

Feynman integral reduction without integration by parts

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We present an interesting study of Feynman integral reduction that does not employ integration-by-parts identities. Our approach proceeds by studying the equivalence relations of integral contours in the Feynman parametrization. We find that the integration contour can take a more general form than that given by the Cheng-Wu theorem. We apply this idea to one-loop integrals and derive universal reduction formulas that can be used to efficiently reduce any one-loop integral. We expect that this approach can be useful in the reduction of multiloop integrals as well.

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I. INTRODUCTION

Scattering amplitudes and Feynman integrals are core components of perturbative quantum field theories. Developing efficient computational methods is crucial for advancing cutting-edge phenomenological applications. To compute multiloop scattering amplitudes, it is often necessary to handle a large number of complicated Feynman integrals. By exploiting the linear relations among these integrals, they can be expressed as linear combinations of a finite set of master integrals (MIs). This process is known as integral reduction, which significantly reduces the computational complexity. Integral reduction is also a key step in the method of differential equations [1–4] for evaluating the MIs.

Currently, the standard method for integral reduction is the Laporta algorithm [5] for solving the integration-by-parts (IBP) identities [6,7] of Feynman integrals. Many program packages implementing this algorithm are available, including FIRE [8], LITERED [9], REDUZE [10] and KIRA [11]. Traditionally, the IBP method has been developed mainly in the momentum representation. It can be formulated in other representations as well, such as in the Baikov representation [12–15] or in the Feynman parametrization [12,16–20]. For complicated multiloop integrals, the system of IBP equations can become very large, and it is rather time-consuming or even impractical to solve

them. Recently, it has been proposed to generate a smaller set of IBP relations by using the algebraic-geometry-based method in NEATIBP [21], or by searching for a block-triangular system in BLADE [22]. In addition to the IBP method, there exist other reduction techniques as well—e.g., the Passarino-Veltman (PV) tensor reduction [23] and its improvements using auxiliary vectors [24–30], the Ossola-Papadopoulos-Pittau (OPP) method [31–33], the unitarity cut method [34–44], generating functions [45–48], *et al.* We will not go into the details of these methods.

An alternative way to formulate the IBP relations is the so-called intersection theory [49–60], where a Feynman integral is regarded as a pairing between a differential form and an integration contour. The IBP relations are formulated as the equivalence relations among differential forms, which live in a so-called twisted cohomology group. The integral reduction can then be performed by calculating the intersection numbers between a pair of differential forms. Such an approach has been extensively developed in the Baikov representation [61,62] of Feynman integrals, and recently has been developed for the Feynman parametrization as well [63].

Within the framework of intersection theory, the equivalence relations can also be established among integration contours. The equivalence classes of contours form a twisted homology group. In principle, these equivalence relations can also be employed for integral reduction, but this approach has not been developed so far.

In this work, we initiate a study that exploits the equivalence of integration contours for integral reduction, based on the Feynman parametrization. One outcome of our study is an improvement of the Cheng-Wu theorem [64–66], such that the delta function in the Feynman parametrization can be modified to a more general form. This essentially corresponds to modifying the integration

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contour.¹ We apply this to one-loop integrals and find that, by splitting the contour and further transforming each part of the contour, we can identify each part with a Feynman integral that is simpler than the original one. As a result, we can construct recursive reduction formulas purely by dealing with integration contours, without solving IBP relations. This approach does not generate any redundant information hidden in the IBP relations, and is therefore highly efficient.

The paper is organized as follows: In Sec. II, we introduce the equivalence relations of integral contours in Feynman parametrization and use several simple examples to demonstrate our reduction method. In Sec. III, we present our general method for one-loop integral reduction, and we provide recursive formulas that can be easily implemented in computer algebra. In Sec. IV, we demonstrate our method through several examples, including a preliminary extension to higher loops. In Sec. V, we provide a summary and discuss the new challenges that may arise in future applications.

II. DOMAIN OF INTEGRATION AND REDUCTION OF FEYNMAN INTEGRALS

A. Feynman parametrization and equivalence classes of integration domains

An L -loop Feynman integral is defined by

$$I(\nu_n) = e^{\epsilon\gamma_E L} \int \frac{d^d k_1}{i\pi^{d/2}} \cdots \frac{d^d k_L}{i\pi^{d/2}} \frac{1}{D_1^{\nu_1} \cdots D_n^{\nu_n}}, \quad (2.1)$$

where $\nu_n \equiv \{\nu_1, \dots, \nu_n\}$, and D_i are propagator denominators or irreducible scalar products. One can convert the above momentum representation into integrals over Feynman parameters. There are many variants of the Feynman parametrization, and one of them was introduced in Ref. [16]. It takes the form (assuming all $\nu_j > 0$)

$$I(\nu_n) = C(\nu_n) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) (\alpha_0 \mathcal{U} + \mathcal{F})^{\lambda_0} \delta \times \left(1 - \sum_{j \in \mathcal{S}} \alpha_j \right), \quad (2.2)$$

where $\mathcal{S}$ is a nonempty subset of $\{0, 1, 2, \dots, n\}$, $\lambda_0 \equiv -d/2$, $\nu_0 \equiv -\nu - (L+1)\lambda_0$, $\nu \equiv \sum_{j=1}^n \nu_j$, and the prefactor is given by

¹Note that the deformation of the integration contour in the Feynman parametrization is a common practice in the method of sector decomposition to avoid singularities in numerical integration [67]. It is also demonstrated in Ref. [68] that generalized cuts can be implemented in the Feynman parametrization by appropriate choices of integral domains. These manipulations, however, have no connection with integral reduction.

$$C(\nu_n) \equiv (-1)^\nu e^{\epsilon\gamma_E L} \frac{\Gamma(-\lambda_0)}{\prod_{i=0}^n \Gamma(\nu_i)}. \quad (2.3)$$

$\mathcal{U}$ and $\mathcal{F}$ are the so-called Symanzik polynomials. We denote

$$\begin{aligned} & \alpha_1 D_1 + \cdots + \alpha_n D_n \\ & \equiv \sum_{i,j=1}^L M_{ij} k_i \cdot k_j - 2 \sum_{i=1}^L k_i \cdot Q_i - J + i0, \end{aligned} \quad (2.4)$$

where Q_i are combinations of external momenta. The two Symanzik polynomials can then be written as

$$\mathcal{U} = \det(M), \quad \mathcal{F} = \det(M) \left[\sum_{i,j=1}^L M_{ij}^{-1} Q_i \cdot Q_j - J - i0 \right]. \quad (2.5)$$

From the above expressions, it is clear that $\mathcal{U}$ and $\mathcal{F}$ are homogeneous polynomials of degree L and $L+1$ in the variables $\{\alpha_1, \dots, \alpha_n\}$, respectively.

Note that the representation (2.2) can be applied to the cases where some $\nu_j \leq 0$ as well. For that, we can introduce a regularization α_j^ρ into the integration measure. After performing the reduction, one takes the limit $\rho \rightarrow 0$ in the end. In the following, we will assume that such regulators are implicitly applied when necessary.

The fact that one can freely choose a subset of Feynman parameters appearing in the δ function in Eq. (2.2) follows from the so-called Cheng-Wu theorem [65]. We now note that Eq. (2.2) can actually be recast into a more general form:

$$I(\nu_n) = C(\nu_n) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) (\alpha_0 \mathcal{U} + \mathcal{F})^{\lambda_0} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}), \quad (2.6)$$

where $\mathcal{X}$ and $\mathcal{Y}$ are homogeneous functions of degree 0 and 1 in the variables α_i , respectively. The integration domain is the hypersurface defined by the equation $\mathcal{X} - \mathcal{Y} = 0$ within the region $\mathbb{R}_+^{n+1}$ (where all $\alpha_j \geq 0$). The hypersurface should satisfy the condition that each ray going from the origin to $\mathbb{R}_+^{n+1}$ should have one and only one intersection point with the surface. For that purpose, $\mathcal{X}/\mathcal{Y}$ should be positive-definite within $\mathbb{R}_+^{n+1}$. To reconcile the sign convention with Eq. (2.2), we require that $\mathcal{X}$ and $\mathcal{Y}$ both be positive-definite within $\mathbb{R}_+^{n+1}$. In practice, this usually requires that we work in a particular region of the kinematic variables. On the other hand, the reduction coefficients are rational functions and are correct in other kinematic regions as well. Therefore, the positivity constraint can be relaxed in the end.

It is easy to see that the original representation (2.2) corresponds to the specific choice $\mathcal{X} = 1$ and $\mathcal{Y} = \sum_{j \in \mathcal{S}} \alpha_j$. Note that by choosing $\mathcal{X} = 1$ and $\mathcal{Y} = \alpha_0$, we can integrate out α_0 to arrive at the so-called Lee-Pomeransky (LP) representation [69]:

$$I(\nu_n) = C(\nu_n) \int_0^\infty \left(\prod_{j=1}^n \alpha_j^{\nu_j-1} d\alpha_j \right) (\mathcal{U} + \mathcal{F})^{\lambda_0}. \quad (2.7)$$

With these standard choices, the singularity structure of the integral within the parameter space can be analyzed using the Newton polytope associated with the polynomial $\alpha_0 \mathcal{U} + \mathcal{F}$. See, e.g., Refs. [70,71]. However, when $\mathcal{X}$ and $\mathcal{Y}$ are rational functions, the Newton polytope analysis needs to be modified to take them into account. We will not perform such analyses here, but simply note that it is always possible to find some regions of λ_0 where the integral converges.

