

# SIDIS at small $x$ at next-to-leading order: Transverse photon

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We calculate the next-to-leading order corrections to single inclusive hadron production in deep inelastic scattering at small  $x$  using the color glass condensate formalism, for the case when the exchanged photon is transversely polarized. We show all UV and soft divergences cancel while collinear and rapidity divergences result in scale evolution of quark-hadron fragmentation function and small- $x$  evolution of dipole amplitudes, respectively.

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## I. INTRODUCTION

The color glass condensate (CGC) (see [1–3] for reviews and references therein) is an effective field theory that describes the high-energy, small- $x$  regime of quantum chromodynamics (QCD), where gluon densities become very large and nonlinear effects dominate. In this regime, the rapid growth of gluon distributions provided by linear QCD evolution is expected to be tamed by gluon recombination effects, leading to the phenomenon of gluon saturation. The CGC formalism provides a framework to study these nonlinear dynamics by describing the high-energy hadron in terms of strong classical color fields. This approach has been instrumental in understanding various phenomena observed in high-energy collisions, including deep inelastic scattering (DIS) at the Hadron-Electron Ring Accelerator (HERA) and proton-nucleus and heavy-ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). By incorporating nonlinear evolution equations, such as the Balitsky-Kovchegov (BK) [4–6] and Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) [7–15] equations, the CGC framework offers a systematic way to explore the transition from a dilute to a saturated system

providing crucial insights into the initial conditions of heavy-ion collisions.

The search for gluon saturation and the verification of its role in high-energy QCD dynamics are among the central objectives of current and future collider experiments. Presently, measurements at RHIC and the LHC have provided tantalizing hints of saturation effects in proton-nucleus and nucleus-nucleus collisions, such as the suppression of forward hadron production and the modification of dihadron correlations. However, definitive confirmation of saturation physics requires more precise experimental data and theoretical advancements. Future high-energy facilities, such as the Electron-Ion Collider (EIC) in the United States and the proposed Large Hadron Electron Collider (LHeC) at CERN, will offer unprecedented opportunities to explore the small- $x$  regime with DIS on heavy nuclei, providing a cleaner and more controlled environment to probe saturation phenomena. In order to fully utilize these experimental programs, the precision of the theoretical calculations have to be increased.

Within the CGC framework, DIS related observables are usually studied using the dipole factorization [16,17]. In this factorization framework, the incoming virtual photon splits into a quark-antiquark pair which then scatters off the dense target. In order to improve the theoretical precision, next-to-leading order (NLO) corrections in strong coupling constant  $\alpha_s$  to DIS related observables have been studied for almost a decade now. Inclusive DIS has been computed at NLO with massless quarks in [18–24] and its fits to HERA data were performed in [25]. The same study was

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also performed for massive quarks in [26–28] with its fit to HERA data provided in [29]. Structure functions in diffractive DIS have been computed at NLO for massless quarks in [30,31]. Apart from the studies that focus on the NLO corrections to the structure functions, inclusive dijet [32–36] and dihadron [37–42] as well as diffractive dijet [43,44] and dihadron [45] production in DIS have been computed at NLO. The computations performed to study the production cross section of dihadrons/dijets at NLO allows one to obtain the NLO cross section for single inclusive hadron and jet production by integrating over one of the produced hadrons/jets. Inclusive and diffractive single hadron and single jet production have been studied both in general kinematics [46–49] and in the large virtuality limit [50–52] at NLO. Apart from the aforementioned studies on inclusive and diffractive jet/hadron production at NLO, various different aspects of single and multijet/hadron production have been studied within the CGC framework [53–69].

In this manuscript, we study the single inclusive hadron production in DIS via transverse photon exchange at NLO starting from the NLO dihadron production cross section. We integrate out the antiquark and let the produced quark fragment into a hadron. The manuscript is organized as follows. In Sec. II, we briefly summarize the leading-order

cross section both for double and single inclusive production in DIS at partonic level. In Sec. III, we compute the NLO corrections to the single inclusive quark production cross section in DIS via transverse photon exchange. Section IV is devoted to the discussion of the cancellation of the divergences that appear at NLO and evolution of the fragmentation functions. Finally, in Sec. V, we summarize our results and provide an outlook. In Appendix A, we provide the quark-antiquark production cross section in DIS via transverse photon at NLO and at finite  $N_c$ . In Appendix B, the  $z \rightarrow 0$  limit of the single inclusive quark production in DIS for the transversely polarized photon is presented. These results are provided at finite  $N_c$  and used to extract the rapidity divergences.

## II. LEADING-ORDER PARTONIC CROSS SECTIONS

In the small  $x$  limit of DIS and in the target rest frame the virtual photon (transverse or longitudinal) splits into a quark-antiquark pair, which then multiply scatters off the target hadron or nucleus. At leading order, the partonic production cross section of a quark with transverse momentum  $\mathbf{p}$ , rapidity  $y_1$ , and an antiquark with transverse momentum  $\mathbf{q}$  and rapidity  $y_2$  reads [70]

$$\frac{d\sigma^{\gamma^* A \rightarrow q\bar{q}X}}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = \frac{e^2 Q^2 N_c}{(2\pi)^7} \delta(1 - z_1 - z_2)(z_1 z_2)^2 \int d^8\mathbf{x} [S_{122'1'} - S_{12} - S_{2'1'} + 1] e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} e^{i\mathbf{q} \cdot \mathbf{x}_{2'2}} \times \left\{ 4z_1 z_2 K_0(|\mathbf{x}_{12}|Q_1) K_0(|\mathbf{x}_{1'2'}|Q_1) + (z_1^2 + z_2^2) \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2'}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2'}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2'}|Q_1) \right\} \quad (1)$$

where the first and second terms in the curly brackets correspond to the contribution of the longitudinal and transverse polarizations of the incoming photon with virtuality  $Q^2$ . Here,  $K_\alpha(\dots)$  is the modified Bessel function of the second type. We have also defined

$$Q_i = Q\sqrt{z_i(1 - z_i)}, \quad \mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \quad d^8\mathbf{x} = d^2\mathbf{x}_1 d^2\mathbf{x}_2 d^2\mathbf{x}_{1'} d^2\mathbf{x}_{2'}. \quad (2)$$

The production cross section is a convolution of the probability for a photon to split into a quark at transverse position  $\mathbf{x}_1$  and an antiquark at position  $\mathbf{x}_2$  represented by the Bessel functions, with the probability for this quark-antiquark pair to multiply scatter from the target which is encoded in the dipoles  $S_{ij}$  and quadrupoles  $S_{ijkl}$ . Dipoles and quadrupoles are two-point and four-point correlators of Wilson lines defined as

$$S_{ij} \equiv S(\mathbf{x}_i, \mathbf{y}_j) = \frac{1}{N_c} \langle \text{tr}[V(\mathbf{x}_i) V^\dagger(\mathbf{y}_j)] \rangle, \\ S_{ijkl} \equiv S(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k, \mathbf{x}_l) \\ = \frac{1}{N_c} \langle \text{tr}[V(\mathbf{x}_i) V^\dagger(\mathbf{x}_j) V(\mathbf{x}_k) V^\dagger(\mathbf{x}_l)] \rangle, \quad (3)$$

where the Wilson line  $V(\mathbf{x}_i)$  is a path ordered exponential of the gluon field in the fundamental representation. It efficiently resums multiple soft scatterings of the quark projectile on the target proton or nucleus which is modeled as a classical color field. We note that this description corresponds to making the well-known eikonal approximation which is not unique to CGC formalism but is also used in calculations of energy loss effects from a dense medium (see e.g., [71,72]). Furthermore, the virtual photon has momentum  $l^\mu$  with  $l^2 = -Q^2$  and we have set the transverse momentum of the photon to zero without any loss of generality. Also,  $p^\mu$  ( $q^\mu$ ) is the momentum of the outgoing quark (antiquark) and  $z_1$  ( $z_2$ ) is its longitudinal momentum fraction relative to the photon. Transverse coordinate of quark (antiquark) is denoted  $\mathbf{x}_1$  ( $\mathbf{x}_2$ ) and primed coordinates are used in the conjugate amplitude.

Quark and antiquark rapidities  $y_1$  and  $y_2$  are related to their momentum fractions  $z_1$  and  $z_2$  via  $dy_i = dz_i/z_i$ . To calculate the single inclusive production cross section,

we integrate over one of the final state partons phase space; we choose it to be the antiquark so that the quark is produced. This sets  $\mathbf{x}_2 = \mathbf{x}_2'$  and gives

$$\frac{d\sigma_T^{\gamma^* A \rightarrow q(p, y_1) X}}{d^2 \mathbf{p} dy_1} = \frac{e^2 Q^2 N_c}{(2\pi)^5} z_1^2 \bar{z}_1 (z_1^2 + \bar{z}_1^2) \int d^6 \mathbf{x} [S_{11'} - S_{12} - S_{1'2} + 1] e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) \quad (4)$$

where we only kept the transverse-photon part and with  $\bar{z}_1 = 1 - z_1$ . The only nontrivial ingredients of the cross section above are the dipoles  $S_{ij}$  that contain the QCD dynamics at small  $x$ . While the LO cross section can provide us with a qualitative and even a semiquantitative QCD dynamics in a proton or nucleus, it is imperative to go beyond a semiquantitative analysis and improve the precision of the theoretical calculations in preparation for the upcoming EIC. There is already a significant amount of work done toward calculating next-to-leading order corrections to the BK-JIMWLK evolution equations which is a major step in this direction. Here we calculate the NLO corrections to the single inclusive hadron production in DIS (SIDIS) for the case when the exchanged photon is transversely polarized.

### III. NEXT-TO-LEADING-ORDER CORRECTIONS

The complete expressions for NLO corrections to double inclusive quark-antiquark production with a transversely polarized exchanged photon were derived in [39] and

written down in the large- $N_c$  limit. Here, we start from those expressions but include the full  $N_c$  structure; they are given in Appendix A. In order to calculate the NLO corrections to SIDIS we start with those and integrate out the antiquark phase space, as was done at LO. There are both real and virtual corrections in the amplitude; real corrections involve radiation of a gluon either before or after scattering from the target where the gluon can be radiated from either a quark or an antiquark in the amplitude as shown in Fig. 1, with  $z$  and  $\mathbf{k}$  denoting the gluon longitudinal momentum fraction and transverse momentum, respectively. The cross section is then obtained from multiplying the amplitude with the complex conjugate amplitude.

The virtual corrections to the amplitude also involve radiation of a gluon from quark or antiquark and either before or after scattering from the target. However in this case the radiated gluon is then absorbed by either the quark or antiquark, before or after scattering from the target at the amplitude (or complex conjugate amplitude) level already. One then obtains the corresponding cross section by

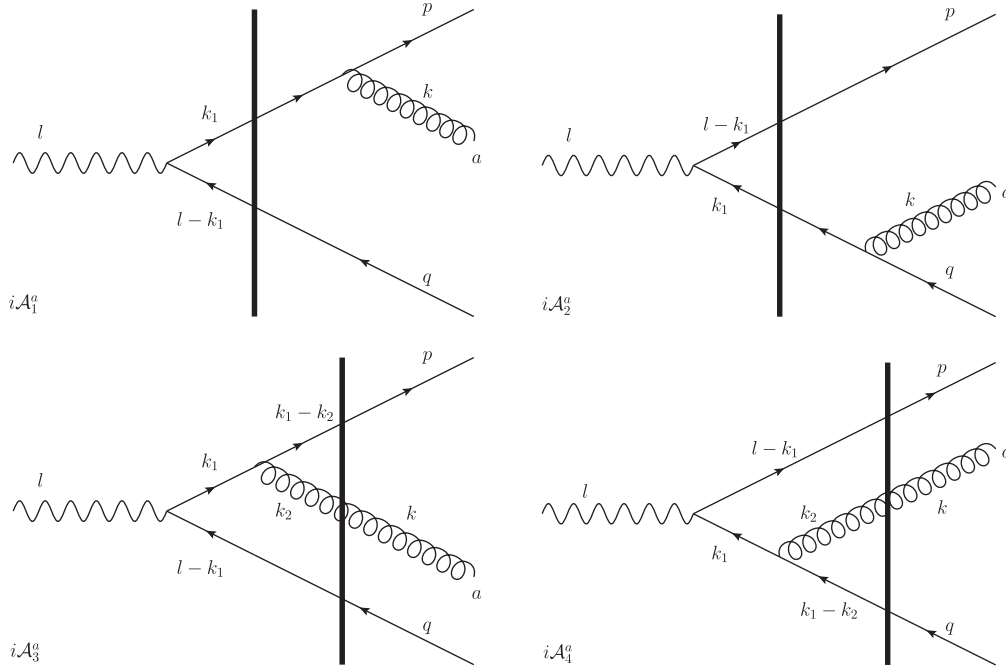


FIG. 1. The real corrections  $i\mathcal{A}_1^a, \dots, i\mathcal{A}_4^a$ . The arrows on the fermion lines indicate fermion number flow; all momenta flow to the right. The thick solid line indicates interaction with the target.

multiplying this amplitude with the LO complex conjugate amplitude (or vice versa). The diagrams for virtual corrections to the amplitude are shown in Fig. 2, where the gluon momentum labeling is now diagram dependent. In our expressions for the virtual diagrams, the transverse part of the gluon momentum will follow those notations but we shall keep using  $z$  to denote the gluon longitudinal momentum fraction.

