

Conformal dispersion relation for mixed correlators

Dean Carmi^{1,*}, Javier Moreno^{2,3,†} and Shimon Sukholuski^{1,‡}

¹*Department of Physics and Haifa Research Center for Theoretical Physics and Astrophysics,
University of Haifa, Haifa 31905, Israel*

²*Departamento de Física, Universidad de Concepción, Casilla, 160-C, Concepción, Chile*

³*Departament de Física Quàntica i Astrofísica, Institut de Ciències del Cosmos,
Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spain*



(Received 19 March 2025; accepted 12 September 2025; published 7 November 2025)

Dispersion relations are nonperturbative formulas that relate the ultraviolet and infrared behavior of an observable with wide-ranging applications in linear response theory, quantum field theory scattering amplitudes, and conformal correlators. We derive a position-space dispersion relation for scalar four-point mixed correlation functions in an arbitrary conformal field theory. This formula expresses the correlator in terms of its integrated double-discontinuity times a kinematic kernel. The kernel is analytically computed, and expressed in a remarkably simple form as a two-variable Appell function. The dispersion kernel is found by solving a coupled partial differential equation that the kernel obeys. Numerical checks of the dispersion relation are successfully performed for generalized free field correlators. Finally, we show that our position-space dispersion relation is equivalent to a Cauchy-type dispersion relation of the Mellin amplitude of the correlator.

DOI: [10.1103/s5sx-9ygz](https://doi.org/10.1103/s5sx-9ygz)

Introduction. Dispersion relations (DRs) are mathematical tools that reconstruct complex functions from their discontinuities. Their first major application in physics was by Kramers and Kronig, who related the real and imaginary parts of the refractive index as a function of frequency [1,2]. There, the imaginary part—which reflects how the material absorbs energy—is enough to determine the full response. This reconstruction relies on causality, which ensures analyticity in the upper half of the complex frequency plane—a key requirement for the validity of DRs. In particular, a Cauchy-type DR expresses the function $\mathcal{M}(s)$ in terms of its discontinuity across the branch cut.

$$\mathcal{M}(s) = \frac{1}{2\pi i} \int_{\text{cut}} ds' \frac{\text{Disc}[\mathcal{M}(s')]}{s' - s}. \quad (1)$$

An important application of DRs arose in the 1950s with the S-matrix bootstrap program, which aimed to determine the spectrum and interactions of strongly coupled hadrons from first principles like analyticity and high-energy

behavior—see, e.g., [3–7]. Recently, the S-matrix bootstrap has been revived through modern analytic and numerical methods [8–10], as well as insights from the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [11–13].

In the context of CFT—which describes physical systems at criticality and finds applications ranging from condensed matter physics to string theory—the bootstrap approach has achieved remarkable success [14–18]. The primary observables in a CFT are correlation functions of local operators, $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$, where each operator $\mathcal{O}_i(x_i)$ carries a scaling dimension Δ_i . A central goal of the conformal bootstrap is to determine these correlators—or, equivalently—to extract the spectrum of anomalous dimensions (critical exponents) and operator product expansion (OPE) coefficients. A key advantage of this approach is its nonperturbative nature, which allows access to precise results even at strong coupling.

In the numerical conformal bootstrap, bounds on conformal data from four-point functions of identical operators become much tighter when including mixed correlators of nonidentical operators. This often produces small allowed “islands” [19–21], enabling precise determination of critical exponents, as in the 3D Ising model.

Dispersive techniques entered the conformal bootstrap in [22], which introduced the double-discontinuity and the Lorentzian inversion formula (LIF)—see also [23,24]. A position-space DR for four-point functions with equal dimensions was later derived in [25], while Mellin-space

*Contact author: deancarmi1@gmail.com

†Contact author: fjaviermoreno@udec.cl

‡Contact author: ssm@technion.ac.il

versions appeared in [26]. Their equivalence was established in [27]. These developments highlight the power of DRs in extracting nonperturbative conformal data from analyticity and symmetry alone, making them a compelling and elegant tool in the ongoing effort to solve CFTs with a nonperturbative approach.

In this work, we derive a position-space conformal DR for scalar four-point mixed correlators, generalizing [25] to unequal scaling dimensions. We also show that our formula is equivalent to a Cauchy-type DR for conformal correlators in Mellin space. Finally, we perform successful checks of the formula, and discuss potential applications of the conformal DR for bounding conformal data via the mixed-correlator dispersive conformal bootstrap.

Lorentz inversion formula. Consider a CFT in $d \geq 2$ space time dimensions, and the four-point correlation functions of operators $\mathcal{O}_i$. The latter can be written in terms of two cross ratios $z, \bar{z}$ defined through the relations $U = z\bar{z} = x_{12}^2 x_{34}^2 / (x_{13}^2 x_{24}^2)$, $V = (1-z)(1-\bar{z}) = x_{23}^2 x_{14}^2 / (x_{13}^2 x_{24}^2)$, with $x_{ij} = x_i - x_j$. Then the four-point function is

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \frac{(x_{14}^2/x_{24}^2)^a (x_{14}^2/x_{13}^2)^b}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \mathcal{G}(z, \bar{z}), \quad (2)$$

where $a \equiv (\Delta_2 - \Delta_1)/2$ and $b \equiv (\Delta_3 - \Delta_4)/2$, and $\mathcal{G}(z, \bar{z})$ is a function that depends on the particular dynamics of the CFT. We will frequently use radial coordinates [28], defined as $\rho_z \equiv (1 - \sqrt{1-z})/(1 + \sqrt{1-z})$, $\bar{\rho}_z \equiv (1 - \sqrt{1-\bar{z}})/(1 + \sqrt{1-\bar{z}})$.

