage ring, in order to mitigate irregularities in the numerically evaluated acceleration that will arise from of the wavelength-dependent spatial discretization of the wave field. We simulate a single $U_2^{+/-}$ ion operated at velocity βc , where $\beta = 0.32$ in a ring with the circumference of the LHC simplified as an array of FODO cells. These consist of an arrangement of a focusing element (F), a weakly focusing element or drift space (O), followed by a defocusing element (D) and another weakly focusing element or drift space (O) [27], in the layout we have used in our previous work [18]. We state the parameters in Table I.

The result is a time series of three-dimensional acceleration vectors at each magnet position, but only for one realization of the time evolution of a wave field. These

acceleration vectors are then used to compute a tracking simulation of a particle. We repeat this procedure and ultimately take the average of the noise power spectra of three statistically independent simulations such that we obtain an estimate of the average noise due to seismic activity of Rayleigh waves. In the above considerations we have neglected the downward force due to the Earth's gravitational field, which we assume to contribute only with a constant term to the acceleration budget of the particle in the storage ring. Therefore, this part is not important for the time delay caused by the fluctuations of the gravitational field.

IV. TIME DELAY FROM GRAVITATIONAL POTENTIAL

Since the gravitational potential is conservative, the velocity of the particle for motion in a closed loop is the same at the beginning and end of the circulation. Due to the velocity difference $\neq 0$ on average, however, a nonzero time delay is measured nonetheless. For obtaining this time delay due to the gravitational force, which is shown below, we require the average velocity difference of the particle during one full circulation, compared to its nominal velocity $\dot{r}_0$, which we then consider as constant for Δt or at least for Δt_{round} . This average of the accumulated velocity change is computed via

$$\overline{\Delta \dot{r}(t)} = \frac{1}{N} \sum_{i \in \text{Ring}} \sum_j^i \frac{F_{\parallel}(t, \alpha_j)}{\gamma^3 m} \Delta t_j. \quad (12)$$

The sums cover all ring elements (drift space, quadrupole, and dipole magnets) located at the angles α_j along the ring and the gravitational force $F_{\parallel}(t, \alpha_j)$, while spatially varying with α_j , is also assumed to be constant in time during Δt . The particle requires the time Δt_j to pass through ring element j with length $\Delta l_j = \Delta t_j \dot{r}_0$.

In Fig. 3 the accumulated velocity change due to an external force is shown for one circulation of the ring and the dashed line indicates the nonzero average velocity difference compared to the particle velocity $\dot{r}_0 = \beta c$ without potential.

Motivated by $\frac{dT(t)}{dt} \sim -\frac{\dot{r}(t)}{\beta c}$ [18] and by observing that

$$\overline{\phi(t)} = -\overline{\Delta \dot{r}(t)} \dot{r}_0 \quad (13)$$

TABLE I. Parameters for a ring with circumference of 26.7 km, similar to the LHC, for $U_2^{+/-}$, operated at $\beta = 0.32$. Each magnetic unit cell consists of radially focusing (QF) and defocusing (QD) quadrupole magnets with magnetic parameters $k^{\text{QD/QF}}$, sector magnets with edge focusing and drift spaces, which have lengths l^{sector} etc.

	β	k^{QF}	k^{QD}	l^{QD}	l^{QF}	l^{sector}	l^{drift}	n_{cells}
$U_2^{+/-}$	0.32	$1.871 \frac{\text{T}}{\text{m}} \cdot \frac{q}{p_0}$	$3.478 \frac{\text{T}}{\text{m}} \cdot \frac{q}{p_0}$	2.950 m	3.400 m	47.10 m	7.500 m	200

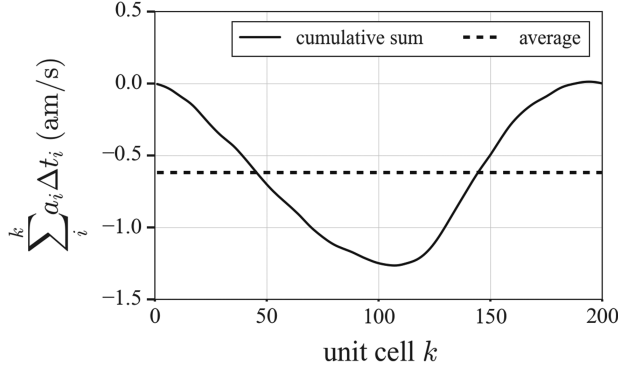


FIG. 3. The accumulated velocity change (relative to the nominal velocity $c\beta$) of a particle under the 3D force of a Rayleigh wave field scaled with the Peterson low noise model with a frequency range from 2.5 mHz up to 0.1 Hz. The average velocity difference during one round along the ring is shown as dashed line.

physically corresponds to a conservative potential, we thus relate

$$\frac{dT^{\text{Ring}}(t)}{dt} = -\eta\gamma^2 \frac{1}{\beta^2} \frac{\overline{\phi(t)}}{c^2} \quad (14)$$

as the change of the time delay that accumulates due to the gravitational force. Here the slip factor η and the relativistic factor γ account for the change of particle trajectory in the storage ring and the relativistic mass increase, both due to changing momentum, respectively [18].

We stress that the units in Eqs. (12)–(14) are consistent with the definition of a potential and the temporal derivative of the time delay. The proportionality of β^{-2} in Eq. (14) seems puzzling at first, but one β^{-1} originates from the gravitational force having more (for small β) or less (for large β) time to act, leading to a correspondingly larger or smaller time delay. We see from Eq. (12) that the accumulated velocity change involves the time the particle spends in each ring element and so a slower particle lingers longer at every point along the ring, such that the same gravitational force accumulates a larger velocity kick. The second β^{-1} occurs in Eq. (14), because the quantity that is measured in the storage ring is the fractional time delay which relates the effect of the gravitational potential to the unperturbed velocity $c\beta$ of the particle. Furthermore, when $\beta \rightarrow 0$, we see that $\frac{dT^{\text{Ring}}(t)}{dt} \rightarrow \infty$, which just means that the time for a circulation of the particle becomes extremely long; a plausible result for a particle that is not moving at all.

We demonstrate the agreement of Eq. (14) with a direct comparison of the integrated potential using the computed force and the time delay computed from a tracking simulation, which we depict both in Fig. 4.