To summarize, by following the same conventions introduced in [40], we write the total NLO contribution to the single inclusive quark production in DIS via the transverse photon as

$$\begin{aligned} d\sigma_{\text{NLO}}^T = & \sum_{i=1}^4 d\sigma_{i \times i} + \sum_{i=2}^4 \sum_{j=1}^{i-1} (d\sigma_{i \times j} + d\sigma_{j \times i}) \\ & + \sum_{i=5}^{14} (d\sigma_i + d\sigma_i^*) \end{aligned} \quad (5)$$

where the first two terms are the real contributions which can be computed as

$$d\sigma_{i \times j} = \frac{1}{\mathcal{F}} \int_{z, \mathbf{k}, z_2, \mathbf{q}} [(i\mathcal{M}_i^a)(i\mathcal{M}_j^a)^*] d\Phi^{(3)}. \quad (6)$$

Here,  $\mathcal{M}_i$  corresponds to the real amplitudes computed in [39] by using the helicity amplitude method and shown in Fig. 1.  $\mathcal{F}$  is the incident flux which is taken to be  $2l^+$  and  $d\Phi^{(3)}$  corresponds to the phase space factor and reads [40]

$$d\Phi^{(3)} = 2l^+ \frac{d^2\mathbf{p} d^2\mathbf{q} d^2\mathbf{k} dy_1 dy_2 dz}{(2\pi)^8 (4l^+)^2 z} \delta(1 - z_1 - z_2 - z). \quad (7)$$

On the other hand, the last term in Eq. (5) corresponds to the virtual contributions and those read

$$d\sigma_i = \frac{1}{\mathcal{F}} \int_{z_2, \mathbf{q}} [(i\mathcal{M}_i)(i\mathcal{M}_0)^*] d\Phi^{(2)} \quad (8)$$

where  $\mathcal{M}_0$  is the leading-order quark-antiquark amplitude. Here,  $\mathcal{M}_i$  corresponds to virtual amplitudes shown in Fig. 2

and the phase factor  $d\Phi^{(2)}$  is given by

$$d\Phi^{(2)} = 2l^+ \frac{d^2\mathbf{p} d^2\mathbf{q} dy_1 dy_2}{2(2\pi)^5 (2l^+)^2} \delta(1 - z_1 - z_2). \quad (9)$$

### A. Single inclusive hadron (quark) production

As mentioned above, to calculate single inclusive hadron production in DIS we start with the expressions for double inclusive quark-antiquark production and integrate out the antiquark momenta ( $\mathbf{q}, z_2$ ) so that the quark is produced<sup>1</sup> and hadronizes. After integrating out the antiquark phase space one can see that there are cancellations among some of the diagrams which become topologically equivalent. Specifically the following diagrams totally cancel after integrating over the antiquark momenta:

$$d\sigma_{2 \times 2} + 2 \text{Re}(d\sigma_{10}) = 0, \quad (10)$$

$$d\sigma_{2 \times 3} + d\sigma_8^* = d\sigma_{3 \times 2} + d\sigma_8 = 0, \quad (11)$$

$$d\sigma_{2 \times 4} + d\sigma_6^* = d\sigma_{4 \times 2} + d\sigma_6 = 0. \quad (12)$$

This cancellation is easy to understand: After integrating out the phase space of the antiquark in the quark-antiquark production cross section, some of the real and virtual diagrams become topologically equivalent except for a relative minus sign that leads to the cancellations. The origin of this relative minus sign is due to the vertex factors  $ig$  in the amplitude and  $-ig$  in the complex conjugate amplitude in the NLO real diagrams vs  $(ig)^2$  in the amplitude or  $(-ig)^2$  in the complex conjugate amplitude in NLO virtual diagrams.

#### 1. Real corrections to single inclusive quark production at NLO

In this subsection, we provide the real NLO corrections to the single inclusive quark production that are obtained by starting from the real NLO corrections to the quark-antiquark production given in [39] and integrating over the antiquark momenta ( $\mathbf{q}, z_2$ ).

$$\begin{aligned} \frac{d\sigma_{1 \times 1}^T}{d^2\mathbf{p} dy_1} = & \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} \frac{(\bar{z}_1 - z)(z_1 + z)}{z_1} [z_1^2 + (z_1 + z)^2][(\bar{z}_1 - z)^2 + (z_1 + z)^2] \\ & \times \int d^8\mathbf{x} C_F [S_{11'} - S_{12} - S_{21'} + 1] K_1(|\mathbf{x}_{12}| Q_{1z}) K_1(|\mathbf{x}_{1'2}| Q_{1z}) \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} e^{i \frac{z_1 + z}{z_1} \mathbf{p} \cdot \mathbf{x}_{1'1}} \Delta_{1'1}^{(3)}. \end{aligned} \quad (13)$$

<sup>1</sup>Note that the result is the same whether one integrates out a quark or an antiquark, up to a trivial relabeling of momenta.

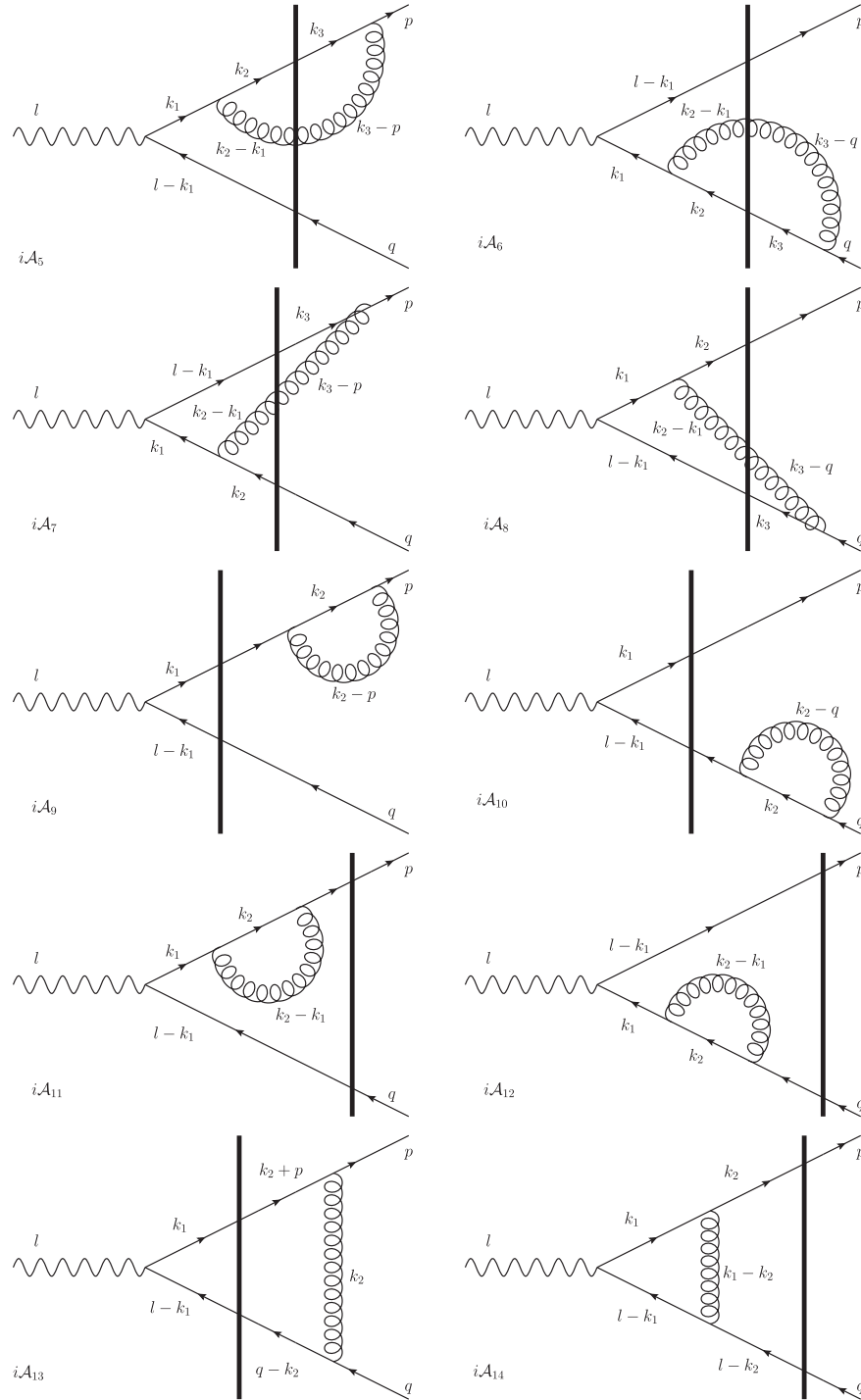


FIG. 2. The ten virtual NLO diagrams  $i\mathcal{A}_5, \dots, i\mathcal{A}_{14}$ . The arrows on fermion lines indicate fermion number flow; all momenta flow to the right, *except* for gluon momenta. The thick solid line indicates interaction with the target.