It is straightforward to show that Eq. (2.6), with an arbitrary choice of $\mathcal{X}$ and $\mathcal{Y}$ (satisfying the conditions thereof), is equivalent to Eq. (2.2). Starting from Eq. (2.6), we insert the identity

$$1 = \int_0^\infty \delta\left(\eta - \sum_{j \in \mathcal{S}} \alpha_j\right) d\eta \quad (2.8)$$

and then perform the rescaling $\alpha_i \rightarrow \eta \alpha_i$. Using the homogeneity of the integrand, as well as the functions $\mathcal{X}$ and $\mathcal{Y}$, Eq. (2.6) becomes

$$I(\nu_n) = C(\nu_n) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) (\alpha_0 \mathcal{U} + \mathcal{F})^{\lambda_0} \times \delta\left(1 - \sum_{j \in \mathcal{S}} \alpha_j\right) \int_0^\infty d\eta \mathcal{X} \delta(\eta \mathcal{X} - \mathcal{Y}). \quad (2.9)$$

The positivity of $\mathcal{X}$ and $\mathcal{Y}$ ensures that the integration over η equals unity. Therefore, Eq. (2.9) is precisely Eq. (2.2).

Equation (2.6), with the freedom of choosing suitable $\mathcal{X}$ and $\mathcal{Y}$ functions, will be our starting point to derive the reduction formulas. Note that we do not need IBP relations in the above derivation. On the other hand, our method is secretly equivalent to IBP reduction. In fact, the independence of Eq. (2.6) on $\mathcal{X}$ and $\mathcal{Y}$ can be associated with Stokes's theorem, as we will demonstrate in the following.

We will follow an approach similar to that in Sec. 2.5.3 of Ref. [72]. The δ function in Eq. (2.6) effectively restricts the integration onto the n -dimensional hypersurface S_n embedded in $\mathbb{R}_+^{n+1}$, determined by $\mathcal{X} - \mathcal{Y} = 0$:

$$S_n \equiv \{(\alpha_0, \dots, \alpha_n) \in \mathbb{R}_+^{n+1} | \mathcal{X} - \mathcal{Y} = 0\}. \quad (2.10)$$

Let us define an integrand function

$$f \equiv C(\nu_n) \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} \right) (\alpha_0 \mathcal{U} + \mathcal{F})^{\lambda_0} \quad (2.11)$$

and an integration measure

$$\omega \equiv \sum_{j=0}^n (-1)^j \alpha_j d\alpha_0 \wedge \dots \wedge \widehat{d\alpha_j} \wedge \dots \wedge d\alpha_n, \quad (2.12)$$

where the hat indicates that the corresponding factor is omitted. When restricted to the surface S_n , the value ω can be parametrized by n out of $n+1$ variables. Picking one of the variables (say, without loss of generality, α_0) appearing in $\mathcal{X} - \mathcal{Y}$, we can express the integration measure ω on S_n as

$$\omega|_{S_n} = - \left| \frac{\partial(\mathcal{X} - \mathcal{Y})}{\partial \alpha_0} \right|_{S_n}^{-1} \left[\sum_{j=0}^n \alpha_j \frac{\partial(\mathcal{X} - \mathcal{Y})}{\partial \alpha_j} \right]_{S_n} \times d\alpha_1 \wedge \dots \wedge d\alpha_n, \quad (2.13)$$

where the subscript S_n indicates that α_0 is replaced by a solution of $\mathcal{X} - \mathcal{Y} = 0$. This expression is obtained by pulling back the integration measure on the hypersurface to the n -dimensional parameter space of a local coordinate chart. Note that the absolute value is necessary to enforce a consistent orientation convention. If the hypersurface cannot be covered by a single coordinate chart, the formula is understood with an implicit summation over the relevant local patches. Next, we use the fact that $\mathcal{X}$ and $\mathcal{Y}$ are homogeneous functions of degree 0 and 1, respectively. By Euler's homogeneous function theorem, we have $\sum_i \alpha_i \partial_{\alpha_i} \mathcal{X} = 0$ and $\sum_i \alpha_i \partial_{\alpha_i} \mathcal{Y} = \mathcal{Y}$, which imply

$$\left[\sum_{j=0}^n \alpha_j \frac{\partial(\mathcal{X} - \mathcal{Y})}{\partial \alpha_j} \right]_{S_n} = -\mathcal{Y}|_{S_n} = -\mathcal{X}|_{S_n}. \quad (2.14)$$

Combining the above information, we arrive at

$$\omega|_{S_n} = \left| \frac{\partial(\mathcal{X} - \mathcal{Y})}{\partial \alpha_0} \right|_{S_n}^{-1} \mathcal{X}|_{S_n} d\alpha_1 \wedge \dots \wedge d\alpha_n = \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) d\alpha_0 \wedge d\alpha_1 \wedge \dots \wedge d\alpha_n. \quad (2.15)$$

Note that when solving for α_0 from the δ function, one may obtain several solutions, which correspond to the different local patches on the hypersurface, as mentioned earlier. From the above expression, we see that the integral in Eq. (2.6) can be written as

$$I(\nu_n) = \int_{S_n} f \omega. \quad (2.16)$$

Suppose that S_n and S'_n are two n -dimensional-oriented hypersurfaces defined by $\mathcal{X} - \mathcal{Y} = 0$ and $\mathcal{X}' - \mathcal{Y}' = 0$,

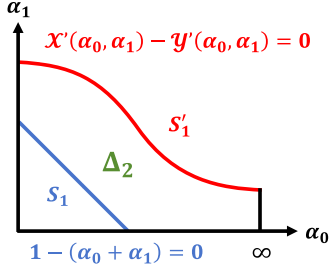


FIG. 1. The independence of the integral on the integration domain for the case of two variables. S_1 is the integration domain defined by $\mathcal{X} = 1$ and $\mathcal{Y} = \alpha_0 + \alpha_1$, while S'_1 is the integration domain defined by the two functions $\mathcal{X}'(\alpha_0, \alpha_1)$ and $\mathcal{Y}'(\alpha_0, \alpha_1)$.

respectively. See Fig. 1 for a simple illustration with $n = 1$. We need to show that the integrals on S_n and S'_n are the same. For that, we will employ Stokes's theorem. Let Δ_{n+1} be the region enclosed by S_n , S'_n , the coordinate hyperplanes defined by $\alpha_j = 0$ ($j = 0, 1, \dots, n$), and possibly the hyperplanes at infinity. Note that f is a homogeneous function of degree $(-n - 1)$ in the variables $\{\alpha_j\}$. This can be used to demonstrate that the n -form $f\omega$ is closed—i.e., $d(f\omega) = df \wedge \omega + f d\omega = 0$:

$$\begin{aligned} df \wedge \omega &= \left(\sum_{j=0}^n \frac{\partial f}{\partial \alpha_j} d\alpha_j \right) \\ &\wedge \left(\sum_{j=0}^n (-1)^j \alpha_j d\alpha_0 \wedge \dots \wedge \widehat{d\alpha_j} \wedge \dots \wedge d\alpha_n \right) \\ &= -(n+1) f d\alpha_0 \wedge d\alpha_1 \wedge \dots \wedge d\alpha_n \\ &= -f d\omega. \end{aligned} \quad (2.17)$$

Therefore, the integration of $d(f\omega)$ in any region Δ_{n+1} in $\mathbb{R}^{n+1}$ is zero. By Stokes's theorem, this means that

$$\iint_{\Delta_{n+1}} d(f\omega) = \oint_{\partial\Delta_{n+1}} f\omega = 0. \quad (2.18)$$

When restricted on the coordinate hyperplanes as part of $\partial\Delta_{n+1}$, ω always vanishes due to the factor of α_j in its definition (2.12). The boundary hyperplanes at infinity correspond to infinitesimally small solid angles. Therefore, if the integrand has no singularities on these boundaries, its integral yields a vanishing contribution. If, on the other hand, the integrand exhibits singularities on the boundaries at infinity, one needs to introduce regulators to define the integral. In dimensional regularization, the integral is regarded as an analytic function of the complex variable $\lambda_0 = -d/2$. Unlike the integrals on S_n and S'_n , the integrals at infinity are simpler to analyze, since the functions $\mathcal{X}$ and $\mathcal{Y}$ no longer come into play. It is enough to realize that $\alpha_0\mathcal{U} + \mathcal{F}$ is a homogeneous polynomial, while the

variables cannot go to infinity uniformly when approaching the boundaries [otherwise, the assumption below Eq. (2.6) would be violated]. Therefore, it is always possible to find some region of λ_0 where $\alpha_0^{-(L+1)\lambda_0}(\alpha_0\mathcal{U} + \mathcal{F})^{\lambda_0}$ goes to zero sufficiently fast: if α_0 is finite at the boundary, we can choose λ_0 to be sufficiently negative; while if $\alpha_0 \rightarrow \infty$ at the boundary, we can choose λ_0 to be sufficiently positive.

The possible introduction of analytic regulators α_j^ρ does not change this conclusion, since ρ should then be regarded as another complex variable on which the integral depends. Putting all the above together, we are now left with only S_n and S'_n in $\partial\Delta_{n+1}$. After taking care of the orientations of the hypersurfaces, we finally arrive at

$$\int_{S_n} f\omega = \int_{S'_n} f\omega. \quad (2.19)$$

In the above, we have shown that the two contours S_n and S'_n give rise to the same integral, and they can be regarded as belonging to the same equivalence class. In the language of twisted homology (see, e.g., Ref. [73] for more details), the equivalence classes of integration contours are elements (cycles) of a twisted homology group determined by the polynomial $\alpha_0\mathcal{U} + \mathcal{F}$. This homology group is dual to the twisted cohomology group of the integrands. In the literature [49,50], there have been extensive discussions on how to perform integral reduction using the vector-space structure of the cohomology groups. This can be done using the techniques of intersection theory [74–83]. The homology groups of integration contours also admit a vector-space structure, and in principle can be used for integral reduction as well. However, this path has not been followed in the literature to the best of our knowledge.