For a ring with fixed radius, we show in the Appendix C by including a gravitational potential in the storage ring metric and solving the corresponding geodesic equations, that the change of the time delay due to the gravitational

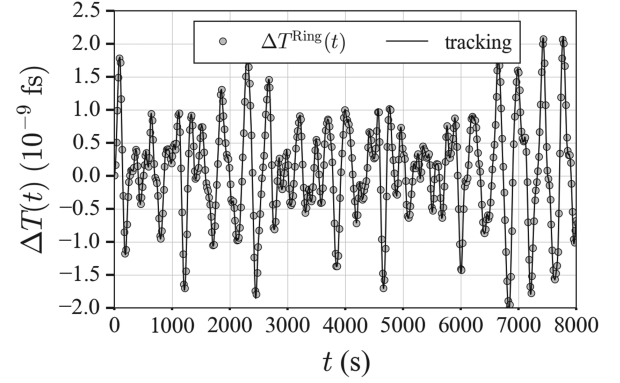


FIG. 4. The accumulated time delay $\Delta T^{\text{Ring}}(t)$ for a Rayleigh wave field. The result of the tracking simulation is shown as a line and the integrated force is shown as gray dots. The wave field has a frequency range from 2.5 mHz up to 0.1 Hz where each bin of width 0.5 mHz contains 20 waves with identical amplitude in frequency steps of 2.5×10^{-2} mHz.

potential in lowest order in ϕ , which corresponds to the first-order treatment of the time delay due to the gravitational wave strain $h(t)$ in [12,18], is given by

$$\frac{dT^{\text{fixed}}}{dt} = \frac{1 + 4\beta^2 - \beta^4}{\beta^2} \left(\frac{\phi(t)}{c^2} - \frac{\phi_0}{c^2} \right). \quad (15)$$

Hence, we see that our classical treatment of the gravitational force on the time delay above correctly accounts for the lowest-order approximation of the fully relativistic result and contains only the nonconstant part. We, therefore, note that our definition of the conservative potential, Eq. (14), includes the fact that the absolute offset of the gravitational potential is nonaccessible by measurement. However, as we have analyzed in our previous paper [18], the time delay computed with a storage ring tracking simulation is obtained relative to a design particle, which defines the coordinate system. Consequently, the constant part in Eq. (15) is not part of $\frac{dT^{\text{Ring}}}{dt}$ in our computation. With this in mind, finally, we integrate the two different expressions for the time delay and relate them via

$$\begin{aligned} \Delta T^{\text{fixed}}(t) = & -\frac{1}{\eta\gamma^2} (1 + 4\beta^2 - \beta^4) \Delta T^{\text{Ring}}(t) \\ & - (1 + 4\beta^2 - \beta^4) \frac{\phi_0}{c^2\beta^2} t, \end{aligned} \quad (16)$$

using η and γ . The resulting expression is remarkably similar to the relation of the time delay from a gravitational wave in a fixed ring and a storage ring which we derived in our previous work [18].

V. NOISE POWER SPECTRAL DENSITY

We employ the method derived by [28], adapted to the time delay computed from our storage ring model, to

estimate the noise power spectral density of the gravitational fluctuations of a Rayleigh wave field analytically. For this, we start with Eq. (4), insert Eq. (2) and write the result as

$$\frac{d\Delta T^{\text{fixed}}}{dt} = \frac{G\omega}{2\pi\beta c} \delta M(\lambda) I_2, \quad (17)$$

where I_2 is the solution to the integral, which is given by Eq. (2) without factors [stated explicitly in Eq. (A2)] and we rewrote $M = \delta M(\lambda)$ to emphasize that the mass causing the gravitational fluctuation originates from a density variation of the soil as a result of seismic waves with wavelength λ . Later we will consider the density variation in frequency space by using a Fourier transform. We are interested in the noise power spectral density for low-frequency seismic waves and so we will focus on the results important for the long wavelength regime where the wavelength λ of the seismic field is much larger than the storage ring radius R . For the numerical computation we will not assume a pointlike storage ring, while we will do so for the analytic derivation in order to obtain closed expressions, independent of ring radius R , but still dependent on its height H above the ground. In both numerical and analytical computations, however, in order to avoid divergences in our analysis also for zero height H , we work with an inner cutoff radius of $\lambda/4$, which is the closest distance of a mass fluctuation and the storage ring. In the next steps of this section, we will focus on the wavelength dependence of this expression and use the result to simplify the expression for $\frac{d\Delta T^{\text{fixed}}}{dt}$.

Following Saulson [22], we divide the ground into pieces that have coherent mass fluctuations on the length of $\lambda/2$. Thus, in order to estimate the contribution to the density variation of a volume element with edge length $\lambda/2$ at location $\mathbf{r}_i$ in the vicinity of the storage ring, we integrate I_2 , using spherical coordinates, over a suitable volume and define the function

$$A(\mathbf{r}_i) = \left(\frac{2}{\lambda}\right)^3 \int_{r_i-\lambda/4}^{r_i+\lambda/4} dr \int_{\theta_i-\frac{\lambda}{4r_i}}^{\theta_i+\frac{\lambda}{4r_i}} d\theta \int_{\phi_i-\frac{\lambda}{4r_i \sin \theta_i}}^{\phi_i+\frac{\lambda}{4r_i \sin \theta_i}} d\phi \times I_2(r_i, \theta_i, \phi_i) r^2 \sin(\theta). \quad (18)$$

The coordinate bounds are motivated by Fig. 5, where we show the blocks in spherical coordinates with the appropriate edge lengths. The corresponding volume integral computes in first order to

$$\int_{r_i-\lambda/4}^{r_i+\lambda/4} dr \int_{\theta_i-\frac{\lambda}{4r_i}}^{\theta_i+\frac{\lambda}{4r_i}} d\theta \int_{\phi_i-\frac{\lambda}{4r_i \sin \theta_i}}^{\phi_i+\frac{\lambda}{4r_i \sin \theta_i}} d\phi r^2 \sin \theta \approx \frac{\lambda^3}{8}, \quad (19)$$

which reproduces the correct volume. We are interested in the level of TGN due to the volume elements of the entire seismic field. Hence, we sum over the square of each of the

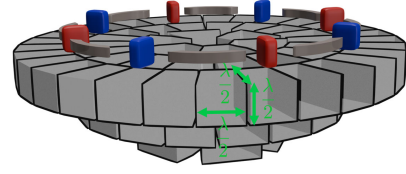


FIG. 5. A schematic of a storage ring above a ground, which is divided into blocks with edge length $\approx \lambda/2$.