$$\begin{aligned}
\frac{d\sigma_{1\times 2}^T}{d^2\mathbf{p}d\mathbf{y}_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} \frac{(\bar{z}_1 - z)^2}{\bar{z}_1^2} \frac{\sqrt{z_1 \bar{z}_1 (\bar{z}_1 - z)(z_1 + z)}}{(\bar{z}_1 - z)} \int d^8\mathbf{x} K_1(|\mathbf{x}_{12}|Q_{1z}) K_1(|\mathbf{x}_{1'2'}|Q_1) \\
&\times \left[ C_F \left( S_{12} S_{2'1'} - S_{12} - S_{2'1'} + 1 \right) - \frac{1}{2N_c} \left( S_{122'1'} - S_{12} S_{2'1'} \right) \right] 4 \operatorname{Re} \left\{ \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2'}|} \right. \\
&\times \left[ \left( z_1^2 + (\bar{z}_1 - z)^2 \right) \bar{z}_1 (z_1 + z) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{2'3} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2 \mathbf{x}_{2'3}^2} + \left( \bar{z}_1^2 + (z_1 + z)^2 \right) z_1 (\bar{z}_1 - z) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)(\mathbf{x}_{2'3} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{31}^2 \mathbf{x}_{2'3}^2} \right] \left. \right\} e^{i\mathbf{p} \cdot (\mathbf{x}_{1'1} + \frac{z}{\bar{z}_1} \mathbf{x}_{31})}. \quad (14)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{3\times 3}^T}{d^2\mathbf{p}d\mathbf{y}_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} z_1 (\bar{z}_1 - z)^2 \int d^8\mathbf{x} \left[ \frac{K_1(QX) K_1(QX')}{XX'} \right]_{\mathbf{x}'_2 = \mathbf{x}_2} e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\
&\times \left[ C_F \left( S_{11'} - S_{13} S_{32} - S_{31'} S_{23} + 1 \right) - \frac{1}{2N_c} \left( S_{13} S_{32} + S_{31'} S_{23} - S_{12} - S_{21'} \right) \right] \\
&\times 4 \operatorname{Re} \left\{ \left[ \left( z_1^2 + (\bar{z}_1 - z)^2 \right) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{31'} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2 \mathbf{x}_{31'}^2} + \left( (z_1 + z)^2 + \frac{z_1^2 (\bar{z}_1 - z)^2}{(z_1 + z)^2} \right) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)(\mathbf{x}_{31'} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{31}^2 \mathbf{x}_{31'}^2} \right] \right. \\
&\times [(\mathbf{z}_1 \mathbf{x}_{12} + z \mathbf{x}_{32}) \cdot \boldsymbol{\epsilon}] [(\mathbf{z}_1 \mathbf{x}_{1'2} + z \mathbf{x}_{32}) \cdot \boldsymbol{\epsilon}^*] \\
&\left. - \frac{z_1^2 z (\bar{z}_1 - z)}{2(z_1 + z)^2} \left[ \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} [(\mathbf{z}_1 \mathbf{x}_{12} + z \mathbf{x}_{32}) \cdot \boldsymbol{\epsilon}] + \frac{(\mathbf{x}_{31'} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{31'}^2} [(\mathbf{z}_1 \mathbf{x}_{1'2} + z \mathbf{x}_{32}) \cdot \boldsymbol{\epsilon}^*] \right] + \frac{z_1^2 z^2}{4(z_1 + z)^2} \right\}. \quad (15)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{4\times 4}^T}{d^2\mathbf{p}d\mathbf{y}_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} z_1^3 \int d^8\mathbf{x} \left[ \frac{K_1(QX) K_1(QX')}{XX'} \right]_{\mathbf{x}'_2 = \mathbf{x}_2} e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\
&\times \left[ C_F \left( S_{11'} - S_{13} S_{32} - S_{31'} S_{23} + 1 \right) - \frac{1}{2N_c} \left( S_{13} S_{32} + S_{31'} S_{23} - S_{12} - S_{21'} \right) \right] \\
&\times 4 \operatorname{Re} \left\{ \left[ \left( z_1^2 + (\bar{z}_1 - z)^2 \right) \frac{(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{32}^2 \mathbf{x}_{32}^2} + \left( \bar{z}_1^2 + \frac{z_1^2 (\bar{z}_1 - z)^2}{\bar{z}_1^2} \right) \frac{(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*)(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{32}^2 \mathbf{x}_{32}^2} \right] \right. \\
&\times \left[ \left( (\bar{z}_1 - z) \mathbf{x}_{21} + z \mathbf{x}_{31} \right) \cdot \boldsymbol{\epsilon} \right] \left[ \left( (\bar{z}_1 - z) \mathbf{x}_{21'} + z \mathbf{x}_{31'} \right) \cdot \boldsymbol{\epsilon}^* \right] \\
&\left. - \frac{(\bar{z}_1 - z)^2 z_1 z}{2\bar{z}_1^2} \left[ \frac{(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{32}^2} \left[ \left( (\bar{z}_1 - z) \mathbf{x}_{21} + z \mathbf{x}_{31} \right) \cdot \boldsymbol{\epsilon} \right] + \frac{(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{32}^2} \left[ \left( (\bar{z}_1 - z) \mathbf{x}_{21'} + z \mathbf{x}_{31'} \right) \cdot \boldsymbol{\epsilon}^* \right] \right] + \frac{(\bar{z}_1 - z)^2 z^2}{4(z_1 + z)^2} \right\}. \quad (16)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{3\times 4}^T}{d^2\mathbf{p}d\mathbf{y}_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} z_1^2 (\bar{z}_1 - z) \int d^8\mathbf{x} \left[ \frac{K_1(QX) K_1(QX')}{XX'} \right]_{\mathbf{x}'_2 = \mathbf{x}_2} e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\
&\times \left[ C_F \left( S_{11'} - S_{13} S_{32} - S_{31'} S_{23} + 1 \right) - \frac{1}{2N_c} \left( S_{13} S_{32} + S_{31'} S_{23} - S_{12} - S_{21'} \right) \right] \\
&\times 4 \operatorname{Re} \left\{ \left[ \left( z_1^2 + (\bar{z}_1 - z)^2 \right) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2 \mathbf{x}_{32}^2} + \frac{z_1 (\bar{z}_1 - z)}{\bar{z}_1 (z_1 + z)} \left( \bar{z}_1^2 + (z_1 + z)^2 \right) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{31}^2 \mathbf{x}_{32}^2} \right] \right. \\
&\times \left[ (\mathbf{z}_1 \mathbf{x}_{12} + z \mathbf{x}_{32}) \cdot \boldsymbol{\epsilon} \right] \left[ \left( (\bar{z}_1 - z) \mathbf{x}_{21'} + z \mathbf{x}_{31'} \right) \cdot \boldsymbol{\epsilon}^* \right] \\
&\left. - \frac{z (\bar{z}_1 - z) (z_1 + z)}{2\bar{z}_1} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} \left[ (\mathbf{z}_1 \mathbf{x}_{12} + z \mathbf{x}_{32}) \cdot \boldsymbol{\epsilon} \right] - \frac{z_1 z \bar{z}_1}{2(z_1 + z)} \frac{(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{32}^2} \left[ \left( (\bar{z}_1 - z) \mathbf{x}_{21'} + z \mathbf{x}_{31'} \right) \cdot \boldsymbol{\epsilon}^* \right] \right\}. \quad (17)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{1\times 3}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -\frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} (\bar{z}_1 - z) \sqrt{(\bar{z}_1 - z)(z_1 + z)} \int d^8\mathbf{x} \left[ \frac{K_1(|\mathbf{x}_{12}|Q_{1z})K_1(QX')}{X'} \right]_{\mathbf{x}'_2=\mathbf{x}_2} e^{i\mathbf{p}\cdot(\mathbf{x}_{1'1}+\frac{z}{\bar{z}_1}\mathbf{x}_{31})} \\
& \times \left[ C_F \left( S_{13}S_{31'} - S_{31'}S_{23} - S_{12} + 1 \right) - \frac{1}{2N_c} \left( S_{11'} - S_{21'} - S_{13}S_{31'} + S_{31'}S_{23} \right) \right] \\
& \times 4 \operatorname{Re} \left\{ (z_1 + z) \left( z_1^2 + (\bar{z}_1 - z)^2 \right) \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{31'} \cdot \boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}|\mathbf{x}_{31'}^2} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon})[(z_1\mathbf{x}_{1'2} + z\mathbf{x}_{32}) \cdot \boldsymbol{\epsilon}^*]}{\mathbf{x}_{31}^2} \right. \\
& \left. + \left( (z_1 + z)^3 + \frac{z_1^2(\bar{z}_1 - z)^2}{z_1 + z} \right) \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{31'} \cdot \boldsymbol{\epsilon})}{|\mathbf{x}_{12}|\mathbf{x}_{31'}^2} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)[(z_1\mathbf{x}_{1'2} + z\mathbf{x}_{32}) \cdot \boldsymbol{\epsilon}^*]}{\mathbf{x}_{31}^2} - \frac{z_1^2(\bar{z}_1 - z)z}{2(z_1 + z)} \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})}{|\mathbf{x}_{12}|} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} \right\}. \quad (18)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{1\times 4}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -\frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} z_1 \sqrt{(\bar{z}_1 - z)(z_1 + z)} \int d^8\mathbf{x} \left[ \frac{K_1(|\mathbf{x}_{12}|Q_{1z})K_1(QX')}{X'} \right]_{\mathbf{x}'_2=\mathbf{x}_2} e^{i\mathbf{p}\cdot(\mathbf{x}_{1'1}+\frac{z}{\bar{z}_1}\mathbf{x}_{31})} \\
& \times \left[ C_F \left( S_{13}S_{31'} - S_{31'}S_{23} - S_{12} + 1 \right) - \frac{1}{2N_c} \left( -S_{13}S_{31'} + S_{31'}S_{23} + S_{11'} - S_{21'} \right) \right] \\
& \times 4 \operatorname{Re} \left\{ (z_1 + z) \left( z_1^2 + (\bar{z}_1 - z)^2 \right) \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}|\mathbf{x}_{32}^2} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon})[(\bar{z}_1 - z)\mathbf{x}_{21'} + z\mathbf{x}_{31'}] \cdot \boldsymbol{\epsilon}^*}{\mathbf{x}_{31}^2} \right. \\
& + \frac{z_1(\bar{z}_1 - z)}{\bar{z}_1} \left( \bar{z}_1^2 + (z_1 + z)^2 \right) \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{32} \cdot \boldsymbol{\epsilon})}{|\mathbf{x}_{12}|\mathbf{x}_{32}^2} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)[(\bar{z}_1 - z)\mathbf{x}_{21'} + z\mathbf{x}_{31'}] \cdot \boldsymbol{\epsilon}^*}{\mathbf{x}_{31}^2} \\
& \left. - \frac{(\bar{z}_1 - z)z(z_1 + z)^2}{2\bar{z}_1} \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})}{|\mathbf{x}_{12}|} \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} \right\}. \quad (19)
\end{aligned}$$

Before we proceed further, a few comments are in order. The shorthand notation  $Q_{1z}$  used in the arguments of the Bessel functions in Eqs. (13), (14), (18), and (19) is given by

$$Q_{1z} \equiv Q\sqrt{(\bar{z}_1 - z)(z_1 + z)}. \quad (20)$$

The transverse photon polarization vector  $\boldsymbol{\epsilon}$  that appears in all NLO contributions is given by

$$\boldsymbol{\epsilon} = \frac{1}{\sqrt{2}}(1, i) \quad (21)$$

and the radiation kernel  $\Delta_{ij}^{(3)}$  used in Eq. (13) is defined as

$$\Delta_{ij}^{(3)} \equiv \frac{\mathbf{x}_{3i} \cdot \mathbf{x}_{3j}}{\mathbf{x}_{3i}^2 \mathbf{x}_{3j}^2}. \quad (22)$$

The shorthand notation  $\mathbf{x}_{2'}$  introduced in Eq. (14) is  $\mathbf{x}_{2'} = \mathbf{x}_2 - \frac{z}{\bar{z}_1}\mathbf{x}_{23}$  so that it satisfies  $\mathbf{x}_{2'3} = \frac{(\bar{z}_1 - z)}{\bar{z}_1}\mathbf{x}_{23}$ . Finally, the notations  $X$  and  $X'$  used in Eqs. (15), (16), (17), (18), and (19) are defined as

$$X = \sqrt{z_1(\bar{z}_1 - z)\mathbf{x}_{12}^2 + z_1 z \mathbf{x}_{13}^2 + (\bar{z}_1 - z)z \mathbf{x}_{23}^2}, \quad (23)$$

$$X' = \sqrt{z_1(\bar{z}_1 - z)\mathbf{x}_{1'2'}^2 + z_1 z \mathbf{x}_{1'3}^2 + (\bar{z}_1 - z)z \mathbf{x}_{2'3}^2}. \quad (24)$$

Apart from the above-listed contributions, a NLO single inclusive quark production cross section receives real corrections from diagrams  $\sigma_{2\times 1}$ ,  $\sigma_{4\times 3}$ ,  $\sigma_{3\times 1}$  and  $\sigma_{4\times 1}$  that can be obtained by taking the complex conjugate of Eqs. (14), (17), (18), and (19), respectively.

## 2. Virtual corrections to single inclusive quark production at NLO

In this subsection, we list the virtual corrections to the single inclusive quark production cross section at NLO that are obtained from the virtual corrections to a double inclusive quark-antiquark cross section at NLO in DIS via transverse photon upon integration over the antiquark momenta  $(z_2, \mathbf{q})$ . The results read