The OPE implies a principle series expansion of $\mathcal{G}(z, \bar{z})$ in terms of conformal partial waves $F_{J,\Delta}$,

$$\mathcal{G}(z, \bar{z}) = 1_{12} 1_{34} + \sum_{J=0}^{\infty} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} c_{J,\Delta} F_{J,\Delta}(z, \bar{z}), \quad (3)$$

where $F_{J,\Delta}$ are a shadow symmetric combination of the conformal blocks $G_{J,\Delta}$. The coefficients of the expansion, $c_{J,\Delta}$, are called OPE functions. The location of poles of $c_{J,\Delta}$ in the complex plane gives the spectrum of scaling dimensions of the theory.

The LIF introduced in [22] inverts the conformal block expansion (3), by extracting the OPE function $c_{J,\Delta}$ from the double discontinuity of $\mathcal{G}(z, \bar{z})$

$$c'_{J,\Delta} = \frac{\kappa}{4} \int_0^1 dw d\bar{w} \mu G_{J,\Delta}^{(\text{inv})}(w, \bar{w}) d\text{Disc}_t[\mathcal{G}(w, \bar{w})], \quad (4)$$

where $c_{J,\Delta} = c'_{J,\Delta} + (-1)^J c''_{J,\Delta}$. The “inverted block” is defined from the conformal block as $G_{J,\Delta}^{(\text{inv})}(w, \bar{w}) \equiv G_{\Delta+1-d, J+d-1}(w, \bar{w})$, while $\mu \equiv \mu_{(d)}^{(a,b)}(w, \bar{w}) = |(w - \bar{w})/(w\bar{w})|^{d-2} [(1-w)(1-\bar{w})]^{a+b} / (w\bar{w})^2$ and $\kappa \equiv \kappa_{J+\Delta} =$

$\Gamma_{\beta/2-a} \Gamma_{\beta/2+a} \Gamma_{\beta/2-b} \Gamma_{\beta/2+b} / (2\pi^2 \Gamma_{\beta-1} \Gamma_{\beta})$ [29]. The double discontinuity around $\bar{z} = 1$ is,

$$\begin{aligned} d\text{Disc}_t[\mathcal{G}(z, \bar{z})] &= \cos[\pi(a+b)] \mathcal{G}(z, \bar{z}) \\ &\quad - \frac{1}{2} e^{i\pi(a+b)} \mathcal{G}(z, \bar{z}^\cup) - \frac{1}{2} e^{-i\pi(a+b)} \mathcal{G}(z, \bar{z}^\cup). \end{aligned} \quad (5)$$

Conformal dispersion relation. We aim to derive a *conformal DR* that expresses the four-point correlator $\mathcal{G}(z, \bar{z})$ as an integral over its (double) discontinuities, in the spirit of (1). A method to derive such a DR was devised in [25]. It consists of combining the LIF (4), inside the OPE expansion (3). Replacing the order of integrations gives the conformal DR,

$$\mathcal{G}'(z, \bar{z}) = \int dw d\bar{w} K(z, \bar{z}, w, \bar{w}) d\text{Disc}[\mathcal{G}(w, \bar{w})] \quad (6)$$

and $G(z, \bar{z}) = \mathcal{G}'(z, \bar{z}) + \mathcal{G}''(z, \bar{z})$. The kernel $K \equiv K^{(a,b)}(z, \bar{z}, w, \bar{w})$ reads

$$K = \frac{\mu}{8\pi i} \sum_{J=0}^{\infty} \int d\Delta \kappa_{J+\Delta} F_{J,\Delta}(z, \bar{z}) G_{J,\Delta}^{(\text{inv})}(w, \bar{w}). \quad (7)$$

The conformal DR (6) is a remarkable nonperturbative formula that bootstraps the correlator $\mathcal{G}(z, \bar{z})$. However in order to apply it efficiently, the kernel K in (7) needs to be computed analytically. We can show numerically that K is independent of the space-time dimension d , just as the Cauchy-type DR (1) is. This intriguing fact arises because the DR should hold for any function $\mathcal{G}(z, \bar{z})$ with analytic properties as the four-point correlator, and these analytic properties do not depend on d . Furthermore, the kernel can be written as a sum of two terms, i.e.,

$$\begin{aligned} K(z, \bar{z}, w, \bar{w}) &= K_B(z, \bar{z}, w, \bar{w}) \theta(\rho_z \bar{\rho}_z \bar{\rho}_w - \rho_w) \\ &\quad + K_C(z, \bar{z}, w, \bar{w}) \frac{d\rho_w}{dw} \delta(\rho_w - \rho_z \bar{\rho}_z \bar{\rho}_w), \end{aligned} \quad (8)$$

where δ is the Dirac delta function, and the unit step function θ implies that the integration region in the DR (6) has support only for $\rho_w < \rho_z \bar{\rho}_z \bar{\rho}_w$.

The contact term K_C was computed [25] for general external scaling dimensions Δ_i , i.e., for any a and b while the bulk term K_B was computed explicitly only for $a = b = 0$, see [30]. In the present work we close this gap, and compute K_B for any a and b , i.e., for general mixed correlation functions.

