

Massive gauge theories from consistency conditions of amplitudes

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Based on the general principles of Lorentz symmetry and unitarity, we introduce two consistency conditions—on-shell gauge symmetry and strong massive-massless continuation—in constructing amplitudes of massive gauge theory with elementary particles. In particular we argue that on-shell gauge symmetry can be understood as a consequence of Lorentz symmetry, unitarity, and massive-massless continuation. Based on these two conditions, combined with the little group transformation and consistent factorization, we construct three-point and four-point vector-boson/scalar amplitudes that correspond to renormalizable interactions, then analyze the underlying theories and models. Given the particle masses, almost all possible vertices, including those involving Goldstone modes, are uniquely fixed. The only exceptions are triple and quartic scalar self-couplings, as well as mixing angles between vacuum expectation values (VEVs) and scalars. In addition, all particle masses must have the same physical origin. If the number of vector bosons is smaller than 3, the underlying theories for the amplitudes are either massive gauge theories with spontaneous symmetry breaking (SSB) or Stueckelberg theory. The necessary condition for the latter is that the scalars have equal masses. We also discuss different models depending on the number of scalars involved. If the number of vector bosons is larger than 3, the underlying theory must be Yang-Mills theory with SSB. In both Abelian and non-Abelian cases, the specific shape of the Higgs potential cannot be determined, which explains the fact that scalar self-couplings are undetermined, and the relations between the masses are generally nonlinear.

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I. INTRODUCTION

The project of constraining viable gauge theories through Lorentz invariance and analytic structure of the S matrix can be traced back to Weinberg in his work of [1–3], which generated many beautiful results and physical insights. For example, by considering the S matrix for a massless spin-1 particle taking part in interactions, Lorentz invariance requires the S matrix to obey on-shell gauge invariance: $k_\mu \mathcal{M}^\mu = 0$. Moreover, by taking the soft limit of the spin-1 particle, charge conservation can be derived. Thus an interacting theory of massless spin-1 particles has to be a gauge theory. Similar considerations for massless spin-2 particles lead us to the linearized Einstein equation and equivalence principle. This project got a new life in the last decade with the development of a modern approach of amplitudes, especially spinor-helicity formalism and the Britto-Cachazo-Feng-Witten (BCFW) recursion relation [4].

In [5–7], it was found that $U(1)$ little group scaling of

three-point amplitudes uniquely fixes all possible three-point amplitudes. Moreover, by demanding consistent factorization of four-point amplitudes at s, t, u channels when they go to 0, massless spin-1 and spin-2 interacting theories are uniquely fixed to be Yang-Mills theory and linearized general relativity, respectively. For a fresh review, see also [8].

Similar projects have also been carried out for massive gauge theory. As early as the 1970s, Refs. [9–12] derived gauge invariance and the upper bound of the Higgs boson by applying perturbative unitarity on four-point vector boson scattering amplitudes. In recent years, Ref. [8] introduced a new framework that generalizes the spinor formalism to massive amplitudes, bringing new life to this project. In that work and [13] before it, the authors constructed three-point massive amplitudes, but left four-point amplitudes untouched. Following [8] there have also been many works that generalize this approach to four-point amplitudes ([14–19]), effective field theory (EFT) ([20–24]), and others (e.g., Refs. [25–29]).

Despite those achievements, the project of constructing massive gauge theories from amplitudes is still not wholly satisfactory. The spinor formalism, though very useful in constructing three-point amplitudes, still essentially makes use of unitarity in constructing four-point amplitudes. Moreover, the perturbative unitarity approach itself neglects some crucial aspects of the underlying physics. First,

