

Systematic analysis of the form factors of B_c to P -wave charmonia and corresponding weak decays

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In this article, the vector, axial vector, and tensor form factors of $B_c \rightarrow \chi_{cJ}$ ($J = 0, 1, 2$) and $B_c \rightarrow h_c$ are analyzed within the framework of three-point QCD sum rules. With the calculated vector and axial vector form factors, we directly study the decay widths and branching ratios of semileptonic decays $B_c^- \rightarrow \chi_{cJ} l \bar{\nu}_l$, $h_c l \bar{\nu}_l$ ($l = e, \mu$ and τ) and analyze the nonleptonic decays $B_c^- \rightarrow \chi_{cJ} \pi^-, \chi_{cJ} K^-, \chi_{cJ} \rho^-, \chi_{cJ} K^{*-}$, $B_c^- \rightarrow h_c \pi^-, h_c K^-, h_c \rho^-, h_c K^{*-}$ by using the naive factorization approach. These results can provide more information to understand the properties of B_c meson and P -wave charmonia and to study the heavy quark dynamics.

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I. INTRODUCTION

Since the B_c meson was first observed by the CDF Collaboration in 1998 [1], most of theoretical and experimental physicists paid much attention to it for the special properties. First, as the only quarkonium with mixed heavy flavor quarks, B_c meson has rich decay channels because each heavy quark in it can decay individually, and the other acts as a spectator. Secondly, the mass of B_c meson lies below the threshold of $B\bar{D}$; thus, the B_c meson can only decay via weak interaction. This provides a suitable place to test the standard model (SM) and find the new physics beyond the SM. Third, both b and c quarks can annihilate and provide new kinds of weak decays with sizable partial decay widths. This pure leptonic or radiative leptonic decay can be used to extract the decay constant of B_c and the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{cb} [2,3]. In addition, the B_c meson includes two different heavy flavor quarks; the spectroscopy may be different with the light mesons or the mesons with only one heavy quark. It offers us a different laboratory to study the strong interactions. Moreover, it was estimated that the inclusive

production cross sections of B_c meson and its resonance states at the LHC is at level of $1 \mu\text{b}$ for $\sqrt{s} = \sqrt{14} \text{ TeV}$. This means that there are about $O(10^9)$ B_c meson can be anticipated in 1 fb^{-1} [4]. So, abundant events in the experiment can provide more information to carry out this research.

The weak decays of B_c to charmonia including semileptonic and nonleptonic processes are interesting and valuable. Taking the B_c^- meson as an example, its weak decay processes can be described as a b quark decaying, while the other $\bar{c}$ quark plays as a spectator at the quark level. In the case of semileptonic decays, the decay mode of b quark is $b \rightarrow c l \bar{\nu}_l$ ($l = e, \mu$, and τ). The final c quark will combine with the spectator $\bar{c}$ quark to produce final charmonium. In the nonleptonic decay processes, the decay modes of the b quark include $b \rightarrow c \bar{u} d$, $c \bar{u} s$, $c d \bar{c}$, and $c s \bar{c}$. Except for the final c quark that combines with spectator $\bar{c}$ quark to produce final charmonium, the rest of the quarks combine with each other to produce another final meson. Theoretical investigations about these decay processes are difficult because quantum chromodynamics (QCD) is nonperturbative in low energy regions. Therefore, some nonperturbative methods have been employed to study the decay processes of B_c to charmonia in recent years, such as lattice QCD (LQCD) [5,6], QCD sum rules (QCDSR) [7–11], light-cone QCD sum rules (LCSR) [12–15], perturbative QCD (pQCD) [16–19], relativistic quark model (RQM) [20–22], nonrelativistic quark model (NRQM) [23], light-front quark model (LFQM) [24–29], nonrelativistic QCD (NRQCD) [30–32], and other methods [33,34]. With the updated high luminosity of the LHC,

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more experimental measurements of B_c to charmonia will become feasible in the coming years; this will provide more information to test various theoretical models.

As one of the most powerful nonperturbative approaches, the QCD sum rules are widely used in studying the properties of hadrons such as the masses, decay constants, light-cone distribution amplitude of hadrons [35–44], form factors of hadron transition [45–52], coupling constants of strong interaction [53–61], and others [62–64]. In our previous work, the scalar, vector, axial vector, and tensor form factors of B_c to S -wave charmonia η_c and J/ψ were analyzed in the framework of three-point QCD sum rules, and the corresponding semileptonic and nonleptonic decays processes were studied [11]. As a continuation of our previous work, we systematically analyze the vector, axial vector, and tensor form factors of B_c to P -wave charmonia χ_{cJ} ($J = 0, 1, 2$) and h_c by using three-point QCD sum rules in the present work. With these form factors, we also study the semileptonic decay processes $B_c^- \rightarrow \chi_{cJ} l \bar{\nu}_l$, $h_c l \bar{\nu}_l$ ($l = e, \mu$ and τ) and the nonleptonic decay processes $B_c^- \rightarrow \chi_{cJ} \pi^-, \chi_{cJ} K^-, \chi_{cJ} \rho^-, \chi_{cJ} K^{*-}, B_c^- \rightarrow h_c \pi^-, h_c K^-, h_c \rho^-, h_c K^{*-}$. The nonleptonic decays $B_c \rightarrow \chi_{cJ} D_{(s)}^{(*)}$ and $h_c D_{(s)}^{(*)}$ are also related to the form factors in this study. However, other form factors involved in these decay processes are beyond the scope of the present work.

This work is organized as follows. After the Introduction, the form factors of B_c to P -wave charmonia are analyzed by three-point QCD sum rules in Sec. II. Based on these form factors, we analyze the corresponding semileptonic and nonleptonic decay widths and branching ratios in Sec. III. Section IV is employed to present the numerical results and discussions, and Sec. V is devoted to a short summary. Some important figures and formulas are shown in Appendixes A–C.

II. THREE-POINT QCD SUM RULES FOR TRANSITION FORM FACTORS

To study the form factors of B_c to P -wave charmonia, we first write down the following three-point correlation function:

$$\Pi(p, p') = i^2 \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \times \langle 0 | T \{ J_X(x) \tilde{J}(y) J_{B_c}^+(0) \} | 0 \rangle, \quad (1)$$

where T denotes the time ordered product, and J_{B_c} and J_X ($X = \chi_{cJ}$ and h_c , $J = 0, 1, 2$) are the interpolating currents of the B_c meson and P -wave charmonia, respectively. $\tilde{J}$ is the transition currents. These currents are taken as the following forms:

$$\begin{aligned} J_{B_c}(0) &= \bar{c}(0) i \gamma_5 b(0), \\ \tilde{J}(y) &= \bar{c}(y) \Gamma b(y), \\ J^{\chi_{c0}}(x) &= \bar{c}(x) c(x), \\ J^{\chi_{c1}}_\mu(x) &= \bar{c}(x) \gamma_\mu \gamma_5 c(x), \\ J^{h_c}_{\mu\nu}(x) &= \bar{c}(x) \sigma_{\mu\nu} c(x), \\ J^{\chi_{c2}}_{\mu\nu}(x) &= \frac{i}{2} \bar{c}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu - \frac{1}{2} g_{\mu\nu} \overleftrightarrow{D} \right) c(x), \end{aligned} \quad (2)$$

where $\Gamma = \gamma_\mu, \gamma_\mu \gamma_5$, and $\sigma_{\mu\nu}$ or $\sigma_{\mu\nu} \gamma_5$ for vector, axial vector and tensor form factors, respectively. $\overleftrightarrow{D}_\mu = \frac{1}{2} (\vec{D}_\mu - \vec{D}_\mu)$, $\vec{D}_\mu = \vec{\partial}_\mu - i g_s t^a G_\mu^a$, and $\vec{D}_\mu = \vec{\partial}_\mu + i g_s t^a G_\mu^a$ denote the covariant derivative.

In the framework of QCD sum rules, the correlation function is calculated at both hadron and quark levels which are called the phenomenological side and QCD side, respectively. Combining these two sides by using quark hadron duality, the sum rules for the form factors are derived.

A. The phenomenological side

In the phenomenological side, complete sets of hadronic states with the same quantum numbers as interpolating currents are inserted in correlation function in Eq. (1). Performing the integration in coordinate space and using the double dispersion relation, the correlation function can be written as the following form [65]:

$$\Pi^{\text{phy}}(p, p') = \frac{\langle 0 | J_X(0) | X(p') \rangle \langle B_c(p) | J_{B_c}^+(0) | 0 \rangle}{(p'^2 - m_X^2)(p^2 - m_{B_c}^2)} \times \langle X(p') | \tilde{J}(0) | B_c(p) \rangle + \dots, \quad (3)$$

where the ellipsis represents the contributions of higher resonances and continuum states. The meson vacuum matrix elements in the above correlation function are defined as

$$\begin{aligned} \langle 0 | J^{\chi_{c0}}(0) | \chi_{c0}(p') \rangle &= f_{\chi_{c0}} m_{\chi_{c0}}, \\ \langle 0 | J^{\chi_{c1}}_\mu(0) | \chi_{c1}(p') \rangle &= f_{\chi_{c1}} m_{\chi_{c1}} \epsilon_\mu^{\chi_{c1}}, \\ \langle 0 | J^{\chi_{c1}}_\mu(0) | \eta_c(p') \rangle &= i f_{\eta_c} p'_\mu, \\ \langle 0 | J^{h_c}_{\mu\nu}(0) | h_c(p') \rangle &= i f_{h_c} \epsilon_{\mu\nu\alpha\beta} \epsilon_\alpha^{h_c} p'_\beta, \\ \langle 0 | J^{h_c}_{\mu\nu}(0) | J/\psi(p') \rangle &= i f_{J/\psi} (p'_\mu \epsilon_\nu^{J/\psi} - p'_\nu \epsilon_\mu^{J/\psi}), \\ \langle 0 | J^{\chi_{c2}}_{\mu\nu}(0) | \chi_{c2}(p') \rangle &= f_{\chi_{c2}} m_{\chi_{c2}}^3 \epsilon_{\mu\nu}^{\chi_{c2}}, \\ \langle B_c(p) | J_{B_c}^+(0) | 0 \rangle &= \frac{f_{B_c} m_{B_c}^2}{m_b + m_c}, \end{aligned} \quad (4)$$

where $f_{\chi_{c0}}, f_{\chi_{c1}}, f_{\eta_c}, f_{h_c}, f_{J/\psi}, f_{\chi_{c2}}$, and f_{B_c} are the decay constants of corresponding mesons, $\epsilon_\mu^{\chi_{c1}}, \epsilon_\alpha^{h_c}$, and $\epsilon_\mu^{J/\psi}$ are the polarization vectors of χ_{c1} , h_c and J/ψ , respectively.

$\epsilon_{\mu\nu}^{\chi_{c2}}$ is the polarization tensor of χ_{c2} , and $\epsilon_{\mu\nu\alpha\beta}$ is the four-dimensional Levi-Civita tensor. It is noted that the current of χ_{c1} ($J^P = 1^+$) can couple with the pseudoscalar charmonium η_c ($J^P = 0^-$). The tensor current $\bar{c}\sigma_{\mu\nu}c$ which is selected to interpolate h_c ($J^P = 1^+$) can also couple with J/ψ ($J^P = 1^-$). In order to eliminate the contamination of the redundant coupling, the projection operator should be induced in the corresponding correlation function which is discussed in the next sections. The transition matrix elements in Eq. (3) can be expressed in terms of various form factors. For the vector and axial vector form factors, the Bauer-Stech-Wirbel forms are more frequently used and defined as [66]

$$\begin{aligned} & \langle \chi_{c0}(p') | \bar{c}(0) \gamma_\mu \gamma_5 b(0) | B_c(p) \rangle \\ &= F_0^{B_c \rightarrow \chi_{c0}}(q^2) \frac{m_{B_c}^2 - m_{\chi_{c0}}^2}{q^2} q_\mu \\ &+ F_1^{B_c \rightarrow \chi_{c0}}(q^2) \left(p_\mu + p'_\mu - \frac{m_{B_c}^2 - m_{\chi_{c0}}^2}{q^2} q_\mu \right), \quad (5) \end{aligned}$$

$$\begin{aligned} & \langle \chi_{c1}[h_c](p') | \bar{c}(0) \gamma_\mu b(0) | B_c(p) \rangle \\ &= -i(m_{B_c} - m_{\chi_{c1}[h_c]}) V_1^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) \epsilon_{\mu}^{\chi_{c1}[h_c]*} \\ &+ i V_2^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) \frac{\epsilon^{\chi_{c1}[h_c]*} \cdot P}{m_{B_c} - m_{\chi_{c1}[h_c]}} P_\mu \\ &+ 2i m_{\chi_{c1}[h_c]} [V_3^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) - V_0^{B_c \rightarrow \chi_{c1}[h_c]}(q^2)] \frac{\epsilon^{\chi_{c1}[h_c]*} \cdot P}{q^2} q_\mu, \\ & \langle \chi_{c1}[h_c](p') | \bar{c}(0) \gamma_\mu \gamma_5 b(0) | B_c(p) \rangle \\ &= -\frac{A^{B_c \rightarrow \chi_{c1}[h_c]}(q^2)}{m_{B_c} - m_{\chi_{c1}[h_c]}} \epsilon_{\mu\alpha\beta\gamma} \epsilon_{\alpha}^{\chi_{c1}[h_c]*} P_\beta q_\gamma, \quad (6) \end{aligned}$$

$$\begin{aligned} & \langle \chi_{c2}(p') | \bar{c}(0) \gamma_\mu b(0) | B_c(p) \rangle \\ &= h^{B_c \rightarrow \chi_{c2}}(q^2) \epsilon_{\mu\beta\rho\sigma} \epsilon_{\beta\lambda}^{\chi_{c2}*} P^\lambda P^\rho q^\sigma, \\ & \langle \chi_{c2}(p') | \bar{c}(0) \gamma_\mu \gamma_5 b(0) | B_c(p) \rangle \\ &= -ik^{B_c \rightarrow \chi_{c2}}(q^2) \epsilon_{\mu\beta}^{\chi_{c2}*} P^\beta - i \epsilon_{\beta\lambda}^{\chi_{c2}*} P^\beta P^\lambda \\ &\times [b_+^{B_c \rightarrow \chi_{c2}}(q^2) P_\mu + b_-^{B_c \rightarrow \chi_{c2}}(q^2) q_\mu], \quad (7) \end{aligned}$$

where $P = p + p'$ and $q = p - p'$. F_0 and F_1 are axial vector form factors of χ_{c0} ; the vector form factors of χ_{c0} do not exist due to the conservation of angular momentum. A and $V_0/V_1/V_2$ are axial vector and vector form factors of χ_{c1} and h_c . V_3 in Eq. (6) has the following form:

$$\begin{aligned} V_3^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) &= \frac{m_{B_c} - m_{\chi_{c1}[h_c]}}{2m_{\chi_{c1}[h_c]}} V_1^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) \\ &- \frac{m_{B_c} + m_{\chi_{c1}[h_c]}}{2m_{\chi_{c1}[h_c]}} V_2^{B_c \rightarrow \chi_{c1}[h_c]}(q^2). \quad (8) \end{aligned}$$