Computing the kernel. We present two complementary methods to compute the bulk kernel K_B in (8).

- (I) Series representation [25]: Since the kernel is independent of the dimension, we set $d = 2$ in (7), in which we have explicit expressions for the conformal blocks: $G_{J,\Delta}(z, \bar{z}) = [k_{\Delta-J}(z)k_{\Delta+J}(\bar{z}) + (z \leftrightarrow \bar{z})]/(1 + \delta_{J,0})$ where $k_\beta(z) \equiv z^{\beta/2} {}_2F_1(\beta/2 + a, \beta/2 + b, \beta, z)$. Now we perform the Δ integral in (7) using the residue theorem. There are four towers of poles coming from the κ factor, and the J sum can be performed. The contribution of the four towers of poles (see [25]) is,

$$\begin{aligned} K_B(z, \bar{z}, w, \bar{w}) &= K_a^{(a,b)} + K_{-a}^{(a,b)} + K_b^{(b,a)} + K_{-b}^{(b,a)} \\ &= \mu \mathcal{D}_2 (S_a^{(a,b)} + S_{-a}^{(a,b)} + S_b^{(b,a)} + S_{-b}^{(b,a)}) \\ &= \mu \mathcal{D}_2 S, \end{aligned} \quad (9)$$

where the differential operator $\mathcal{D}_2$ is [31], and we have,

$$S_{a'}^{(a,b)} = \sum_{m=0}^{\infty} \frac{\sin(2\pi a')}{2\pi m!} \frac{\Gamma_{1+2a'+2m}^2 k_{-2m-2a'}(z) k_{-2m-2a'}(\bar{z}) k_{2m+2a'+2}(w) k_{-2m-2a'}(\bar{w})}{\Gamma_{1+2a'+m} \Gamma_{1+a'-b+m} \Gamma_{1+a'+b+m} \sin[\pi(a-b)] \sin[\pi(a+b)]}. \quad (10)$$

The sum over m in (10) arises from the sum over the residues of poles in the right-hand side of (7). Expression (10) gives a representation of the bulk kernel in terms of an infinite sum of a product of four hypergeometric functions. Our goal is to obtain an analytic expression for this sum. In [25], the case $a = b = 0$ was analyzed, where each of the hypergeometric functions k_β reduces to a Legendre function. This simplification allowed for the explicit evaluation of the sum over the double poles in (7). The bulk kernel K_B is subsequently obtained by applying the differential operator given in (9). The resulting expression for the bulk kernel in the case $a = b = 0$ is [30].

For general mixed correlators with $a, b \neq 0$, an analytic evaluation of the sum in (10) is unknown. We therefore turn to an alternative method for computing the kernel.

- (II) Differential equations: Repeating the computation for $d = 4$ yields a result analogous to Eqs. (9) and (10). Imposing that the kernel in $d = 4$ matches its counterpart in $d = 2$ then leads to the following two constraints:

$$\mathcal{D}_{(z, \bar{z}, w, \bar{w})}^{(24)} S_{a'}^{(a,b)} = 0, \quad \mathcal{D}_{(\bar{w}, \bar{z}, w, z)}^{(24)} S_{a'}^{(a,b)} = 0, \quad (11)$$

with $\mathcal{D}_{(z, \bar{z}, w, \bar{w})}^{(24)} \equiv \mathcal{D}_2 - \mathcal{D}_4$ and $\mathcal{D}_4$ defined in [31].

These constraints arise because S and $\mathcal{D}_{(z, \bar{z}, w, \bar{w})}^{(24)}$ have permutation symmetry in the variables $(z, \bar{z}, \bar{w})$ and $(w, z, \bar{z})$, respectively. Applying (11) to the four-variable function $S(z, \bar{z}, w, \bar{w})$, reduces its dependence two variables, namely [32]

$$S_{a'}^{(a,b)} = \frac{\sqrt{z\bar{z}w\bar{w}}}{y^{\frac{1}{2}+a+b}} \tilde{S}_{a'}^{(a,b)}(x, y), \quad (12)$$

where the relation between the two variables, x and y and the radial coordinates is

$$x \equiv \frac{\frac{\rho_z \bar{\rho}_z \rho_w \bar{\rho}_w}{(1 - \rho_z \bar{\rho}_z \rho_w \bar{\rho}_w)} (1 - \rho_z^2)(1 - \bar{\rho}_z^2)(1 - \rho_w^2)(1 - \bar{\rho}_w^2)}{(\bar{\rho}_z \bar{\rho}_w - \rho_z \rho_w)(\rho_z \bar{\rho}_w - \bar{\rho}_z \rho_w)(\rho_z \bar{\rho}_z - \rho_w \bar{\rho}_w)}, \quad (13)$$

$$y \equiv \frac{(1 - \rho_z)(1 - \bar{\rho}_z)(1 - \rho_w)(1 - \bar{\rho}_w)}{(1 + \rho_z)(1 + \bar{\rho}_z)(1 + \rho_w)(1 + \bar{\rho}_w)}. \quad (14)$$

Note that the dispersion kernel $K(z, \bar{z}, w, \bar{w})$ contains four variables. The problem of obtaining K is thus reduced to that of obtaining the two-variable function $\tilde{S}_{a'}^{(a,b)}(x, y)$.