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precise cancellation between diagrams from perturbative unitarity implies an underlying symmetry. This symmetry, however, is not manifest in the amplitudes, making the cancellation seem *ad hoc*. It is only in the Lagrangian that it is clear the symmetry is gauge symmetry with spontaneous symmetry breaking (SSB). The problem then is how to implement gauge symmetry at the level of amplitudes. It would be also optimal to trace the origin of massive gauge symmetry to first principles such as Lorentz symmetry like the massless case, instead of extracting it from the Lagrangian. Second, the perturbative unitarity approach proves that an elementary scalar can restore the self-consistency of scattering amplitudes and the related theory. However, it's not the only possible solution. In both technicolor theory ([30–32]) and composite Higgs models ([33–36]), unitarity is restored through exchange of many resonances in scattering and ultimately the dissolution of composite particles. This means that we need a clear criterion for the amplitudes to exclude other possibilities in favor of the elementary particle solution. Finally, perturbative unitarity does not apply to all physical scenarios. In Stueckelberg theory, amplitudes have no unitarity violating behavior [37]. It is more complete and satisfying to incorporate this case into the project too.

The purpose of this paper is to find consistency conditions for amplitudes that solve the above problems, then apply them to construct massive amplitudes and the corresponding theories. For the first problem of gauge symmetry, we notice that gauge symmetry with SSB manifests itself in the Goldstone equivalence theorem and its precise form the massive Ward identity $k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi)$ (φ is Goldstone boson) [38–40]. For the second problem of selecting elementary particles, we notice an elementary particle interacts pointlike, which means the coupling is constant at tree level. Based on these analyses we propose the following two conditions in constructing massive amplitudes:

- (i) On-shell gauge symmetry (OGS): the amplitude of massive vector boson with the polarization vector replaced with k^μ equals some corresponding scalar amplitude, i.e., $k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi)$. The vector boson and the scalar have a common mass.
- (ii) Strong massive-massless continuation: the S matrix remains finite when one of the particles' mass is taken to be 0 continuously.

The two conditions can be further understood as reflections of the general principle of massive-massless continuation in different aspects. In particular we will show on-shell gauge symmetry can be understood as emerging from Lorentz symmetry, unitarity, and manifest massive-massless continuation. In [41], a construction of particles as unitary and irreducible representations of Poincaré group that satisfies massive-massless continuation was proposed, providing a solid foundation for this principle.

After finding the consistency conditions that satisfy massive gauge symmetry and select elementary particles,

we proceed to apply those conditions, along with little group transformation and consistent factorization, to construct scattering amplitudes for massive vector bosons and scalars up to four point. We find our approach to be very powerful: we construct all three-point and four-point amplitudes with different mass combinations. All couplings, including those related to Goldstone bosons, are uniquely fixed up to an overall factor, with the sole exceptions of triple and quartic scalar self-couplings, as well as mixing angles between vacuum expectation values (VEVs) and scalars. Most importantly, we find that in constructing four-point amplitudes with $VV \rightarrow VV$, at least one scalar S and the subsequent vertices of VVS are needed to make sure gauge symmetry is satisfied. In addition, all particles' masses go to 0 if the scalar mass goes to 0, implying they all have the same physical origin. Thus we conclude the only theory that is consistent with the conditions of on-shell gauge symmetry and (strong) massive-massless continuation is Yang-Mills theory with Higgs mechanism, in the general sense that symmetry is spontaneously broken to give masses to all particles. But the shape of the scalar potential is not specified. Moreover, we found Higgs theory is not the only possible theory when constructing other amplitudes. For example, the underlying theory for two scalars and one vector boson is scalar QED with the Stueckelberg mechanism, when the two scalars have equal masses. In this case, the vector boson obtains mass without “eating” a dynamic Goldstone mode.

The rest of the paper is organized as follows,

In Sec. II we explain in detail the motivations and justifications of the consistent conditions of on-shell gauge symmetry and strong massive-massless continuation.

In Sec. III we review and introduce the framework of constructing amplitudes, including conventions of polarizations vectors, propagators, and amplitudes.

In Sec. IV, we reconstruct three-point amplitudes for all possible combination of scalars and vector bosons with arbitrary masses.

In Sec. V, we construct four-point amplitudes from three-point amplitudes, determine the conditions for it to succeed, and discuss the underlying theories and models.

Finally in Sec. VI we have conclusions and discussions.