Here, h and $k/b_+/b_-$ in Eq. (7) are vector and axial vector form factors of $B_c \rightarrow \chi_{c2}$. In order to simplify the

calculations of scatter amplitude, the following substitutions are commonly used [25]:

$$\begin{aligned} A^{B_c \rightarrow \chi_{c2}}(q^2) &= -(m_{B_c} - m_{\chi_{c2}}) h^{B_c \rightarrow \chi_{c2}}(q^2), \\ V_1^{B_c \rightarrow \chi_{c2}}(q^2) &= -\frac{k^{B_c \rightarrow \chi_{c2}}(q^2)}{m_{B_c} - m_{\chi_{c2}}}, \\ V_2^{B_c \rightarrow \chi_{c2}}(q^2) &= (m_{B_c} - m_{\chi_{c2}}) b_+^{B_c \rightarrow \chi_{c2}}(q^2), \\ V_0^{B_c \rightarrow \chi_{c2}}(q^2) &= \frac{m_{B_c} - m_{\chi_{c2}}}{2m_{\chi_{c2}}} V_1^{B_c \rightarrow \chi_{c2}}(q^2) \\ &- \frac{m_{B_c} + m_{\chi_{c2}}}{2m_{\chi_{c2}}} V_2^{B_c \rightarrow \chi_{c2}}(q^2) \\ &- \frac{q^2}{2m_{\chi_{c2}}} b_-^{B_c \rightarrow \chi_{c2}}(q^2). \quad (9) \end{aligned}$$

The tensor form factors of $B_c \rightarrow \chi_{cJ}$ and h_c are defined as follows [67]:

$$\langle \chi_{c0}(p') | \bar{c}(0) \sigma_{\mu\nu} b(0) | B_c(p) \rangle = -\frac{T^{B_c \rightarrow \chi_{c0}}(q^2)}{m_{B_c} - m_{\chi_{c0}}} \epsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta, \quad (10)$$

$$\begin{aligned} & \langle \chi_{c1}[h_c](p') | \bar{c}(0) \sigma_{\mu\nu} \gamma_5 b(0) | B_c(p) \rangle \\ &= -T_0^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) \frac{\epsilon^{\chi_{c1}[h_c]*} \cdot q}{(m_{B_c} - m_{\chi_{c1}[h_c]})^2} \epsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta \\ &+ T_1^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) \epsilon_{\mu\nu\alpha\beta} p_\alpha \epsilon_{\beta}^{\chi_{c1}[h_c]*} \\ &+ T_2^{B_c \rightarrow \chi_{c1}[h_c]}(q^2) \epsilon_{\mu\nu\alpha\beta} p'_\alpha \epsilon_{\beta}^{\chi_{c1}[h_c]*}, \quad (11) \end{aligned}$$

$$\begin{aligned} & \langle \chi_{c2}(p') | \bar{c}(0) \sigma_{\mu\nu} b(0) | B_c(p) \rangle \\ &= \frac{q^\tau}{m_{B_c}} \left[T_0^{B_c \rightarrow \chi_{c2}}(q^2) \frac{\epsilon_{\rho\tau}^{\chi_{c2}*} q^\rho}{(m_{B_c} - m_{\chi_{c2}})^2} \epsilon_{\mu\nu\chi\delta} p^\chi p'^\delta \right. \\ &\left. + T_1^{B_c \rightarrow \chi_{c2}}(q^2) \epsilon_{\mu\nu\chi\delta} \epsilon_{\chi\tau}^{\chi_{c2}*} p_\delta + T_2^{B_c \rightarrow \chi_{c2}}(q^2) \epsilon_{\mu\nu\chi\delta} \epsilon_{\chi\tau}^{\chi_{c2}*} p'_\delta \right], \quad (12) \end{aligned}$$

where T and $T_0/T_1/T_2$ are tensor form factors of χ_{c0} and $\chi_{c1}/\chi_{c2}/h_c$.

From these above equations, the correlation functions in the phenomenological side can be obtained and expanded into different tensor structures. In principle, each structure can be used to carry out the calculations of form factors and will lead to the same results. However, the final results obtained by different structures have different uncertainties because of the truncation of the operator product expansion (OPE) in the QCD side and different contributions of higher resonances and continuum states [68]. In this article, we adopt the traditional way to choose the appropriate structure, where the pole contributions should be large than 50%, and the stability of Borel window should be satisfied. The details are discussed in the next sections.

B. The QCD side

In the QCD side, we perform the OPE of correlation function by contracting all of the quark fields operator with Wick's theorem. After this step, the correlation functions of all processes in the QCD side are expressed as follows:

$$\begin{aligned}\Pi_{\mu}^{\chi_{c0}\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) C^{mn}(x-y) \gamma_{\mu} \gamma_5 B^{nk}(y) \gamma_5], \\ \Pi_{\mu\nu}^{\chi_{c0}\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) C^{mn}(x-y) \sigma_{\mu\nu} B^{nk}(y) \gamma_5],\end{aligned}\quad (13)$$

$$\begin{aligned}\Pi_{1\mu\nu}^{\chi_{c1}\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \gamma_{\mu} \gamma_5 C^{mn}(x-y) \gamma_{\nu} B^{nk}(y) \gamma_5], \\ \Pi_{2\mu\nu}^{\chi_{c1}\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \gamma_{\mu} \gamma_5 C^{mn}(x-y) \gamma_{\nu} \gamma_5 B^{nk}(y) \gamma_5], \\ \Pi_{\mu\nu\sigma}^{\chi_{c1}\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \gamma_{\mu} \gamma_5 C^{mn}(x-y) \sigma_{\nu\sigma} \gamma_5 B^{nk}(y) \gamma_5],\end{aligned}\quad (14)$$

$$\begin{aligned}\Pi_{1\mu\nu\sigma}^{h_c\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \sigma_{\mu\nu} C^{mn}(x-y) \gamma_{\sigma} B^{nk}(y) \gamma_5], \\ \Pi_{2\mu\nu\sigma}^{h_c\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \sigma_{\mu\nu} C^{mn}(x-y) \gamma_{\sigma} \gamma_5 B^{nk}(y) \gamma_5], \\ \Pi_{\mu\nu\alpha\beta}^{h_c\text{QCD}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \sigma_{\mu\nu} C^{mn}(x-y) \sigma_{\alpha\beta} \gamma_5 B^{nk}(y) \gamma_5],\end{aligned}\quad (15)$$

$$\begin{aligned}\Pi_{1\mu\nu\sigma}^{\chi_{c2}\text{QCD}}(p, p') &= \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \Gamma_{\mu\nu} C^{mn}(x-y) \gamma_{\sigma} B^{nk}(y) \gamma_5], \\ \Pi_{2\mu\nu\sigma}^{\chi_{c2}\text{QCD}}(p, p') &= \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \Gamma_{\mu\nu} C^{mn}(x-y) \gamma_{\sigma} \gamma_5 B^{nk}(y) \gamma_5], \\ \Pi_{\mu\nu\alpha\beta}^{\chi_{c2}\text{QCD}}(p, p') &= \int d^4x d^4y e^{ip'x} e^{i(p-p')y} \text{Tr}[C^{km}(-x) \Gamma_{\mu\nu} C^{mn}(x-y) \sigma_{\alpha\beta} B^{nk}(y) \gamma_5],\end{aligned}\quad (16)$$

where $\Gamma_{\mu\nu} = \frac{1}{2}(\gamma_{\mu} \overleftrightarrow{D}_{\nu} + \gamma_{\nu} \overleftrightarrow{D}_{\mu} - \frac{1}{2} g_{\mu\nu} \overleftrightarrow{\not{D}})$, and $C^{ij}(x)$ and $B^{ij}(x)$ are the full propagator of c and b quarks. They can uniformly be represented as the following forms [69]:

$$\begin{aligned}Q^{ij}(x) &= \frac{i}{(2\pi)^4} \int d^4k e^{-ik \cdot x} \left\{ \frac{\delta^{ij}}{\not{k} - m_Q} - \frac{g_s G_{\alpha\beta}^n t_{ij}^n}{4} \frac{\sigma^{\alpha\beta}(\not{k} + m_Q) + (\not{k} + m_Q) \sigma^{\alpha\beta}}{(k^2 - m_Q^2)^2} \right. \\ &\quad \left. - \frac{g_s^2 (t^a t^b)_{ij} G_{\alpha\beta}^a G_{\mu\nu}^b (f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\mu\nu\beta})}{4(k^2 - m_Q^2)^5} + \dots \right\}.\end{aligned}\quad (17)$$

Here, Q denotes C or B , $t^n = \frac{\lambda^n}{2}$, λ^n ($n = 1, \dots, 8$) are the Gell-Mann matrices, i and j are color indices, $\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}]$, and $f^{\alpha\beta\mu\nu}$ have the following form:

$$f^{\alpha\beta\mu\nu} = (\not{k} + m_Q) \gamma^{\alpha} (\not{k} + m_Q) \gamma^{\beta} (\not{k} + m_Q) \gamma^{\mu} (\not{k} + m_Q) \gamma^{\nu} (\not{k} + m_Q). \quad (18)$$

Substituting with the full propagator in Eqs. (13)–(16), the correlation functions in the QCD side can also be expanded in different tensor structures,

$$\begin{aligned}\Pi_{\mu}^{\chi_{c0}\text{QCD}}(p, p') &= \Pi_1^{\chi_{c0}\text{QCD}} p_{\mu} + \Pi_2^{\chi_{c0}\text{QCD}} p'_{\mu}, \\ \Pi_{\mu\nu}^{\chi_{c0}\text{QCD}}(p, p') &= \Pi_3^{\chi_{c0}\text{QCD}} \varepsilon_{\mu\nu\alpha\beta} p^{\alpha} p'^{\beta},\end{aligned}\quad (19)$$

$$\begin{aligned}
\tilde{\Pi}_{1\lambda\nu}^{\chi_{c1}\text{QCD}}(p, p') &= \left(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2}\right) \Pi_{1\mu\nu}^{\chi_{c1}\text{QCD}}(p, p'), \\
&= \Pi_1^{\chi_{c1}\text{QCD}} g_{\lambda\nu} + \Pi_2^{\chi_{c1}\text{QCD}} p_\lambda p_\nu + \Pi_3^{\chi_{c1}\text{QCD}} p_\lambda p'_\nu + \dots, \\
\tilde{\Pi}_{2\lambda\nu}^{\chi_{c1}\text{QCD}}(p, p') &= \left(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2}\right) \Pi_{2\mu\nu}^{\chi_{c1}\text{QCD}}(p, p'), \\
&= \Pi_4^{\chi_{c1}\text{QCD}} \varepsilon_{\lambda\nu\alpha\beta} p^\alpha p'^\beta, \\
\tilde{\Pi}_{\lambda\nu\sigma}^{\chi_{c1}\text{QCD}}(p, p') &= \left(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2}\right) \Pi_{\mu\nu\sigma}^{\chi_{c1}\text{QCD}}(p, p'), \\
&= \Pi_5^{\chi_{c1}\text{QCD}} \varepsilon_{\lambda\nu\sigma\alpha} p^\alpha + \Pi_6^{\chi_{c1}\text{QCD}} \varepsilon_{\lambda\nu\sigma\alpha} p'^\alpha + \Pi_7^{\chi_{c1}\text{QCD}} p_\lambda \varepsilon_{\nu\sigma\alpha\beta} p^\alpha p'^\beta + \dots,
\end{aligned} \tag{20}$$

$$\begin{aligned}
\tilde{\Pi}_{1\lambda\chi\sigma}^{h_c\text{QCD}}(p, p') &= \left(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2}\right) \left(g_{\chi\nu} - \frac{p'_\chi p'_\nu}{p'^2}\right) \Pi_{1\mu\nu\sigma}^{h_c\text{QCD}}(p, p'), \\
&= \Pi_1^{h_c\text{QCD}} \varepsilon_{\lambda\sigma\chi\alpha} p^\alpha + \Pi_2^{h_c\text{QCD}} p_\sigma \varepsilon_{\lambda\chi\alpha\beta} p^\alpha p'^\beta + \Pi_3^{h_c\text{QCD}} p'_\sigma \varepsilon_{\lambda\chi\alpha\beta} p^\alpha p'^\beta + \dots, \\
\tilde{\Pi}_{2\lambda\chi\sigma}^{h_c\text{QCD}}(p, p') &= \left(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2}\right) \left(g_{\chi\nu} - \frac{p'_\chi p'_\nu}{p'^2}\right) \Pi_{2\mu\nu\sigma}^{h_c\text{QCD}}(p, p'), \\
&= \Pi_4^{h_c\text{QCD}} p_\lambda g_{\sigma\chi} + \dots, \\
\tilde{\Pi}_{\alpha\chi}^{h_c\text{QCD}}(p, p') &= g^{\lambda\beta} \left(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2}\right) \left(g_{\chi\nu} - \frac{p'_\chi p'_\nu}{p'^2}\right) \Pi_{\mu\nu\alpha\beta}^{h_c\text{QCD}}(p, p'), \\
&= \Pi_5^{h_c\text{QCD}} g_{\alpha\chi} + \Pi_6^{h_c\text{QCD}} p_\alpha p_\chi + \Pi_7^{h_c\text{QCD}} p'_\alpha p'_\chi + \dots,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\Pi_{1\mu\nu\sigma}^{\chi_{c2}\text{QCD}}(p, p') &= \Pi_1^{\chi_{c2}\text{QCD}} p_\mu \varepsilon_{\sigma\nu\alpha\beta} p^\alpha p'^\beta, \\
\Pi_{2\mu\nu\sigma}^{\chi_{c2}\text{QCD}}(p, p') &= \Pi_2^{\chi_{c2}\text{QCD}} p_\mu g_{\nu\sigma} + \Pi_3^{\chi_{c2}\text{QCD}} p_\mu p_\nu p_\sigma + \Pi_4^{\chi_{c2}\text{QCD}} p'_\mu p'_\nu p'_\sigma, \\
\Pi_{\mu\nu\alpha\beta}^{\chi_{c2}\text{QCD}}(p, p') &= \Pi_5^{\chi_{c2}\text{QCD}} g_{\mu\nu} \varepsilon_{\alpha\beta\chi\lambda} p^\chi p'^\lambda + \Pi_6^{\chi_{c2}\text{QCD}} p_\mu \varepsilon_{\nu\alpha\beta\lambda} p'^\lambda + \Pi_7^{\chi_{c2}\text{QCD}} p_\mu \varepsilon_{\nu\alpha\beta\lambda} p'^\lambda + \dots.
\end{aligned} \tag{22}$$