We now show that $\tilde{S}_{a'}^{(a,b)}(x, y)$ satisfies a system of coupled second-order partial differential equations (PDEs). The k_β in (10) obeys $\mathcal{D}_{\rho_z} k_{-2m-2a'}(z) = (m + a')(m + a' + 1) k_{-2m-2a'}(z)$, with similar equations for the coordinates $\bar{z}$ and $\bar{w}$, [31]. By taking the differences, we obtain PDEs for $S_{a'}^{(a,b)}$, namely $(\mathcal{D}_{\rho_z} - \mathcal{D}_{\bar{\rho}_z}) S_{a'}^{(a,b)} = 0$ and $(\mathcal{D}_{\rho_z} - \mathcal{D}_{\bar{\rho}_w}) S_{a'}^{(a,b)} = 0$. After inserting (12) in these two PDEs, the system reduces to

$$\begin{aligned} [x^2(1-x)\partial_x^2 - x^2\partial_x + y^2\partial_y^2 + y\partial_y - x^2(1-y^2)\partial_x\partial_y/2 \\ + (1/4 - a^2 - b^2)] \tilde{S}_{a'}^{(a,b)} = 0, \end{aligned} \quad (15a)$$

$$[(x^2(1-y)^2 + 4xy)\partial_x\partial_y - 2y\partial_y - 4ab] \tilde{S}_{a'}^{(a,b)} = 0. \quad (15b)$$

To obtain an explicit expression for the bulk kernel, one must solve the PDE system. In [25], solutions were obtained only in specific limits—namely, as $x \rightarrow 1$ and $y \rightarrow 0$ or $y \rightarrow \infty$ —and not for general values of a, b , and (x, y) . In this work, we overcome this limitation and provide a solution valid in the full kinematic range.

Solving the PDE system. Now, we derive a novel expression for the kernel with unequal scaling dimensions by solving (15), which, as we will see, can be written in terms of the Appell functions—see Supplemental Material [33]. Our strategy begins with solving the equation in specific, simple cases, and then inferring the general solution based on the insights gained from these special instances—a detailed step-by-step derivation is provided in Supplemental Material [33]. We start by adding and subtracting Eqs. (15a) and (15b). Afterward, we can put one of the two PDEs in its canonical form. Then one notices that there are two special cases, $b = 1/2 \mp a$, for which the PDEs are explicitly solvable. The solutions in these two cases are given by [34],

$$\begin{aligned}\tilde{S}^{(a,b=\frac{1}{2}-a)} &= -\frac{1}{2\pi} {}_2F_1(2a, 1-2a, 1; \xi), \\ \tilde{S}^{(a,b=\frac{1}{2}+a)} &= -\frac{1}{2\pi} {}_2F_1(-2a, 1+2a, 1; \eta),\end{aligned}\quad (16)$$

where the (ξ, η) coordinates are defined as $\xi \equiv (1 + y - \sqrt{4y/x + (1-y)^2})/2$ and $\eta \equiv (1 + y - \sqrt{4y/x + (1-y)^2})/(2y)$ [35].

The special solutions (16) give us a hint that (ξ, η) might be a natural set of coordinates. We thus transform the PDEs in (15) to the new variables (ξ, η) , and get

$$\mathcal{D}_{(\eta,\xi)}^{(a,b)} \tilde{S}_a(\xi, \eta) = 0, \quad \mathcal{D}_{(\xi,\eta)}^{(a,-b)} \tilde{S}_a(\xi, \eta) = 0, \quad (17)$$

where the introduced differential operator reads $\mathcal{D}_{(\eta,\xi)}^{(a,b)} \equiv \eta(1-\eta)\partial_\eta^2 + \xi\partial_\xi\partial_\eta + (1-2\eta)\partial_\eta + ((a+b)^2 - 1/4)$. Remarkably, Eq. (17) are now explicitly in the form of the Appell function PDEs—see Supplemental Material [33]. The solution to the PDEs is given in (18), in terms of the Appel F_3 function, this is

$$\tilde{S}^{(a,b)} = -\frac{1}{2\pi} F_3(\xi, \eta) \equiv -\frac{1}{2\pi} F_3\left(\frac{1}{2} + a - b, \frac{1}{2} - a - b, \frac{1}{2} - a + b, \frac{1}{2} + a + b, 1; \xi, \eta\right). \quad (18)$$

Using (9), the bulk kernel is obtained by action of the differential operator $\mathcal{D}_2$ (written in x variables),

$$K_B(z, \bar{z}, w, \bar{w}) = \frac{\mu(2)}{2\pi} \frac{z\bar{z}(w-\bar{w})sx^2}{8y^{\frac{1}{2}+a+b}} \partial_x F_3(\xi, \eta), \quad (19)$$

where $s \equiv \sqrt{w\bar{w}z\bar{z}}(w^{-1} + \bar{w}^{-1} + z^{-1} + \bar{z}^{-1} - 2)$. Taking the x derivative, the final expression of K_B is a sum of two F_3 functions with shifted parameters.

Expression (18)—and, by extension (19)—is our main result. The function $\tilde{S}$ and the bulk kernel K_B of the DR, turn out to be an Appell F_3 function. The Appell functions are the simplest two-variable generalizations of the ${}_2F_1$ hypergeometric function. The remarkable simplicity of this result is notable. This result holds for and $d \geq 2$. It would be interesting to derive the $d = 1$ dispersion kernel for mixed correlators [36–38].