$$\begin{aligned}
\frac{d\sigma_5^T}{d^2\mathbf{p}d\mathbf{y}_1} = & \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{\bar{z}_1} \frac{dz}{z} \bar{z}_1 \sqrt{z_1 \bar{z}_1} \int d^8\mathbf{x} \frac{K_1(QX_5)K_1(|\mathbf{x}_{1'2}|Q_1)}{X_5|\mathbf{x}_{1'2}|\mathbf{x}_{31}^2} e^{i\mathbf{p}\cdot(\mathbf{x}_{1'1}-\frac{z}{\bar{z}_1}\mathbf{x}_{31})} \\
& \times \left[ C_F \left( S_{13}S_{31'} - S_{32}S_{13} - S_{21'} + 1 \right) - \frac{1}{2N_c} \left( -S_{13}S_{31'} + S_{32}S_{13} + S_{11'} - S_{12} \right) \right] \\
& \times \mathbf{x}_{1'2} \cdot \left\{ (z_1^2 + \bar{z}_1^2)[z_1^2 + (z_1 - z)^2]\mathbf{x}_{32} + (z_1 - z) \left[ z_1[z_1^2 + (z_1 - z)^2] + \bar{z}_1[z_1\bar{z}_1 + (z_1 - z)(\bar{z}_1 + z)] \right] \mathbf{x}_{13} \right\}. \quad (25)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_7^T}{d^2\mathbf{p}dy_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} \int_0^{z_1} \frac{dz}{z} z_1(z_1 - z) \sqrt{z_1 \bar{z}_1} \int d^8\mathbf{x} \frac{K_1(QX_5)K_1(|\mathbf{x}_{1'2}|Q_1)}{X_5|\mathbf{x}_{1'2}|\mathbf{x}_{31}^2} e^{i\mathbf{p}\cdot(\mathbf{x}_{1'1} - \frac{z}{z_1}\mathbf{x}_{31})} \\
&\times \left[ C_F \left( S_{13}S_{31'} - S_{32}S_{13} - S_{21'} + 1 \right) - \frac{1}{2N_c} \left( -S_{13}S_{31'} + S_{32}S_{13} + S_{11'} - S_{12} \right) \right] \\
&\times \left\{ \frac{4\text{Re}}{\mathbf{x}_{32}^2} \left\{ (\mathbf{x}_{1'2} \cdot \boldsymbol{\epsilon}^*) \left[ \left( \mathbf{x}_{31} + \frac{\bar{z}_1}{\bar{z}_1 + z} \mathbf{x}_{23} \right) \cdot \boldsymbol{\epsilon} \right] \left[ \frac{\bar{z}_1(z_1 - z)}{z_1} (z_1^2 + (\bar{z}_1 + z)^2) (\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}) (\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*) \right. \right. \right. \\
&\quad \left. \left. \left. + (\bar{z}_1 + z)(\bar{z}_1^2 + (z_1 - z)^2) (\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}) (\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*) \right] \right\} - \frac{z_1 \bar{z}_1 z}{\bar{z}_1 + z} \mathbf{x}_{31} \cdot \mathbf{x}_{1'2} \right\} \quad (26)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_9^T}{d^2\mathbf{p}dy_1} &= -\frac{e^2 g^2 Q^2 N_c}{2(2\pi)^6} \int_0^{z_1} \frac{dz}{z} \bar{z}_1(z_1^2 + \bar{z}_1^2) \left[ z_1^2 + (z_1 - z)^2 \right] \int d^6\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} \\
&\times \left[ C_F \left( S_{11'} - S_{12} - S_{21'} + 1 \right) \right] \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2}. \quad (27)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{11}^T}{d^2\mathbf{p}dy_1} &= i \frac{e^2 g^2 Q N_c}{(2\pi)^5} \int_0^{z_1} \frac{dz}{z} \sqrt{z_1 \bar{z}_1} (z_1^2 + \bar{z}_1^2) \frac{[(z_1 - z)^2 + z_1^2]}{(z_1 - z)} \int d^6\mathbf{x} \frac{K_1(|\mathbf{x}_{1'2}|Q_1)}{|\mathbf{x}_{1'2}|} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} \\
&\times \left[ C_F \left( S_{11'} - S_{12} - S_{21'} + 1 \right) \right] \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{\mathbf{k}_1 \cdot \mathbf{x}_{1'2}}{[\mathbf{k}_1^2 + Q_1^2][zQ^2 + \frac{z\mathbf{k}_1^2}{z_1\bar{z}_1} + \frac{z_1}{z_1 - z}\mathbf{k}_2^2]} e^{i\mathbf{k}_1 \cdot \mathbf{x}_{12}} \quad (28)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{12}^T}{d^2\mathbf{p}dy_1} &= -i \frac{e^2 g^2 Q N_c}{(2\pi)^5} \int_0^{\bar{z}_1} \frac{dz}{z} \sqrt{\bar{z}_1} z_1 (z_1^2 + \bar{z}_1^2) \frac{[(\bar{z}_1 - z)^2 + \bar{z}_1^2]}{(\bar{z}_1 - z)} \int d^6\mathbf{x} \frac{K_1(|\mathbf{x}_{1'2}|Q_1)}{|\mathbf{x}_{1'2}|} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} \\
&\times \left[ C_F \left( S_{11'} - S_{12} - S_{21'} + 1 \right) \right] \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{\mathbf{k}_1 \cdot \mathbf{x}_{1'2}}{[\mathbf{k}_1^2 + Q_1^2][zQ^2 + \frac{z\mathbf{k}_1^2}{z_1\bar{z}_1} + \frac{\bar{z}_1}{\bar{z}_1 - z}\mathbf{k}_2^2]} e^{i\mathbf{k}_1 \cdot \mathbf{x}_{21}} \quad (29)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{13(1)}^T}{d^2\mathbf{p}dy_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^6} \int_0^{\bar{z}_1} dz z_1 \sqrt{z_1 \bar{z}_1 (z_1 + z)(\bar{z}_1 - z)} \\
&\times \int d^8\mathbf{x} \left[ C_F \left( S_{2'1'}S_{12} - S_{12} - S_{2'1'} + 1 \right) - \frac{1}{2N_c} \left( -S_{2'1'}S_{12} + S_{122'1'} \right) \right] \int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}_{21}} \\
&\times \frac{1}{|\mathbf{x}_{12}|} K_1 \left( |\mathbf{x}_{12}|Q \sqrt{(z_1 + z)(\bar{z}_1 - z)} \right) \frac{1}{|\mathbf{x}_{1'2'}|} K_1(|\mathbf{x}_{1'2'}|Q_1) 4\text{Re} \left\{ \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*)}{(\bar{z}_1 \mathbf{k} - z\mathbf{q})^2} \right. \\
&\times \left[ \frac{\bar{z}_1(\bar{z}_1 - z)[z_1(z_1 + z) + \bar{z}_1(\bar{z}_1 - z)]}{2z} + \frac{1}{[\frac{(z_1 \mathbf{k} - z\mathbf{p})^2}{z_1(z_1 + z)} - \frac{(\bar{z}_1 \mathbf{k} - z\mathbf{q})^2}{\bar{z}_1(\bar{z}_1 - z)}]} \left( f_1(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) + f_2(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) \right) \right] \Big\} \quad (30)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{13(2)}^T}{d^2\mathbf{p}dy_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^6} \int_0^{z_1} dz z_1 \sqrt{z_1 \bar{z}_1 (z_1 - z)(\bar{z}_1 + z)} \\
&\times \int d^8\mathbf{x} \left[ C_F \left( S_{2'1'}S_{12} - S_{12} - S_{2'1'} + 1 \right) - \frac{1}{2N_c} \left( -S_{2'1'}S_{12} + S_{122'1'} \right) \right] \int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}_{12}} \\
&\times \frac{1}{|\mathbf{x}_{12}|} K_1 \left( |\mathbf{x}_{12}|Q \sqrt{(z_1 - z)(\bar{z}_1 + z)} \right) \frac{1}{|\mathbf{x}_{1'2'}|} K_1(|\mathbf{x}_{1'2'}|Q_1) 4\text{Re} \left\{ \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*)}{(z_1 \mathbf{k} - z\mathbf{p})^2} \right. \\
&\times \left[ \frac{z_1(z_1 - z)[z_1(z_1 - z) + \bar{z}_1(\bar{z}_1 + z)]}{2z} + \frac{1}{[\frac{(z_1 \mathbf{k} - z\mathbf{p})^2}{z_1(z_1 - z)} - \frac{(\bar{z}_1 \mathbf{k} - z\mathbf{q})^2}{\bar{z}_1(\bar{z}_1 + z)}]} \left( f_3(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) + f_4(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) \right) \right] \Big\} \quad (31)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{14(1)}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -i \frac{e^2 g^2 Q N_c}{(2\pi)^5} \int_0^{\bar{z}_1} \frac{dz}{z} z_1 \sqrt{z_1 \bar{z}_1} \int d^6\mathbf{x} \left[ C_F \left( S_{11'} - S_{12} - S_{21'} + 1 \right) \right] \frac{1}{|\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\
& \times \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_2 \cdot \mathbf{x}_{12}} \frac{1}{[(\mathbf{k}_1 - \frac{z_1 - \bar{z}_1}{z_1} \mathbf{k}_2)^2 + \frac{z(z_1 - \bar{z}_1)}{z_1 \bar{z}_1^2} \mathbf{k}_2^2 + \frac{z}{z_1} (z_1 - z) Q^2]} \\
& \times \left\{ \frac{[z_1(z_1 - z) + \bar{z}_1(\bar{z}_1 + z) - z(1 - z)](\mathbf{k}_2 \cdot \mathbf{x}_{1'2})}{[\mathbf{k}_2^2 + Q_1^2]} + \frac{(z_1 - z)[1 + z^2 - 2\bar{z}_1(z_1 - z)](\mathbf{k}_1 \cdot \mathbf{x}_{1'2})}{z_1[\mathbf{k}_1^2 + (z_1 - z)(\bar{z}_1 + z)Q^2]} \right. \\
& \left. - \frac{Q^2(z_1 - z)[2z_1 \bar{z}_1 z(\mathbf{k}_1 \cdot \mathbf{x}_{1'2}) + z(z + \bar{z}_1 - z_1)^2(\mathbf{k}_2 \cdot \mathbf{x}_{1'2})]}{z_1[\mathbf{k}_1^2 + (z_1 - z)(\bar{z}_1 + z)Q^2][\mathbf{k}_2^2 + Q_1^2]} \right\} \quad (32)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{14(2)}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -i \frac{e^2 g^2 Q N_c}{(2\pi)^5} \int_0^{\bar{z}_1} \frac{dz}{z} z_1 \sqrt{z_1 \bar{z}_1} \int d^6\mathbf{x} \left[ C_F \left( S_{11'} - S_{12} - S_{21'} + 1 \right) \right] \frac{1}{|\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\
& \times \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_2 \cdot \mathbf{x}_{12}} \frac{1}{[(\mathbf{k}_1 - \frac{\bar{z}_1 - z}{\bar{z}_1} \mathbf{k}_2)^2 + \frac{z(\bar{z}_1 - z)}{z_1 \bar{z}_1^2} \mathbf{k}_2^2 + \frac{z}{\bar{z}_1} (\bar{z}_1 - z) Q^2]} \\
& \times \left\{ \frac{[\bar{z}_1(\bar{z}_1 - z) + z_1(z_1 + z) - z(1 - z)](\mathbf{k}_2 \cdot \mathbf{x}_{1'2})}{[\mathbf{k}_2^2 + Q_1^2]} + \frac{(\bar{z}_1 - z)(1 + z^2 - 2z_1(\bar{z}_1 - z))(\mathbf{k}_1 \cdot \mathbf{x}_{1'2})}{\bar{z}_1[\mathbf{k}_1^2 + (\bar{z}_1 - z)(z_1 + z)Q^2]} \right. \\
& \left. - \frac{Q^2(\bar{z}_1 - z)[2z_1 \bar{z}_1 z(\mathbf{k}_1 \cdot \mathbf{x}_{1'2}) + z(z + z_1 - \bar{z}_1)^2(\mathbf{k}_2 \cdot \mathbf{x}_{1'2})]}{\bar{z}_1[\mathbf{k}_1^2 + (\bar{z}_1 - z)(z_1 + z)Q^2][\mathbf{k}_2^2 + Q_1^2]} \right\}. \quad (33)
\end{aligned}$$

In Eqs. (25) and (26) we have used  $X_5$  which is defined as

$$X_5 = \sqrt{\bar{z}_1(z_1 - z)\mathbf{x}_{12}^2 + z(z_1 - z)\mathbf{x}_{13}^2 + \bar{z}_1 z \mathbf{x}_{23}^2}. \quad (34)$$

Moreover, for convenience the results given in Eqs. (30) and (31) are written in terms of functions  $f_i(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*)$  whose explicit expressions read

$$\begin{aligned}
f_1(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) = & z_1 \bar{z}_1 z \left[ z_1 z \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{k} \cdot \epsilon^*}{z} \right) + z^2 \left( \frac{\mathbf{k} \cdot \epsilon}{z} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{k} \cdot \epsilon^*}{z} \right) \right. \\
& \left. - \bar{z}_1 z \left( \frac{\mathbf{k} \cdot \epsilon}{z} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) - p \cdot q \right], \quad (35)
\end{aligned}$$

$$\begin{aligned}
f_2(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) = & \frac{\bar{z}_1^2 z (\bar{z}_1 - z)}{(z_1 + z)} \left[ z_1 z \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{k} \cdot \epsilon}{z} \right) + z^2 \left( \frac{\mathbf{k} \cdot \epsilon^*}{z} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{k} \cdot \epsilon}{z} \right) \right. \\
& \left. - \bar{z}_1 z \left( \frac{\mathbf{k} \cdot \epsilon^*}{z} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) - p \cdot q \right], \quad (36)
\end{aligned}$$

$$\begin{aligned}
f_3(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) = & z_1 \bar{z}_1 z \left[ z_1 z \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{k} \cdot \epsilon}{z} \right) - z^2 \left( \frac{\mathbf{k} \cdot \epsilon^*}{z} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{k} \cdot \epsilon}{z} \right) \right. \\
& \left. - \bar{z}_1 z \left( \frac{\mathbf{k} \cdot \epsilon^*}{z} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) + p \cdot q \right], \quad (37)
\end{aligned}$$

$$\begin{aligned}
f_4(z, z_1; \mathbf{k}, \mathbf{p}, \mathbf{q}; \epsilon, \epsilon^*) = & \frac{z_1^2 z (z_1 - z)}{\bar{z}_1 + z} \left[ z_1 z \left( \frac{\mathbf{p} \cdot \epsilon}{z_1} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{k} \cdot \epsilon^*}{z} \right) - z^2 \left( \frac{\mathbf{k} \cdot \epsilon}{z} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{k} \cdot \epsilon^*}{z} \right) \right. \\
& \left. - \bar{z}_1 z \left( \frac{\mathbf{k} \cdot \epsilon}{z} - \frac{\mathbf{q} \cdot \epsilon}{\bar{z}_1} \right) \left( \frac{\mathbf{p} \cdot \epsilon^*}{z_1} - \frac{\mathbf{q} \cdot \epsilon^*}{\bar{z}_1} \right) + p \cdot q \right]. \quad (38)
\end{aligned}$$

Finally, the results presented in Eqs. (25)–(33) are the virtual contributions that are computed by considering the virtual diagrams in the amplitude and leading-order diagrams in the complex conjugate amplitude. The virtual contributions that

will come from considering the leading-order diagrams in the amplitude and virtual diagrams in the complex conjugate amplitude, i.e.,  $\sigma_5^*$ ,  $\sigma_7^*$ ,  $\sigma_9^*$ ,  $\sigma_{11}^*$ ,  $\sigma_{12}^*$ ,  $\sigma_{13(1)}^*$ ,  $\sigma_{13(2)}^*$ ,  $\sigma_{14(1)}^*$ , and  $\sigma_{14(2)}^*$  can be trivially obtained by taking the complex conjugate of Eqs. (25)–(33).

#### IV. DIVERGENCES

In this section, we discuss the ultraviolet and infrared divergences that appear in the formal expressions above. We show that they either cancel or get absorbed into redefinitions of the relevant nonperturbative quantities (fragmentation function and dipole scattering amplitude), as expected.

##### A. Ultraviolet divergences

Ultraviolet (UV) divergences in  $D = 2 - 2\epsilon$  arise when the transverse momentum in the loop goes to infinity, or equivalently in coordinate space when the transverse coordinate of the radiated gluon  $\mathbf{x}_3$  approaches that of a quark ( $\mathbf{x}_1, \mathbf{x}'_1$ ) or antiquark ( $\mathbf{x}_2$ ). These divergences need to be regularized in the expressions for each diagram. Using dimensional regularization these UV divergences in  $D = 2 - 2\epsilon$  dimensions appear as poles in  $\frac{1}{\epsilon}$  as  $\epsilon \rightarrow 0$ . There would also be finite pieces which depend on the regularization scheme used, for example,  $\overline{\text{MS}}$ . However once contributions of all diagrams are added together all UV divergences as well as scheme-dependent finite pieces cancel. Due to this fact the exact nature of regularization of UV divergences is not important as it cannot affect the final results. We therefore use power counting to identify and isolate the UV divergences in this work.

In the case of the double-inclusive cross section, these divergences appeared only in virtual corrections, and therefore could be dealt with already at the amplitude level [32,33,39], where complete cancellation occurred. Things work a bit differently in the case of single-inclusive production as some of the diagrams involved in the UV cancellation prior to the antiquark integration ( $\sigma_6$  and  $\sigma_{10}$ ) are no longer present; conversely, a UV divergence now appears in the real correction  $\sigma_{4\times 4}$ , as  $\mathbf{x}_3 \rightarrow \mathbf{x}_2$ .

After combination with those appearing in virtual corrections (namely when  $\mathbf{x}_3 \rightarrow \mathbf{x}_1$  in  $\sigma_5$ ;  $\mathbf{x}_3 \rightarrow \mathbf{x}'_1$  in  $\sigma_5^*$ ;  $|\mathbf{k}| \rightarrow \infty$  in  $\sigma_9$  and  $\sigma_9^*$ ;  $|\mathbf{k}_2| \rightarrow \infty$  in  $\sigma_{11}$ ,  $\sigma_{11}^*$ ,  $\sigma_{12}$ , and  $\sigma_{12}^*$ ; and  $|\mathbf{k}_1| \rightarrow \infty$  in  $\sigma_{14}$  and  $\sigma_{14}^*$ ), we find that these UV divergences all cancel as follows:

$$\begin{aligned} [\text{d}\sigma_5 + \text{d}\sigma_{11}]_{\text{UV}} &= [\text{d}\sigma_5^* + \text{d}\sigma_{11}^*]_{\text{UV}} = 0, \\ [2\text{Re}(\text{d}\sigma_9 + \text{d}\sigma_{12} + \text{d}\sigma_{14(1)} + \text{d}\sigma_{14(2)}) + \text{d}\sigma_{4\times 4}]_{\text{UV}} &= 0, \end{aligned} \quad (39)$$

where the cancellation in the last line happens after integrating over  $z$ , in a similar fashion to the case of SIDIS with a longitudinal photon exchange [49].