The projection operator $(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2})$ in Eq. (20) is introduced to eliminate the coupling of axial vector current $\bar{c}\gamma_\mu\gamma_5 c$ with pseudoscalar charmonium η_c , and $(g_{\lambda\mu} - \frac{p'_\lambda p'_\mu}{p'^2})(g_{\chi\nu} - \frac{p'_\chi p'_\nu}{p'^2})$ in Eq. (21) is to eliminate the coupling of tensor current $\bar{c}\sigma_{\mu\nu}c$ with vector charmonium J/ψ . Π_i^{QCD} in the right side of the above equations are named as the scalar invariant amplitude and can be divided into perturbative and nonperturbative parts,

$$\Pi_i^{\text{QCD}}(p, p') = \Pi_i^{\text{pert}}(p, p') + \Pi_i^{\text{non-pert}}(p, p'), \tag{23}$$

where the nonperturbative part includes the two gluon condensate $\langle g_s^2 GG \rangle$, three gluon condensate $g_s^3 \langle fGGG \rangle$, and higher dimension condensate terms. In our previous work [11,52], the contribution of the three gluon condensate is small, about one-tenth that of the two gluon condensate. The higher dimension terms such as the four gluon condensate are very small and further suppressed by $O(\alpha_s^2)$ ($\alpha_s = g_s^2/4\pi$), so can be safely neglected in our calculations. Thus, we only reserve the two gluon condensate term as the nonperturbative contribution in our calculations. The scalar invariant amplitude can be written as the following form according to the double dispersion relation:

$$\Pi_i^{\text{QCD}}(p, p') = \int_{s_1}^{\infty} ds \int_{u_1}^{\infty} du \frac{\rho_i^{\text{QCD}}(s, u, q^2)}{(s - p^2)(u - p'^2)}, \tag{24}$$

where $\rho^{\text{QCD}}(s, u, q^2)$ is the QCD spectral density and can be expressed as

$$\rho_i^{\text{QCD}}(s, u, q^2) = \rho^{\text{pert}}(s, u, q^2) + \rho^{\langle g_s^2 GG \rangle}(s, u, q^2), \quad (25)$$

with $s = p^2$, $u = p'^2$, and $q = p - p'$. s_1 and u_1 in Eq. (24) are the kinematic limits for the B_c meson and P -wave charmonium, and they are taken as $(m_b + m_c)^2$ and $4m_c^2$, respectively.

The QCD spectral density of the perturbative part and two gluon condensate can be obtained by using Cutkosky's rules [45,47]; the calculation details can be found in Ref. [11]. The corresponding Feynman diagrams are shown in Fig. 1. We take the change of variables $p^2 \rightarrow -P^2$, $p'^2 \rightarrow -P'^2$, and $q^2 \rightarrow -Q^2$ and perform double Borel transforms [70] for the variables P^2 and P'^2 to both phenomenological and QCD sides. The variables P^2 and P'^2 will be replaced by Borel parameters T_1^2 and T_2^2 , respectively. For simplicity, the relations $T^2 = T_1^2$ and $T_2^2 = kT_1^2 = kT^2$ are usually used to reduce the two Borel parameters, where $k = m_X^2/m_{B_c}^2$ [53], and X denotes the P -wave charmonium. The plausibility for this reduction is discussed in the next section. After matching the phenomenological and QCD sides by using quark-hadron duality condition, the QCD sum rules for the form factors are obtained. Taking the axial vector form factors for $B_c \rightarrow \chi_{c0}$ as an example, these form factors can be obtained by combining two invariant amplitudes $\Pi_1^{\chi_{c0}\text{QCD}}$ and $\Pi_2^{\chi_{c0}\text{QCD}}$ in Eq. (19) and have the following forms:

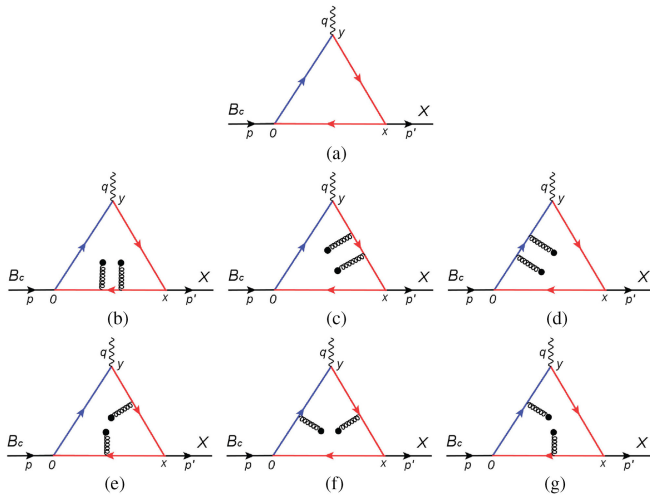


FIG. 1. The Feynman diagrams for the perturbative part (a) and gluon condensate (b)–(g), where the blue and red solid lines denote the b and c quark lines, respectively. The black loop lines are the gluon lines.

$$\begin{aligned} F_0(Q^2) &= -\frac{1}{2\mathcal{X}\mathcal{Y}} e^{\frac{m_{B_c}^2 + m_X^2}{T_1^2 + T_2^2}} \\ &\times \int_{s_1}^{s_0} \int_{u_1}^{u_0} ds du [(\mathcal{X} + 1) \rho_1^{\chi_{c0}\text{QCD}}(s, u, Q^2) \\ &+ (\mathcal{X} - 1) \rho_2^{\chi_{c0}\text{QCD}}(s, u, Q^2)] e^{-\frac{s}{T_1^2} - \frac{u}{T_2^2}}, \\ F_1(Q^2) &= -\frac{1}{2\mathcal{Y}} e^{\frac{m_{B_c}^2 + m_X^2}{T_1^2 + T_2^2}} \\ &\times \int_{s_1}^{s_0} \int_{u_1}^{u_0} ds du [\rho_1^{\chi_{c0}\text{QCD}}(s, u, Q^2) \\ &+ \rho_2^{\chi_{c0}\text{QCD}}(s, u, Q^2)] e^{-\frac{s}{T_1^2} - \frac{u}{T_2^2}}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} \mathcal{X} &= -\frac{m_{B_c}^2 - m_{\chi_{c0}}^2}{Q^2}, \\ \mathcal{Y} &= \frac{f_{\chi_{c0}} f_{B_c} m_{\chi_{c0}} m_{B_c}^2}{m_b + m_c}. \end{aligned} \quad (27)$$

In Eq. (26), s_0 and u_0 are the threshold parameters which are introduced to eliminate the contributions of higher resonances and continuum states. They usually fulfill the relations $s_0 = (m_{B_c} + \delta)^2$ and $u_0 = (m_X + \delta)^2$, where δ is the mass gap between the ground and first excited states and is commonly taken as the value of 0.4–0.6 GeV [53].

For the full heavy quark system, the quark condensate does not exist, and major nonperturbative contributions are gluon condensates. In addition, the next-to-leading order correction for the perturbative contribution is also important for the accuracy of final results. However, the rigorous calculation for the loop diagrams in three-point QCD sum rules is difficult. The Coulomb-like α_s/v correlation is proposed to ameliorate this problem [8], and it is illustrated in Fig. 2. In a nonrelativistic approximation, this correction will contribute a renormalization coefficient for the spectral density of the perturbative part,

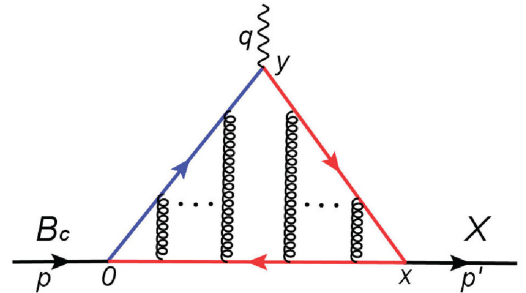


FIG. 2. The Feynman diagram for the Coulomb-like interaction.

$$\begin{aligned} \rho_c^{\text{pert}}(s, u, Q^2) &= \sqrt{\frac{4\pi\alpha_s^C}{3v_1} \left[1 - \exp\left(-\frac{4\pi\alpha_s^C}{3v_1}\right) \right]^{-1}} \\ &\times \sqrt{\frac{4\pi\alpha_s^C}{3v_2} \left[1 - \exp\left(-\frac{4\pi\alpha_s^C}{3v_2}\right) \right]^{-1}} \\ &\times \rho^{\text{pert}}(s, u, Q^2), \end{aligned} \quad (28)$$

where $\alpha_s^C \approx \alpha_s(\mu)$, and v_1 and v_2 are the relative velocities of heavy quarks in the B_c meson and P -wave charmonia and can be expressed as,

$$\begin{aligned} v_1 &= \sqrt{1 - \frac{4m_b m_c}{s - (m_b - m_c)^2}}, \\ v_2 &= \sqrt{1 - \frac{4m_c^2}{u}}. \end{aligned} \quad (29)$$

In this study, we take into account the contribution of this correction and briefly discuss its impact on the results.

III. WEAK DECAY OF B_c TO P-WAVE CHARMONIA

This section is devoted to analyzing the semileptonic and nonleptonic decay processes of the B_c meson by using the vector and axial vector form factors obtained in the previous section.

A. Semileptonic decay

The semileptonic decay $B_c \rightarrow X l \bar{l}$ ($l = e, \mu$ and τ) can be described by the following effective Hamiltonian:

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{l=e,\mu,\tau} V_{cb} \bar{c} \gamma_\mu (1 - \gamma_5) b \bar{l} \gamma_\mu (1 - \gamma_5) l, \quad (30)$$

where $G_F = 1.16637 \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi constant, and V_{cb} is CKM matrix element. The transition matrix element can be written as

$$\begin{aligned} T &= \langle X(p') l(l) \bar{l}(v) | H_{\text{eff}} | B_c(p) \rangle, \\ &= \frac{G_F}{\sqrt{2}} V_{cb} \langle X(p') | \bar{c} \gamma_\mu (1 - \gamma_5) b | B_c(p) \rangle \\ &\times \langle l(l) \bar{l}(v) | \bar{l} \gamma_\mu (1 - \gamma_5) l | 0 \rangle. \end{aligned} \quad (31)$$

In the above equation, the $B_c \rightarrow X$ transition matrix element can be represented as the vector and axial vector form factors. The leptonic matrix element can be formulated as the following form by using perturbative field theory approach:

$$\langle l(l) \bar{l}(v) | \bar{l} \gamma_\mu (1 - \gamma_5) l | 0 \rangle = \bar{u}_{v,s} \gamma_\mu (1 - \gamma_5) u_{-l,s'}, \quad (32)$$

where $u_{v,s}$ and $u_{-l,s'}$ are the spinor wave functions of $\bar{l}$ and l , and the subscripts $v(l)$ and $s(s')$ denote the four momentum and spin of corresponding particles.

B. Nonleptonic decay

The nonleptonic decays are $B_c \rightarrow XP$ and $B_c \rightarrow XV$, where X represent the P -wave charmonia χ_{cJ} ($J = 0, 1, 2$) and h_c, P denotes the light pseudoscalar meson π or K , and V denotes the light vector meson ρ or K^* that can be uniformly described by the following effective Hamiltonian:

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{q=d,s} V_{cb} V_{uq}^* a_1 \bar{c} \gamma_\mu (1 - \gamma_5) b \bar{u} \gamma_\mu (1 - \gamma_5) q, \quad (33)$$

where V_{cb} and V_{uq} are CKM matrix elements, and a_1 is the Wilson coefficient. The transition matrix for these decay processes can be expressed as the following form by using the naive factorization approach (NFA) [71,72],

$$\begin{aligned} T &= \langle X(p') P[V](q) | H_{\text{eff}} | B_c(p) \rangle, \\ &= \frac{G_F}{\sqrt{2}} V_{cb} V_{uq}^* a_1 \langle X(p') | \bar{c} \gamma_\mu (1 - \gamma_5) b | B_c(p) \rangle \\ &\times \langle P[V](q) | \bar{u} \gamma_\mu (1 - \gamma_5) q | 0 \rangle. \end{aligned} \quad (34)$$

The meson vacuum matrix elements can be defined as the following forms:

$$\begin{aligned} \langle P(q) | \bar{u} \gamma_\mu (1 - \gamma_5) q | 0 \rangle &= -i f_P q_\mu, \\ \langle V(q) | \bar{u} \gamma_\mu (1 - \gamma_5) q | 0 \rangle &= f_V m_V \epsilon_\mu^*, \end{aligned} \quad (35)$$

where f_P and f_V are the decay constants of pseudoscalar and vector meson, and ϵ_μ is the polarization vector of vector meson.

IV. NUMERICAL RESULTS AND DISCUSSIONS

A. Numerical results of form factors

As important input parameters, the heavy quark masses are energy scale dependent, which can be expressed as the following renormalization group equation (RGE):

$$\begin{aligned} m_Q(\mu) &= m_Q(m_Q) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_Q)} \right]^{\frac{12}{33-2N_f}}, \\ \alpha_s(\mu) &= \frac{1}{b_0 t} \left[1 - \frac{b_1 \log t}{b_0^2 t} \right. \\ &\quad \left. + \frac{b_1^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right], \end{aligned} \quad (36)$$

where $t = \log(\frac{\mu^2}{\Lambda_{\text{QCD}}^2})$, $b_0 = \frac{33-2N_f}{12\pi}$, $b_1 = \frac{153-19N_f}{24\pi^2}$, and $b_2 = \frac{2857 - \frac{5033}{9}N_f + \frac{325}{27}N_f^2}{128\pi^3}$. $\Lambda_{\text{QCD}} = 213 \text{ MeV}$ for the flavors $N_f = 5$ in this work [73]. The $\overline{\text{MS}}$ masses of c and b

TABLE I. Input parameters (IP) used to calculate the form factors.