cells of Eq. (18). We use as the mass fluctuations in a volume element $\delta M(\lambda)^2 = \frac{\rho_0^2 |\Delta z(\lambda, t)|^2 \lambda^4}{16}$, where we have used the vertical displacement $\Delta z(\lambda, t)$ of the Rayleigh waves at the surface to describe the density variation within all surface and subsurface elements of the soil. We use this approximation, because we are only interested in the order of magnitude and accept a possible overestimation of the TGN in our analysis and we obtain

$$\left(\frac{d\Delta T^{\text{fixed}}}{dt}\right)^2 = \left(\frac{G\omega_{\text{Ring}}}{2\pi\beta c}\right)^2 \frac{\rho_0^2 |\Delta z(\lambda, t)|^2}{16} \lambda^4 \sum_i A(\mathbf{r}_i)^2. \quad (20)$$

We point out that i counts the volume elements that have at least the distance $\lambda/4$ to the storage ring center, but the sum depends on the radius R and height H . Numerically, this expression can now be evaluated and the result for the λ -dependent part $S(\lambda) := \lambda^4 \sum_i A(\mathbf{r}_i)^2$ is shown in Fig. 6 as dots and crosses for different ring sizes and heights above the surface, where, following the discussion of Eq. (9), we use the longitudinal Rayleigh wave velocity $c_R = 1100$ m/s and density $\rho_0 = 2.0$ g/cm³ to scale the frequency axis. The frequency dependence of this result changes when the wavelength of the Rayleigh wave field is on the order of magnitude of the storage ring size. Hence, for smaller storage rings, this occurs at higher frequencies than for larger storage rings.

For obtaining an analytic expression describing the limit for very small volume elements, we assume that the square of the integral Eq. (18) can be approximated by $A(\mathbf{r}_i)^2 \approx I_2(\mathbf{r}_i)^2 \frac{8}{\lambda^3} dV$ as average of the mass fluctuation over the entire volume element dV and the sum in Eq. (20) can be replaced by an integral over the lower hemisphere. We will approximate $I_2(\mathbf{r}_i)$ at a large distance $r \gg R$, where the wavelength λ of the seismic field is much larger than the storage ring radius R . For $r \ll R$ the following integrals do not converge, so we focus on the case where the wavelengths are large—in any case the more interesting regime for low-frequency detection. Therefore, the analytical derivation treats the storage ring as pointlike detector without inner structure, but with height H above the ground. The numerical result, however, resolves the inner structure, leading to finite size effects, which will be discussed at the end of this section.

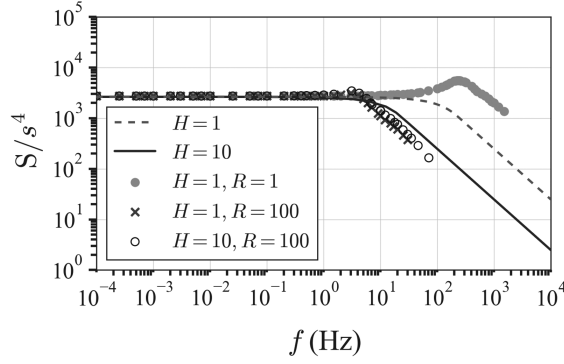


FIG. 6. Numerical (dots) and analytical (lines) result of the summation of volume elements, valid for the low-frequency limit where $\lambda \gg R$. For all curves $\omega_{\text{Ring}} = 1$ Hz. For the theoretical results the frequency behavior changes where $H \sim \lambda/4$, whereas for the numerical results this point depends on R . The slope at higher frequencies is the same for all curves.

Going forth, we replace $r \rightarrow R/g$, where $g \ll 1$. Next we find the Taylor expansion around $g \approx 0$ and replace again $g \rightarrow R/r$, leading to

$$I_2 \approx \frac{2\pi \cos \phi \sin \theta}{r^2 \omega_{\text{Ring}}^2}. \quad (21)$$

The frequency ω_{Ring} is reserved for the revolution frequency of the particle in the ring and not the frequency of the wave field, which we denote f . To avoid confusion we work with λ and replace the f by the wavelength $\lambda = c/f$.

Again, similar to Saulson and others [22,28], we introduce the inner cutoff radius of $\lambda/4$ to facilitate convergence of the integrals also for zero height H . Next, with Eq. (21), a distinction is made between $H < \lambda/4$ and $H > \lambda/4$.

For the first case the inner cutoff leaves out some of the soil beneath the ring, because the elevation above the ground is small compared to the wavelength of the seismic waves. Therefore the θ integration is split into two parts: one where some of the soil beneath the ring is cut, for small θ and the other, for $\theta \geq \arccos \frac{H}{\lambda/4}$, where in radial direction the surface of the ground is further away than $\lambda/4$. Starting with this part we find

$$\left(\frac{2}{\lambda}\right)^3 \int_{\lambda/4}^{\infty} dr \int_0^{2\pi} d\phi \int_{\arccos \frac{H}{\lambda/4}}^{\pi/2} d\theta \left(\frac{2\pi \cos \phi \sin \theta}{r^2 \omega_{\text{Ring}}^2}\right)^2 \times \sin \theta r^2 = \frac{256\pi^3 (-4H + \lambda)^2 (2H + \lambda)}{3\lambda^7 \omega_{\text{Ring}}^4}. \quad (22)$$

Next, the part is integrated, where the θ is large enough and the soil beneath the ring starts where $r \leq \lambda/4$. This leads to

$$\left(\frac{2}{\lambda}\right)^3 \int_{\arccos 4H/\lambda}^{\pi/2} d\theta \int_0^{2\pi} d\phi \int_{H \cos \theta}^{\infty} dr \left(\frac{2\pi \cos \phi \sin \theta}{r^2 \omega_{\text{Ring}}^2}\right)^2 \times \sin \theta r^2 = \frac{256H\pi^3 (-8H^2 + \lambda^2)}{\lambda^7 \omega_{\text{Ring}}^4}, \quad (23)$$

which is added to the previous result. For the other case where $H > \lambda/4$, we can directly integrate

$$\left(\frac{2}{\lambda}\right)^3 \int_{4H/\lambda}^{\infty} dr \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \left(\frac{2\pi \cos \phi \sin \theta}{r^2 \omega_{\text{Ring}}^2}\right)^2 \sin \theta r^2 = \frac{8\pi^3}{H\lambda^3 \omega_{\text{Ring}}^4}. \quad (24)$$

Thus, we find for the final result

$$S(\lambda) = \Theta(\lambda/4 < H) \frac{8\pi^3 \lambda}{H\omega_{\text{Ring}}^4} + \Theta(\lambda/4 > H) \left(\frac{2048\pi^3 H^3}{3\lambda^3 \omega_{\text{Ring}}^4} - \frac{256H\pi^3}{\lambda\omega_{\text{Ring}}^4} + \frac{256\pi^3}{3\omega_{\text{Ring}}^4} \right). \quad (25)$$