In the following, we will exploit the equivalence of integration domains to set up recursion relations that can be used to reduce one-loop integrals. We will write the relations in terms of index-raising and -lowering operators, defined as

$$\begin{aligned} \hat{j}^+ I(\dots, \nu_j, \dots) &\equiv \nu_j I(\dots, \nu_j + 1, \dots), \\ \hat{j}^- I(\dots, \nu_j, \dots) &\equiv I(\dots, \nu_j - 1, \dots). \end{aligned} \quad (2.20)$$

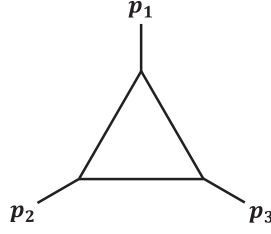
It is also useful to define an operator that sets an index to zero:

$$\hat{j}_0 I(\dots, \nu_j, \dots) \equiv I(\dots, 0, \dots). \quad (2.21)$$

Before going into more general formalities, we first study a few simple examples.

B. A reducible sector

To illustrate our approach, we first consider a simple case where a sector is reducible to one of its subsectors. A

FIG. 2. The massless triangle with $p_1^2 = s$ and $p_2^2 = p_3^2 = 0$.

specific example is the massless triangle diagram shown in Fig. 2. The corner integral in this sector corresponding to $\nu_3 = \mathbf{1}_3 = \{1, 1, 1\}$ is given by

$$I(\mathbf{1}_3) = C(\mathbf{1}_3) \int_0^\infty d\alpha_0 d\alpha_1 d\alpha_2 d\alpha_3 \alpha_0^{d-4} [\alpha_0 \mathcal{U} + \mathcal{F}]^{-d/2} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}), \quad (2.22)$$

where $\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3$ and $\mathcal{F} = -\alpha_1 \alpha_2 s$. The above integral is reducible to a bubble integral. One can observe that α_3 is present in $\mathcal{U}$ but absent in $\mathcal{F}$. We will show that there is a general rule: whenever a variable is present in $\mathcal{U}$ but absent in $\mathcal{F}$, the corresponding integral is reducible to the subsector without the variable.

Let us consider an n -point one-loop integral, and denote such a variable as α_l . Defining $\mathcal{U}_{l,0} \equiv \mathcal{U}|_{\alpha_l \rightarrow 0}$, we can write $\mathcal{U} = \alpha_l + \mathcal{U}_{l,0}$. The corner integral in this sector can then be written as

$$I(\mathbf{1}_n) = C(\mathbf{1}_n) \int_0^\infty \left(\prod_{j=0}^n d\alpha_j \right) \alpha_0^{d-n-1} [\alpha_0(\alpha_l + \mathcal{U}_{l,0}) + \mathcal{F}]^{\lambda_0} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}). \quad (2.23)$$

We choose

$$\mathcal{X} = \frac{\mathcal{U}_{l,0}}{\alpha_l + \mathcal{U}_{l,0}}, \quad \mathcal{Y} = \alpha_0 \quad (2.24)$$

and integrate out α_0 using the δ function. After that, the variable α_l only appears in a power of $\alpha_l + \mathcal{U}_{l,0}$. The integration over α_l can then be performed using the formula

$$I(\nu_1, \nu_2) = C(\nu_1, \nu_2) \int_0^\infty d\alpha_0 d\alpha_1 d\alpha_2 \alpha_0^{\nu_0-1} \alpha_1^{\nu_1-1} \alpha_2^{\nu_2-1} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \times [(\alpha_1 + \alpha_2)\alpha_0 + \alpha_1 \alpha_2 (m_1^2 + m_2^2 - s) + \alpha_1^2 m_1^2 + \alpha_2^2 m_2^2]^{\lambda_0}. \quad (2.30)$$

We consider the reduction of $I(1, 2)$. Note that the integral shown in Eq. (2.30) for positive ν_1 and ν_2 does not possess the property of being reducible to subsectors as described in Sec. II B. Therefore, we need to split the integral into two parts: one part that is reducible to subsectors, and the other part that can be eventually related to $I(1, 1)$. To this end, we consider

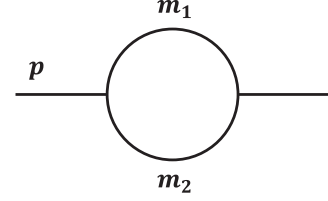


FIG. 3. The bubble diagram with two masses.

$$\int_0^\infty dx x^{\beta-1} \left(1 + \frac{x}{\Lambda}\right)^{\gamma-1} = \Lambda^\beta B(\beta, 1 - \beta - \gamma), \quad (\Lambda > 0), \quad (2.25)$$

where the beta function is

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}. \quad (2.26)$$

After integrating over α_l , we arrive at

$$I(\mathbf{1}_n) = C(\mathbf{1}_n) \int_0^\infty \left(\prod_{j \neq 0, l} d\alpha_j \right) \mathcal{U}_{l,0} (\mathcal{U}_{l,0} + \mathcal{F})^{\lambda_0}. \quad (2.27)$$

As promised, the integral now manifestly has the form of the LP representation in the subsector without α_l . Using the index-changing operators, the above result can be written as

$$I(\mathbf{1}_n) = \frac{1}{d - n - 1} \sum_{k \neq l} \hat{k}^+ \hat{l}_0 I(\mathbf{1}_n). \quad (2.28)$$

Applying the above general formula to the triangle integral in Fig. 2, we have

$$I(1, 1, 1) = \frac{1}{d - 4} [I(2, 1, 0) + I(1, 2, 0)]. \quad (2.29)$$

C. The bubble family

We now turn to the reduction in an irreducible sector: the one-loop bubble with two masses shown in Fig. 3. The kinematic variables are m_1^2 , m_2^2 , and $p^2 = s$. The integrals in this family can be represented as

the auxiliary integral

$$g(1, 1) = \frac{1}{d-3} \left(\frac{2m_1^2}{m_1^2 - m_2^2 + s} \right) I(2, 0). \quad (2.32)$$

$$g(1, 1) \equiv C(1, 1) \int_0^\infty d\alpha_0 d\alpha_1 d\alpha_2 \alpha_0^{d-3} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \times \{[\alpha_1 + (1 - q_1)\alpha_2]\alpha_0 + \alpha_1^2 m_1^2\}^{\lambda_0}, \quad (2.31)$$

where $q_1 \equiv (m_1^2 + m_2^2 - s)/(2m_1^2)$. The motivation for considering the above integral comes from two perspectives. First of all, the integral itself is reducible. According to the result in Sec. II B, one can see that $g(1, 1)$ is reducible to the subsector without α_2 . Applying the method from Sec. II B, we obtain

On the other hand, the integral can be related to a linear combination of $I(1, 1)$ and $I(1, 2)$ through a change of variables. We perform the variable change $\alpha_1 \rightarrow \alpha_1 + q_1 \alpha_2$ and $\alpha_2 \rightarrow \alpha_2(1 - W\alpha_2/\alpha_0)$ in Eq. (2.31), where $W \equiv \lambda(m_1^2, m_2^2, s)/[2(m_1^2 - m_2^2 + s)]$ and the Källén function is given by $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. For convenience, we will work in the kinematic region where $W < 0$, such that the integration domain of α_2 remains $[0, \infty]$ after the variable change. The reduction coefficients, being rational functions of kinematic variables, are

insensitive to this choice. After the change of variables, we arrive at²

$$\begin{aligned} g(1, 1) &= C(1, 1) \int_0^\infty d\alpha_0 d\alpha_2 \int_{-q_1 \alpha_2}^\infty d\alpha_1 \alpha_0^{d-3} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \left(1 - 2W \frac{\alpha_2}{\alpha_0} \right) \\ &\quad \times [(\alpha_1 + \alpha_2)\alpha_0 + \alpha_1 \alpha_2 (m_1^2 + m_2^2 - s) + \alpha_1^2 m_1^2 + \alpha_2^2 m_2^2]^{\lambda_0} \\ &= I(1, 1) + \frac{2W}{d-3} I(1, 2) + h_1(1, 1), \end{aligned} \quad (2.33)$$

where we have split the integration range of α_1 and define

$$\begin{aligned} h_1(1, 1) &\equiv C(1, 1) \int_0^\infty d\alpha_0 d\alpha_2 \int_{-q_1 \alpha_2}^0 d\alpha_1 \alpha_0^{d-3} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \left(1 - 2W \frac{\alpha_2}{\alpha_0} \right) \\ &\quad \times [(\alpha_1 + \alpha_2)\alpha_0 + \alpha_1 \alpha_2 (m_1^2 + m_2^2 - s) + \alpha_1^2 m_1^2 + \alpha_2^2 m_2^2]^{\lambda_0}. \end{aligned} \quad (2.34)$$

The $h_1(1, 1)$ function is also reducible to subsectors. This can be seen by further applying the variable change $\alpha_0 \rightarrow \alpha_0 \alpha_2 / (\alpha_2 + \alpha_1 / q_1)$, $\alpha_1 \rightarrow -\alpha_1 \alpha_2 / (\alpha_2 + \alpha_1 / q_1)$. Denoting $\alpha'_1 \equiv \alpha_1 [1 - W(2\alpha_2 + \alpha_1 / q_1) / \alpha_0]$, we have

$$\begin{aligned} h_1(1, 1) &= C(1, 1) \int_0^\infty d\alpha_0 d\alpha'_1 d\alpha_2 \alpha_0^{d-3} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \left\{ \left[(1 - q_1) \frac{\alpha'_1}{q_1} + \alpha_2 \right] \alpha_0 + \alpha_2^2 m_2^2 \right\}^{\lambda_0} \\ &= \frac{1}{d-3} \frac{m_1^2 + m_2^2 - s}{m_1^2 - m_2^2 + s} I(0, 2), \end{aligned} \quad (2.35)$$

where we have again utilized the method of Sec. II B (assuming $0 < q_1 < 1$). Combining Eqs. (2.32), (2.33), and (2.35), we arrive at

$$I(1, 2) = (3 - d) \frac{m_1^2 - m_2^2 + s}{\lambda(m_1^2, m_2^2, s)} I(1, 1) + \frac{2m_1^2}{\lambda(m_1^2, m_2^2, s)} I(2, 0) - \frac{m_1^2 + m_2^2 - s}{\lambda(m_1^2, m_2^2, s)} I(0, 2), \quad (2.36)$$

which precisely agrees with the result of IBP reduction.