IP	Values (GeV)	IP	Values
m_{B_c}	6.274 [73]	$f_{\chi_{c0}}$	0.343 GeV [74]
$m_{\chi_{c0}}$	3.414 [73]	$f_{\chi_{c1}}$	0.338 GeV [74]
$m_{\chi_{c1}}$	3.511 [73]	f_{h_c}	0.235 GeV [75]
m_{h_c}	3.525 [73]	$f_{\chi_{c2}}$	0.0111 ± 0.0062 [76]
$m_{\chi_{c2}}$	3.556 [73]	$\langle g_s^2 GG \rangle$	$0.88 \pm 0.15 \text{ GeV}^4$ [77–79]
f_{B_c}	0.371 ± 0.037 [44]		

quarks are taken from the Particle Data Group [73], where $m_c(m_c) = 1.275 \pm 0.025 \text{ GeV}$, and $m_b(m_b) = 4.18 \pm 0.03 \text{ GeV}$. Based on our previous work [44], the energy scale $\mu = 2 \text{ GeV}$ works well in the B_c meson system. Thus, this value is still employed in the present work. The energy scale dependence of the form factors is shown below. The other parameters like masses, decay constants of the B_c meson and P-wave charmonia, and the standard value of vacuum condensate parameters are all listed in Table I.

We first discuss how the numerical results of form factors are obtained. From Eq. (26), the form factors depend on some input parameters such as the Borel parameters T_1^2 and T_2^2 , the continuum threshold parameters s_0 and u_0 , and the square momentum Q^2 . To analyze the corresponding decay processes of B_c to charmonia, the values of the form factors in timelike regions ($Q^2 = -q^2 < 0$) need to be obtained. However, the calculations of three-point QCD sum rules are carried out in spacelike regions ($Q^2 > 0$) or $Q^2 = 0$, because of the geometric constraints in Delta function integrals. Therefore, we uniformly get the values of form factors in spacelike regions in this work and extrapolating them into timelike regions.

To obtain reliable sum rule results, two conditions should be satisfied, which are the pole dominance and convergence of OPE. The pole contribution is defined as [47]

$$\text{PC} = \frac{\int_{s_1}^{s_0} ds \int_{u_1}^{u_0} du \rho^{\text{QCD}}(s, u, Q^2) e^{-\frac{s}{T_1^2} - \frac{u}{T_2^2}}}{\int_{s_1}^{\infty} ds \int_{u_1}^{\infty} du \rho^{\text{QCD}}(s, u, Q^2) e^{-\frac{s}{T_1^2} - \frac{u}{T_2^2}}}. \quad (37)$$

The condition of pole dominance requires $\text{PC} > 50\%$. An appropriate work region of Borel parameters should be selected to determine the values of form factors, where the final results are less dependent on the Borel parameters, and at the same time the condition of pole dominance is also satisfied. This work region is commonly named as “Borel platform.” Taking the axial vector form factors of $B_c \rightarrow \chi_{c0}$ as an example, by fixing $Q^2 = 1 \text{ GeV}^2$ in Eq. (26), we plot the color figure of form factors F_0 and F_1 with variation of Borel parameters T_1^2 and T_2^2 which are shown in Fig. 3. The ranges of T_1^2 and T_2^2 are taken as 2–18 GeV^2 and 1–7 GeV^2 , respectively. From these figures, one can find that the form factors will decrease as the Borel parameters

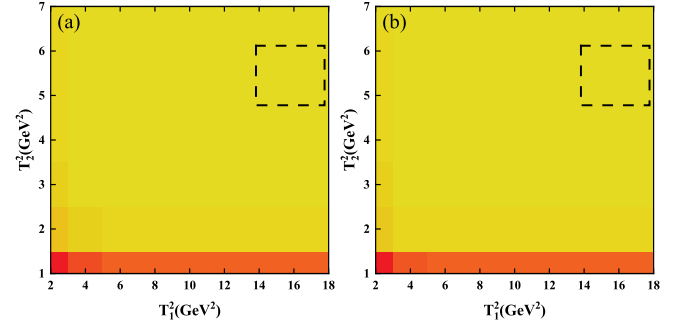


FIG. 3. F_0 (a) and F_1 (b) in $Q^2 = 1 \text{ GeV}^2$ as functions of the Borel parameters T_1^2 and T_2^2 , where T_1^2 and T_2^2 are taken as free parameters. The larger the form factors are, the darker the color is. The 2D Borel platforms are enclosed by the dashed contours (the pole contribution in this region is about 75%–55%).

increase and tend to stabilize. Meantime, the pole contribution will also decrease as the Borel parameters increase. By considering the conditions of pole dominance, the Borel platforms are determined and enclosed by the dashed contours in Fig. 3. The pole contribution in this region is about 75%–55%. It also can be found that the Borel parameters T_1^2 and T_2^2 approximately satisfy the relation $T_2^2/T_1^2 \approx m_{\chi_{c0}}^2/m_{B_c}^2 \approx 0.3$ in this region. Thus, the relation $T_2^2 = kT_1^2 = kT^2$ mentioned above is feasible to reduce the double Borel parameters in our analysis.

According to RGE in Eq. (36), the masses of the heavy quarks will change with the energy scale, which will affect the results of the dispersion integral in sum rules, which in turn will change the position of the Borel platform and the values of the form factors. Taking axial vector form factors F_0 and F_1 of $B_c \rightarrow \chi_{c0}$ as an example, we get the values of the form factors in $Q^2 = 1 \text{ GeV}^2$ in different energy scales, which are shown in Fig. 4. From these figures, one can find that the form factors increase with the increase of the energy scale, where the range of the scale energy is 1.5–3.5 GeV . In principle, if we only consider the leading order contribution, a suitable energy scale cannot be obtained. We need to consider the higher order corrections to reduce the energy scale dependence of the results. Although, we consider the Coulomb-like α_s/v correction in our analysis. However, the energy scale dependence of the results seem

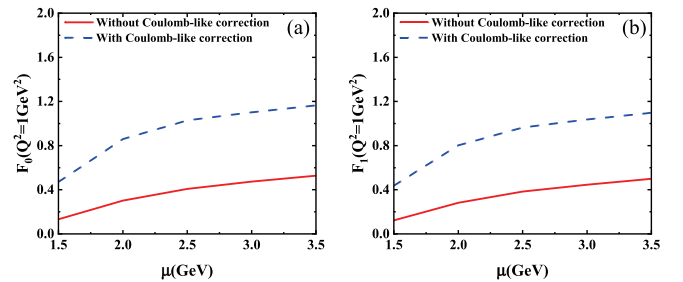


FIG. 4. The form factors F_0 (a) and F_1 (b) in $Q^2 = 1 \text{ GeV}^2$ with variation of energy scale μ .

to not be significantly weakened. Therefore, we can only determine the energy scale according to phenomenological considerations. In Ref. [44], we analyzed the mass and decay constant of the B_c meson in two-point QCD sum rules by considering the next-to-leading order contribution, and we got a suitable energy scale $\mu = 2$ GeV for the B_c meson. In this study, we continue to use this energy scale to obtain the form factors and compare them with other theoretical models.

Taking $Q^2 = 1$ GeV², the Borel parameters are determined, and the Borel platform is chosen after repeated trial and contrast. The Borel platforms for all form factors are

$$\sqrt{\frac{4\pi\alpha_s^C}{3v_1} \left[1 - \exp\left(-\frac{4\pi\alpha_s^C}{3v_1}\right) \right]^{-1}} \sqrt{\frac{4\pi\alpha_s^C}{3v_2} \left[1 - \exp\left(-\frac{4\pi\alpha_s^C}{3v_2}\right) \right]^{-1}} = 1 + \alpha_s^C \left(\frac{\pi}{3v_1} + \frac{\pi}{3v_2} \right) + (\alpha_s^C)^2 \left[\frac{\pi^2}{54} \left(\frac{1}{v_1} \right)^2 + \frac{\pi^2}{9} \frac{1}{v_1 v_2} + \frac{\pi^2}{54} \left(\frac{1}{v_2} \right)^2 \right] + O[(\alpha_s^C)^3] + \dots \quad (38)$$

TABLE II. The Borel platform (BP), pole contributions (PC), and values of form factors (FF) in $Q^2 = 1$ GeV² for different decay modes.

Mode	FF	BP (GeV ²)	PC (%)	F	F^a
$B_c \rightarrow \chi_{c0}$	F_0	15–17	64–58	$0.30^{+0.02}_{-0.04}$	$0.84^{+0.06}_{-0.08}$
	F_1	15–17	63–57	$0.28^{+0.02}_{-0.03}$	$0.78^{+0.06}_{-0.07}$
	T	18–20	73–69	$0.35^{+0.03}_{-0.04}$	$0.99^{+0.07}_{-0.08}$
$B_c \rightarrow \chi_{c1}$	A	19–21	71–67	$0.19^{+0.02}_{-0.02}$	$0.53^{+0.04}_{-0.04}$
	V_0	23–25	79–76	$0.060^{+0.013}_{-0.012}$	$0.15^{+0.03}_{-0.03}$
	V_1	18–20	64–59	$0.51^{+0.05}_{-0.06}$	$1.42^{+0.11}_{-0.13}$
	V_2	20–22	64–59	$0.088^{+0.005}_{-0.007}$	$0.26^{+0.01}_{-0.02}$
	T_0	19–21	76–72	$0.087^{+0.006}_{-0.008}$	$0.25^{+0.02}_{-0.02}$
	T_1	17–19	54–50	$-0.44^{+0.05}_{-0.05}$	$-1.23^{+0.11}_{-0.13}$
	T_2	15–17	55–50	$1.23^{+0.12}_{-0.15}$	$3.48^{+0.31}_{-0.34}$
	A	9–11	55–46	$0.075^{+0.010}_{-0.011}$	$0.19^{+0.03}_{-0.02}$
$B_c \rightarrow h_c$	V_0	13–15	67–61	$0.80^{+0.07}_{-0.07}$	$2.30^{+0.16}_{-0.18}$
	V_1	13–5	61–56	$2.46^{+0.23}_{-0.26}$	$7.05^{+0.56}_{-0.65}$
	V_2	9–11	57–50	$0.16^{+0.02}_{-0.02}$	$0.46^{+0.05}_{-0.06}$
	T_0	16–18	90–87	$0.16^{+0.01}_{-0.02}$	$0.45^{+0.03}_{-0.03}$
	T_1	13–15	77–72	$-0.039^{+0.001}_{-0.002}$	$-0.13^{+0.01}_{-0.01}$
	T_2	13–15	84–78	$0.49^{+0.02}_{-0.02}$	$1.48^{+0.04}_{-0.05}$
	h	18–20	70–65	$0.031^{+0.003}_{-0.003}$	$0.089^{+0.005}_{-0.007}$
$B_c \rightarrow \chi_{c2}$	k	13–15	76–69	$1.91^{+0.14}_{-0.17}$	$5.40^{+0.34}_{-0.30}$
	b_+	13–15	64–55	$-0.0099^{+0.0001}_{-0.0005}$	$-0.031^{+0.001}_{-0.001}$
	b_-	13–15	66–57	$0.011^{+0.001}_{-0.001}$	$0.034^{+0.002}_{-0.001}$
	T_0	15–17	66–53	$-0.085^{+0.007}_{-0.010}$	$-0.22^{+0.01}_{-0.03}$
	T_1	13–15	66–58	$0.026^{+0.002}_{-0.002}$	$0.085^{+0.006}_{-0.007}$
	T_2	17–19	81–76	$2.00^{+0.17}_{-0.18}$	$5.63^{+0.37}_{-0.43}$

^aIt denotes the results with Coulomb-like correction.

shown in Figs. 10 and 11 in Appendix A. From Figs. 10 and 11, we can see that the nonperturbative contribution is tiny, and the convergence of OPE is also satisfied. The Borel platform, pole contributions in the Borel platform, and values of form factors in $Q^2 = 1$ GeV² are listed in Table II.

It is noted that after considering the Coulomb-like correction, the values of the form factors are about three times of that without a Coulomb-like correction. From Eq. (28), we can find that the Coulomb-like term relies on a strong coupling constant α_s^C . Doing the power series expansion on α_s^C of the Coulomb-like term in Eq. (28), we can obtain the following relation:

According to this equation, we can obtain the ratios among the perturbative contribution, leading order Coulomb-like correction (α_s^C), next-to-leading order Coulomb-like correction [$(\alpha_s^C)^2$], and the higher order contributions [$(\alpha_s^C)^n (n > 2)$] for all form factors in $Q^2 = 1$ GeV², where the total Coulomb-like contributions are taken as 1 (see Table III). From Table. III, the contribution of the leading order Coulomb-like corrections for all form factors are about 1–1.5 times as large as that of perturbative term, and the Coulomb-like corrections decrease quickly with increase of orders of α_s^C . The final results are convergent for the power expansion of α_s^C .

Taking different values of Q^2 , the form factors in space-like regions ($Q^2 > 0$) can be obtained, where the range of Q^2 is taken as 1–5 GeV², uniformly. The values of form factors in timelike regions are obtained by fitting the results in spacelike regions with appropriate functions and by extrapolating these results into the time-like regions. The z series parametrization approach was proposed in Ref. [80] and was widely employed to fit different form factors [14,49,81–83]. With this method, the form factors can be expanded as the following series:

$$F(Q^2) = \frac{1}{1 + Q^2/m_R^2} \times \sum_{k=0}^{N-1} b_k \left[z(Q^2, t_0)^k - (-1)^{k-N} \frac{k}{N} z(Q^2, t_0)^N \right], \quad (39)$$

where m_R is the mass of low-lying B_c resonance [14] which is taken as 6.75 GeV in the present work. b_k is the fitting parameter, and the function $z(Q^2, t_0)$ is taken as the following form:

TABLE III. The ratios among the perturbative contributions, leading order Coulomb-like corrections (α_s^C) , next-to-leading order Coulomb-like corrections $[(\alpha_s^C)^2]$, and the higher order contributions $[(\alpha_s^C)^n (n > 2)]$ for all form factors in $Q^2 = 1 \text{ GeV}^2$, where the total Coulomb-like contributions are taken as 1.