II. CONSISTENCY CONDITIONS

In this section we will demonstrate in detail that the two consistency conditions proposed in Item (I) come from combined consideration of Lorentz symmetry, the general principle of massive-massless continuation and that all particles involved are elementary.

A. On-shell gauge symmetry

To summarize in advance, we will show on-shell gauge symmetry $k^\mu \mathcal{M}_\mu = i m_V \mathcal{M}(\varphi)$ emerges naturally from the

principles of Lorentz symmetry, unitarity, and massive-massless continuation.

It is known that particles can be understood as unitary, irreducible representations of the Poincaré group. The vector representation includes both a spin-1 block and a spin-0 block, which are positive and negative in norms, respectively. In the standard approach, the spin-0 block with negative norm is eliminated from the spectrum by imposing the condition $k \cdot \epsilon = 0$, or equivalently $\partial_\mu V^\mu = 0$ in the position space. Thus only the spin-1 block is taken to be physical. However, a massive vector boson constructed in this way does not have a smooth massless limit. As $m_V \rightarrow 0$, the longitudinal polarization vector goes as $\epsilon_L^\mu \propto \frac{k^\mu}{m_V} \propto \infty$. Here we will make an alternative construction that has manifest massive-massless continuation.

We start with the vector representation without constraint, including both the spin-1 and the spin-0 block. To eliminate the negative norm of the spin-0 block, we introduce a scalar φ to mix with the spin-0 block of the vector boson V^μ , which together with φ is subject to the mixing condition of

$$\partial^\mu V_\mu = m_V \varphi. \quad (1)$$

Notice because of this condition, the number of degrees of freedom (d.o.f.) after adding φ remains the same as $5 - 1 = 4$. Furthermore, noticing that the left side of Eq. (1) only selects the spin-0 block in V^μ , the condition simply relates one scalar in the vector representation with one scalar (φ) in the trivial representation. So the number of scalars after the mixing condition also remains 1.

We can further identify the spin-0 block of V^μ with φ by writing it as $V_{s=0}^\mu = \frac{\partial^\mu \varphi}{m_V}$. In this case, Eq. (1) becomes the equation of motion for the scalar: $\partial^2 \varphi = m_V^2 \varphi$. φ is then simply an auxiliary mode, the purpose of which is to project out the scalar in V^μ from a vector representation to the trivial (scalar) representation.

Equation (1) serves two purposes at the same time. First, it makes sure the negative norm of the spin-0 block is canceled by that of the scalar φ . Applying the condition on the Hilbert space, it indicates that the spin-0 block ($|V_0\rangle$) must always be combined with the scalar $|\varphi\rangle$ to form a physical state, the norm of which is

$$\begin{aligned} (\langle V_0| + \langle \varphi|)(|V_0\rangle + |\varphi\rangle) &= (-1) \frac{k^\mu}{m_V} \cdot \frac{k_\mu}{m_V} + 1 \\ &= -1 + 1 = 0. \end{aligned} \quad (2)$$

The Hilbert space is now positive semidefinite, satisfying the requirement of quantum mechanics. Second, the condition enables us to eliminate the $\frac{k^\mu}{m_V}$ term by adding a zero-norm state $|V_0\rangle + |\varphi\rangle$, thus ensuring that the polarization vectors have smooth massless limits.

The above conclusions can also be translated on polarization vectors. Equation (1) in momentum space can be written as

$$\begin{aligned} k_\mu \epsilon_\sigma^\mu &= \pm i m_V \epsilon^\sigma \delta_{\sigma 0}, & +\text{for incoming} - \text{for outgoing} \\ \epsilon^\sigma &= \mp i, & -\text{for incoming} + \text{for outgoing} \end{aligned} \quad (3)$$

ϵ^σ is the ‘‘polarization’’ of the scalar φ . Equation (3) is also invariant under

$$\epsilon^\mu \rightarrow \epsilon^\mu + \xi k^\mu / m_V, \quad \epsilon^\sigma \rightarrow \epsilon^\sigma \mp \xi i, \quad (4)$$

meaning the polarization vectors defined by Eq. (3) have a gauge redundancy. Moreover, keeping in mind the relative minus sign between the two norms, giving zero norm for the combination of the shift of ϵ_σ^μ and the shift of ϵ^σ : $(-1) \frac{k^\mu}{m_V} \cdot \frac{k_\mu}{m_V} + 1 = -1 + 1 = 0$ as in Eq. (2).