From the above examples, one may find that our procedure mainly consists of two steps: (i) transformation of the integration contours either by explicit choices of the $\mathcal{X}$ and $\mathcal{Y}$ functions, or by appropriate variable changes; and (ii) splitting the integration contour into several parts, and identifying each part manifestly as a Feynman integral. One may then wonder

²It is easy to see that $\mathcal{X}$ and $\mathcal{Y}$ still satisfy the conditions below Eq. (2.6) in terms of the new variables. This is guaranteed, since (i) the variable changes are homogeneous, and (ii) $\mathcal{X}$ and $\mathcal{Y}$ are positive-definite in the original integration domain and must remain so in the transformed integral domain, which maps bijectively to the original domain via the variable changes. The same consideration applies to variable changes employed later.

how these contour transforms are constructed. In the next section, we will present the general method and the explicit recursive reduction formula for one-loop integrals.

III. THE GENERAL METHOD FOR ONE-LOOP INTEGRAL REDUCTION

Based on the examples in the previous section, we now describe our reduction approach for generic one-loop integrals. The final outcome of this section is a system of relations that express a more complicated integral (with higher powers of propagators and/or higher ranks of numerators) as a linear combination of simpler integrals (with lower powers of propagators and/or lower ranks of numerators). By iteratively applying these relations, it gives a shortcut to reduce a integral to corner integrals, with indices being either 1 or 0.

In essence, our iterative relations are equivalent to a symbolic solution to IBP equations, which at one loop can be obtained through other methods as well (see, e.g., Ref. [26] and references therein). In the language of linear systems, these iterative relations automatically generate a system in an upper triangular form (row echelon form). In other words, we have effectively performed the forward elimination in the Gaussian elimination method of solving linear systems with no cost, and applying the iterative relations is equivalent to the step of back-substitution. It is well known that the computational complexity of Gaussian elimination is dominated by that of forward elimination, which scales as N^3 , where N is the number of equations. On the other hand, the complexity of back-substitution only scales as N^2 . Therefore, it is natural to expect that our method provides better efficiency than the traditional IBP methods.

A. More powers vs more variables

The goal of one-loop integral reduction is to reduce the indices ν_j to either 0 or 1. We now introduce an interesting technique to lower an index by 1, at the cost of adding an auxiliary integration variable. While this seems to be meaningless at first sight, it will be employed in the derivation of the final reduction formula.

For an arbitrary function $g(u)$, we can derive the following integral relation:

$$\begin{aligned} & \frac{1}{\Gamma(n+1)} \int_0^\infty du u^n g(u) \\ &= \frac{1}{\Gamma(n+1)} \int_0^\infty du \left[n \int_0^u dx x^{n-1} \right] g(u) \\ &= \frac{1}{\Gamma(n)} \int_0^\infty dx \int_x^\infty du [x^{n-1} g(u)] \\ &= \frac{1}{\Gamma(n)\Gamma(1)} \int_0^\infty dx \int_0^\infty dy [x^{n-1} g(x+y)]. \end{aligned} \quad (3.1)$$

Here, we emphasize that a regularization $n \rightarrow n + \rho$ is implicitly assumed if $n \leq 0$. We may apply the above relation to the Feynman parametric integrals. We define

$$\begin{aligned} & I(\nu_1, \dots, [\nu_l, 1], \dots, \nu_n) \\ & \equiv C(\nu_n, 1) \int_0^\infty d\beta_l \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \\ & \mathcal{X}\delta(\mathcal{X} - \mathcal{Y}) \left[(\alpha_0 \mathcal{U} + \mathcal{F})|_{\alpha_l \rightarrow \alpha_l + \beta_l} \right]^{\lambda_0}. \end{aligned} \quad (3.2)$$

A few words are needed to explain a subtlety in the above definition. Both the prefactor C and the power of α_0 in the integrand involves ν , which is defined as the sum of all indices in the argument of I and C , as can be seen below Eq. (2.2). It then follows that, in the above expression, we have $\nu = 1 + \sum_{j=1}^n \nu_j$. Note that the value of $\nu_0 \equiv -\nu - (L+1)\lambda_0$ is determined accordingly. Similar considerations apply for the functions $g(\nu)$ and $h_i(\nu)$ that will be introduced later.

Applying Eq. (3.1) to Eq. (3.2), we obtain

$$I(\nu_1, \dots, \nu_l + 1, \dots, \nu_n) = I(\nu_1, \dots, [\nu_l, 1], \dots, \nu_n). \quad (3.3)$$

From a different point of view, the above relation can also be obtained by splitting $D_l^{-\nu_l-1}$ as $D_l^{-\nu_l} D_l^{-1}$, and introducing two Feynman parameters α_l and β_l for the two factors. This relation will play a crucial role in subsequent derivations. It is worth noting that it also holds for general L -loop integrals.

B. Reduction for irreducible sectors

We now consider a general one-loop integral in the Feynman representation:

$$\begin{aligned} & I(\nu_n) = C(\nu_n) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \\ & \mathcal{X}\delta(\mathcal{X} - \mathcal{Y}) [(\mathbf{1}^T \boldsymbol{\alpha}) \alpha_0 + \boldsymbol{\alpha}^T \mathbf{Z} \boldsymbol{\alpha}]^{\lambda_0}, \end{aligned} \quad (3.4)$$

where $\mathbf{1}$ denotes a column vector of length n with all elements equal to 1, and $\boldsymbol{\alpha} \equiv (\alpha_1, \dots, \alpha_n)^T$. It is evident that

$$\mathcal{U} = \mathbf{1}^T \boldsymbol{\alpha}, \quad \mathcal{F} = \boldsymbol{\alpha}^T \mathbf{Z} \boldsymbol{\alpha}, \quad (3.5)$$

where the Gram matrix $\mathbf{Z}$ is symmetric. For an irreducible sector, we have $\det(\mathbf{Z}) \neq 0$ (but the reverse is not true, as we will see in the “magic relations” discussed later).

In the following, we will aim for decreasing the power $\nu_l > 1$ of the variable α_l , where $l \in \{1, \dots, n\}$. For that, we will need to study the submatrix of $\mathbf{Z}$ obtained by removing the l th row and l th column. We will denote this submatrix as $\mathbf{Z}(\hat{l}, \hat{l})$, and refer to it as “the (l, l) -submatrix.” A closely related concept is the (i, j) -minor of $\mathbf{Z}$, which is the

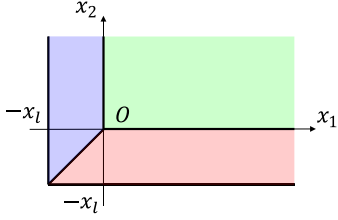


FIG. 4. The contour-splitting in two dimensions.

determinant of the submatrix of $\mathbf{Z}$ without the i th row and j th column. The (i, j) -cofactor is further defined by multiplying the (i, j) -minor with $(-1)^{i+j}$. We will denote the (i, j) -cofactor of $\mathbf{Z}$ as $Z_{i,j}$.

The (l, l) -submatrix can be either nonsingular with rank $n - 1$, or singular with a smaller rank. In the following, we will discuss the two situations separately.

1. When the (l, l) -submatrix is nonsingular

Similar to the idea of Eq. (2.31),³ we introduce an auxiliary integral

$$g(\nu_n, \mathbf{1}_n) \equiv C(\nu_n, \mathbf{1}_n) \int_0^\infty \left(\prod_{j=1}^n d\beta_j \right) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ \times \left\{ [\mathbf{1}^T(\boldsymbol{\alpha} + \boldsymbol{\beta}^{(l)} - q\boldsymbol{\beta}_l)]\alpha_0 + (\boldsymbol{\alpha} + \boldsymbol{\beta}^{(l)})^T \mathbf{Z}(\boldsymbol{\alpha} + \boldsymbol{\beta}^{(l)}) \right\}^{\lambda_0}, \quad (3.6)$$

where $\boldsymbol{\beta}^{(l)}$ denotes the vector obtained by replacing the l th element (i.e., β_l) of the vector $\boldsymbol{\beta} \equiv (\beta_1, \dots, \beta_n)^T$ with 0, and the elements of the vector $\mathbf{q}$ are given by

$$q_k \equiv -\frac{Z_{k,l}}{Z_{l,l}}, \quad (3.7)$$

where we recall that $Z_{i,j}$ is the (i, j) -cofactor of $\mathbf{Z}$.

Using the method from Sec. II B, we can extract β_l and integrate it out with an appropriate choice of $\mathcal{X}$ and $\mathcal{Y}$. We arrive at

$$g(\nu_n, \mathbf{1}_n) = \frac{1}{\nu_0 - 1} \left(-\frac{1}{q} \right) \sum_{j=1}^n [\hat{j}^+ \hat{l}^- I(\nu_n + \mathbf{1}_n)], \quad (3.8)$$

where $q \equiv \sum_{k=1}^n q_k$, and we have used Eq. (3.3) to absorb the extra β_j variables at the cost of increasing the powers of α_j . Note that ν_0 is defined in terms of the sequence of indices $(\nu_n, \mathbf{1}_n)$, as explained below Eq. (3.2).