Mode	Form factor	Perturbative	α_s^C	$(\alpha_s^C)^2$	$(\alpha_s^C)^n (n > 2)$
$B_c \rightarrow \chi_{c0}$	F_0	0.35	0.49	0.11	0.05
	F_1	0.35	0.49	0.11	0.05
	T	0.35	0.49	0.11	0.05
$B_c \rightarrow \chi_{c1}$	A	0.35	0.49	0.11	0.05
	V_0	0.40	0.45	0.066	0.064
	V_1	0.36	0.48	0.10	0.06
	V_2	0.34	0.49	0.12	0.05
	T_0	0.35	0.49	0.11	0.05
	T_1	0.36	0.48	0.10	0.06
	T_2	0.35	0.48	0.10	0.07
$B_c \rightarrow h_c$	A	0.39	0.46	0.076	0.074
	V_0	0.35	0.49	0.11	0.05
	V_1	0.35	0.48	0.11	0.06
	V_2	0.34	0.47	0.093	0.097
	T_0	0.35	0.49	0.11	0.05
	T_1	0.30	0.51	0.15	0.04
	T_2	0.33	0.51	0.13	0.03
$B_c \rightarrow \chi_{c2}$	h	0.35	0.49	0.11	0.05
	k	0.35	0.49	0.11	0.05
	b_+	0.32	0.53	0.15	0
	b_-	0.32	0.53	0.15	0
	T_0	0.39	0.45	0.09	0.07
	T_1	0.31	0.45	0.045	0.005
	T_2	0.36	0.49	0.11	0.04

$$z(Q^2, t_0) = \frac{\sqrt{t_+ + Q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ + Q^2} + \sqrt{t_+ - t_0}}, \quad (40)$$

where $t_{\pm} = (m_{B_c} \pm m_X)^2$, and $t_0 = t_+ - \sqrt{t_+(t_+ - t_-)}$ [14,83]. The z series are truncated at $N = 3$ in our calculation, and the values of fitting parameters b_k for all form factors are listed in Table IV. The fitting diagrams of these form factors are shown in Figs. 5–8.

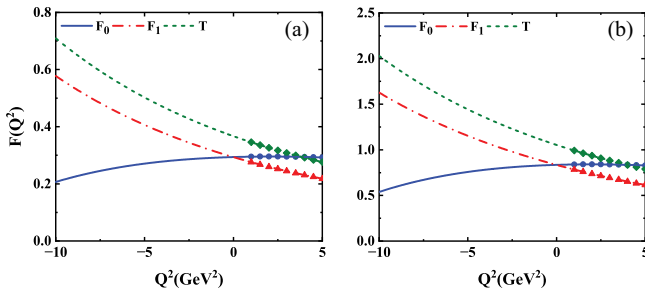


FIG. 5. Fitting results of form factors for $B_c \rightarrow \chi_{c0}$ without considering Coulomb-like correction (a) and considering Coulomb-like correction (b).

TABLE IV. Fitting parameters of different form factors (FF) in z series expand approach. b_k^* denotes the fitting parameters with Coulomb-like correction.

Mode	FF	b_0	b_1	b_2	b_0^*	b_1^*	b_2^*
$B_c \rightarrow \chi_{c0}$	F_0	0.25	4.3	-53	0.70	14	-200
	F_1	0.35	-5.2	34	0.99	-14	75
	T	0.43	-6.2	43	1.24	-17	97
$B_c \rightarrow \chi_{c1}$	A	0.23	-3.3	24	0.65	-8.8	48
	V_0	0.073	-0.94	4.2	0.19	-2.7	14
	V_1	0.42	9.0	-96	1.2	26	-310
	V_2	0.098	-0.71	2.9	0.29	-2.1	2.4
	T_0	0.11	-2.0	17	0.32	-5.7	47
	T_1	-0.55	8.9	-65	-1.6	26	-190
	T_2	1.5	-18	101	4.2	-55	320
$B_c \rightarrow h_c$	A	0.099	-1.9	14	0.26	-5.2	37
	V_0	1.0	-16	110	2.9	-48	330
	V_1	2.9	-36	200	8.6	-120	750
	V_2	0.20	-3.5	22	0.61	-13	96
	T_0	0.21	-4.2	38	0.60	-12	110
	T_1	-0.044	0.26	7.7	-0.18	3.9	-37
	T_2	0.69	-16	130	2.1	-51	420
$B_c \rightarrow \chi_{c2}$	h	0.041	-0.80	6.6	0.12	-2.4	24
	k	2.3	-26	149	6.3	-66	274
	b_+	-0.0088	-0.15	3.2	-0.028	-0.39	9.5
	b_-	0.010	0.10	-2.6	0.033	0.29	-8.5
	T_0	-0.12	2.4	-13	-0.31	8.0	-77
	T_1	0.025	0.30	-11	0.093	-0.34	-11
	T_2	2.6	-46	407	7.1	-118	890

The values of form factors at $Q^2 = 0$ are shown in Table V. As a contrast, the results of other collaborations are also shown in this table. One can find that the form factors with Coulomb-like correction are about three times that of no Coulomb-like correction. For the axial vector form

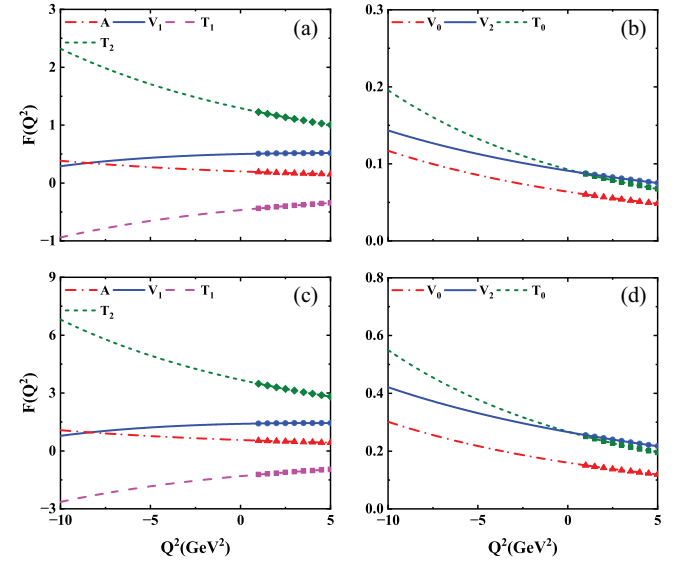


FIG. 6. Fitting results of form factors for $B_c \rightarrow \chi_{c1}$ without considering Coulomb-like correction (a), (b) and considering Coulomb-like correction (c), (d).

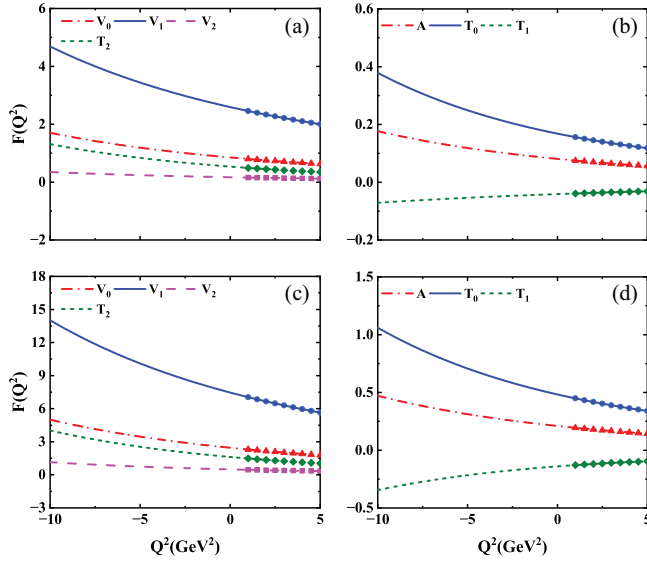


FIG. 7. Fitting results of form factors for $B_c \rightarrow h_c$ without considering Coulomb-like correction (a), (b) and considering Coulomb-like correction (c), (d).

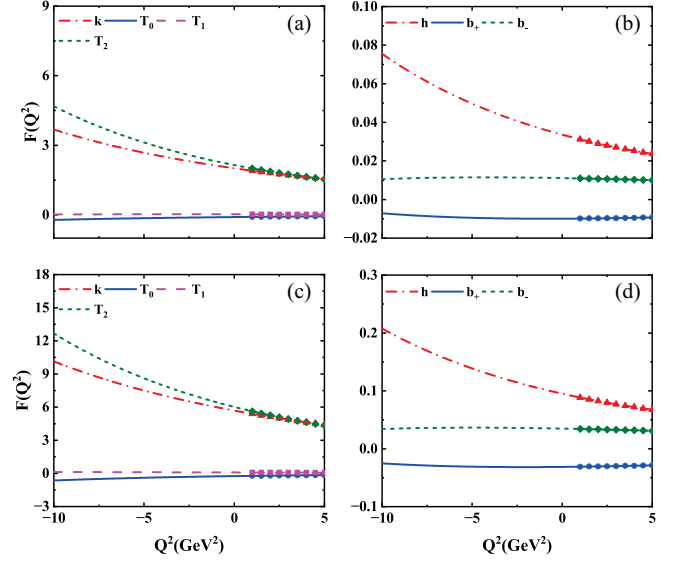


FIG. 8. Fitting results of form factors for $B_c \rightarrow \chi_{c2}$ without considering Coulomb-like correction (a), (b) and considering Coulomb-like correction (c), (d).

TABLE V. The values of the form factors at $Q^2 = 0$.

Modes	Form factors	This work	This work ^a	[9,10]	[25]	[26]	[27]	[28]	[31]
$B_c \rightarrow \chi_{c0}$	F_0	$0.29^{+0.03}_{-0.03}$	$0.84^{+0.06}_{-0.08}$	0.67	0.47	0.33	0.337	...	1.65
	F_1	$0.29^{+0.03}_{-0.03}$	$0.83^{+0.07}_{-0.07}$	0.67	0.47	0.33	0.337	...	1.65
	T	$0.37^{+0.03}_{-0.04}$	$1.05^{+0.07}_{-0.08}$	...	...	...	...	...	...
$B_c \rightarrow \chi_{c1}$	A	$0.20^{+0.02}_{-0.02}$	$0.57^{+0.03}_{-0.05}$	0.13	0.36	0.24	0.215	0.21	1.24
	V_0	$0.064^{+0.013}_{-0.012}$	$0.16^{+0.03}_{-0.03}$	0.03	0.13	0.15	0.025	0.07	0.17
	V_1	$0.50^{+0.05}_{-0.55}$	$1.40^{+0.12}_{-0.12}$	0.28	0.85	0.72	0.339	0.47	2.87
	V_2	$0.091^{+0.005}_{-0.006}$	$0.27^{+0.01}_{-0.02}$	0.059	0.15	0.10	0.078	0.08	0.57
	T_0	$0.093^{+0.006}_{-0.008}$	$0.27^{+0.01}_{-0.02}$	...	...	...	...	...	...
	T_1	$-0.47^{+0.06}_{-0.06}$	$-1.31^{+0.11}_{-0.14}$	...	...	...	...	...	...
	T_2	$1.29^{+0.13}_{-0.14}$	$3.68^{+0.31}_{-0.35}$	...	...	...	...	...	...
	A	$0.081^{+0.010}_{-0.012}$	$0.21^{+0.02}_{-0.02}$	0.14	0.07	0.06	0.039	0.04	0.07
$B_c \rightarrow h_c$	V_0	$0.85^{+0.08}_{-0.07}$	$2.46^{+0.16}_{-0.18}$	0.03	0.64	0.41	0.390	0.35	2.11
	V_1	$2.59^{+0.24}_{-0.26}$	$7.47^{+0.57}_{-0.65}$	0.29	0.50	0.42	0.298	0.30	0.54
	V_2	$0.17^{+0.02}_{-0.02}$	$0.49^{+0.07}_{-0.05}$	0.059	-0.32	-0.18	-0.196	-0.16	-0.99
	T_0	$0.17^{+0.01}_{-0.02}$	$0.48^{+0.03}_{-0.03}$	...	...	...	...	...	...
	T_1	$-0.041^{+0.002}_{-0.002}$	$-0.14^{+0.00}_{-0.01}$	...	...	...	...	...	...
	T_2	$0.54^{+0.01}_{-0.03}$	$1.62^{+0.04}_{-0.05}$	...	...	...	...	...	...
	h	$0.034^{+0.002}_{-0.003}$	$0.095^{+0.005}_{-0.007}$	-1.70×10^{-4}	0.022	...	...	...	...
	k	$2.06^{+0.10}_{-0.21}$	$5.67^{+0.37}_{-0.38}$	0.18	1.27	...	...	...	...
$B_c \rightarrow \chi_{c2}$	b_+	$-0.010^{+0.001}_{-0.004}$	$-0.031^{+0.002}_{-0.001}$	-0.038	-0.011	...	...	...	...
	b_-	$0.011^{+0.001}_{-0.000}$	$0.035^{+0.001}_{-0.001}$	-0.056	0.020	...	...	...	...
	T_0	$-0.097^{+0.003}_{-0.015}$	$-0.24^{+0.01}_{-0.03}$	...	...	...	...	...	...
	T_1	$0.027^{+0.002}_{-0.003}$	$0.089^{+0.006}_{-0.007}$	...	...	...	...	...	...
	T_2	$2.18^{+0.14}_{-0.22}$	$6.02^{+0.38}_{-0.42}$	...	...	...	...	...	...
	h	$0.034^{+0.002}_{-0.003}$	$0.095^{+0.005}_{-0.007}$	-1.70×10^{-4}	0.022	...	...	...	...
	k	$2.06^{+0.10}_{-0.21}$	$5.67^{+0.37}_{-0.38}$	0.18	1.27	...	...	...	...
	b_+	$-0.010^{+0.001}_{-0.004}$	$-0.031^{+0.002}_{-0.001}$	-0.038	-0.011	...	...	...	...