The analytical result is plotted as lines alongside the numerical result in Fig. 6, where we depict the results for various radii and heights. The most striking feature is that the result of the sum is constant in the low-frequency limit, where both numeric and analytical results coincide, and does not depend on the ring size or height above the surface and amounts to $\frac{256\pi^3}{3\omega_{\text{Ring}}^4}$. The numerical evaluation of the sum exhibits finite size effects, as the result depends on the ring radius and slightly on the elevation H and, thus, the inner structure of the storage ring. For the theoretical results, however, from the Taylor approximation performed in Eq. (21) onward, the ring is assumed pointlike and hence only H plays a role in the frequency-behavior. We stress, that our modeling is only valid for very low frequencies, where both analytical and numerical results coincide, and therefore we write down the result for a ring with fixed radius in the case of low frequency seismic waves as

$$\left(\frac{d\Delta T^{\text{fixed}}}{dt}\right)^2 = \left(\frac{G\rho_0}{\beta c}\right)^2 \frac{4|\Delta z(\lambda, t)|^2 \pi}{3\omega_{\text{Ring}}^2}. \quad (26)$$

In a last step, we perform the Fourier transform, such that $\text{FT}[\Delta z(\lambda, t)] = \Delta z(f)$ and replace the expectation value of the vertical displacement by the displacement PSD $D(f)$ of the Peterson low-noise model, such that we substitute $\frac{1}{T}|\Delta z(f)|^2 \approx D(f)$ and find for the one-sided noise PSD for the storage ring detector [18]

$$S_n(f) = \frac{2}{T} \left| \text{FT} \left[\frac{d\Delta T^{\text{fixed}}}{dt} \right] \right|^2 = \left(\frac{G\rho_0}{\beta c} \right)^2 \frac{8D(f)\pi}{3\omega_{\text{Ring}}^2}, \quad (27)$$

which we depict as characteristic noise strain $h_n(f) = \sqrt{fS_n(f)}$ with a red line in Fig. 7, and denote it “analytic.”

For the numerical result we have used $D(f)$ as the low noise model of Peterson [24], which we describe in Sec. III. For the simulation of the tracking, we employ the procedure also described Sec. III. The result of the tracking simulation is the time series $\Delta T(t_j)$, which we compute for $1 < j < N_k$ time steps with the temporal resolution Δt . For our simulation we choose $N_k = 10400$, which, for a temporal resolution of $\Delta t = 0.835$ s, corresponding to $k = 3000$ circulations of the particle in the storage ring, amounts to $T = 8684$ s time interval of simulation. We use a similar consideration as the result for the arrival time simulation power spectral density that we described in the Appendix of our last paper [18]. We additionally employ Eq. (14) and find for the noise power spectral density

$$S_n(f_j) = \frac{2\Delta t}{N_k} \left\langle \left| \sum_{j=1}^{N_k-1} e^{-i2\pi f_j t} \frac{\phi(t_j)}{c^2 \beta^2} \right|^2 \right\rangle, \quad (28)$$

where $f_j = \frac{j}{\Delta t N_k}$, indicating that it is basically just given by the Fourier transform of the longitudinal velocity change due to gravitational force. In this Fourier transform, we use the entire unwrapped time series to find the one-sided noise PSD, and we show the numerically obtained $\sqrt{fS_n(f)}$ as average over the resulting power spectral density from three different Rayleigh wave fields in Fig. 7 and denote it “numeric.”

Comparing the analytical prediction in Fig. 7 to the numerical results in the frequency range 2.5 mHz to 0.1 Hz we find that especially at the higher-frequency end the analytical prediction and numerical result deviate slightly, but altogether agree well. We see that, using Eq. (9), the long wavelength approximation holds for the entire considered frequency interval, which permits us to directly compare our numerical characteristic noise strain data with the analytical result in the $\lambda \gg R$ limit. To this end, similar to how the numerical data was obtained, we evaluate the analytical characteristic noise strain inferred from Eq. (27), in steps of 0.5 mHz covering the same frequency range and, for every frequency step, compute the relative difference for every pair of analytical and numerical data points. From the resulting distribution, which resembles a skew normal distribution, we determine the median to be 0.06, a mean of about 0.14 and that the 1σ -interval around the median is given from -0.38 to 0.70 . We consider this small deviation as a confirmation of our order-of-magnitude estimate and of the validity of our approximation in the target frequency interval.

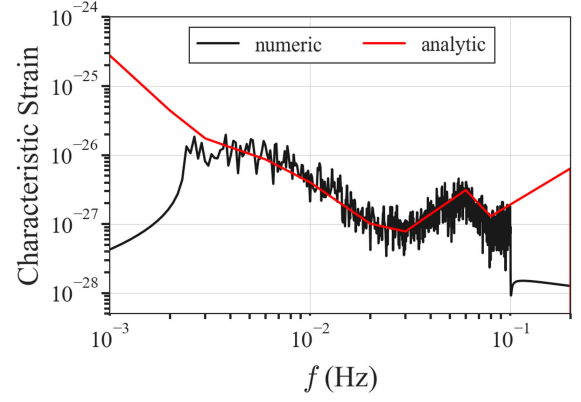


FIG. 7. The average of three noise power spectral densities, depicted as characteristic noise strain for Rayleigh wave fields scaled with the Peterson low noise model. The frequency range of the wave field is 2.5 mHz up to 0.1 Hz, where each bin of width 0.5 mHz contains 20 waves with identical amplitude in frequency steps of 2.5×10^{-2} mHz.

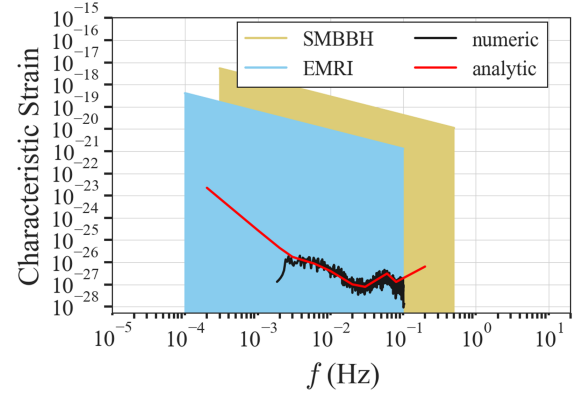


FIG. 8. The average of three noise power spectral densities, depicted as characteristic noise strain for Rayleigh wave fields scaled with the Peterson low noise model. The frequency range of the wave field is 2.5 mHz up to 0.1 Hz, where each bin of width 0.5 mHz contains 20 waves with identical amplitude in frequency steps of 2.5×10^{-2} mHz. As a comparison to the strength of the sources in this frequency band, the typical characteristic strain of extreme mass-ratio inspirals (EMRI) and super massive black hole binaries (SMBBH) are shown [18].