On the other hand, Eq. (3.6) can be transformed in another way with the variable change

$$\beta_j \rightarrow \beta_j + q_j \beta_l, \quad (\forall j \neq l), \quad \beta_l \rightarrow \beta_l \left(1 - W \frac{\beta_l + 2\alpha_l}{\alpha_0} \right), \quad (3.9)$$

where

$$W \equiv -\frac{\det(\mathbf{Z})}{\sum_{j=1}^n Z_{j,l}} = \frac{\det(\mathbf{Z})}{q Z_{l,l}}. \quad (3.10)$$

This leads to

$$g(\nu_n, \mathbf{1}_n) = C(\nu_n, \mathbf{1}_n) \int_0^\infty d\beta_l \left(\prod_{j \neq l} \int_{-q_j \beta_l}^\infty d\beta_j \right) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ \times \left(1 - 2W \frac{\alpha_l + \beta_l}{\alpha_0} \right) [\mathbf{1}^T(\boldsymbol{\alpha} + \boldsymbol{\beta})\alpha_0 + (\boldsymbol{\alpha} + \boldsymbol{\beta})^T \mathbf{Z}(\boldsymbol{\alpha} + \boldsymbol{\beta})]^{\lambda_0}. \quad (3.11)$$

³As in Sec. II C, the kinematic invariants appearing in the derivation of this section are implicitly assumed to lie in an appropriate domain.

We can now split the integration domain of β_j . First, we introduce the contour-splitting identity

$$\prod_{j \neq l} \int_{-x_l}^{\infty} dx_j = \prod_{j \neq l} \int_0^{\infty} dx_j + \sum_{k \neq l} \int_{-x_l}^0 dx_k \prod_{j \neq l, k} \int_{x_k}^{\infty} dx_j, \quad (3.12)$$

whose geometric meaning is natural and straightforward. In two dimensions, it corresponds to the domain decomposition illustrated in Fig. 4. After a rescaling of the variables, we arrive at

$$g(\nu_n, \mathbf{1}_n) = I(\nu_n + \mathbf{1}_n) + \frac{2W}{\nu_0 - 1} \hat{l}^+ I(\nu_n + \mathbf{1}_n) + \sum_{k \neq l} h_k(\nu_n, \mathbf{1}_n), \quad (3.13)$$

where we have defined

$$\begin{aligned} h_k(\nu_n, \mathbf{1}_n) &\equiv C(\nu_n, \mathbf{1}_n) \int_0^{\infty} d\beta_l \int_{-q_k \beta_l}^0 d\beta_k \left(\prod_{j \neq l, k} \int_{q_j \beta_k / q_k}^{\infty} d\beta_j \right) \int_0^{\infty} \left(\prod_{j=0}^n \alpha_j^{\nu_j - 1} d\alpha_j \right) \\ &\times \left(1 - 2W \frac{\alpha_l + \beta_l}{\alpha_0} \right) [\mathbf{1}^T (\boldsymbol{\alpha} + \boldsymbol{\beta}) \alpha_0 + (\boldsymbol{\alpha} + \boldsymbol{\beta})^T \mathbf{Z} (\boldsymbol{\alpha} + \boldsymbol{\beta})]^{\lambda_0} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}). \end{aligned} \quad (3.14)$$

Our next task is to transform $h_k(\nu_n)$ into Feynman integrals. For that, we perform the variable change

$$\alpha_j \rightarrow \alpha_j \frac{\beta_l}{\beta_l + \frac{\beta_k}{q_k}}, \quad (\forall j), \quad \beta_k \rightarrow -\beta_k \frac{\beta_l}{\beta_l + \frac{\beta_k}{q_k}}, \quad \beta_j \rightarrow \left(\beta_j - q_j \frac{\beta_k}{q_k} \right) \frac{\beta_l}{\beta_l + \frac{\beta_k}{q_k}}, \quad (\forall j \neq l, k). \quad (3.15)$$

We can then write

$$\begin{aligned} h_k(\nu_n, \mathbf{1}_n) &= C(\nu_n, \mathbf{1}_n) \int_0^{\infty} d\beta'_k \int_0^{\infty} \left(\prod_{j \neq k} d\beta_j \right) \int_0^{\infty} \left(\prod_{j=0}^n \alpha_j^{\nu_j - 1} d\alpha_j \right) \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ &\times \left\{ \left[\mathbf{1}^T (\boldsymbol{\alpha} + \boldsymbol{\beta}^{(k)}) - q \frac{\beta'_k}{q_k} \right] \alpha_0 + (\boldsymbol{\alpha} + \boldsymbol{\beta}^{(k)})^T \mathbf{Z} (\boldsymbol{\alpha} + \boldsymbol{\beta}^{(k)}) \right\}^{\lambda_0}, \end{aligned} \quad (3.16)$$

where

$$\beta'_k \equiv \beta_k \left(1 - W \frac{2\alpha_l + 2\beta_l + \beta_k / q_k}{\alpha_0} \right). \quad (3.17)$$

Therefore,

$$h_k(\nu_n, \mathbf{1}_n) = \frac{1}{\nu_0 - 1} \left(-\frac{q_k}{q} \right) \sum_{j=1}^n [\hat{j}^+ \hat{k}^- I(\nu_n + \mathbf{1}_n)]. \quad (3.18)$$

Combining Eqs. (3.8), (3.13), and (3.18), we finally obtain

$$2 \det(\mathbf{Z}) \hat{l}^+ I(\nu_n) = \sum_{j=1}^n [Z_{j,l}(\nu_0 - \nu_j)] I(\nu_n) - \sum_{k=1}^n Z_{k,l} J_k(\nu_n), \quad (3.19)$$

where we have performed a simple shift of indices $\nu_i \rightarrow \nu_i - 1$, and

$$J_k(\nu_n) \equiv \sum_{j \neq k} \hat{j}^+ \hat{k}^- I(\nu_n). \quad (3.20)$$

In the context of integral reduction, we regard the integral which has smaller total positive indices as “simpler.” In this sense, it can be seen that the integrals appearing on the right-hand side of Eq. (3.19) are simpler than the one on the left-hand side. Since we are dealing with an irreducible sector, we have $\det(\mathbf{Z}) \neq 0$. Therefore, we can use Eq. (3.19) to recursively reduce a particular Feynman integral to simpler integrals.

2. When the (l, l) -submatrix is singular

In this case, the (l, l) -submatrix is noninvertible, which implies that $Z_{l,l} = 0$. Equation (3.7) then becomes singular. However, we find that the correct reduction rules can be obtained by taking the limit $Z_{l,l} \rightarrow 0$ in the final expression, Eq. (3.19). Therefore, in this case, we have

$$2 \det(\mathbf{Z}) \hat{l}^+ I(\nu_n) = \sum_{j \neq l} [Z_{j,l}(\nu_0 - \nu_j)] I(\nu_n) - \sum_{k \neq l} [Z_{k,l} J_k(\nu_n)], \quad (3.21)$$

which can be used to reduce $\hat{l}^+ I(\nu_n)$.

A special case occurs when the matrix $\mathbf{Z}$ has only one element, meaning that $\mathbf{Z}(\hat{l}, \hat{l})$ is an empty matrix. In this case, we have $\nu = \nu_1$ and $\nu_0 = d - \nu$. The procedure can still be carried out by defining $Z_{1,1} = 1$, and we obtain the correct reduction formula for single-propagator integrals:

$$I(\nu + 1) = \frac{d - 2\nu}{2\nu \det(\mathbf{Z})} I(\nu). \quad (3.22)$$

After the variable change, an integral can be written as

$$\begin{aligned} I(\nu_n) &= C(\nu_n) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) [\alpha_0 \mathcal{U} + \mathcal{F}]^{\lambda_0} \\ &= C(\nu_n) \int_0^\infty \alpha_l^{\nu_l-1} d\alpha_l \left(\prod_{j \neq 0, l} \int_{-\xi_j \alpha_l}^\infty (\alpha_j + \xi_j \alpha_l)^{\nu_j-1} d\alpha_j \right) \\ &\quad \times \int_0^\infty d\alpha_0 \alpha_0^{\nu_0-1} [(\mathcal{U}_{l,0} + \xi \alpha_l) \alpha_0 + \mathcal{F}_{l,0}]^{\lambda_0} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}), \end{aligned} \quad (3.25)$$

where $\xi \equiv \sum_{j=1}^n \xi_j$, $\mathcal{F}_{l,0} \equiv \mathcal{F}|_{\alpha_l \rightarrow 0}$ and we recall that $\mathcal{U}_{l,0} \equiv \mathcal{U}|_{\alpha_l \rightarrow 0}$. We can again split the integration domain using Eq. (3.12) and perform a variable rescaling similar to that leading to Eq. (3.13). We define

$$\begin{aligned} G_l(\nu_n) &\equiv C(\nu_n) \int_0^\infty d\alpha_l \alpha_l^{\nu_l-1} \int_0^\infty d\alpha_0 \alpha_0^{\nu_0-1} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ &\quad \times \int_0^\infty \left(\prod_{j \neq 0, l} d\alpha_j \left(\alpha_j + \xi_j \frac{\alpha_l}{\xi_l} \right)^{\nu_j-1} \right) \left[\left(\mathcal{U}_{l,0} + \xi \frac{\alpha_l}{\xi_l} \right) \alpha_0 + \mathcal{F}_{l,0} \right]^{\lambda_0} \end{aligned} \quad (3.26)$$

C. The reducible sectors

We now turn to the case where $\det(\mathbf{Z}) = 0$, which corresponds to reducible sectors according to the discussions in Sec. II B. Note that this criterion is equivalent to those found in earlier literature [84]. The Gram matrix $\mathbf{Z}$ is singular in this case, and it implies that the degrees of freedom in the polynomial $\mathcal{F}$ are fewer than the number of variables. In other words, we can find a vector $\xi = (\xi_1, \dots, \xi_n)$ with $\xi_l = 1$, such that

$$\mathcal{F}(\alpha_1 + \xi_1 \alpha_l, \dots, \alpha_l, \dots, \alpha_n + \xi_n \alpha_l) = \mathcal{F}(\alpha_1, \dots, 0, \dots, \alpha_n). \quad (3.23)$$

The vector ξ is in the kernel of $\mathbf{Z}$ —i.e., $\mathbf{Z}\xi = 0$.