^aIt denotes the results with Coulomb-like correction.

factors $F_{0,1}$ of $B_c \rightarrow \chi_{c0}$, the axial vector form factor A , and vector form factor V_2 of $B_c \rightarrow \chi_{c1}$, our results with no Coulomb-like correction are consistent well with those calculated using the light-front quark model (LFQM) [26–28], but more small for the results given by other LFQM [25]. For the vector form factors V_0 and V_1 of χ_{c1} , our results are well consistent with Ref. [28]. After the Coulomb-like correction is considered, our results are larger than their predictions but smaller than the NRQCD predictions [31]. For the transitions of $B_c \rightarrow \chi_{c2}$, our predictions of h , $b_{+,-}$ without Coulomb-like corrections are compatible with Ref. [25], but much larger than Ref. [10]. For other form factors, various theory predictions are not consistent with each other especially for $B_c \rightarrow h_c$ transitions. In Table V, we can find that our result for the vector form factor V_1 of $B_c \rightarrow h_c$ is much larger than other predictions. Moreover, the results for the vector form factor V_2 are negative in LFQM and NRQCD calculations, but they are positive in QCD sum rules predictions. From Table V, one can also find that our results are not well compatible with other sum rules predictions. By detailed analysis, we find the main discrepancies originate from different values of the decay constant for the hadron, of the energy scale, the type of interpolating currents of P -wave charmonia, etc. In Ref. [9], the vector and axial vector form factors of $B_c \rightarrow \chi_{c0,1}$ and h_c are analyzed with three-point QCD sum rules. In this article, the central values of decay constants for B_c and all P -wave charmonia are taken as 0.4 and 0.34 GeV; the energy scale of heavy quarks is taken as m_c . In addition, the axial vector current $\bar{c}\gamma_\mu\gamma_5 c$ is selected to interpolate both χ_{c1} and h_c . In our present work, the decay constants of B_c are taken as 0.371 ± 0.037 GeV which originates from two-point QCD sum rules calculation considering the next-to-leading contribution of the perturbative term [44], and the values of decay constants for χ_{c0} , χ_{c1} , and h_c are taken from Refs. [74,75] which are 0.343, 0.338, and 0.235 GeV. The heavy quark energy scale $\mu = 2$ GeV is also taken from Ref. [44]. In the present work, we choose the tensor current $\bar{c}\sigma_{\mu\nu}c$ instead of axial vector current to interpolate h_c because the quantum number J^{PC} of h_c is 1^{+-} , but the C parity of the axial current is $+$ [84]. In Ref. [10], the vector and axial vector form factors of $B_c \rightarrow \chi_{c2}$ are analyzed by three-point QCD sum rules; the authors get the QCD spectral density by using Feynman parametrization approach in QCD side. In our analysis, the QCD spectral density is obtained by Cutkosky's rules. This is the main reason for the disagreement.

B. Numerical results of decay widths and branching ratios

With these above form factors, the decay widths and branching ratios of the semileptonic and nonleptonic decays related to B_c to P -wave charmonia transition are

TABLE VI. Input parameters (IP) in analysis the decay processes of B_c meson.

IP	Values [73]	IP	Values
m_π	0.140 GeV	f_K	0.160 GeV [73]
m_K	0.494 GeV	f_ρ	0.216 GeV [73]
m_ρ	0.775 GeV	f_{K^*}	0.217 GeV [73]
m_{K^*}	0.892 GeV	V_{cb}	0.041 [73]
m_e	0.511 MeV	V_{us}	0.224 [73]
m_μ	105.7 MeV	V_{ud}	0.974 [73]
m_τ	1.78 GeV	a_1	1.07 [85]
f_π	0.131 GeV		

calculated in this section. The relevant input parameters are listed in Table VI.

For three-body semileptonic decay $B_c \rightarrow X l \bar{\nu}_l$ ($l = e, \mu$, and τ), the differential decay width can be expressed as

$$d\Gamma(B_c \rightarrow X l \bar{\nu}_l) = \frac{1}{2J+1} \sum \frac{1}{2m_{B_c}} d\Phi(p \rightarrow p', l, v) |T|^2, \\ d\Phi(p \rightarrow p', l, v) = (2\pi)^4 \delta^4(p - p' - l - v) \\ \times \frac{d^3 \vec{p}'}{(2\pi)^3 2p'_0} \frac{d^3 \vec{l}}{(2\pi)^3 2l_0} \frac{d^3 \vec{v}}{(2\pi)^3 2v_0}, \quad (41)$$

where J is the total angular momentum of B_c meson, Σ denotes the summation of all the polarization, $d\Phi(p \rightarrow p', l, v)$ is the three-body phase space, p , p' , l , and v are four momentum for B_c , X , l , and $\bar{\nu}_l$, and T denotes the transition matrix. By inserting intermediate state momentum $q = l + v$, the three-body phase space can be decomposed as a product of two-body phase space,

$$d\Phi(p \rightarrow p', l, v) = \frac{dq^2}{2\pi} d\Phi(p \rightarrow p', q) d\Phi(q \rightarrow l, v), \quad (42)$$

where the two-body space phase $d\Phi(p \rightarrow p', q)$ can be written as

$$d\Phi(p \rightarrow p', q) = (2\pi)^4 \delta^4(p - p' - q) \frac{d^3 \vec{p}'}{(2\pi)^3 2p'_0} \frac{d^3 \vec{q}}{(2\pi)^3 2q_0}. \quad (43)$$

With the Eqs. (31), (32), and (41)–(43), the differential decay widths of $B_c \rightarrow X l \bar{\nu}_l$ can be obtained. The full expressions are shown in Appendix B. The differential decay widths with variations of q^2 are shown in Fig. 9. The decay widths and branching ratios for the three-body semileptonic decays can be derived by finishing the integration of q^2 . The numerical results are collected in Table VII. From this table, one can find that $\Gamma(B_c \rightarrow h_c l \bar{\nu}_l) > \Gamma(B_c \rightarrow \chi_{c2} l \bar{\nu}_l) > \Gamma(B_c \rightarrow \chi_{c0} l \bar{\nu}_l) > \Gamma(B_c \rightarrow \chi_{c1} l \bar{\nu}_l)$ in our prediction, the decay processes of $B_c \rightarrow h_c l \bar{\nu}_l$ and $B_c \rightarrow \chi_{c2} l \bar{\nu}_l$ have large branching ratios. Comparing with results in Refs. [9,10,21,23,25,27,33] for the $B_c \rightarrow \chi_{cJ} l \bar{\nu}_l$ ($J = 0, 1, 2$) and $h_c l \bar{\nu}_l$ decays which are

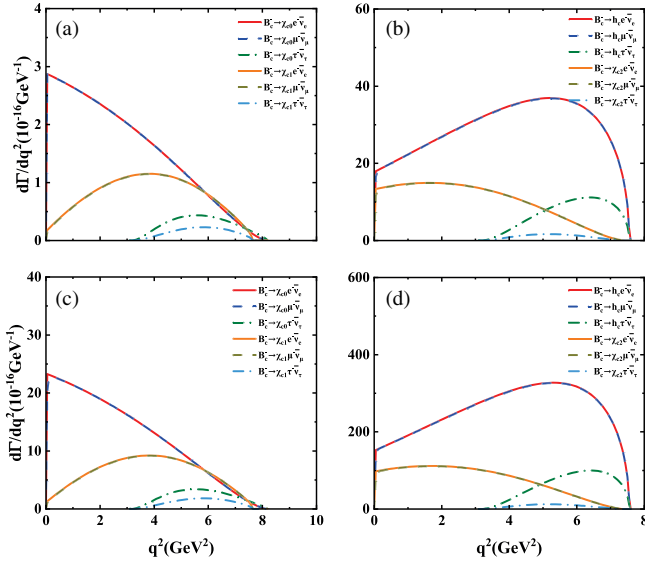


FIG. 9. The differential decay width $d\Gamma/dq^2$ with variations of q^2 for semileptonic decay processes $B_c \rightarrow \chi_{c0} l \nu_l$, $\chi_{c1} l \nu_l$ and $B_c \rightarrow h_c l \nu_l$, $\chi_{c2} l \nu_l$ without considering the Coulomb-like correction (a), (b) and considering the Coulomb-like correction (c), (d).

also collected in Table VII, we can see that most of our results for branching ratios of $B_c \rightarrow \chi_{c0} l \bar{\nu}_l$ and $B_c \rightarrow \chi_{c1} l \bar{\nu}_l$ without Coulomb-like correction are comparable with their predictions. For the branching ratios of $B_c \rightarrow h_c l \bar{\nu}_l$ and $B_c \rightarrow \chi_{c2} l \bar{\nu}_l$, our predictions are larger than other collaborations. When we consider the Coulomb-like correction, these results are much larger than other predictions. This is due to the fact that the form factors will be about three times larger after considering the Coulomb-like correction, which will result in about nine times the decay widths and branching ratios. As the next leading order contribution,

the Coulomb-like correction is too large. Thus, the results without Coulomb-like correction seem more reasonable. In conclusion, the rigorous calculation of next-to-leading order correction for perturbative contribution of form factors eagerly needs to be carried out, and the discrepancies of different theoretical results will be tested by more experiments in the future.

For two-body nonleptonic decay $B_c \rightarrow XP[V]$, the decay width can be expressed by the following standard two-body decay formula,

$$\Gamma = \frac{1}{2J+1} \sum \int \frac{1}{2m_{B_c}} d\Phi(p \rightarrow p', q) |T|^2 \quad (44)$$

With Eqs. (34), (35), and (44), the decay widths of $B_c \rightarrow XP[V]$ can be obtained. The full expressions of decay widths are shown in Appendix C. The numerical results of these two-body decay widths and branching ratios and those of other collaborations are explicitly shown in Table VIII. It can be seen from Table VIII that the theoretical results of different collaborations are not consistent well with each other. The experimental data for the weak decay of B_c to P -wave charmonia are scarce. Only the branching ratios of decay channel $B_c \rightarrow \chi_{c0} \pi$ is given as $\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{Br}(B_c^+ \rightarrow \chi_{c0} \pi^+) = (9.8^{+3.4}_{-3.0}(\text{stat}) \pm 0.8(\text{syst})) \times 10^{-6}$ by the LHCb Collaboration [86]. With our results $\text{Br}(B_c \rightarrow \chi_{c0} \pi) = 0.24^{+0.05}_{-0.04} \times 10^{-3}$ without Coulomb-like correction and $1.96^{+0.21}_{-0.30} \times 10^{-3}$ with Coulomb-like correction, the cross section ratio $\frac{\sigma(B_c^+)}{\sigma(B^+)}$ can be extracted to be about $(2-7) \times 10^{-2}$ and $(3-8) \times 10^{-3}$, respectively, which is useful to study the total cross section of B_c meson. Recently, a systematic analysis about the B_c nonleptonic decay ratios is obtained by using the factorization approach [87]. For pseudoscalar

TABLE VII. Decay widths (in 10^{-6} eV) and branching ratios (%) of B_c semileptonic decays. Branching ratios are calculated at $\tau_{B_c} = 0.51$ ps [73].

Decay channels	Decay widths		Branching ratios								
	This work	This work ^a	This work	This work ^a	[25]	[33]	[21]	[23]	[9,10]	[27]	[28]
$B_c^- \rightarrow \chi_{c0} e \bar{\nu}_e$	$1.25^{+0.20}_{-0.24}$	$9.99^{+1.29}_{-1.44}$	$0.097^{+0.016}_{-0.019}$	$0.78^{+0.09}_{-0.12}$	0.21	0.12	0.17	0.11	0.182	0.11	...
$B_c^- \rightarrow \chi_{c0} \mu \bar{\nu}_\mu$	$1.24^{+0.20}_{-0.24}$	$9.93^{+1.26}_{-1.45}$	$0.096^{+0.016}_{-0.018}$	$0.77^{+0.10}_{-0.11}$	0.21	0.12	0.17	0.11	0.182	0.11	...
$B_c^- \rightarrow \chi_{c0} \tau \bar{\nu}_\tau$	$0.14^{+0.02}_{-0.02}$	$1.05^{+0.20}_{-0.08}$	$0.011^{+0.002}_{-0.002}$	$0.082^{+0.015}_{-0.007}$	0.024	0.017	0.013	0.013	0.049	0.0094	...
$B_c^- \rightarrow \chi_{c1} e \bar{\nu}_e$	$0.61^{+0.15}_{-0.10}$	$4.85^{+0.71}_{-0.83}$	$0.047^{+0.012}_{-0.007}$	$0.38^{+0.05}_{-0.07}$	0.14	0.15	0.092	0.066	0.146	0.030	0.052
$B_c^- \rightarrow \chi_{c1} \mu \bar{\nu}_\mu$	$0.60^{+0.15}_{-0.09}$	$4.82^{+0.70}_{-0.82}$	$0.047^{+0.011}_{-0.007}$	$0.37^{+0.06}_{-0.06}$	0.14	0.15	0.092	0.066	0.146	0.029	0.052
$B_c^- \rightarrow \chi_{c1} \tau \bar{\nu}_\tau$	$0.064^{+0.018}_{-0.007}$	$0.51^{+0.08}_{-0.08}$	$0.0049^{+0.0014}_{-0.0005}$	$0.039^{+0.007}_{-0.005}$	0.015	0.024	0.0089	0.0072	0.0147	0.0026	0.0057
$B_c^- \rightarrow h_c e \bar{\nu}_e$	$22.15^{+3.18}_{-3.43}$	$194.23^{+23.08}_{-24.12}$	$1.72^{+0.24}_{-0.27}$	$15.06^{+1.79}_{-1.87}$	0.31	0.18	0.27	0.17	0.142	0.11	0.105
$B_c^- \rightarrow h_c \mu \bar{\nu}_\mu$	$22.05^{+3.16}_{-3.41}$	$193.31^{+22.98}_{-23.96}$	$1.71^{+0.24}_{-0.27}$	$14.98^{+1.79}_{-1.85}$	0.31	0.18	0.27	0.17	0.142	0.11	0.104
$B_c^- \rightarrow h_c \tau \bar{\nu}_\tau$	$3.03^{+0.36}_{-0.46}$	$26.92^{+3.08}_{-2.66}$	$0.24^{+0.02}_{-0.04}$	$2.09^{+0.24}_{-0.21}$	0.022	0.025	0.017	0.015	0.0137	0.0051	0.0088
$B_c^- \rightarrow \chi_{c2} e \bar{\nu}_e$	$7.50^{+0.79}_{-0.81}$	$55.92^{+7.83}_{-6.64}$	$0.58^{+0.06}_{-0.06}$	$4.34^{+0.51}_{-0.52}$	0.17	0.19	0.17	0.13	0.130	...	...
$B_c^- \rightarrow \chi_{c2} \mu \bar{\nu}_\mu$	$7.43^{+0.78}_{-0.80}$	$55.38^{+7.74}_{-6.58}$	$0.57^{+0.07}_{-0.06}$	$4.29^{+0.60}_{-0.51}$	0.17	0.19	0.17	0.13	0.130	...	...
$B_c^- \rightarrow \chi_{c2} \tau \bar{\nu}_\tau$	$0.41^{+0.04}_{-0.01}$	$3.01^{+0.44}_{-0.29}$	$0.031^{+0.004}_{-0.001}$	$0.23^{+0.04}_{-0.02}$	0.0092	0.029	0.0082	0.0093	0.020	...	...