##### B. Collinear divergences and evolution of fragmentation functions

Collinear divergences occur when the angle  $\theta$  between the radiated gluon 3-momentum and the final-state quark 3-momentum goes to zero at finite  $\mathbf{k}$  and  $z$ . They are only present in  $\sigma_{1\times 1}$ ,  $\sigma_9$ , and  $\sigma_9^*$ , where, in momentum space expressions, they appear in the denominator  $1/(z_1\mathbf{k} - z\mathbf{p})^2$  which behaves as  $\sim 1/\theta^2$  when  $z_1\mathbf{k} \rightarrow z\mathbf{p}$ . In coordinate space, due to the following expressions,

$$\int d^2\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{x}_{1'1}}}{(\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2} = e^{i\frac{z}{z_1}\mathbf{p}\cdot\mathbf{x}_{1'1}} \int d^2\mathbf{x}_3 \Delta_{11'}^{(3)}, \quad (40)$$

$$\int d^2\mathbf{k} \frac{1}{(\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2} = \int d^2\mathbf{x}_3 \frac{1}{\mathbf{x}_3^2}, \quad (41)$$

this limit translates to  $|\mathbf{x}_3| \rightarrow \infty$ , but it should be kept in mind that those collinear divergences do not correspond to the gluon transverse momentum vanishing (there are no such singularities in our expressions, unless  $z$  also vanishes, as we will see in the following section).

The collinear divergences do not cancel and must be absorbed into a redefinition of the quark-to-hadron fragmentation function  $D_{h/q}$ . The procedure is identical to that in the case of dihadrons and is detailed e.g., in [40]; here we shall highlight the main steps. First, one writes the hadronic cross section in terms of the partonic cross section as

$$\frac{\text{d}\sigma^{\gamma^*A \rightarrow h(\mathbf{p}_h, y_h)X}}{\text{d}^2\mathbf{p}_h \text{d}y_h} = \int_{z_h}^1 \frac{\text{d}z_1}{z_h} D_{h/q}(z_h/z_1) \frac{\text{d}\sigma^{\gamma^*A \rightarrow q((z_1/z_h)\mathbf{p}_h, y_1)X}}{\text{d}^2\mathbf{p}_h \text{d}y_1}, \quad (42)$$

where  $z_h$  and  $\mathbf{p}_h$  stand for the fraction of photon longitudinal momentum carried by the produced hadron and its transverse momentum, respectively. Then, the divergent part in  $\sigma_{1\times 1} + 2\text{Re}(\sigma_9)$  can be isolated with the help of a factorization scale  $\mu$ ; it reads

$$\begin{aligned} \text{d}\sigma_{\text{LO}} \times \frac{\alpha_s C_F}{2\pi} \left( \int_0^{1-z_h} \frac{\text{d}z}{z} \frac{1 + (1-z)^2}{1-z} D_{h/q} \left( \frac{z_h}{1-z} \right) \right. \\ \left. - \int_0^1 \frac{\text{d}z}{z} [1 + (1-z)^2] D_{h/q}(z_h) \right) \int_{1/\mu^2} \frac{\text{d}^2\mathbf{x}}{\mathbf{x}^2}, \end{aligned} \quad (43)$$

where in that expression  $\text{d}\sigma_{\text{LO}}$  means the leading-order hadron-level cross section stripped of its fragmentation function  $D_{h/q}(z_h)$ . This divergent part is then absorbed into the leading-order cross section using a redefinition of the bare  $D_{h/q}$  into a fragmentation function renormalized at the scale  $\mu$ :

$$D_{h/q}(z, \mu^2) = D_{h/q}(z) + \mathcal{O}(\alpha_s \ln \mu^2) \quad (44)$$

where the  $\mathcal{O}(\alpha_s)$  term is precisely the one in (43). What is left of  $\sigma_{1\times 1} + 2\text{Re}(\sigma_9)$  is a finite piece that depends on  $\mu$ ,

while the fragmentation function in the LO term is obtained from (44) as the solution of the DGLAP evolution equation  $(d/d \ln \mu^2) D_{h/q}(z, \mu^2) = \dots$  where the right-hand side is now free of the infrared divergence. Finally, in the leftover finite piece of  $\sigma_{1 \times 1} + 2\text{Re}(\sigma_9)$ , as well as in all the other NLO corrections, the bare fragmentation function can be replaced by the renormalized one, up to corrections of even higher order beyond the accuracy of our calculation.

### C. Soft divergences

We refer to soft divergences as those that arise when both the transverse and longitudinal momenta in the loop go to zero simultaneously, i.e.,  $\mathbf{k} \rightarrow \mathbf{0}, z \rightarrow 0$ . In our coordinate space expression, this translates to  $|\mathbf{x}_3| \rightarrow \infty, z \rightarrow 0$ . Those fully cancel as follows:

$$\begin{aligned}
 & [\text{d}\sigma_{1 \times 1} + 2 \text{Re}(\text{d}\sigma_9)]_{\text{soft}} = 0, \\
 & [\text{d}\sigma_{1 \times 2} + \text{d}\sigma_{13(1)} + \text{d}\sigma_{13(2)}]_{\text{soft}} = [\text{d}\sigma_{2 \times 1} + \text{d}\sigma_{13(1)}^* + \text{d}\sigma_{13(2)}^*]_{\text{soft}} = 0, \\
 & [\text{d}\sigma_{3 \times 3} + \text{d}\sigma_{4 \times 4} + 2 \text{Re}(\text{d}\sigma_{3 \times 4})]_{\text{soft}} = 0, \\
 & [\text{d}\sigma_{1 \times 3} + \text{d}\sigma_{1 \times 4}]_{\text{soft}} = [\text{d}\sigma_{3 \times 1} + \text{d}\sigma_{4 \times 1}]_{\text{soft}} = 0, \\
 & [\text{d}\sigma_5 + \text{d}\sigma_7]_{\text{soft}} = [\text{d}\sigma_5^* + \text{d}\sigma_7^*]_{\text{soft}} = 0, \\
 & [\text{d}\sigma_{11} + \text{d}\sigma_{14(1)}]_{\text{soft}} = [\text{d}\sigma_{11}^* + \text{d}\sigma_{14(1)}^*]_{\text{soft}} = 0, \\
 & [\text{d}\sigma_{12} + \text{d}\sigma_{14(2)}]_{\text{soft}} = [\text{d}\sigma_{12}^* + \text{d}\sigma_{14(2)}^*]_{\text{soft}} = 0.
 \end{aligned} \tag{45}$$

### D. Rapidity divergences and small $x$ evolution

Rapidity divergences appear when  $z \rightarrow 0$  at nonzero transverse momentum. In that case, none of the soft cancellations of Sec. IV C survive, except

$$\begin{aligned}
 & [\text{d}\sigma_{1 \times 2} + \text{d}\sigma_{13(1)} + \text{d}\sigma_{13(2)}]_{z=0} \\
 & = [\text{d}\sigma_{2 \times 1} + \text{d}\sigma_{13(1)}^* + \text{d}\sigma_{13(2)}^*]_{z=0} = 0.
 \end{aligned} \tag{46}$$

We note that when considering the double-inclusive cross section, this was not the case:  $\sigma_{1 \times 2}$ ,  $\sigma_{13}$ ,  $\sigma_{2 \times 1}$ , and  $\sigma_{13}^*$  also had to be considered in what follows.

The surviving rapidity divergences can be isolated by introducing a rapidity factorization scale  $z_f$  and writing the  $z$  integral as

$$\int_0^1 \frac{dz}{z} f(z) = \left\{ \int_0^{z_f} \frac{dz}{z} + \int_{z_f}^1 \frac{dz}{z} \right\} f(z) \tag{47}$$

where the first integral contains the  $z \rightarrow 0$  divergence while the second integral contains the finite NLO pieces. When putting together all the divergent terms, we obtain

$$\begin{aligned}
 & \frac{\text{d}\sigma_{\text{NLO}}^T}{d^2 \mathbf{p} dy_1} \Big|_{z \rightarrow 0} = \frac{e^2 Q^2 N_c}{(2\pi)^5} (1 - z_1) z_1^2 [z_1^2 + (1 - z_1)^2] \int d^6 \mathbf{x} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{\mathbf{x}_{12}^2 \mathbf{x}_{1'2}^2} e^{i \mathbf{p} \cdot \mathbf{x}_{1'1}} \\
 & \times \frac{\alpha_s N_c}{2\pi^2} \int_0^{z_f} \frac{dz}{z} \int d^2 \mathbf{x}_3 \left[ \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{1'3}^2} (S_{31'} S_{13} - S_{11'}) - \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} (S_{31} S_{23} - S_{12}) - \frac{\mathbf{x}_{21'}^2}{\mathbf{x}_{23}^2 \mathbf{x}_{1'3}^2} (S_{31'} S_{23} - S_{21'}) \right]
 \end{aligned} \tag{48}$$

similarly to the case of SIDIS with a longitudinal photon exchange [49], although we have now obtained that result at finite  $N_c$ .

The individual contributions of all the diagrams to (48) are listed in Appendix B. As for contributions of diagrams  $\sigma_{1 \times 1}$ ,  $\sigma_9$ , and  $\sigma_9^*$ , their  $z \rightarrow 0$  limits have been obtained prior to the extraction of (43). Because the collinear divergence in (43) cancels in the  $z \rightarrow 0$  limit, their contribution to (48) may also be extracted from  $[\sigma_{1 \times 1} + 2\text{Re}(\sigma_9)]_{\text{finite}}$ , while

ignoring the  $\mu$  dependence, providing  $z_f$  is small enough. Behind the question of how small  $z_f$  should be lies the issue of large Sudakov logarithms discussed in [50].

Comparing (48) to the LO result in Eq. (4) it is clear that the terms inside the square brackets correspond to the BK/JIMWLK evolution [4,5,8–10] of the dipoles that appear in the LO cross section. In other words, this rapidity-divergent contribution can be absorbed into the LO cross section by renormalizing the dipole amplitudes at the scale  $z_f$ :

$$S_{ij}(z_f) = S_{ij} + \mathcal{O}(\alpha_s \ln(z_f)) \quad (49)$$

and obtaining them as the solution of the BK equation  $d/d \ln(z_f) S_{ij}(z_f) = \dots$  where the right-hand side is now divergence-free.

The contribution of the  $\int_{z_f}^1 dz$  region is finite and  $z_f$  dependent. Along with the other NLO corrections not affected by the rapidity divergence, they constitute the NLO correction to the LO result. Up to higher-order corrections beyond the accuracy of our calculation, we are free to promote all the bare dipole amplitudes involved to renormalized ones. The full result for the single inclusive cross section at next-to-leading order can be written schematically as

$$d\sigma^{A \rightarrow hX} = d\sigma_{\text{LO}}[S_{ij}(z_f), D_{h/q}(z_h, \mu^2)] + d\sigma_{\text{NLO}}^{\text{finite}}(z_f, \mu^2). \quad (50)$$

## V. SUMMARY AND OUTLOOK

We have calculated the next-to-leading-order corrections to single inclusive hadron production in DIS at small  $x$  in the color glass condensate formalism when the exchanged photon is transversely polarized. We have shown that all UV and soft divergences cancel. There are collinear divergences that are absorbed into a quark-hadron fragmentation function and cause it to evolve with DGLAP

evolution equation. On the other hand the rapidity divergences are absorbed into rapidity evolution of dipoles with the BK equation. The final result is finite and given at full  $N_c$ .

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## DATA AVAILABILITY

No data were created or analyzed in this study.