1. Reduction for general reducible sectors

In Sec. II B, we have discussed a special case where the $\mathcal{F}$ polynomial already takes the form of the right-hand side of Eq. (3.23), without introducing the transformation induced by ξ . In other words, it corresponds to the case where all elements in ξ , except $\xi_l = 1$, are zero. In that case, integrals in this sector can be reduced to a single subsector. More generically, we need to apply the variable change

$$\alpha_j \rightarrow \alpha_j + \xi_j \alpha_l, \quad (\forall j \neq 0, l) \quad (3.24)$$

to transform the $\mathcal{F}$ polynomial into the desired form. Consequently, the integrals will be reduced to linear combinations of different subsectors, as we will demonstrate in the following.

and

$$H_k(\nu_n) \equiv C(\nu_n) \int_0^\infty d\alpha_l \alpha_l^{\nu_l-1} \int_{-\xi_k \alpha_l}^0 d\alpha_k (\alpha_k + \xi_k \alpha_l)^{\nu_k-1} \int_0^\infty d\alpha_0 \alpha_0^{\nu_0-1} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ \times \left(\prod_{j \neq 0, l, k} \int_{\xi_j \alpha_k / \xi_k}^\infty d\alpha_j (\alpha_j + \xi_j \alpha_l)^{\nu_j-1} \right) [(\mathcal{U}_{l,0} + \xi \alpha_l) \alpha_0 + \mathcal{F}_{l,0}]^{\lambda_0}. \quad (3.27)$$

Let S_ξ be the subset of $\{1, \dots, n\}$ such that $\xi_k \neq 0$ if $k \in S_\xi$; we can then write

$$I(\nu_n) = G_l(\nu_n) + \sum_{k \in S_\xi / \{l\}} H_k(\nu_n). \quad (3.28)$$

We can further show that $H_k(\nu_n)$ can actually be rewritten as $G_k(\nu_n)$. For that, we introduce the variable change

$$\alpha_0 \rightarrow \alpha_0 \frac{\alpha_l}{\alpha_l - \alpha_k / \xi_k}, \quad \alpha_k \rightarrow -\xi_k \alpha_l \frac{\alpha_l}{\alpha_l - \alpha_k / \xi_k}, \\ \alpha_j \rightarrow (\alpha_j - \xi_j \alpha_l) \frac{\alpha_l}{\alpha_l - \alpha_k / \xi_k}, \quad (\forall j \neq 0, k, l), \quad (3.29)$$

and therefore

$$H_k(\nu_n) = C(\nu_n) \int_0^\infty d\alpha_k \alpha_k^{\nu_k-1} \int_0^\infty d\alpha_0 \alpha_0^{\nu_0-1} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ \times \int_0^\infty \left(\prod_{j \neq 0, k} d\alpha_j \left(\alpha_j + \xi_j \frac{\alpha_k}{\xi_k} \right)^{\nu_j-1} \right) \left[\left(\mathcal{U}_{k,0} + \xi \frac{\alpha_k}{\xi_k} \right) \alpha_0 + \mathcal{F}_{k,0} \right]^{\lambda_0}. \quad (3.30)$$

Using the method in Sec. II B, we can reduce $G_k(\nu_n)$ (for both $k = l$ and $k \neq l$) to integrals in a subsector. Combining these results, we arrive at the reduction of $I(\nu_n)$:

$$I(\nu_n) = \sum_{k \in S_\xi} \frac{(\xi_k)^{\nu_k}}{(\nu_k - 1)!} \hat{k}_0 \hat{P}_k I(\nu_n), \quad (3.31)$$

where the operator $\hat{P}_k$ is a combination of index-raising and -lowering operators:

$$\hat{P}_k = \left(\prod_{j \in S_\xi} \sum_{\mu_j=0}^{\nu_j-1} \right) \left[B(-\nu_0 + 1, \mu + \nu_k) \left(\frac{\sum_{i \neq k} \hat{i}^+}{-\xi} \right)^{\mu + \nu_k} \prod_{m \in S_\xi}^{m \neq k} \frac{(\xi_m \hat{m}^-)^{\mu_m}}{\mu_m!} \right], \quad (3.32)$$

where $\mu \equiv \sum_{j \in S_\xi}^{j \neq k} \mu_j$. Obviously, the above formula only applies if $\xi \neq 0$. The case for $\xi = 0$ will be discussed in the following.

2. Degenerate limits

The case $\xi = 0$ corresponds to certain degenerate limits of Feynman integrals. As a specific example, we consider the bubble diagram shown in Fig. 3 with $m_1^2 = m_2^2$ and $s = 0$. Here, we find that $\xi = \{1, -1\}$, and hence $\xi = 0$. We now discuss how to deal with this kind of situation.

When $\xi = 0$, both $\mathcal{U}$ and $\mathcal{F}$ satisfy

$$\mathcal{O}(\alpha_1, \dots, \alpha_l, \dots, \alpha_n) = \mathcal{O}(\alpha_1 - \xi_1 \alpha_l, \dots, 0, \dots, \alpha_n - \xi_n \alpha_l) \quad (3.33)$$

for certain l , where $\mathcal{O} \in \{\mathcal{U}, \mathcal{F}\}$. Therefore, $\alpha_0 \mathcal{U} + \mathcal{F}$ satisfies the above property as well. We now consider the integral $I(\nu_n + \mathbf{1}_n)$. By introducing the β variables as in Sec. III A, we can write it as

$$I(\nu_n, \mathbf{1}_n) = C(\nu_n, \mathbf{1}_n) \int_0^\infty \left(\prod_{j=1}^n d\beta_j \right) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \mathcal{G}_0(\boldsymbol{\alpha}, \boldsymbol{\beta})^{\lambda_0} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}), \quad (3.34)$$

where

$$\mathcal{G}_0(\boldsymbol{\alpha}, \boldsymbol{\beta}) \equiv [\alpha_0 \mathcal{U} + \mathcal{F}]_{\boldsymbol{\alpha} \rightarrow \boldsymbol{\alpha} + \boldsymbol{\beta}} = [\mathbf{1}^T(\boldsymbol{\alpha} + \boldsymbol{\beta})] \alpha_0 + (\boldsymbol{\alpha} + \boldsymbol{\beta})^T \mathbf{Z}(\boldsymbol{\alpha} + \boldsymbol{\beta}). \quad (3.35)$$

It follows from the property of $\mathcal{U}$ and $\mathcal{F}$ that

$$\mathcal{G}_0(\boldsymbol{\alpha}, \beta_1, \dots, \beta_l, \dots, \beta_n) = \mathcal{G}_0(\boldsymbol{\alpha}, \beta_1 - \xi_l \beta_l, \dots, 0, \dots, \beta_n - \xi_n \beta_l). \quad (3.36)$$

We now show that this property can be used for reducing the integral to subsectors.

For convenience, we consider a function $g(y, \mathbf{x}_{n-1})$ which satisfies

$$g(y, \mathbf{x}_{n-1}) = g(0, \mathbf{x}_{n-1} + y \boldsymbol{\eta}_{n-1}), \quad (3.37)$$

where $\mathbf{x}_{n-1}$ denotes the sequence of $n-1$ variables, and $\boldsymbol{\eta}_{n-1}$ is the sequence of $n-1$ constants. Using the relation

$$\int_0^\infty dx_l \int_{x_l}^\infty \left(\prod_{j \neq l} dx_j \right) = \sum_{k \neq l} \int_0^\infty dx_k \int_{x_k}^\infty \left(\prod_{j \neq k, l} dx_j \right) \int_0^{x_k} dx_l, \quad (3.38)$$

we can show that

$$\begin{aligned} \int_0^\infty dy \int_0^\infty \left(\prod_{j=1}^{n-1} dx_j \right) g(y, \mathbf{x}_{n-1}) &= \int_0^\infty dy \left(\prod_{j=1}^{n-1} \int_{y\eta_j}^\infty dx'_j \right) g(0, \mathbf{x}'_{n-1}) \\ &= \sum_{k=1}^{n-1} \int_0^\infty dx'_k \left(\prod_{j=1}^{n-1} \int_{\eta_j x'_k / \eta_k}^\infty dx'_j \right) \int_0^{x'_k} dy g(0, \mathbf{x}'_{n-1}) \\ &= \sum_{k=1}^{n-1} \eta_k \int_0^\infty dY \int_0^\infty \left(\prod_{j \neq k} dX_j \right) Y g(Y, \mathbf{X}_{n-1}), \end{aligned} \quad (3.39)$$

where $\mathbf{x}'_{n-1} \equiv \mathbf{x}_{n-1} + y \boldsymbol{\eta}_{n-1}$, $Y \equiv x'_k / \eta_k = x_k / \eta_k + y$, and $X_j \equiv x'_j - \eta_j x'_k / \eta_k = x_j - \eta_j x_k / \eta_k$. Note that in the k th term of the above result, we have $X_k = 0$ in g , and we do not have to integrate over X_k . Hence, we have eliminated one variable at the cost of increasing the power of Y .