^aIt denotes the results obtained by considering Coulomb-like correction.

TABLE VIII. Decay widths (in 10^{-7} eV) and branching ratios (10^{-3}) of B_c nonleptonic decays.

Decay channels	Decay widths		Branching ratios							
	This work	This work ^a	This work	This work ^a	[22]	[23]	[26]	[17]	[31]	[16]
$B_c^- \rightarrow \chi_{c0}\pi^-$	$3.11^{+0.65}_{-0.58}$	$25.30^{+3.99}_{-4.07}$	$0.24^{+0.05}_{-0.04}$	$1.96^{+0.21}_{-0.30}$	0.21	0.26	0.66	1.6	6.47	9.8
$B_c^- \rightarrow \chi_{c0}K^-$	$0.24^{+0.05}_{-0.04}$	$1.96^{+0.31}_{-0.31}$	$0.019^{+0.004}_{-0.004}$	$0.15^{+0.03}_{-0.02}$	0.016	0.02	0.052	0.12	0.49	...
$B_c^- \rightarrow \chi_{c0}\rho^-$	$8.23^{+1.46}_{-1.67}$	$66.60^{+8.88}_{-11.31}$	$0.64^{+0.11}_{-0.13}$	$5.16^{+0.69}_{-0.87}$	0.58	0.67	1.69	5.8	...	33
$B_c^- \rightarrow \chi_{c0}K^{*-}$	$0.43^{+0.08}_{-0.09}$	$3.49^{+0.46}_{-0.59}$	$0.033^{+0.006}_{-0.006}$	$0.27^{+0.04}_{-0.04}$	0.04	0.037	0.096	0.33	...	...
$B_c^- \rightarrow \chi_{c1}\pi^-$	$0.14^{+0.06}_{-0.05}$	$0.91^{+0.31}_{-0.34}$	$0.011^{+0.005}_{-0.004}$	$0.070^{+0.025}_{-0.026}$	0.2	0.0014	0.13	0.51	0.064	0.089
$B_c^- \rightarrow \chi_{c1}K^-$	$0.011^{+0.004}_{-0.004}$	$0.070^{+0.024}_{-0.026}$	$0.00082^{+0.00038}_{-0.00028}$	$0.0054^{+0.0019}_{-0.0020}$	0.015	0.00011	0.010	0.038	0.0049	...
$B_c^- \rightarrow \chi_{c1}\rho^-$	$0.88^{+0.23}_{-0.23}$	$6.74^{+1.05}_{-1.74}$	$0.068^{+0.018}_{-0.018}$	$0.52^{+0.08}_{-0.13}$	0.15	0.10	0.43	2.8	...	4.6
$B_c^- \rightarrow \chi_{c1}K^{*-}$	$0.050^{+0.017}_{-0.010}$	$0.41^{+0.07}_{-0.10}$	$0.0041^{+0.0011}_{-0.0010}$	$0.032^{+0.005}_{-0.008}$	0.01	0.0073	0.027	0.18	...	...
$B_c^- \rightarrow h_c\pi^-$	$24.07^{+4.74}_{-3.74}$	$199.98^{+28.92}_{-25.76}$	$1.87^{+0.36}_{-0.29}$	$15.50^{+2.24}_{-1.99}$	0.46	0.53	0.96	0.54	9.73	16
$B_c^- \rightarrow h_cK^-$	$1.86^{+0.37}_{-0.29}$	$15.48^{+2.20}_{-1.59}$	$0.14^{+0.03}_{-0.02}$	$1.20^{+0.17}_{-0.15}$	0.035	0.041	0.075	0.043	0.74	...
$B_c^- \rightarrow h_c\rho^-$	$61.36^{+12.36}_{-9.43}$	$523.62^{+65.30}_{-78.40}$	$4.76^{+0.95}_{-0.73}$	$40.59^{+5.06}_{-6.08}$	1.0	1.3	2.42	2.3	...	53
$B_c^- \rightarrow h_cK^{*-}$	$3.44^{+0.68}_{-0.53}$	$29.20^{+3.64}_{-4.35}$	$0.27^{+0.05}_{-0.04}$	$2.26^{+0.29}_{-0.33}$	0.07	0.071	0.13	0.13	...	...
$B_c^- \rightarrow \chi_{c2}\pi^-$	$19.43^{+1.94}_{-3.97}$	$142.31^{+19.87}_{-20.76}$	$1.51^{+0.15}_{-0.31}$	$11.03^{+1.54}_{-1.61}$	0.38	0.22	...	4.0	4.37	8.3
$B_c^- \rightarrow \chi_{c2}K^-$	$1.45^{+0.15}_{-0.29}$	$10.63^{+1.50}_{-1.53}$	$0.11^{+0.01}_{-0.02}$	$0.82^{+0.12}_{-0.11}$	0.028	0.017	...	0.31	0.33	...
$B_c^- \rightarrow \chi_{c2}\rho^-$	$52.70^{+5.31}_{-9.90}$	$387.82^{+54.56}_{-53.93}$	$4.09^{+0.41}_{-0.77}$	$30.06^{+4.23}_{-4.18}$	1.10	0.65	...	16	...	33
$B_c^- \rightarrow \chi_{c2}K^{*-}$	$2.80^{+0.28}_{-0.51}$	$20.61^{+2.90}_{-2.82}$	$0.22^{+0.02}_{-0.04}$	$1.60^{+0.22}_{-0.22}$	0.074	0.038	...	0.96	...	...

^aIt denotes the results obtained by considering Coulomb-like correction.

mesons π and K , their values about the decay ratios are $B_{\chi_{c2}}^{\chi_{c0}}(\pi) = 0.658$, $B_{\chi_{c2}}^{h_c}(\pi) = 1.597$, $B_{\chi_{c2}}^{\chi_{c0}}(K) = 0.663$ and $B_{\chi_{c2}}^{h_c}(K) = 1.645$, where $B_{\chi_{c2}}^{\chi_{c0}}(\pi)$ denotes $\frac{\text{Br}(B_c \rightarrow \chi_{c0}\pi)}{\text{Br}(B_c \rightarrow \chi_{c2}\pi)}$. Our results are 0.159(0.178), 1.238(1.405), 0.172(0.183), and 1.273(1.463), where the values in parentheses are obtained by considering the Coulomb-like correction. It can be found that our predictions for $B_{\chi_{c2}}^{\chi_{c0}}(\pi/K)$ are smaller, while the results for $B_{\chi_{c2}}^{h_c}(\pi/K)$ are compatible with theirs. For vector mesons ρ and K^* , their values about the decay ratios are $B_{\chi_{c0}}^{\chi_{c1}}(\rho) = 0.206$, $B_{\chi_{c2}}^{\chi_{c0}}(\rho) = 0.590$, $B_{\chi_{c2}}^{h_c}(\rho) = 1.312$, $B_{\chi_{c0}}^{\chi_{c1}}(K^*) = 0.276$, $B_{\chi_{c2}}^{\chi_{c0}}(K^*) = 0.570$ and $B_{\chi_{c2}}^{h_c}(K^*) = 1.231$. Our results are 0.106(0.101), 0.156(0.172), 1.163(1.350), 0.124(0.119), 0.150(0.169), and 1.227(1.413), respectively. It is indicted that our results for $B_{\chi_{c0}}^{\chi_{c1}}(\rho/K^*)$ and $B_{\chi_{c2}}^{\chi_{c0}}(\rho/K^*)$ are smaller, while the results for $B_{\chi_{c2}}^{h_c}(\rho/K^*)$ are close to theirs. In short, both predictions show the same conclusion that the nonleptonic decay width of B_c to P -wave charmonia satisfy the relation $\chi_{c2} > h_c > \chi_{c0} > \chi_{c1}$, while the predictions for ratios are different with each other. Overall, further experimental measurements are eagerly awaited to testify the discrepancies of different theoretical predictions.

It is noted that the NFA is adopted to analyze the two-body nonleptonic decays in our present work. However, as the simplest model to study the two-body weak decays of the heavy flavor hadron, NFA does not take into account the hard gluon exchange between the energetic meson emitted from the weak vertex and the B_c transition form factors. Thus, the two-body decay width formulas in the framework of NFA are not renormalization invariant; this

will lead to systematic uncertainties which are difficult to quantify. Some new factorization techniques such as the QCD factorization (QCDF) [88,89] and the perturbative QCD factorization (pQCD) [90,91] methods, etc., are employed to solve this problem. In future works, we can try to use these new techniques to improve the accuracy of analysis for B_c two-body weak decays.

V. CONCLUSIONS

In this work, the vector, axial vector, and tensor form factors of B_c to P -wave charmonia in spacelike regions are first analyzed in three-point QCD sum rules where the contributions of gluon condensate and Coulomb-like correction are considered. Then, these form factors in zero point and timelike regions are obtained by fitting the results within the z series parametrization approach. With calculated vector and axial vector form factors, we directly analyze the three-body semileptonic decay processes of B_c to P -wave charmonia. In addition, the two-body nonleptonic decay processes of B_c to P -wave charmonia and light pseudoscalar or vector mesons are also analyzed by using NFA. We hope these results can help to shed more light on the properties of B_c meson and P -wave charmonia and provide useful information to research the heavy-flavor physics.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: THE GRAPHICS OF BOREL PLATFORMS FOR ALL FORM FACTORS

See Figs. 10 and 11.

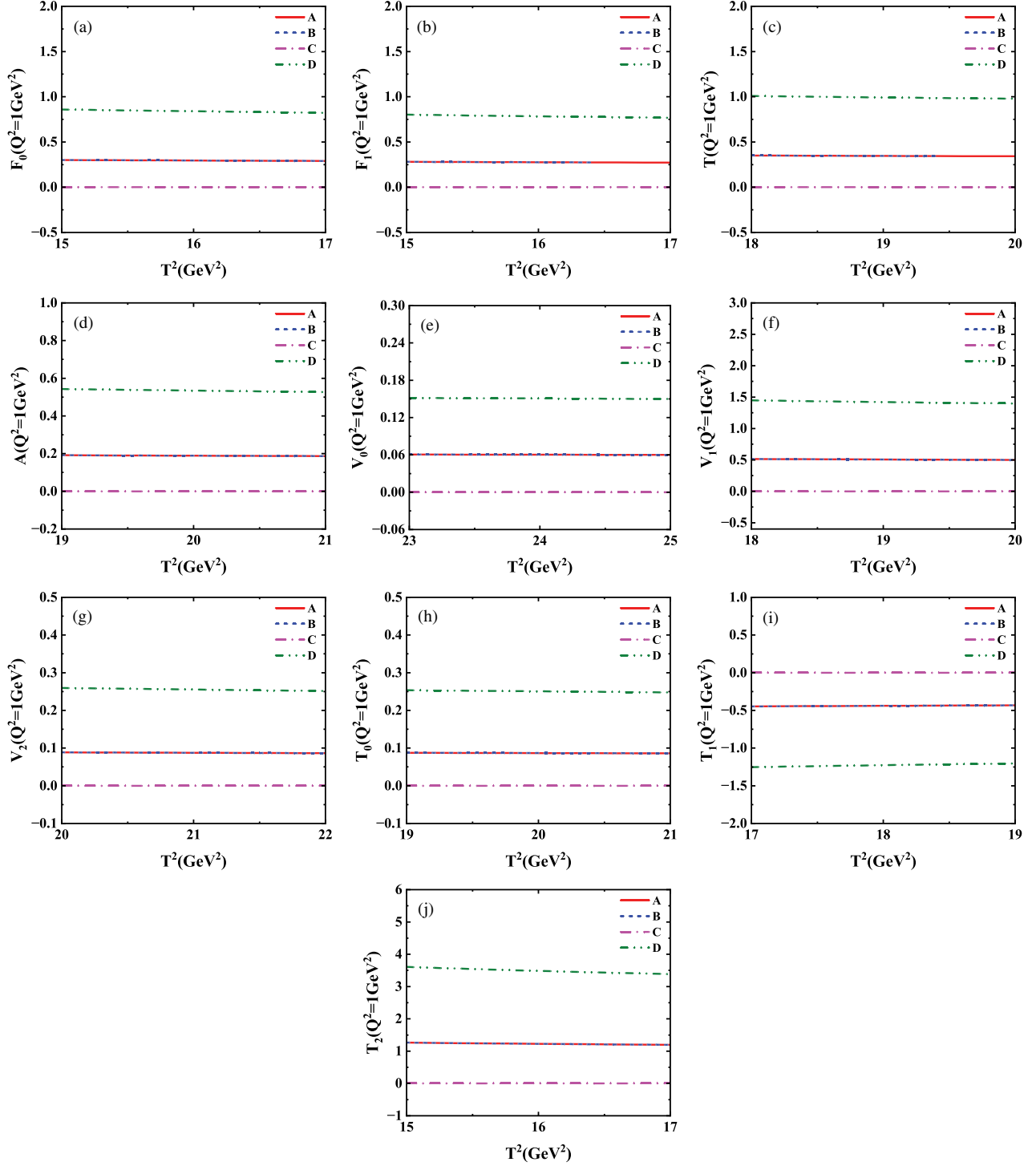


FIG. 10. The contributions of perturbative and nonperturbative terms with variation of Borel parameter T^2 . (a)–(c) Axial vector and tensor form factors of $B_c \rightarrow \chi_{c0}$, (d)–(j) Axial vector, vector, and tensor form factors of $B_c \rightarrow \chi_{c1}$. The legends A–C denote the total, perturbative, gluon condensate contributions, and D represents the total contribution by considering the Coulomb-like correction.