## APPENDIX A: QUARK-ANTIQUARK PRODUCTION CROSS SECTION AT NLO

Here we write the quark antiquark production cross section including the full  $N_c$  structure,

$$\begin{aligned} \frac{d\sigma_{1 \times 1}^T}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} &= \frac{e^2 g^2 Q^2 N_c z_2^2 (1 - z_2) [z_1^2 z_2^2 + (z_1^2 + z_2^2)(1 - z_2)^2 + (1 - z_2)^4]}{(2\pi)^{10} z_1} \int d^{10} \mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \\ &\times \int \frac{dz}{z} e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} e^{i\mathbf{q} \cdot \mathbf{x}_{2'2}} e^{i\frac{z}{z_1} \mathbf{p} \cdot \mathbf{x}_{1'1}} K_1(|\mathbf{x}_{12}|Q_2) K_1(|\mathbf{x}_{1'2'}|Q_2) \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2'}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2'}|} \Delta_{1'1}^{(3)}. \end{aligned} \quad (A1)$$

$$\begin{aligned} \frac{d\sigma_{2 \times 2}^T}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} &= \frac{e^2 g^2 Q^2 N_c z_1^2 (1 - z_1) [z_1^2 z_2^2 + (z_1^2 + z_2^2)(1 - z_1)^2 + (1 - z_1)^4]}{(2\pi)^{10} z_2} \int d^{10} \mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \\ &\times \int \frac{dz}{z} e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} e^{i\mathbf{q} \cdot \mathbf{x}_{2'2}} e^{i\frac{z}{z_2} \mathbf{q} \cdot \mathbf{x}_{2'2}} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2'}|Q_1) \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2'}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2'}|} \Delta_{2'2}^{(3)}. \end{aligned} \quad (A2)$$

$$\begin{aligned} \frac{d\sigma_{1 \times 2}^T}{d^2 \mathbf{p} d^2 \mathbf{q} dy_1 dy_2} &= \frac{e^2 g^2 Q^2 N_c \sqrt{z_1 z_2 (1 - z_1)(1 - z_2)}}{(2\pi)^{10}} \int d^{10} \mathbf{x} \left\{ \frac{N_c}{2} S_{12} S_{2'1'} + C_F [1 - S_{12} - S_{2'1'}] - \frac{1}{2N_c} S_{122'1'} \right\} \\ &\times \int \frac{dz}{z} e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} e^{i\mathbf{q} \cdot \mathbf{x}_{2'2}} e^{i\frac{z}{z_1} \mathbf{p} \cdot \mathbf{x}_{31}} e^{i\frac{z}{z_2} \mathbf{q} \cdot \mathbf{x}_{2'3}} K_1(|\mathbf{x}_{12}|Q_2) K_1(|\mathbf{x}_{1'2'}|Q_1) \\ &\times 4 \text{Re} \left[ \frac{(\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2'}|} \left\{ (z_1^2 + z_2^2)(1 - z_1)(1 - z_2) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{2'3} \cdot \boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2 \mathbf{x}_{2'3}^2} + z_1 z_2 ((1 - z_1)^2 \right. \right. \\ &\left. \left. + (1 - z_2)^2) \frac{(\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*)(\mathbf{x}_{2'3} \cdot \boldsymbol{\epsilon})}{\mathbf{x}_{31}^2 \mathbf{x}_{2'3}^2} \right\} \right]. \end{aligned} \quad (A3)$$