Applying Eq. (3.39) to $I(\nu_n, \mathbf{1}_n)$, we get

$$\begin{aligned} I(\nu_n, \mathbf{1}_n) &= - \sum_{k \neq l} \xi_k C(\nu_n, \mathbf{1}_n) \int_0^\infty \left(\prod_{j \neq k} d\beta_j \right) \int_0^\infty \left(\prod_{j=0}^n \alpha_j^{\nu_j-1} d\alpha_j \right) \\ &\quad \times \beta_l \mathcal{G}_0(\boldsymbol{\alpha}, \boldsymbol{\beta}^{(k)})^{\lambda_0} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}). \end{aligned} \quad (3.40)$$

Absorbing the extra β_j variables using Eq. (3.3), we finally obtain the reduction formula that works in the degenerate limits:

$$I(\dots, \nu_l, \dots) = - \sum_{k \neq l} \xi_k I(\dots, \nu_k - 1, \dots, \nu_l + 1, \dots). \quad (3.41)$$

It can be observed that by fixing l , we can recursively decrease other indices at the cost of increasing ν_l . This eventually leads to integrals in the subsectors.

Now, let us consider the example mentioned earlier: the bubble diagram with $m_1^2 = m_2^2$ and $s = 0$. By choosing $l = 1$ and repeatedly applying Eq. (3.41), we can arrive at

$$I(\nu_1, \nu_2) = I(\nu_1 + \nu_2, 0). \quad (3.42)$$

Up to now, we have exhausted all possibilities and have presented iterative formulas to reduce any one-loop integral with indices larger than 1 to linear combinations of simpler integrals. The remaining task is to reduce the negative indices corresponding to numerators in the momentum representation [Eq. (2.1)].

D. Reduction of integrals with numerators

If any index $\nu_j < 0$, the corresponding D_j is in the numerator of Eq. (2.1). Such integrals can always be reduced to integrals with $\nu_j = 0$. In the following, we show how to achieve this reduction within our approach.

1. Rewriting the integrals in shifted dimensions

It is well known that integrals with $\nu_j < 0$ can be rewritten as integrals with $\nu_j = 0$ in shifted spacetime dimensions. This can be easily seen in the LP representation:

$$I^{(d)}(\nu_n) = C^{(d)}(\nu_n) \int_0^\infty \mathcal{G}^{-d/2} \prod_{j=1}^n \alpha_j^{\nu_j-1} d\alpha_j, \quad (3.43)$$

where $\hat{\mathcal{O}}_j^+$ (for $\mathcal{O} \in \{\mathcal{U}, \mathcal{F}\}$) represents an operator obtained by substituting each α_k in $(\partial\mathcal{O}/\partial\alpha_j)$ with $\hat{k}^+$. Note that Eq. (3.45) holds for general L -loop integrals, and for the purpose of this work, we can set $L = 1$.

With Eq. (3.45), we can recursively increase a negative index to zero, at the cost of employing integrals in shifted dimensions. Let us consider a simple example: $I(-1, 2)$ from the bubble integral family in Fig. 3. Applying Eq. (3.45) with $j = 1$, we obtain

$$I^{(d)}(-1, 2) = (1-d)I^{(d+2)}(0, 2) + 2(m_1^2 + m_2^2 - s)I^{(d+2)}(0, 3). \quad (3.46)$$

2. Dimensional recurrence relations

We now need to transform the integrals in shifted dimensions to the d -dimensional ones. These are the so-called dimensional recurrence relations [85–88]. We show how such relations can be naturally derived within our approach.

In the Feynman parametrization, it is easy to increase the dimension:

where $\mathcal{G} \equiv \mathcal{U} + \mathcal{F}$. We have used the superscript (d) to specify the dependence on the spacetime dimension.

If $\nu_j < 0$, we can increase ν_j with a simple integration by parts:

$$\begin{aligned} & \int_0^\infty d\alpha_j \frac{\alpha_j^{\rho+\nu_j-1}}{\Gamma(\rho+\nu_j)} \mathcal{G}^{-d/2} \\ &= \frac{d}{2} \int_0^\infty d\alpha_j \frac{\alpha_j^{\rho+\nu_j}}{\Gamma(\rho+\nu_j+1)} \left(\frac{\partial \mathcal{G}}{\partial \alpha_j} \right) \mathcal{G}^{-(d+2)/2}, \end{aligned} \quad (3.44)$$

where we have explicitly introduced the regulator ρ as discussed in Sec. II A. One may see that the right-hand side of the above equation corresponds to integrals in $d+2$ dimensions, but with an increased ν_j index. Note that $\mathcal{U}$ and $\mathcal{F}$ are homogeneous polynomials of degree L and $L+1$, respectively. The above relation can be rewritten in terms of index-raising operators as

$$I^{(d)}(\dots, \nu_j, \dots) = (-1)^L \left[\left((L+1) \frac{d}{2} - \sum_{i=1}^n \nu_i \right) \hat{\mathcal{U}}_j^+ - \hat{\mathcal{F}}_j^+ \right] I^{(d+2)}(\dots, \nu_j + 1, \dots), \quad (3.45)$$

$$I^{(d)}(\nu_n) = - \sum_{k=1}^n \hat{k}^+ I^{(d+2)}(\nu_n). \quad (3.47)$$

We need to find the inverse relation, to express $I^{(d+2)}$ in terms of $I^{(d)}$. Note that the indices here are all non-negative. Since we can apply the reduction rules that we have constructed in both d and $d+2$ dimensions, we only need to find the relations among corner integrals. Combining the above equation with Eq. (3.19), we obtain

$$\begin{aligned} 2 \det(\mathbf{Z}) I^{(d)}(\mathbf{1}_n) &= -2 \det(\mathbf{Z}) \sum_{l=1}^n \hat{l}^+ I^{(d+2)}(\mathbf{1}_n) \\ &= \zeta(n-1-d) I^{(d+2)}(\mathbf{1}_n) - \sum_{k=1}^n \zeta_k \hat{k}_0 I^{(d)}(\mathbf{1}_n), \end{aligned} \quad (3.48)$$

where $\zeta_k \equiv \sum_{j=1}^n Z_{k,j}$, $\zeta \equiv \sum_{k=1}^n \zeta_k$, and we have used

$$\sum_{j \neq k} \hat{j}^+ \hat{k}_0 I^{(d+2)}(\mathbf{1}_n) = -\hat{k}_0 I^{(d)}(\mathbf{1}_n). \quad (3.49)$$

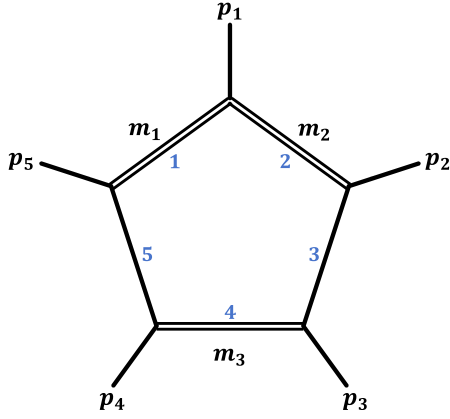


FIG. 5. The pentagon diagram with three masses.

If $\zeta \neq 0$, Eq. (3.48) gives rise to the dimension recurrence relation that we will need:

$$I^{(d+2)}(\mathbf{1}_n) = \frac{2 \det(\mathbf{Z})}{\zeta(n-1-d)} I^{(d)}(\mathbf{1}_n) + \frac{1}{\zeta(n-1-d)} \sum_{k=1}^n \zeta_k \hat{k}_0 I^{(d)}(\mathbf{1}_n). \quad (3.50)$$

Combining with Eq. (3.45), we complete the reduction of integrals with numerators.

Applying the above relation, we can continue the reduction of $I(-1, 2)$ starting from Eq. (3.46). We first reduce the right-hand side of Eq. (3.46) to corner integrals in $d+2$ dimensions, and we obtain

$$I^{(d)}(-1, 2) = -\frac{d}{2m_2^2} \frac{2m_2^2 - (d-2)(m_1^2 - m_2^2 - s)}{2m_2^2} I^{(d+2)}(0, 1). \quad (3.51)$$

We then use Eq. (3.50) to derive (recall that we have defined $Z_{1,1} = 1$ when the matrix $\mathbf{Z}$ has only one element)

$$I^{(d+2)}(0, 1) = -\frac{2m_2^2}{d} I^{(d)}(0, 1). \quad (3.52)$$

Finally, we have

$$I^{(d)}(-1, 2) = \frac{2m_2^2 - (d-2)(m_1^2 - m_2^2 - s)}{2m_2^2} I^{(d)}(0, 1). \quad (3.53)$$

Note that if $\zeta = 0$, Eq. (3.48) does not lead to a dimension recurrence, but rather gives rise to an interesting reduction rule of the corner integral to subsectors:

$$I^{(d)}(\mathbf{1}_n) = -\frac{1}{2 \det(\mathbf{Z})} \sum_{k=1}^n \zeta_k \hat{k}_0 I^{(d)}(\mathbf{1}_n). \quad (3.54)$$

This is the so-called “magic relation.” In the usual IBP reduction, it can only be observed when IBP relations involving certain supersectors are included [11,89]. For example, for the bubble diagram in Fig. 3, we find $\zeta = s$. Therefore, when $s = 0$, the integral $I(1, 1)$ can be reduced to subsectors:

$$I^{(d)}(1, 1) = \frac{1}{m_1^2 - m_2^2} [I^{(d)}(1, 0) - I^{(d)}(0, 1)]. \quad (3.55)$$

In KIRA, this reduction rule can only be found when embedding the bubble integrals in a triangle supersector.

IV. EXAMPLES

A. One-loop integrals

The recursive formulas derived in the previous section allow a straightforward computer implementation. We have written a proof-of-concept *Mathematica* code to automatically reduce one-loop integrals. We have tested the program with various examples and confirmed its correctness.