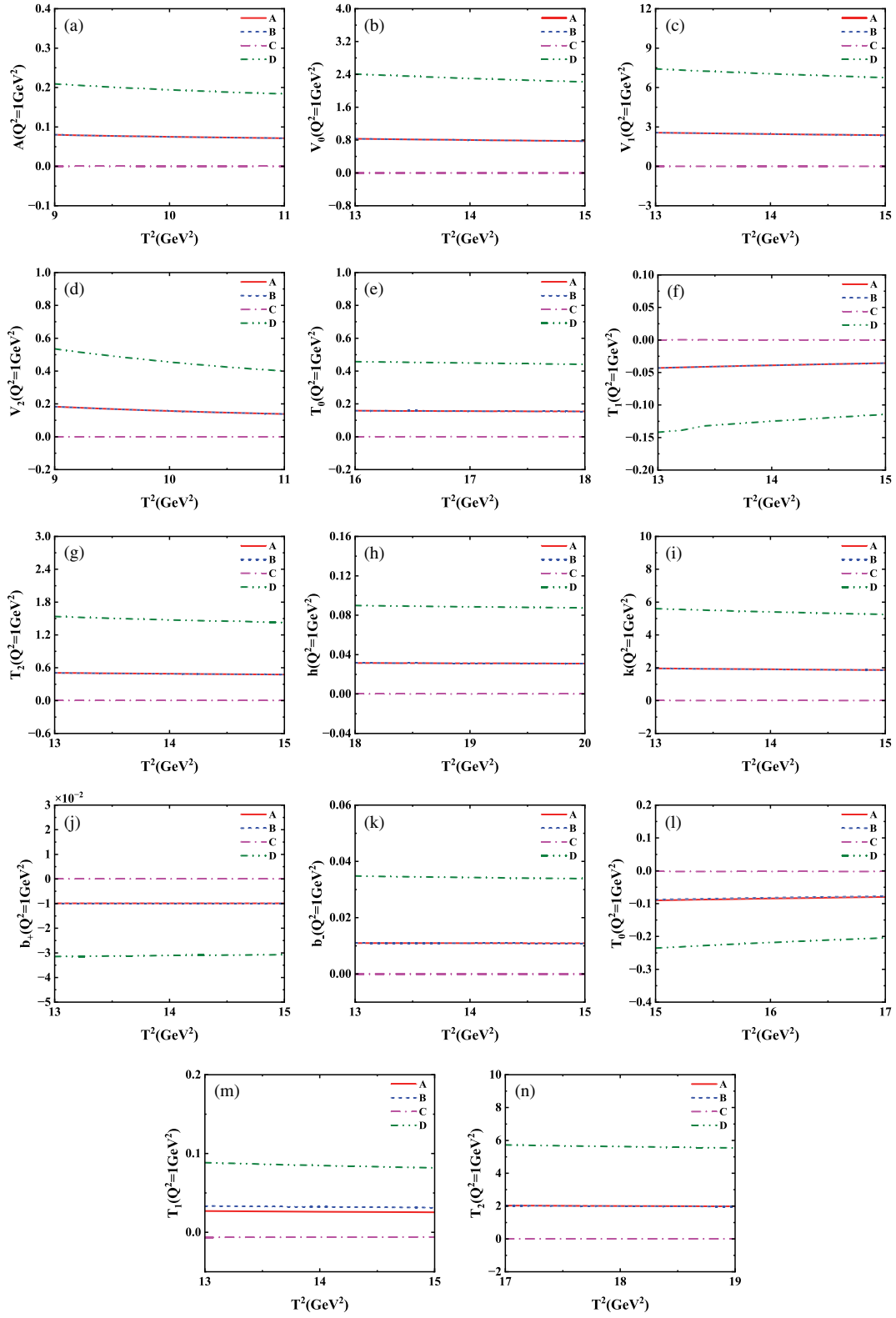


FIG. 11. The contributions of perturbative and nonperturbative terms with variation of Borel parameter T^2 . (a)–(g) Axial vector, vector, and tensor form factors of $B_c \rightarrow h_c$. (h)–(n) Axial vector, vector, and tensor form factors of $B_c \rightarrow \chi_{c2}$. The legends are same as Fig. 10.

APPENDIX B: THE FULL EXPRESSIONS OF DIFFERENTIAL DECAY WIDTHS FOR B_c SEMILEPTONIC DECAYS

The differential decay width of $B_c \rightarrow \chi_{c0} l \bar{\nu}_l$ can be expressed as

$$\frac{d\Gamma(B_c \rightarrow \chi_{c0} l \bar{\nu}_l)}{dq^2} = \left(\frac{q^2 - m_l^2}{q^2} \right)^2 \frac{G_F^2 V_{cb}^2 \lambda(m_{B_c}^2, m_{\chi_{c0}}^2, q^2)^{1/2}}{384\pi^3 m_{B_c}^3 q^2} [(m_l^2 + 2q^2) \lambda(m_{B_c}^2, m_{\chi_{c0}}^2, q^2) F_1(q^2)^2 + 3m_l^2 (m_{B_c}^2 - m_{\chi_{c0}}^2)^2 F_0(q^2)^2], \quad (\text{B1})$$

where the lambda function can be defined as

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc). \quad (\text{B2})$$

The expressions of the differential decay width for $B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l$ is as follows:

$$\frac{d\Gamma(B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l)}{dq^2} = \frac{d\Gamma^L(B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l)}{dq^2} + \frac{d\Gamma^+(B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l)}{dq^2} + \frac{d\Gamma^-(B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l)}{dq^2}, \quad (\text{B3})$$

where

$$\begin{aligned} \frac{d\Gamma^L(B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l)}{dq^2} &= \left(\frac{q^2 - m_l^2}{q^2} \right)^2 \frac{G_F^2 V_{cb}^2 \lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, q^2)^{1/2}}{384\pi^3 m_{B_c}^3 q^2} \left\{ 3m_l^2 \lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, q^2) V_0(q^2)^2 + \frac{m_l^2 + 2q^2}{4m_{\chi_{c1}[h_c]}^2} \right. \\ &\quad \times \left[(m_{B_c}^2 - m_{\chi_{c1}[h_c]}^2 - q^2)(m_{B_c} - m_{\chi_{c1}[h_c]}) V_1(q^2) - \frac{\lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, q^2)}{m_{B_c} - m_{\chi_{c1}[h_c]}} V_2(q^2) \right]^2 \Big\}, \\ \frac{d\Gamma^\pm(B_c \rightarrow \chi_{c1}[h_c] l \bar{\nu}_l)}{dq^2} &= \left(\frac{q^2 - m_l^2}{q^2} \right)^2 \frac{G_F^2 V_{cb}^2 \lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, q^2)^{1/2}}{384\pi^3 m_{B_c}^3} \left\{ (m_l^2 + 2q^2) \lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, q^2) \right. \\ &\quad \times \left[\frac{A(q^2)}{m_{B_c} - m_{\chi_{c1}[h_c]}} \mp \frac{(m_{B_c} - m_{\chi_{c1}[h_c]}) V_1(q^2)}{\lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, q^2)^{1/2}} \right]^2 \Big\}. \end{aligned} \quad (\text{B4})$$

The differential decay width of $B_c \rightarrow \chi_{c2} l \bar{\nu}_l$ can be expressed as

$$\frac{d\Gamma(B_c \rightarrow \chi_{c2} l \bar{\nu}_l)}{dq^2} = \frac{d\Gamma^L(B_c \rightarrow \chi_{c2} l \bar{\nu}_l)}{dq^2} + \frac{d\Gamma^+(B_c \rightarrow \chi_{c2} l \bar{\nu}_l)}{dq^2} + \frac{d\Gamma^-(B_c \rightarrow \chi_{c2} l \bar{\nu}_l)}{dq^2}, \quad (\text{B5})$$

where

$$\begin{aligned} \frac{d\Gamma^L(B_c \rightarrow \chi_{c2} l \bar{\nu}_l)}{dq^2} &= \left(\frac{q^2 - m_l^2}{q^2} \right)^2 \frac{G_F^2 V_{cb}^2 \lambda(m_{B_c}^2, m_{\chi_{c2}}^2, q^2)^{3/2}}{3072\pi^3 m_{B_c}^3 m_{\chi_{c2}}^2 q^2} \left\{ 3m_l^2 \lambda(m_{B_c}^2, m_{\chi_{c2}}^2, q^2) V_0(q^2)^2 + \frac{m_l^2 + 2q^2}{4m_{\chi_{c2}}^2} \right. \\ &\quad \times \left[(m_{B_c}^2 - m_{\chi_{c2}}^2 - q^2)(m_{B_c} - m_{\chi_{c2}}) V_1(q^2) - \frac{\lambda(m_{B_c}^2, m_{\chi_{c2}}^2, q^2)}{m_{B_c} - m_{\chi_{c2}}} V_2(q^2) \right]^2 \Big\}, \\ \frac{d\Gamma^\pm(B_c \rightarrow \chi_{c2} l \bar{\nu}_l)}{dq^2} &= \left(\frac{q^2 - m_l^2}{q^2} \right)^2 \frac{G_F^2 V_{cb}^2 \lambda(m_{B_c}^2, m_{\chi_{c2}}^2, q^2)^{3/2}}{2304\pi^3 m_{\chi_{c2}}^2 m_{B_c}^3} \left\{ (m_l^2 + 2q^2) \lambda(m_{B_c}^2, m_{\chi_{c2}}^2, q^2) \right. \\ &\quad \times \left[\frac{A(q^2)}{m_{B_c} - m_{\chi_{c2}}} \mp \frac{(m_{B_c} - m_{\chi_{c2}}) V_1(q^2)}{\lambda(m_{B_c}^2, m_{\chi_{c2}}^2, q^2)^{1/2}} \right]^2 \Big\}. \end{aligned} \quad (\text{B6})$$

The definitions of form factors A , V_0 , V_1 , and V_2 for $B_c \rightarrow \chi_{c2}$ are given by Eq. (9).

APPENDIX C: THE FULL EXPRESSIONS OF DECAY WIDTHS FOR B_c NONLEPTONIC DECAYS

The full expressions of decay widths for two-body nonleptonic decays $B_c \rightarrow XP$ and $B_c \rightarrow XV$ are as follows:

$$\begin{aligned}\Gamma(B_c \rightarrow \chi_{c0}P) &= \frac{G_F^2 f_P^2 V_{cb}^2 V_{uq}^2 \lambda(m_{B_c}^2, m_{\chi_{c0}}^2, m_P^2)^{1/2}}{32\pi m_{B_c}^3} a_1^2 F_0(m_P^2) (m_{B_c}^2 - m_{\chi_{c0}}^2)^2, \\ \Gamma(B_c \rightarrow \chi_{c0}V) &= \frac{G_F^2 f_V^2 V_{cb}^2 V_{uq}^2 \lambda(m_{B_c}^2, m_{\chi_{c0}}^2, m_V^2)^{3/2}}{32\pi m_{B_c}^3} a_1^2 F_1(m_V^2),\end{aligned}\quad (C1)$$

$$\begin{aligned}\Gamma(B_c \rightarrow \chi_{c1}[h_c]P) &= \frac{G_F^2 f_P^2 V_{cb}^2 V_{uq}^2 \lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, m_P^2)^{3/2}}{32\pi m_{B_c}^3} a_1^2 V_0(m_P^2)^2, \\ \Gamma(B_c \rightarrow \chi_{c1}[h_c]V) &= \Gamma^L(B_c \rightarrow \chi_{c1}[h_c]V) + \Gamma^+(B_c \rightarrow \chi_{c1}[h_c]V) + \Gamma^-(B_c \rightarrow \chi_{c1}[h_c]V),\end{aligned}\quad (C2)$$

where

$$\begin{aligned}\Gamma^L(B_c \rightarrow \chi_{c1}[h_c]V) &= \frac{\lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, m_V^2)^{1/2}}{32\pi m_{B_c}^3} G_F^2 V_{cb}^2 V_{uq}^2 f_V^2 m_V^2 a_1^2 \left[(m_{B_c} - m_{\chi_{c1}[h_c]}) \frac{m_{B_c}^2 - m_{\chi_{c1}[h_c]}^2 - m_V^2}{2m_{\chi_{c1}[h_c]} m_V} V_1(m_V^2) \right. \\ &\quad \left. - \frac{\lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, m_V^2)}{2m_{\chi_{c1}[h_c]} m_V (m_{B_c} - m_{\chi_{c1}[h_c]})} V_2(m_V^2) \right]^2, \\ \Gamma^\pm(B_c \rightarrow \chi_{c1}[h_c]V) &= \frac{\lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, m_V^2)^{1/2}}{32\pi m_{B_c}^3} G_F^2 V_{cb}^2 V_{uq}^2 f_V^2 m_V^2 a_1^2 \left[(m_{B_c} - m_{\chi_{c1}}) V_1(m_V^2) \right. \\ &\quad \left. \mp \frac{\lambda(m_{B_c}^2, m_{\chi_{c1}[h_c]}^2, m_V^2)^{1/2}}{2(m_{B_c} - m_{\chi_{c1}})} A(m_V^2) \right]^2,\end{aligned}\quad (C3)$$

$$\begin{aligned}\Gamma(B_c \rightarrow \chi_{c2}P) &= \frac{G_F^2 f_P^2 V_{cb}^2 V_{uq}^2 \lambda(m_{B_c}^2, m_{\chi_{c2}}^2, m_P^2)^{5/2}}{192\pi m_{B_c}^3 m_{\chi_{c2}}^2} a_1^2 V_0(m_P^2)^2, \\ \Gamma(B_c \rightarrow \chi_{c2}V) &= \Gamma^L(B_c \rightarrow \chi_{c2}V) + \Gamma^+(B_c \rightarrow \chi_{c2}V) + \Gamma^-(B_c \rightarrow \chi_{c2}V),\end{aligned}\quad (C4)$$

where

$$\begin{aligned}\Gamma^L(B_c \rightarrow \chi_{c2}V) &= \frac{\lambda(m_{B_c}^2, m_{\chi_{c2}}^2, m_V^2)^{3/2}}{192\pi m_{B_c}^3 m_{\chi_{c2}}^2} G_F^2 V_{cb}^2 V_{uq}^2 f_V^2 m_V^2 a_1^2 \left[(m_{B_c} - m_{\chi_{c2}}) \frac{m_{B_c}^2 - m_{\chi_{c2}}^2 - m_V^2}{2m_{\chi_{c2}} m_V} V_1(m_V^2) \right. \\ &\quad \left. - \frac{\lambda(m_{B_c}^2, m_{\chi_{c2}}^2, m_V^2)}{2m_{\chi_{c2}} m_V (m_{B_c} - m_{\chi_{c2}})} V_2(m_V^2) \right]^2, \\ \Gamma^\pm(B_c \rightarrow \chi_{c2}V) &= \frac{\lambda(m_{B_c}^2, m_{\chi_{c2}}^2, m_V^2)^{3/2}}{256\pi m_{B_c}^3 m_{\chi_{c2}}^2} G_F^2 V_{cb}^2 V_{uq}^2 f_V^2 m_V^2 a_1^2 \left[(m_{B_c} - m_{\chi_{c2}}) V_1(m_V^2) \right. \\ &\quad \left. \mp \frac{\lambda(m_{B_c}^2, m_{\chi_{c2}}^2, m_V^2)^{1/2}}{2(m_{B_c} - m_{\chi_{c2}})} A(m_V^2) \right]^2.\end{aligned}\quad (C5)$$

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