$$\begin{aligned}
\frac{d\sigma_{3\times 3}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = & \frac{e^2g^2Q^2N_cz_1z_2^3}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{11'}S_{22'} - S_{13}S_{32} - S_{31'}S_{2'3}] + C_F - \frac{1}{2N_c} [S_{122'1'} - S_{12} - S_{2'1'}] \right\} \\
& \times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(QX)K_1(QX')}{XX'} 4\text{Re} \left[ (z_1^2 + z_2^2) \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2\mathbf{x}_{31'}^2} [(\mathbf{z}_1\mathbf{x}_{12} + \mathbf{z}\mathbf{x}_{32})\cdot\boldsymbol{\epsilon}] [(\mathbf{z}_1\mathbf{x}_{1'2'} + \mathbf{z}\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*] \right. \\
& + \left( (1-z_2)^2 + \frac{(z_1z_2)^2}{(1-z_2)^2} \right) \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{31}^2\mathbf{x}_{31'}^2} [(\mathbf{z}_1\mathbf{x}_{12} + \mathbf{z}\mathbf{x}_{32})\cdot\boldsymbol{\epsilon}] [(\mathbf{z}_1\mathbf{x}_{1'2'} + \mathbf{z}\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*] \\
& \left. - \frac{z_1^2z_2z}{2(1-z_2)^2} \left\{ \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} [(\mathbf{z}_1\mathbf{x}_{12} + \mathbf{z}\mathbf{x}_{32})\cdot\boldsymbol{\epsilon}] + \frac{(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{31'}^2} [(\mathbf{z}_1\mathbf{x}_{1'2'} + \mathbf{z}\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*] \right\} + \frac{z_1^2z^2}{4(1-z_2)^2} \right]. \quad (\text{A4})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{4\times 4}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = & \frac{e^2g^2Q^2N_cz_2z_1^3}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{11'}S_{22'} - S_{13}S_{32} - S_{31'}S_{2'3}] + C_F - \frac{1}{2N_c} [S_{122'1'} - S_{12} - S_{2'1'}] \right\} \\
& \times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(QX)K_1(QX')}{XX'} 4\text{Re} \left[ (z_1^2 + z_2^2) \frac{(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{32}^2\mathbf{x}_{32'}^2} [(\mathbf{z}_2\mathbf{x}_{21} + \mathbf{z}\mathbf{x}_{31})\cdot\boldsymbol{\epsilon}] [(\mathbf{z}_2\mathbf{x}_{2'1'} + \mathbf{z}\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*] \right. \\
& + \left( (1-z_1)^2 + \frac{(z_1z_2)^2}{(1-z_1)^2} \right) \frac{(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon}^*)(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{32}^2\mathbf{x}_{32'}^2} [(\mathbf{z}_2\mathbf{x}_{21} + \mathbf{z}\mathbf{x}_{31})\cdot\boldsymbol{\epsilon}] [(\mathbf{z}_2\mathbf{x}_{2'1'} + \mathbf{z}\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*] \\
& \left. - \frac{z_2^2z_1z}{2(1-z_1)^2} \left\{ \frac{(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{32}^2} [(\mathbf{z}_2\mathbf{x}_{21} + \mathbf{z}\mathbf{x}_{31})\cdot\boldsymbol{\epsilon}] + \frac{(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{32'}^2} [(\mathbf{z}_2\mathbf{x}_{2'1'} + \mathbf{z}\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*] \right\} + \frac{z_2^2z^2}{4(1-z_1)^2} \right]. \quad (\text{A5})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{3\times 4}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = & \frac{e^2g^2Q^2N_c(z_1z_2)^2}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{11'}S_{22'} - S_{13}S_{32} - S_{31'}S_{2'3}] + C_F - \frac{1}{2N_c} [S_{122'1'} - S_{12} - S_{2'1'}] \right\} \\
& \times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(QX)K_1(QX')}{XX'} 4\text{Re} \left[ (z_1^2 + z_2^2) \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2\mathbf{x}_{32'}^2} [(\mathbf{z}_1\mathbf{x}_{12} + \mathbf{z}\mathbf{x}_{32})\cdot\boldsymbol{\epsilon}] [(\mathbf{z}_2\mathbf{x}_{2'1'} + \mathbf{z}\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*] \right. \\
& + \frac{z_1z_2}{(1-z_1)(1-z_2)} [(1-z_1)^2 + (1-z_2)^2] \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{31}^2\mathbf{x}_{32'}^2} [(\mathbf{z}_1\mathbf{x}_{12} + \mathbf{z}\mathbf{x}_{32})\cdot\boldsymbol{\epsilon}] [(\mathbf{z}_2\mathbf{x}_{2'1'} + \mathbf{z}\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*] \\
& \left. - \frac{z_2z(1-z_2)}{2(1-z_1)} \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} [(\mathbf{z}_1\mathbf{x}_{12} + \mathbf{z}\mathbf{x}_{32})\cdot\boldsymbol{\epsilon}] - \frac{z_1z(1-z_1)}{2(1-z_2)} \frac{(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{32'}^2} [(\mathbf{z}_2\mathbf{x}_{2'1'} + \mathbf{z}\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*] \right]. \quad (\text{A6})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{1\times 3}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} = & \frac{-e^2g^2Q^2N_cz_2^{5/2}\sqrt{1-z_2}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{122'3}S_{31'} - S_{31'}S_{2'3}] + C_F [1 - S_{12}] - \frac{1}{2N_c} [S_{122'1'} - S_{2'1'}] \right\} \\
& \times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{i\frac{z}{z_1}\mathbf{p}\cdot\mathbf{x}_{31}} \frac{K_1(|\mathbf{x}_{12}|Q_2)K_1(QX')}{X'} \\
& \times 4\text{Re} \left[ (1-z_2)(z_1^2 + z_2^2) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}|\mathbf{x}_{31'}^2} \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon})}{\mathbf{x}_{31}^2} [(\mathbf{z}_1\mathbf{x}_{1'2'} + \mathbf{z}\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*] \right. \\
& + \left( (1-z_2)^3 + \frac{(z_1z_2)^2}{1-z_2} \right) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon})}{|\mathbf{x}_{12}|\mathbf{x}_{31'}^2} \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} [(\mathbf{z}_1\mathbf{x}_{1'2'} + \mathbf{z}\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*] - \frac{z_1^2z_2z}{2(1-z_2)} \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})}{|\mathbf{x}_{12}|} \frac{(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)}{\mathbf{x}_{31}^2} \left. \right]. \quad (\text{A7})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{1\times 4}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-e^2g^2Q^2N_cz_1z_2^{3/2}\sqrt{1-z_2}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{122'3}S_{31'} - S_{31'}S_{2'3}] + C_F[1 - S_{12}] - \frac{1}{2N_c} [S_{122'1'} - S_{2'1'}] \right\} \\
&\times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{i\frac{z}{z_1}\mathbf{p}\cdot\mathbf{x}_{31}} \frac{K_1(|\mathbf{x}_{12}|Q_2)K_1(QX')}{X'} \\
&\times 4 \operatorname{Re} \left[ (1-z_2)(z_1^2+z_2^2) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon}^*)(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon})[(z_2\mathbf{x}_{2'1'}+z\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*]}{|\mathbf{x}_{12}|\mathbf{x}_{32'}^2\mathbf{x}_{31}^2} \right. \\
&+ \frac{z_1z_2}{1-z_1} ((1-z_1)^2 + (1-z_2)^2) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)[(z_2\mathbf{x}_{2'1'}+z\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*]}{|\mathbf{x}_{12}|\mathbf{x}_{32'}^2\mathbf{x}_{31}^2} \\
&\left. - \frac{z_2z(1-z_2)^2}{2(1-z_1)} \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31}\cdot\boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}|\mathbf{x}_{31}^2} \right]. \quad (\text{A8})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{2\times 3}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_cz_2z_1^{3/2}\sqrt{1-z_1}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{1231'}S_{2'3} - S_{31'}S_{2'3}] + C_F[1 - S_{12}] - \frac{1}{2N_c} [S_{122'1'} - S_{2'1'}] \right\} \\
&\times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{i\frac{z}{z_2}\mathbf{q}\cdot\mathbf{x}_{32}} \frac{K_1(|\mathbf{x}_{12}|Q_1)K_1(QX')}{X'} \\
&\times 4 \operatorname{Re} \left[ (1-z_1)(z_1^2+z_2^2) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon}^*)(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon})[(z_1\mathbf{x}_{1'2'}+z\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*]}{|\mathbf{x}_{12}|\mathbf{x}_{31'}^2\mathbf{x}_{32}^2} \right. \\
&+ \frac{z_1z_2}{1-z_2} ((1-z_1)^2 + (1-z_2)^2) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{31'}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon}^*)[(z_1\mathbf{x}_{1'2'}+z\mathbf{x}_{32'})\cdot\boldsymbol{\epsilon}^*]}{|\mathbf{x}_{12}|\mathbf{x}_{31'}^2\mathbf{x}_{32}^2} \\
&\left. - \frac{z_1z(1-z_1)^2}{2(1-z_2)} \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}|\mathbf{x}_{32}^2} \right]. \quad (\text{A9})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{2\times 4}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_cz_1^{5/2}\sqrt{1-z_1}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c^2}{2} [S_{1231'}S_{2'3} - S_{31'}S_{2'3}] + C_F[1 - S_{12}] - \frac{1}{2N_c} [S_{122'1'} - S_{2'1'}] \right\} \\
&\times \int \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{i\frac{z}{z_2}\mathbf{q}\cdot\mathbf{x}_{32}} \frac{K_1(|\mathbf{x}_{12}|Q_1)K_1(QX')}{X'} \\
&\times 4 \operatorname{Re} \left[ (1-z_1)(z_1^2+z_2^2) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon}^*)(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon})[(z_2\mathbf{x}_{2'1'}+z\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*]}{|\mathbf{x}_{12}|\mathbf{x}_{32'}^2\mathbf{x}_{32}^2} \right. \\
&+ \left( (1-z_1)^3 + \frac{(z_1z_2)^2}{1-z_1} \right) \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32'}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon}^*)[(z_2\mathbf{x}_{2'1'}+z\mathbf{x}_{31'})\cdot\boldsymbol{\epsilon}^*]}{|\mathbf{x}_{12}|\mathbf{x}_{32'}^2\mathbf{x}_{32}^2} \\
&\left. - \frac{z_2^2z_1z}{2(1-z_1)} \frac{(\mathbf{x}_{12}\cdot\boldsymbol{\epsilon})(\mathbf{x}_{32}\cdot\boldsymbol{\epsilon}^*)}{|\mathbf{x}_{12}|\mathbf{x}_{32}^2} \right]. \quad (\text{A10})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_5^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_cz_2^{5/2}\sqrt{z_1}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{13}S_{322'1'} - S_{32}S_{13}] + C_F[1 - S_{2'1'}] - \frac{1}{2N_c} [S_{122'1'} - S_{12}] \right\} \\
&\times \int_0^{z_1} \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{-i\frac{z}{z_1}\mathbf{p}\cdot\mathbf{x}_{31}} \frac{K_1(QX_5)K_1(|\mathbf{x}_{1'2'}|Q_1)}{X_5\mathbf{x}_{31}^2|\mathbf{x}_{1'2'}|} \\
&\times \mathbf{x}_{1'2'} \cdot [(z_1^2+z_2^2)(z_1^2+(z_1-z)^2)\mathbf{x}_{32} + (z_1-z)(z_1(z_1^2+(z_1-z)^2) + z_2[z_1z_2 + (z_1-z)(z_2+z)])\mathbf{x}_{13}]. \quad (\text{A11})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_6^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= -\frac{e^2g^2Q^2N_cz_1^{5/2}\sqrt{z_2}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{2'1'13}S_{32} - S_{13}S_{32}] + C_F[1 - S_{2'1'}] - \frac{1}{2N_c} [S_{122'1'} - S_{12}] \right\} \\
&\times \int_0^{z_2} \frac{dz}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{-i\frac{z}{z_2}\mathbf{q}\cdot\mathbf{x}_{32}} \frac{K_1(QX_6)K_1(|\mathbf{x}_{1'2'}|Q_1)}{X_6|\mathbf{x}_{1'2'}|\mathbf{x}_{32}^2} \\
&\times \mathbf{x}_{1'2'} \cdot [(z_1^2+z_2^2)(z_2^2+(z_2-z)^2)\mathbf{x}_{31} + (z_2-z)[z_2(z_2^2+(z_2-z)^2) + z_1(z_1z_2 + (z_2-z)(z_1+z))]\mathbf{x}_{23}]. \quad (\text{A12})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_7^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c(z_1z_2)^{3/2}}{(2\pi)^{10}} \int d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{13}S_{322'1'} - S_{32}S_{13}] + C_F[1 - S_{2'1'}] - \frac{1}{2N_c} [S_{122'1'} - S_{12}] \right\} \\
&\times \int_0^{z_1} \frac{dz(z_1 - z)}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{-i\frac{z}{z_1}\mathbf{p}\cdot\mathbf{x}_{31}} \frac{K_1(QX_5)K_1(|\mathbf{x}_{1'2'}|Q_1)}{X_5\mathbf{x}_{31}^2|\mathbf{x}_{1'2'}|} \\
&\times \left[ \frac{4\text{Re}}{\mathbf{x}_{32}^2} \left\{ (\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*) \left[ \left( \mathbf{x}_{31} + \frac{z_2}{z_2 + z} \mathbf{x}_{23} \right) \cdot \boldsymbol{\epsilon} \right] \left[ \frac{z_2(z_1 - z)}{z_1} (z_1^2 + (z_2 + z)^2) (\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}) (\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*) \right. \right. \right. \\
&\left. \left. \left. + (z_2 + z)(z_2^2 + (z_1 - z)^2) (\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}) (\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*) \right] \right\} - \frac{z_1 z_2 z}{z_2 + z} \mathbf{x}_{31} \cdot \mathbf{x}_{1'2'} \right]. \quad (\text{A13})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_8^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-e^2g^2Q^2N_c(z_1z_2)^{3/2}}{(2\pi)^{10}} d^{10}\mathbf{x} \left\{ \frac{N_c}{2} [S_{2'1'13}S_{32} - S_{13}S_{32}] + C_F[1 - S_{2'1'}] - \frac{1}{2N_c} [S_{122'1'} - S_{12}] \right\} \\
&\times \int_0^{z_2} \frac{dz(z_2 - z)}{z} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} e^{-i\frac{z}{z_2}\mathbf{q}\cdot\mathbf{x}_{32}} \frac{K_1(QX_6)K_1(|\mathbf{x}_{1'2'}|Q_1)}{X_6|\mathbf{x}_{1'2'}|\mathbf{x}_{32}^2} \\
&\times \left[ \frac{4\text{Re}}{\mathbf{x}_{31}^2} \left\{ (\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*) \left[ \left( \mathbf{x}_{32} + \frac{z_1}{z_1 + z} \mathbf{x}_{13} \right) \cdot \boldsymbol{\epsilon} \right] \left[ (z_1 + z)(z_1^2 + (z_2 - z)^2) (\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}) (\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}^*) \right. \right. \right. \\
&\left. \left. \left. + \frac{z_1(z_2 - z)}{z_2} (z_2^2 + (z_1 + z)^2) (\mathbf{x}_{32} \cdot \boldsymbol{\epsilon}) (\mathbf{x}_{31} \cdot \boldsymbol{\epsilon}^*) \right] \right\} - \frac{z_1 z_2 z}{z_1 + z} \mathbf{x}_{32} \cdot \mathbf{x}_{1'2'} \right]. \quad (\text{A14})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_9^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-e^2g^2Q^2N_c(z_1z_2)^2(z_1^2 + z_2^2)}{2(2\pi)^8} \int d^8\mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2'}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2'}|} \\
&\times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2'}|Q_1) \int_0^{z_1} \frac{dz}{z} \left[ \frac{z_1^2 + (z_1 - z)^2}{z_1^2} \right] \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k} - \frac{z}{z_1}\mathbf{p})^2}. \quad (\text{A15})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{10}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-e^2g^2Q^2N_c(z_1z_2)^2(z_1^2 + z_2^2)}{2(2\pi)^8} \int d^8\mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2'}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2'}|} \\
&\times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2'}|Q_1) \int_0^{z_2} \frac{dz}{z} \left[ \frac{z_2^2 + (z_2 - z)^2}{z_2^2} \right] \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k} - \frac{z}{z_2}\mathbf{q})^2}. \quad (\text{A16})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{11}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{ie^2g^2QN_cz_2^{3/2}\sqrt{z_1}(z_1^2 + z_2^2)}{(2\pi)^7} \int d^8\mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \\
&\times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(|\mathbf{x}_{1'2'}|Q_1)}{|\mathbf{x}_{1'2'}|} \int_0^{z_1} \frac{dz}{z^2} \frac{[(z_1 - z)^2 + z_1^2]}{(z_1 - z)} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{\mathbf{k}_1 \cdot \mathbf{x}_{1'2'}}{[\mathbf{k}_1^2 + Q_1^2][Q^2 + \frac{\mathbf{k}_1^2}{z_1 z_2} + \frac{z_1}{z(z_1 - z)}\mathbf{k}_2^2]} e^{i\mathbf{k}_1 \cdot \mathbf{x}_{12}}. \quad (\text{A17})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{12}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-ie^2g^2QN_cz_1^{3/2}\sqrt{z_2}(z_1^2 + z_2^2)}{(2\pi)^7} \int d^8\mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \\
&\times e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(|\mathbf{x}_{1'2'}|Q_1)}{|\mathbf{x}_{1'2'}|} \int_0^{z_2} \frac{dz}{z^2} \frac{[(z_2 - z)^2 + z_2^2]}{(z_2 - z)} \int \frac{d^2\mathbf{k}_2}{(2\pi)^2} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{\mathbf{k}_1 \cdot \mathbf{x}_{1'2'}}{[\mathbf{k}_1^2 + Q_1^2][Q^2 + \frac{\mathbf{k}_1^2}{z_1 z_2} + \frac{z_2}{z(z_2 - z)}\mathbf{k}_2^2]} e^{i\mathbf{k}_1 \cdot \mathbf{x}_{21}}. \quad (\text{A18})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{13(1)}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c(z_1z_2)^{3/2}}{(2\pi)^8} \int d^8\mathbf{x} \left\{ \frac{N_c}{2} [S_{2'1'}S_{12}] + C_F[1 - S_{2'1'} - S_{12}] - \frac{1}{2N_c} [S_{122'1'}] \right\} \\
&\times \int_0^{z_2} dz \sqrt{(z_1+z)(z_2-z)} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(|\mathbf{x}_{12}|Q\sqrt{(z_1+z)(z_2-z)})}{|\mathbf{x}_{12}|} \frac{K_1(|\mathbf{x}_{1'2'}|Q_1)}{|\mathbf{x}_{1'2'}|} \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}_{21}} \\
&\times 4 \operatorname{Re} \left[ (\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*) \left\{ \frac{\frac{z_2(z_2-z)[z_1(z_1+z)+z_2(z_2-z)]}{2z}}{(z_2\mathbf{k}-z\mathbf{q})^2} \right. \right. \\
&+ z_1z_2z \left( \frac{z_1z(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}) + z^2(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}) - z_2z(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2}) - p \cdot q}{(z_2\mathbf{k}-z\mathbf{q})^2 \left[ \frac{(z_1\mathbf{k}-z\mathbf{p})^2}{z_1(z_1+z)} - \frac{(z_2\mathbf{k}-z\mathbf{q})^2}{z_2(z_2-z)} \right]} \right) \\
&\left. \left. + z_2^2z(z_2-z) \left( \frac{z_1z(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}) + z^2(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}) - z_2z(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2}) - p \cdot q}{(z_1+z)(z_2\mathbf{k}-z\mathbf{q})^2 \left[ \frac{(z_1\mathbf{k}-z\mathbf{p})^2}{z_1(z_1+z)} - \frac{(z_2\mathbf{k}-z\mathbf{q})^2}{z_2(z_2-z)} \right]} \right) \right\} \right]. \quad (\text{A19})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{13(2)}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{e^2g^2Q^2N_c(z_1z_2)^{3/2}}{(2\pi)^8} \int d^8\mathbf{x} \left\{ \frac{N_c}{2} [S_{2'1'}S_{12}] + C_F[1 - S_{2'1'} - S_{12}] - \frac{1}{2N_c} [S_{122'1'}] \right\} \\
&\times \int_0^{z_1} dz \sqrt{(z_1-z)(z_2+z)} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \frac{K_1(|\mathbf{x}_{12}|Q\sqrt{(z_1-z)(z_2+z)})}{|\mathbf{x}_{12}|} \frac{K_1(|\mathbf{x}_{1'2'}|Q_1)}{|\mathbf{x}_{1'2'}|} \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}_{12}} \\
&\times 4 \operatorname{Re} \left[ (\mathbf{x}_{12} \cdot \boldsymbol{\epsilon})(\mathbf{x}_{1'2'} \cdot \boldsymbol{\epsilon}^*) \left\{ \frac{\frac{z_1(z_1-z)[z_1(z_1-z)+z_2(z_2+z)]}{2z}}{(z_1\mathbf{k}-z\mathbf{p})^2} \right. \right. \\
&+ z_1z_2z \left( \frac{z_1z(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}) - z^2(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}) - z_2z(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2}) + p \cdot q}{(z_1\mathbf{k}-z\mathbf{p})^2 \left[ \frac{(z_1\mathbf{k}-z\mathbf{p})^2}{z_1(z_1-z)} - \frac{(z_2\mathbf{k}-z\mathbf{q})^2}{z_2(z_2+z)} \right]} \right) \\
&\left. \left. + z_1^2z(z_1-z) \left( \frac{z_1z(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}) - z^2(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}^*}{z}) - z_2z(\frac{\mathbf{k}\cdot\boldsymbol{\epsilon}}{z}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}}{z_2})(\frac{\mathbf{p}\cdot\boldsymbol{\epsilon}^*}{z_1}-\frac{\mathbf{q}\cdot\boldsymbol{\epsilon}^*}{z_2}) + p \cdot q}{(z_2+z)(z_1\mathbf{k}-z\mathbf{p})^2 \left[ \frac{(z_1\mathbf{k}-z\mathbf{p})^2}{z_1(z_1-z)} - \frac{(z_2\mathbf{k}-z\mathbf{q})^2}{z_2(z_2+z)} \right]} \right) \right\} \right]. \quad (\text{A20})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{14(1)}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-ie^2g^2QN_c(z_1z_2)^{3/2}}{(2\pi)^7} \int d^8\mathbf{x} C_F \{ S_{122'1'} - S_{12} - S_{2'1'} + 1 \} \frac{K_1(|\mathbf{x}_{1'2'}|Q_1)}{|\mathbf{x}_{1'2'}|} e^{i\mathbf{p}\cdot\mathbf{x}_{1'1}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'2}} \\
&\times \int_0^{z_1} \frac{dz}{z} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{d^2\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_2\cdot\mathbf{x}_{12}} \left\{ \frac{[z_1(z_1-z) + z_2(z_2+z) - z(1-z)](\mathbf{k}_2 \cdot \mathbf{x}_{1'2'})}{[\mathbf{k}_2^2 + Q_1^2][(\mathbf{k}_1 - \frac{z_1-z}{z_1}\mathbf{k}_2)^2 + \frac{z(z_1-z)}{z_2z_1^2}\mathbf{k}_2^2 + \frac{z}{z_1}(z_1-z)Q^2]} \right. \\
&+ \frac{\frac{(z_1-z)}{z_1}(1+z^2-2z_2(z_1-z))(\mathbf{k}_1 \cdot \mathbf{x}_{1'2'})}{[\mathbf{k}_1^2 + (z_1-z)(z_2+z)Q^2][(\mathbf{k}_1 - \frac{z_1-z}{z_1}\mathbf{k}_2)^2 + \frac{z(z_1-z)}{z_2z_1^2}\mathbf{k}_2^2 + \frac{z}{z_1}(z_1-z)Q^2]} \\
&\left. - \frac{Q^2 \frac{(z_1-z)}{z_1} [2z_1z_2z(\mathbf{k}_1 \cdot \mathbf{x}_{1'2'}) + z(z+z_2-z_1)^2(\mathbf{k}_2 \cdot \mathbf{x}_{1'2'})]}{[\mathbf{k}_1^2 + (z_1-z)(z_2+z)Q^2][\mathbf{k}_2^2 + Q_1^2][(\mathbf{k}_1 - \frac{z_1-z}{z_1}\mathbf{k}_2)^2 + \frac{z(z_1-z)}{z_2z_1^2}\mathbf{k}_2^2 + \frac{z}{z_1}(z_1-z)Q^2]} \right\}. \quad (\text{A21})
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{14(2)}^T}{d^2\mathbf{p}d^2\mathbf{q}dy_1dy_2} &= \frac{-ie^2g^2QN_c(z_1z_2)^{3/2}}{(2\pi)^7} \int d^8\mathbf{x} C_F \{S_{122'1'} - S_{12} - S_{2'1'} + 1\} \frac{K_1(|\mathbf{x}_{1'2'}|Q_1)}{|\mathbf{x}_{1'2'}|} e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} e^{i\mathbf{q}\cdot\mathbf{x}_{2'}} \\
&\times \int_0^{z_2} \frac{dz}{z} \int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \frac{d^2\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_2\cdot\mathbf{x}_{12}} \left\{ \frac{[z_2(z_2-z) + z_1(z_1+z) - z(1-z)](\mathbf{k}_2\cdot\mathbf{x}_{1'2'})}{[\mathbf{k}_2^2 + Q_1^2][(\mathbf{k}_1 - \frac{z_2-z}{z_2}\mathbf{k}_2)^2 + \frac{z(z_2-z)}{z_1z_2^2}\mathbf{k}_2^2 + \frac{z}{z_2}(z_2-z)Q^2]} \right. \\
&+ \frac{\frac{(z_2-z)}{z_2}(1+z^2-2z_1(z_2-z))(\mathbf{k}_1\cdot\mathbf{x}_{1'2'})}{[\mathbf{k}_1^2 + (z_2-z)(z_1+z)Q^2][(\mathbf{k}_1 - \frac{z_2-z}{z_2}\mathbf{k}_2)^2 + \frac{z(z_2-z)}{z_1z_2^2}\mathbf{k}_2^2 + \frac{z}{z_2}(z_2-z)Q^2]} \\
&\left. - \frac{Q^2\frac{(z_2-z)}{z_2}[2z_1z_2z(\mathbf{k}_1\cdot\mathbf{x}_{1'2'}) + z(z+z_1-z_2)^2(\mathbf{k}_2\cdot\mathbf{x}_{1'2'})]}{[\mathbf{k}_1^2 + (z_2-z)(z_1+z)Q^2][\mathbf{k}_2^2 + Q_1^2][(\mathbf{k}_1 - \frac{z_2-z}{z_2}\mathbf{k}_2)^2 + \frac{z(z_2-z)}{z_1z_2^2}\mathbf{k}_2^2 + \frac{z}{z_2}(z_2-z)Q^2]} \right\} \quad (A22)
\end{aligned}$$