Alternative expression for the kernel. For practical numerical applications one would like to evaluate the $F_3(\xi, \eta)$ via its Taylor expansion around $\xi = \eta = 0$. This series converges in the range $|\xi|, |\eta| < 1$. However, in the DR (6) the variables run in the range $-\infty < \xi, \eta < 0$, outside of the series convergence range. In order to overcome this, we will now obtain an alternative expression for the bulk kernel, which has a convergent series representation in the full range of the DR. The key is an identity that relates the Appell $F_3(\xi, \eta)$ function with a linear combination of Appell $F_2(\eta/(\eta + \xi - \xi\eta), \xi/(\eta + \xi - \xi\eta))$ functions—see Supplemental Material [33] for more details. Thus, instead of (18), it will be useful for practical applications to use the following result for $\tilde{S}_a^{(a,b)}$ [39]:

$$\begin{aligned}\tilde{S}_a^{(a,b)} &= \frac{(\cos(2\pi a) + \cos(2\pi b)) \sin(2\pi a) \Gamma_{2a+1} \Gamma_{2b-2a} \Gamma_{-2b-2a} \Gamma_{\frac{1}{2}+a-b} \Gamma_{\frac{1}{2}+a+b}}{4\pi^4 \left(\frac{\xi+\eta-\xi\eta}{\eta}\right)^{1+2a} \left(\frac{\eta}{\xi}\right)^{\frac{1}{2}+a+b}} \\ &\times F_2\left(1+2a; \frac{1}{2}+a-b, \frac{1}{2}+a+b; 1+2a-2b, 1+2a+2b; \frac{\eta}{\xi+\eta-\xi\eta}, \frac{\xi}{\xi+\eta-\xi\eta}\right).\end{aligned}\quad (20)$$

The F_2 function above has a Taylor series expansion—see Supplemental Material [33], that converges in the range $0 < |\eta/(\eta + \xi - \xi\eta)| + |\xi/(\eta + \xi - \xi\eta)| < 1$. The DR has support in the range $-\infty < \xi, \eta < 0$, and the Taylor series of the F_2 converges in this range. Thus, for numerical application of the DR, one can use the expression (20).

Special cases and numerical checks. Having obtained the full kernel of the conformal DR, we would now like to perform some checks. In the case $b = 0$ and arbitrary a , the Appell function in (18) reduces to a hypergeometric function,

$$\tilde{S}^{(a,0)} = -\frac{x^{a+\frac{1}{2}}}{2\pi} {}_2F_1\left(a + \frac{1}{2}, a + \frac{1}{2}; 1; 1 - x\right), \quad (21)$$

where we used identities displayed in Supplemental Material [33]. In the case $a = b = 0$, we get $\tilde{S}^{(0,0)} = -x^{1/2}/(2\pi) {}_2F_1(1/2, 1/2; 1; 1 - x)$ reproducing the bulk kernel obtained in 3.23 of [25]. Furthermore, for the family of cases with $b = 1/2 - a + n$ with n being a non-negative integer, the result is given in terms of associated Legendre functions.

Numerical checks: Recalling the conformal DR (6), we test our expression for the bulk kernel numerically by applying it to test correlation functions with the structure of generalized free-field correlators, i.e.,

$$\mathcal{G}_{\text{GFF}}(z, \bar{z}) = (z\bar{z})^{r_1} [(1-z)(1-\bar{z})]^{r_2} = U^{r_1} V^{r_2}, \quad (22)$$

where $r_{1,2}$ are parameters that we are free to choose. Taking the t -channel double discontinuity, we obtain $\text{dDisc}_t[\mathcal{G}_{\text{GFF}}] = 2U^{r_1} V^{r_2} \sin(\pi r_2) \sin[\pi(a + b + r_2)]$, which we plug into (6)—see Supplemental Material [33]. Using the bulk kernel arising from the Appell F_2 function of (20), we numerically perform the w and $\bar{w}$ integrals. The integrals converge provided that the values of r_1 and r_2 are inside a specific finite range. For many values of the parameters (a, b, r_1, r_2) , we compare the right-hand side and left-hand side of (6), and obtain perfect matching in each case. These are highly nontrivial checks, implying the correctness of the DR.

Mellin-space dispersion relation. In this section we will establish that our position-space DR (6) is equivalent to a Cauchy-type DR, (1), in Mellin-space. In other words, the two DRs are related by a (double) Mellin transform. A similar equivalency was shown in [27] for the case of equal scaling dimensions $a = b = 0$, and we extend this to the mixed-correlator case.

Consider the four-point correlator in position space $\mathcal{G}(U, V) = \mathcal{G}'(U, V) + \mathcal{G}''(U, V)$. Conformal correlators

have a useful representation in Mellin-space [40–42], in terms of the Mellin amplitude $M(t, u) = M^t(t, u) + M^u(t, u)$,

$$\mathcal{G}'(U, V) = \int \frac{dt du}{(4\pi i)^2} \Gamma_{\Delta_i}^{t,u} U^{\frac{\Delta_1-u}{2}} V^{\frac{t-\Delta_2-\Delta_3}{2}} M^t(t, u), \quad (23)$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$, the product of gamma functions $\Gamma_{\Delta_i}^{t,u}$ is defined in [43], and t and u are analogs of the Mandelstam variables. The Mellin amplitude shares many properties with flat-space scattering amplitudes, in particular it obeys a single-variable Cauchy-type DR [26,27,44,45], similar to (1):

$$M^t(t, u) = \int_{t+\epsilon-i\infty}^{t+\epsilon+i\infty} \frac{dt'}{2\pi i} \frac{M^t(t', u')}{t - t'}. \quad (24)$$