Now let us count the number of d.o.f. in our construction. The vector representation and the scalar φ give 5 d.o.f., while Eqs. (3) and (4) give two constraints, leaving the final number of d.o.f. to be $5 - 2 = 3$, the same as the standard construction. Indeed, the polarization vectors of the standard construction can be understood as a special case in our construction. To see this, we can eliminate the scalar φ from the polarization vectors by making use of Eq. (3).

Our construction has manifest massive-massless continuation. By taking $m_V \rightarrow 0$, our construction reduces to a massless spin-1 particle. First, the constraint in Eq. (4) [Eq. (1)] reduces to $k \cdot \epsilon = 0$ ($\partial^\mu V_\mu = 0$). Second, the symmetry in Eq. (4) reduces to $\epsilon \rightarrow \epsilon + \lambda k$. Combining the two conditions, we obtain exactly the polarization vectors for massless spin-1 particle.

Now we are ready to extract the physical implication of our construction based on Eq. (1) on amplitudes. In our construction the amplitude of a massive vector boson can be written as

$$\mathcal{M}(V) = \epsilon_\sigma^\mu \mathcal{M}_\mu(V) + \epsilon_\sigma^\sigma \mathcal{M}(\varphi). \quad (5)$$

Polarization vectors are equivalent under the redefinition in Eq. (4), which means the gauge redundancy has no contribution to amplitudes. Therefore we must have

$$k^\mu \mathcal{M}_\mu = \pm i m_V \mathcal{M}(\varphi). \quad (6)$$

This is precisely the on-shell gauge symmetry in Item (I).

Finally, the kinetic Lagrangian for a massive vector boson based on Eq. (1) can be derived by making use of Euler-Lagrangian equations. The results are

$$\begin{aligned} \mathcal{L}_{V^2} &= -\frac{1}{4} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2} m_V^2 V_\mu^2, \\ \mathcal{L}_{V-\varphi} &= m_V V_\mu \partial^\mu \varphi, \\ \mathcal{L}_{\varphi^2} &= \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi. \end{aligned} \quad (7)$$

This is exactly the Lagrangian of gauge-Goldstone fields in SSB, with the gauge-Goldstone mixing term $\mathcal{L}_{V-\varphi}$.

Moreover, the mass term for the scalar is zero, which means it has zero mass without the mixing, thus justifying calling it Goldstone boson. The details of the derivation are in the Appendix.

B. Strong massive massless continuation

On-shell gauge symmetry originates from Lorentz symmetry generally, thus does not distinguish between elementary and composite particles, but our purpose is to select theories of elementary particles only. This means that we need another consistent condition.

Particles are elementary in the sense that they remain pointlike in interactions at arbitrary energy scale. This is in contrast with composite particles, whose form factors change drastically with increased energy. When the energy is high enough, a composite particle would eventually fragment into more elementary components. From the perspective of effective field theory, the theory based on the composite particle is only effective up to certain energies. When the energy increases to a certain point, the effective theory becomes invalid, there needs to be another theory that is based on more elementary degrees of freedom to replace the former. In amplitudes, one of the main signals of the process of effective theory becoming invalid is the energy increasing behavior that violates perturbative unitarity. The fact that amplitudes eventually blow up to infinity as $E \rightarrow \infty$, reflects the process of the composite particle dissolving into its elementary components.