As can be seen in Fig. 8, where we depict the characteristic noise strain alongside the characteristic source strain of supermassive black hole binaries and extreme mass-ratio inspirals, our estimate for the TGN is a uniquely low value in comparison with other Earth-based GW detectors, where the TGN is usually the limiting factor for the detection of even single Hz GW, let alone lower-frequency GW [1]. We attribute this result to the fact that the particle stored in the ring is to large extent insensitive to local perturbations of the higher frequency Rayleigh waves, due to the averaging over the large ring circumference. The lower-frequency perturbations have a small effect due to the conservative nature of the gravitational potential, because the particles

pass on a closed loop and, hence, no work can be applied on the particle from this conservative gravitational field. An incoming gravitational wave, however, can apply work on the particles, which coherently accelerates the particle over several minutes as it circulates in the ring, quickly surpassing the effective time delay in magnitude originating from the Rayleigh wave field causing terrestrial gravity noise.

VI. CONCLUSION

We derive analytically the time delay that a particle in a storage ring builds up over the duration of a single round trip due to a mass at a given position. We discuss our method of simulating a Rayleigh wave field with mHz frequencies numerically. We relate the corresponding gravitational force, and, thus, acceleration of such a field, to the time delay which we obtain in the particle tracking simulations. Finally, we use analytical considerations to estimate the noise power spectral density due to a Rayleigh wave field scaled with the Peterson low-noise displacement power.

We find that a particle is affected only by the average of the local low-frequency gravitational perturbations of the Rayleigh wave field, when it circulates in a storage ring. As is expected, the local acceleration of the particle at a point in the ring due to such a wave field is orders of magnitude larger than that due to the gravitational wave in the local frame of reference of the particle [18].

However, due to the conservative nature of the gravitational potential, the trajectory of the particle along a closed loop ends with the same velocity it started with. Therefore, the corresponding time delay measurement is caused only by the velocity change during one circulation of the particle and, consequently, a significant effect builds up only after many rounds. As a consequence the TGN is suppressed in comparison to the effect of a gravitational wave that applies work on the particle and, thus, changes its velocity after each round coherently. This reflects a fundamental difference between the pseudoforce caused by an inhomogeneous spacetime and a flat spacetime with a background force field.

In our quantitative estimate we have only considered the seismic TGN due to Rayleigh waves in a homogeneous half space far away from any of their sources, and we neglected seismic waves of frequencies above 0.1 Hz and below 2.5 mHz, scaled with the Peterson low-noise model. However, considering the expected source strain in this frequency range, based on our order-of-magnitude analysis, we are confident that even for higher seismic activity with larger displacement amplitudes and by considering seismic waves with higher or lower frequencies, the TGN from seismic activity should remain on low enough levels to facilitate GW detection. Of course, there is a large number of other sources of TGN, such as atmospheric and

infrasound disturbances, local density perturbations due to human activity and peculiarities due to topographic effects. All of these require a new analysis to give an order-of-magnitude estimate for the corresponding characteristic noise strain. For the TGN of seismic fields, however, which we consider in this work, we cover the frequency range which is most interesting to the storage ring-based GW detector [18]. We argue that the time of flight of the stored particle in the ring is extremely sensitive to the storage ring geometry, which is strongly affected by, e.g., shaking of the experimental apparatus, also caused by seismic activity. Hence, while we find a relatively low characteristic noise strain of TGN from seismic waves, the same perturbation could couple to the storage ring mechanically and lead to a larger seismic characteristic noise strain than that of the TGN as found in our analysis. Mechanical vibration, however, can be mitigated in principle by, e.g., suspension of the magnets or some form of active noise cancellation via additional magnetic fields, in contrast to TGN, which cannot be shielded.

As seismic noise can be measured by other means of detection, another approach could be to employ seismic vetoes to remove parts of the measurement data or model the impact of the seismic noise on the detector data streams.

The performance of methods generally suited for glitch removal have been optimized for LIGO-Virgo-KAGRA data [29] and applied to remove seismic noise in LIGO GW searches, specifically [30], and we expect it can likely be adapted to the application for mHz GW detection. We conclude that even with our very simplistic analysis of gravitational perturbations we find a strong indication that seismic TGN originating from Rayleigh waves causes a much smaller characteristic noise strain in a storage ring-based GW detector than the expected strain of GW sources with mHz frequencies, which makes the concept of using storage rings to measure mHz gravitational waves very promising.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: SOLUTION TO THE TIME DELAY INTEGRALS

The solution to the integrals is given as

$$I_1 = \int_0^\tau \frac{d\tau'}{\omega_{\text{Ring}}} \frac{r \sin \theta \sin(\tau' - \phi)}{\sqrt{R^2 + r^2 \sin^2 \theta - 2rR \cos(\phi - \tau') \sin \theta + (r \cos \theta - H)^2}} \\ = -\frac{1}{\omega_{\text{Ring}} R} \frac{1}{\sqrt{H^2 + r^2 + R^2 - 2r(H \cos \theta + R \cos(\phi - \tau) \sin \theta)}} + \frac{1}{\omega_{\text{Ring}} R} \frac{1}{\sqrt{H^2 + r^2 + R^2 - 2r(H \cos \theta + R \cos \phi \sin \theta)}}. \quad (\text{A1})$$