As a more complicated example, consider the pentagon integral family with three masses shown in Fig. 5. We set all external momenta to be lightlike. The kinematic variables are m_1^2 , m_2^2 , m_3^2 , $(p_1 + p_2)^2 = s_{12}$, $(p_2 + p_3)^2 = s_{23}$, $(p_3 + p_4)^2 = s_{34}$, $(p_4 + p_5)^2 = s_{45}$, and $(p_5 + p_1)^2 = s_{51}$. The matrix $\mathbf{Z}$ in this family is

$$\begin{pmatrix} m_1^2 & \frac{1}{2}(m_1^2 + m_2^2) & \frac{1}{2}(m_1^2 - s_{12}) & \frac{1}{2}(m_1^2 + m_3^2 - s_{45}) & \frac{1}{2}m_1^2 \\ \frac{1}{2}(m_1^2 + m_2^2) & m_2^2 & \frac{1}{2}m_2^2 & \frac{1}{2}(m_2^2 + m_3^2 - s_{23}) & \frac{1}{2}(m_2^2 - s_{51}) \\ \frac{1}{2}(m_1^2 - s_{12}) & \frac{1}{2}m_2^2 & 0 & \frac{1}{2}m_3^2 & -\frac{1}{2}s_{34} \\ \frac{1}{2}(m_1^2 + m_3^2 - s_{45}) & \frac{1}{2}(m_2^2 + m_3^2 - s_{23}) & \frac{1}{2}m_3^2 & m_3^2 & \frac{1}{2}m_3^2 \\ \frac{1}{2}m_1^2 & \frac{1}{2}(m_2^2 - s_{51}) & -\frac{1}{2}s_{34} & \frac{1}{2}m_3^2 & 0 \end{pmatrix}. \quad (4.1)$$

We consider the reduction of $I(1, 2, 1, 1, 1)$. Since $\mathbf{Z}$ is nonsingular, we should apply Eq. (3.19) with $\nu_n = (1, 1, 1, 1, 1)$ and $l = 2$, which leads to

$$I(1, 2, 1, 1, 1) = (d-6) \frac{\det(\mathbf{M})}{2 \det(\mathbf{Z})} I(1, 1, 1, 1, 1) + \text{subsector integrals}, \quad (4.2)$$

where $\mathbf{M}$ represents the matrix formed by replacing all elements in the second row of $\mathbf{Z}$ with 1. The subsector integrals in the above expression may still contain integrals that need to be reduced [e.g., $I(2, 1, 1, 1, 0)$, $I(0, 1, 2, 1, 1)$, etc.]. We can apply the relevant reduction formulas recursively until only master integrals remain. We have verified that the results agree with those provided by KIRA.

We can also consider a reducible case, with $m_1^2 = m_2^2 = m_3^2 = s_{12} = 0$. At this kinematic point, the $\mathbf{Z}$ matrix

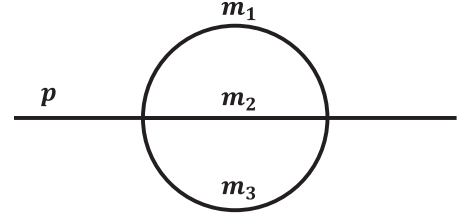


FIG. 6. The sunrise diagram with $p^2 = s$.

is singular. Therefore, we need to apply Eq. (3.31) when performing the reduction of $I(1, 2, 1, 1, 1)$. The ξ vector can be derived as

$$\xi = \left\{ -\frac{s_{23}}{s_{45}}, 1, -\frac{s_{51}}{s_{34}}, 0, 0 \right\}. \quad (4.3)$$

It follows that $I(1, 2, 1, 1, 1)$ can be reduced to integrals in three subsectors:

$$\begin{aligned} I(1, 2, 1, 1, 1) = & \frac{1}{(d-7)(d-8)} \frac{2s_{34}^2 s_{45}^2}{(s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{51})^2} [I(2, 0, 2, 1, 1) + I(2, 0, 1, 2, 1) \\ & + I(2, 0, 1, 1, 2) + I(1, 0, 2, 2, 1) + I(1, 0, 2, 1, 2) + I(1, 0, 1, 2, 2) \\ & + I(3, 0, 1, 1, 1) + I(1, 0, 3, 1, 1) + I(1, 0, 1, 3, 1) + I(1, 0, 1, 1, 3)] \\ & + \text{integrals in the other two subsectors}, \end{aligned} \quad (4.4)$$

where the other two subsectors are those where the first or the third propagator is removed [e.g., $I(0, 1, 1, 2, 2)$, $I(1, 1, 0, 2, 2)$, etc.].

Finally, we consider a degenerate limit, where $p_1 = p_2$, in addition to $m_1^2 = m_2^2 = m_3^2 = 0$. In this case, we have $s_{12} = 0$, $s_{23} = s_{45}/2$, and $s_{51} = s_{34}/2$. Clearly, the top sector is reducible, but we cannot apply Eq. (3.31) because $\xi = \{1, -2, 1, 0, 0\}$, which implies $\xi = 0$. Therefore, in this case, we need to apply Eq. (3.41). By choosing $l = 1$, we ultimately obtain

$$I(1, 2, 1, 1, 1) = 4I(3, 0, 1, 1, 1) - 2I(3, 1, 0, 1, 1) - I(2, 2, 0, 1, 1). \quad (4.5)$$

B. A two-loop example

Given the success in one-loop problems, it is natural to extend our approach to multi-loop integrals. A new ingredient at higher loops is that there can be more than one MI within a sector, and it is no longer guaranteed that all indices can be reduced to 1 or 0. Correspondingly, in our approach, we find that when we perform the variable changes, certain variables may appear in the denominators of the transformed integrands. Such transformations are therefore only valid if the corresponding indices are greater than 1. In addition, the structure of degenerate limits is also much more complicated than the one-loop case.

As a simple example, we consider the massless sunrise family shown in Fig. 6, with $m_1 = m_2 = m_3 = 0$. The integrals in this family can be represented as

$$\begin{aligned} I(\nu_1, \nu_2, \nu_3) = & C(\nu_1, \nu_2, \nu_3) \int_0^\infty d\alpha_0 d\alpha_1 d\alpha_2 d\alpha_3 \alpha_0^{\nu_0-1} \alpha_1^{\nu_1-1} \alpha_2^{\nu_2-1} \alpha_3^{\nu_3-1} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \\ & \times [(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) \alpha_0 - \alpha_1 \alpha_2 \alpha_3 s]^{\lambda_0}. \end{aligned} \quad (4.6)$$

Following the method in Sec. III B, we define the auxiliary integral

$$g(\nu_1, \nu_2, \nu_3) = C(\nu_1, \nu_2, \nu_3) \int_0^\infty d\alpha_0 d\alpha_1 d\alpha_2 d\alpha_3 \alpha_0^{\nu_0-1} \alpha_1^{\nu_1-1} \alpha_2^{\nu_2-1} \alpha_3^{\nu_3-1} \mathcal{X} \delta(\mathcal{X} - \mathcal{Y}) \times [(\alpha_1 \alpha_3 + \alpha_2 \alpha_3) \alpha_0]^{\lambda_0}, \quad (4.7)$$

which is itself a scaleless integral and should therefore vanish. By performing a variable change $\alpha_1 \rightarrow \alpha_1 (1 + \alpha_2/\alpha_3 - s\alpha_2/\alpha_0)$, we can express $g(\nu_1, \nu_2, \nu_3)$ in terms of integrals in the top sector, thus establishing a reduction relation for top-sector integrals. Unlike the case in one-loop integrals, the variable change now introduces the variable α_3 in the denominator. As a result, the auxiliary integral must have $\nu_3 \geq 2$ in order to yield a valid reduction relation. For instance, from the auxiliary integral $g(1, 1, 2)$, we obtain the correct reduction relation

$$I(1, 2, 2) = \frac{10 - 3d}{2s} [I(1, 2, 1) + I(1, 1, 2)]. \quad (4.8)$$

From IBP reduction, we know that $I(1, 2, 1)$ and $I(1, 1, 2)$ can be further reduced to $I(1, 1, 1)$. Such relations cannot be obtained through the auxiliary integral (4.7). In fact, this is one of the degenerate limits, which deserves a systematic investigation in the future. Furthermore, in the above example, we have only applied variable transformations similar to the one-loop method. There is still significant flexibility in the integration contours yet to be fully explored.

V. SUMMARY AND OUTLOOK

In this paper, we have explored interesting properties of the Feynman parametrization that are useful for Feynman integral reduction without employing IBP relations. In particular, we have derived an extension of the Cheng-Wu theorem, such that the delta function in the Feynman parametrization can take a more general form. This can be regarded as a specific implementation of the equivalence of integration contours, which is related to the twisted homology of Feynman integrals. Leveraging these properties, we have derived universal recursive formulas for the reduction of one-loop integrals in both irreducible and reducible sectors, and our method works in degenerate limits as well. The reduction relations can be easily implemented in a computer program. We have applied them to various examples and observed remarkable performance.

We have also demonstrated the validity of our method at two loops. In an explicit two-loop example, the straightforward application of the one-loop method can yield the correct reduction relations. However, we also emphasize that not all reduction relations can be obtained in this way, since additional structures arise at higher loops. This warrants further investigation, including more ways to perform variable changes and contour deformations. In particular, our manipulations strongly depend on the structure of the matrix $\mathbf{Z}$, which is a Gram matrix of external momenta, with entries being kinematic invariants. Its determinant, $\det(\mathbf{Z})$, corresponds to a Landau singularity [90] and appears in the denominator of the reduction coefficients. The relation between Landau singularities and reduction coefficients has been studied in, e.g., Ref. [91]. These singularities can be obtained systematically with dedicated packages such as BAIKOVLETTER [92] or SOFIA [93]. It remains to be investigated whether the knowledge of the Landau singularities can help us to design variable transformations for integral reduction.

Furthermore, in this work we have only exploited the equivalence of integration contours in a simple way. There can be deeper mathematical structures behind these equivalence relations. For example, it is possible to employ the concept of intersection numbers between two integration contours to directly compute the reduction coefficients. This provides a particularly intriguing future perspective.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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