The above analysis can be captured by the partial wave analysis of amplitudes. However, we would like to have an approach that has the above property directly reflected in the analyticity of the S matrix as function of momenta, polarization vectors, as well as masses and couplings. Noticing for longitudinal vector boson, $E \rightarrow \infty$ is equivalent to $m_V \rightarrow 0$ to certain extents. The condition of unitarity for amplitudes can be expressed as follows: amplitudes remain finite when the vector boson's mass goes to 0. We further generalize the condition from the vector boson's mass to any particle's mass in the process. This finally gives the condition of strong massive-massless continuation in Item (I). In mathematical form, it can be written as

$$\lim_{m_i \rightarrow 0} \mathcal{M}(\{p_i, \epsilon_i, m_i\}, \lambda_j) = \text{finite} \quad (8)$$

i being any massive particle.

Finally we note that, although both strong massive-massless continuation and perturbative unitarity can be understood as concrete realizations of the physical differences between elementary and composite particles, the two cannot be guaranteed to be equivalent in every situation. It is an open question that can only be determined by the results of construction from the conditions.

III. FRAMEWORK OF CONSTRUCTING AMPLITUDES

A. Five-component formalism

Since in our construction, a vector boson (V) obtains mass by mixing with a scalar (φ) to form a single physical object, it is natural and convenient to define a unified five-component object as $V^M = (V^\mu, \varphi)$, with $M = 4$ corresponding to the Goldstone component ([42]). The amplitudes for vector bosons, stripping off energy-momentum conservation, are then written as

$$\mathcal{M}(p_1, p_2, \dots) = \epsilon_{s_1}^{M_1}(p_1) \epsilon_{s_2}^{M_2}(p_2) \dots \mathcal{M}_{M_1 M_2 \dots}(p_1, p_2, \dots), \quad (9)$$

in which $\epsilon_{s_i}^{M_i}(p_i)$ ($i = 1, 2, \dots$) is a polarization vector with spin s_i and momentum p_i .

Subsequently, polarization vectors can also be written in five-component form: $\epsilon_\sigma^M = (\epsilon_\sigma^\mu, \epsilon_\sigma^\varphi)$. We can also define the five-component momentum k^M : $k^M = (k^\mu, \pm i m_V)$. We introduce a five-component “matrix” $g_{MN} = \text{diag}(g_{\mu\nu}, -1)$, with the last -1 coming from the positive norm of the scalar φ . This convention ensures on-shell condition ($k^2 = m_V^2$) and the transverse condition ($k^\mu \epsilon_{\mu\sigma} = \pm i \epsilon_\sigma^\varphi$) follows from the five-component form. To eliminate the $\frac{k^\mu}{m_V}$ term, the polarization vectors and five-component momentum are chosen as

$$\begin{aligned} \text{initial: } \epsilon_\pm^M &= (\epsilon_\pm^\mu, 0)^T & \text{final: } \epsilon_\pm^{*M} &= (\epsilon_\pm^{*\mu}, 0)^T, \\ \text{initial: } \epsilon_L^M &= (\epsilon_L^\mu, +i)^T & \text{final: } \epsilon_L^{*M} &= (\epsilon_L^\mu, -i)^T, \\ \text{initial: } k^M &= (k^\mu, -i m_V) & \text{final: } k^{*M} &= (k^\mu, +i m_V) \end{aligned} \quad (10)$$

with $\epsilon_n^\mu(k) = -\frac{m}{n \cdot k} n^\mu$, n being the null vector that imposes light-cone gauge on ϵ^μ ,

$$n \cdot \epsilon(k, r) = 0 \quad \text{with} \quad n^2 = 0. \quad (11)$$

If we choose $n^\mu = (1, -\frac{\vec{k}}{|\vec{k}|})$, the corresponding polarization vectors are precisely the standard ones in textbooks.

The on-shell condition, transverse condition, and orthogonal conditions can be written in five-component form as

$$k^{*M} k_M = 0 \quad k^{*M} \epsilon_M^\sigma = 0 \quad \epsilon_\sigma^{*M} \epsilon_M^{\sigma'} = -\delta_{\sigma\sigma'}. \quad (12)$$