And then

$$I_2 = \int_0^{2\pi} \frac{d\tau}{\omega_{\text{Ring}}} I_1 = \frac{1}{R\omega_{\text{Ring}}} \frac{2\pi}{\sqrt{H^2 + r^2 + R^2 - 2Hr \cos \theta - 2rR \cos \phi \sin \theta}} \\ - \frac{1}{R\omega_{\text{Ring}}^2} \frac{2F\left(\frac{2\pi - \phi}{2}, -\frac{4Rr \sin \theta}{H^2 + r^2 + R^2 - 2r(H \cos \theta + R \sin \theta)}\right) + 2F\left(\frac{\phi}{2}, -\frac{4Rr \sin \theta}{H^2 + r^2 + R^2 - 2r(H \cos \theta + R \sin \theta)}\right)}{\sqrt{H^2 + r^2 + R^2 - 2rH \cos \theta - 2rR \sin \theta}}, \quad (\text{A2})$$

where $F(\phi, k^2) := \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}$ is the incomplete elliptic integral of the first kind. The difference of the elliptic integrals will not change [31], when a shift is introduced in the first coordinate of both functions, which allows us to replace

$$F\left(\frac{\phi}{2}, k^2\right) + F\left(\frac{2\pi - \phi}{2}, k^2\right) = 2K(k^2), \quad (\text{A3})$$

where $K(k^2)$ is the complete elliptic integral of the first kind, leading to the final result

$$\Delta T_{\text{round}} = -\frac{GM}{\beta c} \frac{1}{R\omega_{\text{Ring}}^2} \left[\frac{2\pi}{\sqrt{H^2 + r^2 + R^2 - 2Hr \cos \theta - 2rR \cos \phi \sin \theta}} - \frac{4K\left(-\frac{4Rr \sin \theta}{H^2 + r^2 + R^2 - 2r(H \cos \theta + R \sin \theta)}\right)}{\sqrt{H^2 + r^2 + R^2 - 2rH \cos \theta - 2rR \sin \theta}} \right]. \quad (\text{A4})$$

APPENDIX B: DERIVING 1ST ORDER EQUATIONS OF MOTION FOR PARTICLE IN A STORAGE RING SUBJECT TO A CONSTANT FORCE IN THREE DIMENSIONS

In the following the equations of motions are derived for a particle in a magnetic field under a three-dimensional external force. This derivation leads to a state vector, which is transformed by a transfer matrix. The state vector is given by the coordinates $\{x, x', y, y', l, \delta\}$, where the first two components are the radial position offset and radial velocity, the third and fourth component are the axial position offset and velocity, and the remaining two are the longitudinal offset and momentum deviation from a design particle with momentum p_0 . As in our previous work [18] the external force of the gravitational potential leads to radial and longitudinal position and momentum shifts, which will have to be applied after the transfer matrix.

1. Derivation of transversal equations of motion

With our derivation we follow the standard textbook procedure and especially [27] closely. The storage ring is assumed to be in the x, z -plane, where the two transversal directions are x and y . The transversal equations of motion for a particle in a magnetic field are given by [27]

$$\ddot{x} - \omega^2(\rho_0 + x) = -\frac{q}{p} v_z^2 B_y + a_r \quad (\text{B1})$$

$$\ddot{y} = +\frac{q}{p} v_z^2 B_x + a_{\text{ax}}. \quad (\text{B2})$$

The second term on the left of Eq. (B1) denotes the centripetal force, whereas the first term on the right-hand side on both Eq. (B1) and (B2) denote the Lorentz force. The terms a_r and a_{ax} denote the radial and axial acceleration, respectively. ρ_0 is the radius of the curved path, ω the circular frequency, x the radial displacement, v_z the

longitudinal velocity, p the corresponding momentum, q the charge of the particle and $B_{x/y}$ the components of the magnetic field, respectively.

It is customary to substitute the temporal coordinate t by the arc length s and the inclusion of the longitudinal force has been performed already in [18] resulting in the longitudinal velocity $v_z = (\rho_0 + x)\omega + A \cdot z$, where $A := \frac{\langle F \rangle}{m_0 \gamma^3 \beta c}$. Using $\frac{d}{ds} = \frac{1}{v}$ as the derivative for the path length instead of time, the equation of motion in linear order for x reads

$$x'' + \frac{1-n}{\rho_0^2} x = \frac{\delta}{\rho_0} - \frac{2Az}{\rho_0^2 \omega} + \frac{a_r}{\rho_0^2 \omega^2}. \quad (\text{B3})$$

Eq. (B3) is the result of our previous work for a longitudinal force, with an additional term on the right-hand side. This new term does also not depend on the momentum δ and it will be necessary to include it as an additional shift. For the y -component it is by analogous computation

$$y'' + \frac{ny}{\rho_0^2} = \frac{a_{ax}}{\rho_0^2 \omega^2}, \quad (\text{B4})$$

which contains an addition constant term due to the external axial force.

2. Solution of radial equation of motion for sector magnet

We start with the equation of motion for the radial direction. The homogeneous part (left-hand side) of Eq. (B3) remains unchanged from the textbook result and is solved via sine- and cosinelike functions $s_x(s) := \frac{\sin \sqrt{k}s}{\sqrt{k}}$ and $c_x(s) = \cos \sqrt{k}s$, where s denotes the path length and $k := \frac{1}{\rho_0^2}$ [27]. As the full solution to Hill's equation requires the particular solution $d_x(s)$ added to the homogeneous solution, the Green's function approach is employed to express the general solution as

$$x(s) = x_0 c_x(s) + x'_0 s_x(s) + \delta d_{x1}(s) + d_{x2}(s) \quad (\text{B5})$$

$$x'(s) = x_0 c'_x(s) + x'_0 s'_x(s) + \delta d'_{x1}(s) + d'_{x2}(s), \quad (\text{B6})$$

where the new terms are given as

$$d_{x2}(s) := \frac{a_r}{\omega^2} [1 - \cos \sqrt{k}s] - \frac{2A}{\rho_0^2 \omega} \left[\frac{s}{k} - \frac{\sin \sqrt{k}s}{k^2} \sqrt{k} \right] \quad (\text{B7})$$

and

$$d'_{x2}(s) := \frac{a_r}{\omega^2} \frac{1}{\rho_0} \sin \sqrt{k}s - \frac{2A}{\rho_0^2 \omega} \left[\frac{1}{k} - \frac{\cos \sqrt{k}s}{k} \right]. \quad (\text{B8})$$

In both cases the first term originates from the radial force and the second from the longitudinal force.

3. Solution of axial equation of motion for sector magnet

We repeat the above procedure for the other transversal equation of motion. We write analogously to the previous section

$$y(s) = y_0 c_y(s) + y'_0 s_y(s) + d_y(s) \quad (\text{B9})$$

$$y'(s) = y_0 c'_y(s) + y'_0 s'_y(s) + d'_y(s), \quad (\text{B10})$$

where

$$d_y(s) := \frac{a_{ax}}{\omega^2} [1 - \cos \sqrt{k}s] \quad (\text{B11})$$

$$d'_y(s) := \frac{a_{ax}}{\omega^2} \frac{1}{\rho_0} \sin \sqrt{k}s. \quad (\text{B12})$$

4. Derivation of the longitudinal and transversal equation of motion for drift space and quadrupole magnet

When an external force in longitudinal direction is present, both the longitudinal and momentum components for a particle in a drift space [27] are affected. For a free particle traveling in longitudinal direction, only the shift $\Delta l(s)$ has to be accounted for, which was already derived in our previous work [18].