A similar relation holds for $M^u(t, u)$, and we have that $t + u = t' + u'$. Combining expressions (23) and (24) gives,

$$\mathcal{G}' = \int \frac{dt du dt'}{4(2\pi i)^3} \Gamma_{\Delta_i}^{t,u} U^{\frac{\Delta_1-u}{2}} V^{\frac{t-\Delta_2-\Delta_3}{2}} \frac{M^t(t', u')}{t - t'}. \quad (25)$$

Following a procedure similar to that done in [27], we write the t -channel Mellin amplitude as an integral over the double-discontinuity $\text{dDisc}_t[\mathcal{G}(U, V)]$,

$$M^t(t', u') = \int \frac{dU' dV' U'^{\frac{t'+u'-\Delta}{2}} V'^{\frac{\Delta_2+\Delta_3-t'}{2}}}{U' V' \Gamma_{\Delta_i}^{t',u'}} \times \frac{\text{dDisc}_t[\mathcal{G}(U', V')]}{2 \sin\left(\frac{\pi(t'-\Delta_1-\Delta_3)}{2}\right) \sin\left(\frac{\pi(t'-\Delta_2-\Delta_4)}{2}\right)}. \quad (26)$$

Now we insert (26) into (25), and swap the order of the position and Mellin-space integrations. This now has the form of a position-space DR

$$\mathcal{G}'(U, V) = \int dU' dV' K_{\text{Mellin}} \text{dDisc}_t[\mathcal{G}(U', V')], \quad (27)$$

with a kernel $K_{\text{Mellin}} \equiv K_{\text{Mellin}}(U, V, U', V')$ given by,

$$K_{\text{Mellin}}(U, V, U', V') = \int \frac{dt du dt'}{U' V' (2\pi i)^3} \frac{U^{\frac{\Delta_1-u}{2}} V^{\frac{t-\Delta_2-\Delta_3}{2}}}{U'^{\frac{\Delta_1-u'}{2}} V'^{\frac{t'-\Delta_2-\Delta_3}{2}}} \frac{1}{t - t'} \frac{\Gamma_{\Delta_i}^{t,u} / \Gamma_{\Delta_i}^{t',u'}}{8 \sin\left(\frac{\pi(t'-\Delta_1-\Delta_3)}{2}\right) \sin\left(\frac{\pi(t'-\Delta_2-\Delta_4)}{2}\right)}. \quad (28)$$

Now, we compute (28) using the residue theorem. First we close the integration contour over t' to the left and then we close the integration contour of t and u to the right. This leaves us with four towers of poles,

$$K_{\text{Mellin}} = H_a^{(a,b)} + H_{-a}^{(a,b)} + H_b^{(b,a)} + H_{-b}^{(b,a)}, \quad (29)$$

where each tower of poles is given by [46]. We numerically compute the right-hand side of (29), and find perfect

matching with our position-space bulk kernel given in (19). In Supplemental Material [33] we contextualize this result with the one derived in the collinear limit in [45].

Discussion. In this work, we derived a conformal DR for mixed correlators. This new entry in the successful conformal bootstrap program enables the application of dispersive techniques to a broader class of CFT observables. It thus offers a flexible analytic tool with potential impact across CFT and AdS/CFT contexts.

Among the possible immediate applications, one could construct dispersive sum rules using either the position-space or Mellin-space representation of the conformal DR, and apply them to bootstrap the conformal data of mixed correlators in specific theories, such as $\mathcal{N} = 4$ super Yang–Mills or the 3d Ising model. In the $\mathcal{N} = 4$ case, a numerical analysis using dispersive functionals was done in [47,48], obtaining two-sided bounds on the OPE coefficients of the Konishi operator as well as on the correlator itself. It would be intriguing to push this program forward and study bounds on other observables and models.

It should also be possible to derive the mixed-correlator conformal DR via a contour deformation argument, and to formulate DRs with subtractions [49]. Such a contour deformation proof could facilitate to generalize the DR

to higher-point correlation functions. The four-point (and putative higher-point) conformal DRs can be viewed as novel mathematical theorems in complex analysis of multi-variable functions. Taking a flat-space limit of the correlator DRs could result in new derivations of S-matrix DRs, which themselves play a significant role in the S-matrix bootstrap program. It would be interesting to derive the full conformal DR in the special case of one-dimensional CFTs, as well as for two-point functions in defect or boundary CFTs. Finally, a connection to thermal correlator DRs could follow by taking the heavy-heavy-light-light limit of our conformal DR. We leave these applications for future work.

Acknowledgments. The work of D. C. and S. S. is supported by the Israeli Science Foundation (ISF) Grant No. 1487/21 and by the MOST NSF/BSF physics Grant No. 2022726. The work of S. S. is also partially supported by a PhD fellowship from the Israel Scholarship Education Foundation. The work of J. M. is partially supported by the ISF, Grant No. 1487/21 and by ANID FONDECYT Postdoctorado Grant No. 3230626.

Data availability. The data are not publicly available. The data are available from the authors upon reasonable request.