On-shell gauge symmetry can also be written in five-component form as

$$k^M \mathcal{M}_M = 0. \quad (13)$$

Finally, we discuss how crossing symmetry acts on polarization vectors in five-component form. It is known

the S matrix is invariant under crossing symmetry, which takes the initial/final wave function with momentum k and helicity s to the final/initial wave function with momentum $-k$ and helicity $-s$, i.e., $u_s(k)/\bar{u}_s(k) \rightarrow \bar{v}_{-s}(-k)/v_{-s}(-k)$. Acting on the five-component polarization vectors, a transverse polarization vector simply takes the Hermitian conjugate: $\epsilon_{\pm}^M(k) \rightarrow \epsilon_{\pm}^{*M}(-k) = \epsilon_{\pm}^{*M}(k)$, since they do not depend on momentum. The situation is different for the longitudinal, $\epsilon_L^M(k) \rightarrow \epsilon_L^{*M}(-k)$. From Eq. (10), we have $\epsilon^\mu(-k) = -\epsilon^\mu(k)$, which means that $\epsilon_L^{*M}(-k) = (-\epsilon^\mu(k), -\epsilon^\varphi) = -\epsilon_L^M(k)$. Equivalently, $\epsilon_L^{*M}(k) = -\epsilon_L^M(-k)$. Thus, the complex conjugate for the longitudinal is equivalent to crossing symmetry up to an unphysical, overall minus sign. This result can be useful in doing concrete, analytic calculations in five-component formalism.

B. Approach to constructing amplitudes

Having set up the consistent conditions, we are now ready to construct massive amplitudes with them and other fundamental principles. Here we give a sketch of the method, conveying the basic idea of our approach. In this paper we focus on constructing four-point amplitudes, which can give us enough information to understand the underlying theory. Our method can be divided into two steps: first we construct three-point amplitudes from their massless limit by making use of the consistent conditions; then we construct four-point amplitudes from three-point amplitudes by consistent factorization.

First we analyze three-point amplitudes. Taking the massless limit of a massive vector amplitude, its transverse and longitudinal parts decouple from each other and form Lorentz invariant blocks separately. Taking an amplitude with one vector boson as an example,

$$\mathcal{M}_s(V) = \epsilon_{\pm,L}^M \mathcal{M}_M(p, m_V) \xrightarrow{m_V \rightarrow 0} \begin{cases} \epsilon_s^\mu \mathcal{M}_\mu(p) s = \pm \\ \mathcal{M}^\varphi(p) s = L \end{cases}. \quad (14)$$

Massless amplitudes can be constructed from Lorentz symmetry. Moreover, the massless vector amplitude can be extended naturally back to massive case to include the ϵ_n part (0 in massless limit) of the longitudinal amplitude according to Eq. (10). Now we have all the elements ready to construct the massive amplitude. Based on above analysis, we propose that a massive vector amplitude can be written as a linear combination of massless amplitudes,

$$\mathcal{M}_s^V(p, m_V) = a \epsilon_s^\mu \mathcal{M}_\mu^V(p) + b \mathcal{M}^\varphi(p). \quad (15)$$

a, b are coefficients to be fixed by the two consistent conditions.

It should be noted, however, that there is a potential problem when applying on-shell gauge symmetry on three-point amplitudes. Generally it is impossible to make all

three particles on shell simultaneously, except for the special case of one particle mass larger than the sum of the two other masses. In other words, one of them has to be off shell (in general kinematics). This seems to imply on-shell gauge symmetry cannot apply on three-point amplitudes.

However, after careful consideration, we show OGS is still applicable to a three-point amplitude when one particle is off shell. As will be clear in subsequent sections, when applying on-shell gauge symmetry by replacing ϵ^M with k^M , we first make use of energy-momentum conservation to substitute k with other momenta in the amplitude, then we apply the on-shell condition, transverse condition, and orthogonal conditions [Eq. (12)] for the other particles. In other words, we do not make use of any information that is relevant to the on shellness of the particle we apply OGS on. Therefore, on-shell gauge symmetry applies to an amplitude with one particle being off shell, if OGS is being applied on this off-shell particle. This result also eliminates a motivation for complexifying the momenta. Since there is no need, we will keep the momenta real without complexification throughout the paper.