In addition, the radial and axial force modify the equations of motion, which are

$$x(s) = x'_0 s + \frac{a_r}{2v^2} s^2 + x_0 \quad (\text{B13})$$

$$x'(s) = x'_0 + \frac{a_r}{v^2} s \quad (\text{B14})$$

and for the axial direction we have in analogy

$$y(s) = y'_0 s + \frac{a_{ax}}{v^2} s^2 + y_0 \quad (\text{B15})$$

$$y'(s) = y'_0 + \frac{a_{ax}}{2v^2} s^2 + y_0. \quad (\text{B16})$$

With this result, the influence of an external force is also modeled as a shift, which cannot be included in a transfer matrix. The transfer matrix for a drift space (and also for a quadrupole magnet) remains unaltered, because the force is of external origin and does not depend on any of the coordinates of the particle (but on its velocity v_0). For the quadrupole magnets this result is identical, because only the transversal equations of motion differ compared to a drift space. As a consequence the same argumentation is valid and the displacement and momentum shifts Δl and $\Delta \delta$

are formally the same, such that only the length of the magnet or drift space matters.

5. Shift vectors to include in the numerical simulation

The transformation is given by the regular transfer matrix $R^{\text{sec}}(l)$ [18,27], acting on the incoming vector and the shift. Combining all the results, together with the longitudinal case from the previous paper, the three-dimensional shift for the sector magnets is given by

$$x(l) = R^{\text{sec}}(l) \cdot x(0) + \Delta x^{\text{sec}}(l), \quad (\text{B17})$$

where l is the length of the sector magnet and the external force exerts a shift on the particle given by

$$\Delta x^{\text{sec}}(l) = \begin{pmatrix} \frac{1}{\beta^2 c^2 m_0 \gamma} \left[\frac{1}{k} - \frac{\cos \sqrt{k} l}{k} \right] - \frac{2}{\rho_0 \beta c m_0 \gamma^3 \beta c} \left[\frac{l}{k} - \frac{\sin \sqrt{k} l}{k^2} \sqrt{k} \right] \\ \frac{1}{\beta^2 c^2 m_0 \gamma} \frac{\sin \sqrt{k} l}{\sqrt{k}} - \frac{2}{\rho_0 \beta c m_0 \gamma^3 \beta c} \left[\frac{1}{k} - \frac{\cos \sqrt{k} l}{k} \right] \\ \frac{1}{\beta^2 c^2 m_0 \gamma} \left[\frac{1}{k} - \frac{\cos \sqrt{k} l}{k} \right] \\ \frac{1}{\beta^2 c^2 m_0 \gamma} \frac{\sin \sqrt{k} l}{\sqrt{k}} \\ \frac{\langle F_{\parallel} \rangle}{2 m_0 \gamma^3 \beta^2 c^2} l^2 + \frac{2}{\rho_0^2 c^2 \beta^2 m_0 \gamma^3} \left[\frac{l^2}{2k} + \left(\frac{\cos \sqrt{k} l}{k^2} - \frac{1}{k^2} \right) \right] \\ \frac{\langle F_{\parallel} \rangle}{\gamma^2 p_0 c \beta} l \end{pmatrix}, \quad (\text{B18})$$

where $F_{\parallel}$ is the longitudinal force and $F_{\perp}$ the radial and F_z the axial force, respectively. Due to the simpler differential geometry for the drift space, the correction to the transformation from the state vector $x(0)$ to the vector $x(s)$ is simpler and only the longitudinal shift and momentum shift are nonzero. As only the transversal equations of motion from a quadrupole magnet differ from the drift space, this result is valid in both cases. One obtains the transformation for the drift space as

$$x(l) = R^{\text{drift}}(l) \cdot x(0) + \Delta x^{\text{drift}}(l) \quad (\text{B19})$$

and for the quadrupole magnet as

$$x(l) = R^{\text{QD/QF}}(l) \cdot x(0) + \Delta x^{\text{drift}}(l) \quad (\text{B20})$$

and in both cases the displacement vector is given by

$$\Delta x^{\text{drift}}(l) = \begin{pmatrix} \frac{\langle F_{\perp} \rangle}{2 m_0 \gamma} \frac{l^2}{\beta^2 c^2} \\ \frac{\langle F_{\perp} \rangle}{m_0 \gamma} \frac{l}{(\beta c)^2} \\ \frac{\langle F_z \rangle}{2 m_0 \gamma} \frac{l^2}{\beta^2 c^2} \\ \frac{\langle F_z \rangle}{m_0 \gamma} \frac{l}{(\beta c)^2} \\ \frac{\langle F_{\parallel} \rangle}{2 m_0 \gamma^3 \beta^2 c^2} l^2 \\ \frac{\langle F_{\parallel} \rangle}{p_0 c \beta} l \end{pmatrix}. \quad (\text{B21})$$

APPENDIX C: SOLUTION TO THE GEODESIC EQUATIONS

Similar to the approach performed in [12] for computing the time delay due to the gravitational wave strain in the storage ring frame of reference, the Newtonian gravitational potential is included and the equations of motion solved. For a conservative potential $\phi(\alpha, t)$ and gravitational wave strain $h(\alpha, t)$ in cylindrical coordinates the metric takes the form [32]

$$ds^2 = - \left(1 + 2 \frac{\phi(\alpha, t)}{c^2} \right) c^2 dt^2 + \left(1 - 2 \frac{\phi(\alpha, t)}{c^2} + h(\alpha, t) \right) R^2 d\alpha^2, \quad (\text{C1})$$

where R is the storage ring radius and α the angular coordinate. We obtain the set of geodesic equations

$$\frac{d^2 t}{d\tau^2} + \frac{1}{1 + 2 \frac{\phi(\alpha, t)}{c^2}} \frac{d\phi(\alpha, t)}{dt} \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{2c^2} \frac{-2 \frac{\dot{\phi}(\alpha, t)}{c^2} + \dot{h}(\alpha, t)}{1 + 2 \frac{\phi(\alpha, t)}{c^2}} \left(\frac{dl}{d\tau} \right)^2 = 0 \quad (\text{C2})$$

$$\frac{d^2 l}{d\tau^2} + \frac{-2 \frac{\dot{\phi}(\alpha, t)}{c^2} + \dot{h}(\alpha, t)}{1 - 2 \frac{\phi(\alpha, t)}{c^2} + h(\alpha, t)} \frac{dt}{d\tau} \frac{dl}{d\tau} = 0 \quad (\text{C3})$$

$$- \left(1 + 2 \frac{\phi(\alpha, t)}{c^2} \right) c^2 \left(\frac{dt}{d\tau} \right)^2 + \left(1 - 2 \frac{\phi(\alpha, t)}{c^2} + h(\alpha, t) \right) \left(\frac{dl}{d\tau} \right)^2 = -c^2. \quad (\text{C4})$$