## APPENDIX B: $z \rightarrow 0$ LIMIT OF THE NLO SIDIS CROSS SECTION WITH TRANSVERSE PHOTON

### 1. Real corrections

$$\begin{aligned}
\frac{d\sigma_{1\times 1}^T}{d^2\mathbf{p}dy_1} &= 2 \frac{e^2g^2Q^2N_c}{(2\pi)^8} z_1^2z_2(z_1^2+z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} \\
&\times C_F [S_{11'} - S_{12} - S_{21'} + 1] \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{1'3}^2} - \frac{1}{2} \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{13}^2\mathbf{x}_{1'3}^2} \right). \quad (B1)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{1\times 2}^T}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2g^2Q^2N_c}{(2\pi)^8} z_1^2z_2(z_1^2+z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} \\
&\times [C_F [S_{12}S_{21'} - S_{12} - S_{21'} + 1] - \frac{1}{2N_c} (S_{11'} - S_{12}S_{21'})] \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2\mathbf{x}_{23}^2} \right). \quad (B2)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{2\times 1}^T}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2g^2Q^2N_c}{(2\pi)^8} z_1^2z_2(z_1^2+z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} \\
&\times [C_F [S_{21}S_{1'2} - S_{21} - S_{1'2} + 1] - \frac{1}{2N_c} (S_{1'1} - S_{21}S_{1'2})] \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2\mathbf{x}_{23}^2} \right). \quad (B3)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{3\times 3}^T}{d^2\mathbf{p}dy_1} &= 2 \frac{e^2g^2Q^2N_c}{(2\pi)^8} z_1^2z_2(z_1^2+z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} \\
&\times \left\{ C_F [S_{11'} - S_{13}S_{32} - S_{31'}S_{23} + 1] - \frac{1}{2N_c} [S_{13}S_{32} + S_{31'}S_{23} - S_{12} - S_{21'}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{1'3}^2} - \frac{1}{2} \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{13}^2\mathbf{x}_{1'3}^2} \right). \quad (B4)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{4\times 4}^T}{d^2\mathbf{p}dy_1} &= 2 \frac{e^2g^2Q^2N_c}{(2\pi)^8} z_1^2z_2(z_1^2+z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} \\
&\times \left\{ C_F [S_{11'} - S_{13}S_{32} - S_{31'}S_{23} + 1] - \frac{1}{2N_c} [S_{13}S_{32} + S_{31'}S_{23} - S_{12} - S_{21'}] \right\} \frac{1}{\mathbf{x}_{23}^2}. \quad (B5)
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma_{3\times 4}^T}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2g^2Q^2N_c}{(2\pi)^8} z_1^2z_2(z_1^2+z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12}\cdot\mathbf{x}_{1'2}}{|\mathbf{x}_{12}||\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p}\cdot\mathbf{x}_{1'}} \\
&\times \left\{ C_F [S_{11'} - S_{13}S_{32} - S_{31'}S_{23} + 1] - \frac{1}{2N_c} [S_{13}S_{32} + S_{31'}S_{23} - S_{12} - S_{21'}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2\mathbf{x}_{23}^2} \right). \quad (B6)
\end{aligned}$$

$$\begin{aligned} \frac{d\sigma_{4\times 3}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{1'1} - S_{31} S_{23} - S_{1'3} S_{32} + 1] - \frac{1}{2N_c} [S_{31} S_{23} + S_{1'3} S_{32} - S_{21} - S_{1'2}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \quad (\text{B7}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_{1\times 3}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{13} S_{31'} - S_{31'} S_{23} - S_{12} + 1] - \frac{1}{2N_c} [S_{11'} - S_{21'} - S_{13} S_{31'} + S_{31'} S_{23}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{1'3}^2} - \frac{1}{2} \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{1'3}^2} \right). \quad (\text{B8}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_{3\times 1}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{31} S_{1'3} - S_{1'3} S_{32} - S_{21} + 1] - \frac{1}{2N_c} [S_{1'1} - S_{1'2} - S_{31} S_{1'3} + S_{1'3} S_{32}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{1'3}^2} - \frac{1}{2} \frac{\mathbf{x}_{11'}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{1'3}^2} \right). \quad (\text{B9}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_{1\times 4}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & 2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{13} S_{31'} - S_{31'} S_{23} - S_{12} + 1] - \frac{1}{2N_c} [S_{11'} - S_{21'} - S_{13} S_{31'} + S_{31'} S_{23}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \quad (\text{B10}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_{4\times 1}^T}{d^2\mathbf{p}d\mathbf{y}_1} = & 2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{31} S_{1'3} - S_{1'3} S_{32} - S_{21} + 1] - \frac{1}{2N_c} [S_{1'1} - S_{1'2} - S_{31} S_{1'3} + S_{1'3} S_{32}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \quad (\text{B11}) \end{aligned}$$

## 2. Virtual corrections

$$\begin{aligned} \frac{d\sigma_5^T}{d^2\mathbf{p}d\mathbf{y}_1} = & 2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{13} S_{31'} - S_{32} S_{13} - S_{21'} + 1] - \frac{1}{2N_c} [S_{11'} - S_{12} - S_{13} S_{31'} + S_{32} S_{13}] \right\} \frac{1}{\mathbf{x}_{31}^2}. \quad (\text{B12}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_5^{*,T}}{d^2\mathbf{p}d\mathbf{y}_1} = & 2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{31} S_{1'3} - S_{23} S_{31} - S_{1'2} + 1] - \frac{1}{2N_c} [S_{1'1} - S_{21} - S_{31} S_{1'3} + S_{23} S_{31}] \right\} \frac{1}{\mathbf{x}_{31}^2}. \quad (\text{B13}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_7^T}{d^2\mathbf{p}d\mathbf{y}_1} = & -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{13} S_{31'} - S_{32} S_{13} - S_{21'} + 1] - \frac{1}{2N_c} [S_{11'} - S_{12} - S_{13} S_{31'} + S_{32} S_{13}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \quad (\text{B14}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_7^{*,T}}{d^2\mathbf{p}d\mathbf{y}_1} = & -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'}} \\ & \times \left\{ C_F [S_{31} S_{1'3} - S_{23} S_{31} - S_{1'2} + 1] - \frac{1}{2N_c} [S_{1'1} - S_{21} - S_{31} S_{1'3} + S_{23} S_{31}] \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \quad (\text{B15}) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_9^T}{d^2\mathbf{p}dy_1} &= -\frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\ &\quad \times C_F \left[ S_{11'} - S_{12} - S_{21'} + 1 \right] \frac{1}{\mathbf{x}_3^2}. \end{aligned} \quad (\text{B16})$$

$$\begin{aligned} \frac{d\sigma_9^{*,T}}{d^2\mathbf{p}dy_1} &= -\frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ &\quad \times C_F \left[ S_{1'1} - S_{21} - S_{1'2} + 1 \right] \frac{1}{\mathbf{x}_3^2}. \end{aligned} \quad (\text{B17})$$

$$\begin{aligned} \frac{d\sigma_{11}^T}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\ &\quad \times C_F \left[ S_{11'} - S_{12} - S_{21'} + 1 \right] \frac{1}{\mathbf{x}_3^2}. \end{aligned} \quad (\text{B18})$$

$$\begin{aligned} \frac{d\sigma_{11}^{*,T}}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ &\quad \times C_F \left[ S_{1'1} - S_{21} - S_{1'2} + 1 \right] \frac{1}{\mathbf{x}_3^2}. \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \frac{d\sigma_{12}^T}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_2} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\ &\quad \times C_F \left[ S_{11'} - S_{12} - S_{21'} + 1 \right] \frac{1}{\mathbf{x}_3^2}. \end{aligned} \quad (\text{B20})$$

$$\begin{aligned} \frac{d\sigma_{12}^{*,T}}{d^2\mathbf{p}dy_1} &= -2 \frac{e^2 g^2 Q^2 N_c}{(2\pi)^5} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_2} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ &\quad \times C_F \left[ S_{1'1} - S_{21} - S_{1'2} + 1 \right] \frac{1}{\mathbf{x}_3^2}. \end{aligned} \quad (\text{B21})$$

$$\begin{aligned} \frac{d\sigma_{13(1)}^T}{d^2\mathbf{p}dy_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_2} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\ &\quad \times \left\{ C_F [S_{21'} S_{12} + 1 - S_{12} - S_{21'}] - \frac{1}{2N_c} (S_{11'} - S_{21'} S_{12}) \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \end{aligned} \quad (\text{B22})$$

$$\begin{aligned} \frac{d\sigma_{13(1)}^{*,T}}{d^2\mathbf{p}dy_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_2} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ &\quad \times \left\{ C_F [S_{1'2} S_{21} + 1 - S_{21} - S_{1'2}] - \frac{1}{2N_c} (S_{1'1} - S_{1'2} S_{21}) \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \end{aligned} \quad (\text{B23})$$

$$\begin{aligned} \frac{d\sigma_{13(2)}^T}{d^2\mathbf{p}dy_1} &= \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}|Q_1) K_1(|\mathbf{x}_{1'2}|Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{1'1}} \\ &\quad \times \left\{ C_F [S_{21'} S_{12} + 1 - S_{12} - S_{21'}] - \frac{1}{2N_c} (S_{11'} - S_{21'} S_{12}) \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \end{aligned} \quad (\text{B24})$$

$$\frac{d\sigma_{13(2)}^{*,T}}{d^2\mathbf{p}dy_1} = \frac{e^2 g^2 Q^2 N_c}{(2\pi)^6} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ \times \left\{ C_F [S_{1'2} S_{21} + 1 - S_{21} - S_{1'2}] - \frac{1}{2N_c} (S_{1'1} - S_{1'2} S_{21}) \right\} \left( \frac{1}{2} \frac{1}{\mathbf{x}_{13}^2} + \frac{1}{2} \frac{1}{\mathbf{x}_{23}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{13}^2 \mathbf{x}_{23}^2} \right). \quad (\text{B25})$$

$$\frac{d\sigma_{14(1)}^T}{d^2\mathbf{p}dy_1} = \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ \times C_F [S_{11'} - S_{12} - S_{21'} + 1] \left[ \frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2 \mathbf{x}_{32}^2} \right]. \quad (\text{B26})$$

$$\frac{d\sigma_{14(1)}^{*,T}}{d^2\mathbf{p}dy_1} = \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_1} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ \times C_F [S_{1'1} - S_{21} - S_{1'2} + 1] \left[ \frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2 \mathbf{x}_{32}^2} \right]. \quad (\text{B27})$$

$$\frac{d\sigma_{14(2)}^T}{d^2\mathbf{p}dy_1} = \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_2} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ \times C_F [S_{11'} - S_{12} - S_{21'} + 1] \left[ \frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2 \mathbf{x}_{32}^2} \right]. \quad (\text{B28})$$

$$\frac{d\sigma_{14(2)}^{*,T}}{d^2\mathbf{p}dy_1} = \frac{e^2 g^2 Q^2 N_c}{(2\pi)^8} z_1^2 z_2 (z_1^2 + z_2^2) \int_0^{z_2} \frac{dz}{z} \int d^8\mathbf{x} \frac{\mathbf{x}_{12} \cdot \mathbf{x}_{1'2}}{|\mathbf{x}_{12}| |\mathbf{x}_{1'2}|} K_1(|\mathbf{x}_{12}| Q_1) K_1(|\mathbf{x}_{1'2}| Q_1) e^{i\mathbf{p} \cdot \mathbf{x}_{11'}} \\ \times C_F [S_{1'1} - S_{21} - S_{1'2} + 1] \left[ \frac{1}{\mathbf{x}_{31}^2} + \frac{1}{\mathbf{x}_{32}^2} - \frac{1}{2} \frac{\mathbf{x}_{12}^2}{\mathbf{x}_{31}^2 \mathbf{x}_{32}^2} \right]. \quad (\text{B29})$$

Note in the  $z \rightarrow 0$  limit there is further cancellation,

$$\sigma_{1 \times 2} + \sigma_{13(1)} + \sigma_{13(2)} = 0, \quad (\text{B30})$$

$$\sigma_{2 \times 1} + \sigma_{13(1)}^* + \sigma_{13(2)}^* = 0. \quad (\text{B31})$$

This cancellation happens for any  $\mathbf{k}$  ( $\mathbf{x}_3$ ) so these terms do not contribute to BK/JIMWLK.

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