-
- [1] R. De L. Kronig, *J. Opt. Soc. Am.* **12**, 547 (1926).
 - [2] H. A. Kramers, Atti Cong. Intern. Fisica (Transactions of Volta Centenary Congress) Como **2**, 545 (1927), https://www.lorentz.leidenuniv.nl/IL-publications/sources/Kramers_27.pdf.
 - [3] S. Mandelstam, *Phys. Rev.* **112**, 1344 (1958).
 - [4] G. F. Chew and S. Mandelstam, *Phys. Rev.* **119**, 467 (1960).
 - [5] A. Martin, *Scattering Theory: Unitarity, Analyticity and Crossing* (Springer-Verlag Berlin, Heidelberg, 1969), Vol. 3.
 - [6] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, England, 1966).
 - [7] G. R. Sreaton, *Dispersion Relations* (Oliver and Boyd, Edinburgh, 1961), Vol. 1.
 - [8] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, and P. Vieira, *J. High Energy Phys.* **11** (2017) 133.
 - [9] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, and P. Vieira, *J. High Energy Phys.* **12** (2019) 040.
 - [10] M. Kruczenski, J. Penedones, and B. C. van Rees, *arXiv*: 2203.02421.
 - [11] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
 - [12] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Phys. Lett. B* **428**, 105 (1998).
 - [13] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
 - [14] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *J. High Energy Phys.* **12** (2008) 031.
 - [15] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Phys. Rev. D* **86**, 025022 (2012).
 - [16] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *J. Stat. Phys.* **157**, 869 (2014).
 - [17] D. Simmons-Duffin, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings* (World Scientific Publishing Co. Pte Ltd., Singapore, 2017), pp. 1–74.
 - [18] D. Poland, S. Rychkov, and A. Vichi, *Rev. Mod. Phys.* **91**, 015002 (2019).
 - [19] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, *J. High Energy Phys.* **08** (2016) 036.
 - [20] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, *J. High Energy Phys.* **11** (2015) 106.
 - [21] F. Kos, D. Poland, and D. Simmons-Duffin, *J. High Energy Phys.* **11** (2014) 109.
 - [22] S. Caron-Huot, *J. High Energy Phys.* **09** (2017) 078.
 - [23] D. Simmons-Duffin, D. Stanford, and E. Witten, *J. High Energy Phys.* **07** (2018) 085.
 - [24] P. Kravchuk and D. Simmons-Duffin, *J. High Energy Phys.* **11** (2018) 102.
 - [25] D. Carmi and S. Caron-Huot, *J. High Energy Phys.* **09** (2020) 009.
 - [26] J. Penedones, J. A. Silva, and A. Zhiboedov, *J. High Energy Phys.* **08** (2020) 031.

- [27] S. Caron-Huot, D. Mazac, L. Rastelli, and D. Simmons-Duffin, *J. High Energy Phys.* **05** (2021) 243.
 [28] M. Hogervorst and S. Rychkov, *Phys. Rev. D* **87**, 106004 (2013).
 [29] We are employing the notation $\Gamma_x \equiv \Gamma(x)$ for the gamma function.
 [30] The bulk kernel and contact kernel are given by [25],

$$K_B^{(a=0,b=0)} = \left(\frac{z\bar{z}}{w\bar{w}} \right)^{3/2} \frac{(w - \bar{w})(w^{-1} + \bar{w}^{-1} + z^{-1} + \bar{z}^{-1} - 2)}{((1-z)(1-\bar{z})(1-w)(1-\bar{w}))^{3/4}} \frac{x^3}{64\pi^2} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, 1-x\right), \quad (30)$$

$$K_C^{(a,b)} = \left(\frac{(1-w)(1-\bar{w})}{(1-z)(1-\bar{z})} \right)^{\frac{a+b}{2}} \frac{4}{\pi} \frac{1}{\bar{w}^2} \left(\frac{1 - \rho_z^2 \bar{\rho}_z^2 \bar{\rho}_w^2}{(1 - \rho_z^2)(1 - \bar{\rho}_z^2)(1 - \bar{\rho}_w^2)} \right)^{1/2} \frac{1 - \rho_z \bar{\rho}_z \bar{\rho}_w^2}{(1 - \rho_z \bar{\rho}_w)(1 - \bar{\rho}_z \bar{\rho}_w)}, \quad (31)$$

where ${}_2F_1$ is the hypergeometric function, and x and y are defined in (13) and (14).

- [31] We define the first-order differential operators $\mathcal{D}_2 \equiv zw/(z-w)[(1-w)\partial_w - (1-z)\partial_z] + (z \leftrightarrow \bar{z}) - 1$, $\mathcal{D}_4 \equiv z\bar{z}(w - \bar{w})/[(z - \bar{z})w\bar{w}][zw[(1-w)\partial_w - (1-z)\partial_z]/(z-w) + (z \leftrightarrow \bar{z})]$, and the second-order one $\mathcal{D}_{\rho_z} \equiv \rho_z^2 \partial_{\rho_z}^2 + 2\rho_z^2(2a + 2b + \rho_z)\partial_{\rho_z}/(\rho_z^2 - 1) - 4ab\rho_z/(\rho_z + 1)^2$.
 [32] This is because we have $\mathcal{D}_{(z,\bar{z},w,\bar{w})}^{(24)}[\sqrt{z\bar{z}w\bar{w}}] = \mathcal{D}_{(\bar{w},\bar{z},w,z)}^{(24)}[\sqrt{z\bar{z}w\bar{w}}] = 0$ as well as $\mathcal{D}_{(z,\bar{z},w,\bar{w})}^{(24)}[f(x,y)] = \mathcal{D}_{(\bar{w},\bar{z},w,z)}^{(24)}[f(x,y)] = -f(x,y)$, where $f(x,y)$ is any function of variables x and y in (13) and (14).
 [33] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/s5sx-9yz> for details on (i) the Appell functions, (ii) their role in the partial differential equations for the bulk kernel, (iii) numerical checks of the dispersion relation for generalized free fields, and (iv) the collinear limit of the dispersion kernel.
 [34] Here $\tilde{S}^{(a,b)} = \tilde{S}_a^{(a,b)} + \tilde{S}_{-a}^{(b,a)} + \tilde{S}_b^{(b,a)} + \tilde{S}_{-b}^{(a,b)}$.
 [35] In terms of the variables $(w, \bar{w}, z, \bar{z})$ or (U', V', U, V) , we have,