We replace the terms $\frac{dt}{d\tau} = \kappa$ and $\frac{dl}{d\tau} = \kappa v$. Then we insert Eq. (C4) into Eq. (C2) to eliminate the explicit v^2 term and to obtain a differential equation for κ only, which reads

$$\dot{\kappa} \kappa + \kappa^2 \left[\frac{\frac{\dot{\phi}(\alpha, t)}{c^2}}{1 + 2 \frac{\phi(\alpha, t)}{c^2}} + \frac{1}{2} \frac{-2 \frac{\dot{\phi}(\alpha, t)}{c^2} + \dot{h}(\alpha, t)}{1 - 2 \frac{\phi(\alpha, t)}{c^2} + h(\alpha, t)} \right] + \frac{1}{2} \frac{-2 \frac{\dot{\phi}(\alpha, t)}{c^2} + \dot{h}(\alpha, t)}{(1 + 2 \frac{\phi(\alpha, t)}{c^2})(1 - 2 \frac{\phi(\alpha, t)}{c^2} + h(\alpha, t))} = 0 \quad (\text{C5})$$

and which has the solution

$$\kappa(t) = \left(\sqrt{C + 2 \int_1^t \left(-e^{2 \int_1^{\bar{\tau}} A(\bar{\tau}) d\bar{\tau}} \right) B(\bar{\tau}) d\bar{\tau}} \right) \times e^{-\int_1^t A(\tau) d\tau}, \quad (\text{C6})$$

where $A(t) = \frac{\dot{\phi}(t)}{1+2\frac{\phi(t)}{c^2}} + \frac{1}{2} \frac{-2\frac{\dot{\phi}(t)}{c^2} + h(t)}{1-2\frac{\phi(t)}{c^2} + h(t)}$ and $B(t) = \frac{1}{2} \frac{-2\frac{\dot{\phi}(t)}{c^2} + h(t)}{(1+2\frac{\phi(t)}{c^2})(1-2\frac{\phi(t)}{c^2} + h(t))}$ and C is a constant. Here and in the following we absorb the α -dependence and denote $\phi_0 = \phi(0)$ and $h_0 = h(0)$, etc.

We use the result under the condition $\kappa^2(t=0) = \kappa_0^2$ to fix the constant C and obtain our solution

$$\kappa^2(t) = \frac{\kappa_0^2(1 + 2\frac{\phi_0}{c^2})(1 - 2\frac{\phi_0}{c^2} + h_0) - (h(t) - h_0) + \frac{2\phi(t)}{c^2} - \frac{2\phi_0}{c^2}}{(1 + 2\frac{\phi(t)}{c^2})(1 - 2\frac{\phi(t)}{c^2} + h(t))}. \quad (\text{C7})$$

We replace $\kappa := \frac{1}{\sqrt{1 - \beta^2 + 2\frac{\phi}{c^2}(1 + \beta^2) - \beta^2 h}}$ (and κ_0 analogously), which follows from our metric. With this we rewrite Eq. (C7) in terms of β and insert this into the third geodesic equation written as

$$dl^2 = \frac{c^2\beta^2(t) + 2\frac{\phi(t)}{c^2}}{1 - 2\frac{\phi(t)}{c^2} + h(t)} dt^2. \quad (\text{C8})$$

We use this equation again to relate the velocity $\beta_0 c$ in an unperturbed storage ring with the velocity at the beginning of the measurement with background potential ϕ_0 and gravitational wave strain h_0 , given as $c^2\beta^2(t=0) = c^2\beta_0^2(1 - 2\frac{\phi_0}{c^2} + h_0) - 2\phi_0$.

With this, we expand the fraction in Eq. (C8) in a Taylor series, and then take the square root using another Taylor series, while keeping only first order terms in h and ϕ . We find

$$dl = \beta_0 c \left[1 - \left(\frac{3}{2} - \frac{\beta_0^2}{2} \right) (h(t) - h_0) - \frac{-1 - 4\beta_0^2 + \beta_0^4}{\beta_0^2} \left(\frac{\phi(t)}{c^2} - \frac{\phi_0}{c^2} \right) \right] dt. \quad (\text{C9})$$

Here, the first two terms lead to the nominal circulation time $T_0 = \frac{L}{c\beta_0}$ and the time delay due to the gravitational wave ΔT_{GW} . Hence, we note that for $\phi(t) = 0$, Eq. (C9) is almost consistent with the solution obtained in [12] and the difference originates from our treatment of the full expression for $\frac{dl}{d\tau} = \kappa$, whereas the literature result was obtained by using the approximation of special relativity $\frac{dl}{d\tau} = \gamma$. The last and new term in Eq. (C9) leads to the time delay due to the Newtonian gravitational potential, which reads

$$\Delta T = - \frac{-1 - 4\beta_0^2 + \beta_0^4}{\beta_0^2} \int_0^t \left(\frac{\phi(t')}{c^2} - \frac{\phi_0}{c^2} \right) dt'. \quad (\text{C10})$$

If we keep the initial velocity $c\beta(0)$ and do not replace it with the velocity of the unperturbed storage ring $c\beta_0$ using the third geodesic equation, we find

$$dl = \beta(0)c \left[1 - \left[\left(\frac{3}{2} - \frac{\beta(0)^2}{2} \right) h(t) - \frac{1}{2} (2 - \beta(0)^2) h_0 \right] - \left[\frac{-1 - 4\beta(0)^2 + \beta(0)^4}{\beta(0)^2} \frac{\phi(t)}{c^2} - \frac{-3\beta(0)^2 + \beta(0)^4}{\beta(0)^2} \frac{\phi_0}{c^2} \right] \right] dt. \quad (\text{C11})$$

Here, with the first two terms we again almost reproduce the solution for the time delay due to the gravitational wave obtained in [13] and the difference amounts again to our choice of $\frac{dl}{d\tau} = \kappa$ instead of $\frac{dl}{d\tau} = \gamma$. For the new term, we see that the effect of $\phi(t)$ on the time evolution is the same as before, but the impact of the initial potential ϕ_0 is slightly different.

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