$$\xi \equiv \frac{1}{2} \left(1 + y - \frac{s}{2} \right) = \frac{1}{2} \left(1 + \sqrt{VV'} + \frac{U(1-V') + U'(1-V)}{2\sqrt{UU'}} \right), \quad \frac{\eta}{\xi + \eta - \xi\eta} = \frac{2}{1 + y + \frac{s}{2}} = \frac{2}{1 + \frac{q_1}{2}} \quad (32)$$

$$\eta \equiv \frac{1}{2y} \left(1 + y - \frac{s}{2} \right) = \frac{1}{2\sqrt{VV'}} \left(1 + \sqrt{VV'} + \frac{U(1-V') + U'(1-V)}{2\sqrt{UU'}} \right), \quad \frac{\xi}{\xi + \eta - \xi\eta} = \frac{2y}{1 + y + \frac{s}{2}} = \frac{2}{1 + p_2^{-1}} \quad (33)$$

where $U' \equiv w\bar{w}$, $V' \equiv (1-w)(1-\bar{w})$, $U \equiv z\bar{z}$, $V \equiv (1-z)(1-\bar{z})$, and $s \equiv \sqrt{w\bar{w}z\bar{z}}(w^{-1} + \bar{w}^{-1} + z^{-1} + \bar{z}^{-1} - 2)$, and $y \equiv \sqrt{(1-z)(1-\bar{z})(1-w)(1-\bar{w})}$.

- [36] M. F. Paulos, *J. High Energy Phys.* **08** (2021) 166.
 [37] D. Bonomi and V. Forini, *J. High Energy Phys.* **10** (2024) 181.
 [38] D. Carmi, S. Ghosh, and T. Sharma, *J. High Energy Phys.* **12** (2024) 119.
 [39] Note that from (20) one can get all four terms that go into (9), using $\tilde{S}_{-a}^{(a,b)} = \tilde{S}_a^{(-a,-b)}$.
 [40] G. Mack, [arXiv:0907.2407](https://arxiv.org/abs/0907.2407).
 [41] G. Mack, *Bulgarian J. Phys.* **36**, 214 (2009), <https://www.osti.gov/etdweb/biblio/21490711>.
 [42] J. Penedones, *J. High Energy Phys.* **03** (2011) 025.
 [43] $\Gamma_{\Delta_i}^{t,u} \equiv \Gamma_{\frac{\Delta_1+\Delta_3-1}{2}} \Gamma_{\frac{\Delta_2+\Delta_4-1}{2}} \Gamma_{\frac{\Delta_2+\Delta_3-u}{2}} \Gamma_{\frac{\Delta_1+\Delta_4-u}{2}} \Gamma_{\frac{t+u-\Delta_1-\Delta_2}{2}} \Gamma_{\frac{t+u-\Delta_3-\Delta_4}{2}}$.
 [44] D. Carmi, J. Penedones, J. A. Silva, and A. Zhiboedov, *SciPost Phys.* **10**, 145 (2021).
 [45] A.-K. Trinh, *J. High Energy Phys.* **03** (2022) 032.
 [46]

$$H_{a'}^{(a,b)} = \sum_{t,u=0}^{\infty} \frac{(-1)^{t+u+1} \Gamma_{-\frac{ba'}{a}+a'+1} \Gamma_{\frac{ba'}{a}-a'-t} \Gamma_{-\frac{ba'}{a}+u+a'}}{2\pi \Gamma_{t+1} \Gamma_{u+1} \Gamma_{-u-2a'-1} \Gamma_{\frac{ba'-a(u+a'+1)}{a}} \sin\left(\frac{\pi(a-b)a'}{a}\right)} \times \frac{{}_4\tilde{F}_3\left(1, -\frac{ba'}{a}+a'+1, u+2a'+2, u+a'-\frac{ba'}{a}+2; -\frac{ba'}{a}+a'+2, 1-t, -t+a'-\frac{ba'}{a}+1; \frac{1}{v'}\right)}{u^{a'+t+u} u'^{-a'-t-u+1} v^{\frac{(a+b)(a-a')}{2a}-u} v'^{\frac{b}{2}(-\frac{ba'}{a}+3a'-a-b+2u+4)}} \quad (34)$$

where ${}_4\tilde{F}_3$ is the regularized ${}_4F_3$ function.

- [47] S. Caron-Huot, F. Coronado, A.-K. Trinh, and Z. Zahraee, *J. High Energy Phys.* **02** (2023) 083.
 [48] S. Caron-Huot, F. Coronado, and Z. Zahraee, *J. High Energy Phys.* **05** (2025) 220.
 [49] For the case of identical operators, such results were obtained in [25,27].