After constructing three-point amplitudes, we then proceed to construct four-point amplitudes. According to consistent factorization, when an internal line is put on-shell, amplitudes of an n -point amplitude factorize into amplitudes of fewer number of external points,

$$\mathcal{M}_4 \xrightarrow{k_a^2 \rightarrow m_a^2} \sum_{s,a} \mathcal{M}_{3,a}^s \frac{i}{k_a^2 - m_a^2} \mathcal{M}_{3,a}^{-s} \quad (16)$$

s is spin index. a labels the intermediate state of which the full four-point amplitude requires summing over all possible numbers.

Now we would like to reverse the factorization process and obtain the full amplitude from lower-point ones. A common way is to complexify the momenta and BCFW recursion relation. However, as already stated above, we do not complexify momenta. Instead we look for a natural off-shell continuation for the vector boson propagator. This “off-shell continuation” can be found by writing the summation over polarization vectors [42] as

$$\sum_{s=\pm,0} \epsilon_s^M \epsilon_s^{*N} = -g^{MN} + \frac{k^M n^N + n^M k^{*N}}{n \cdot k}. \quad (17)$$

The rhs of the above equation has the same form when both on shell and off shell, thus can be naturally used for off-shell continuation.

Next notice that k^M terms give 0 to amplitudes by on-shell gauge symmetry, even when off shell. Thus we can drop the k^M terms for the vector propagator. The off-shell continuation for the propagator of the massive vector boson simplifies to,

$$\frac{i \sum_{s=\pm,0} \epsilon_s^M \epsilon_s^{*N} k_i^2 \neq m_i^2}{k_i^2 - m_i^2} \rightarrow \frac{-ig^{MN}}{k_i^2 - m_i^2}. \quad (18)$$

To sum up, we can construct amplitudes from lower points by off-shell continuation in the following way

$$\sum_{s,a} \mathcal{M}_{3,a}^s \frac{i}{k_a^2 - m_a^2} \mathcal{M}_{3,a}^{-s} \xrightarrow{\text{off-shell}} \mathcal{M}_4 = \sum_a \mathcal{M}_{3,a}^M \frac{-ig_{MN}}{k_a^2 - m_a^2} \mathcal{M}_{3,a}^N. \quad (19)$$

Finally, the coefficients in constructing four-point amplitudes should also be fixed by the two consistent conditions.

IV. THREE-POINT AMPLITUDES

In this section we set out to construct all possible three-point amplitudes involving massive vector bosons. We follow the method outlined in Sec. III: construct massless amplitudes first, then construct massive amplitudes from massless amplitudes by the two consistent conditions.

A. Massless three-point amplitudes

Let us first construct massless three-point amplitudes as basis for massive amplitudes. Lorentz invariance implies that a scattering amplitude should be written as

$$\mathcal{M}(1, 2, \dots, n) = \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \dots \epsilon_n^{\mu_n} \mathcal{M}(1, 2, \dots, n)_{\mu_1 \mu_2 \dots \mu_n}. \quad (20)$$

Polarization vectors ϵ_i^μ undertake little group transformations, which include $U(1)$ scaling under rotation around the reference momentum, as well as the momentum shift $\epsilon^\mu \rightarrow \epsilon^\mu + \xi k^\mu$ [2] under the noncompact members of little group transformation. The latter gives massless Ward identity,

$$k_1^{\mu_1} \epsilon_2^{\mu_2} \dots \epsilon_n^{\mu_n} \mathcal{M}(1, 2, \dots, n)_{\mu_1 \mu_2 \dots \mu_n} = 0. \quad (21)$$

$\mathcal{M}(1, 2, \dots, n)_{\mu_1 \mu_2 \dots \mu_n}$ is a function of momenta $k_i^{\mu_i}$ and their contraction with each other through tensor $g^{\mu\nu}$. We neglect $\epsilon^{\mu\nu\rho\sigma}$ by assuming CP symmetry is conserved. In addition, in this paper we fix the energy dimension of the three-point amplitudes to 1, corresponding to renormalizable Lagrangian terms only. Finally, we use S to denote scalar, V to denote vector boson.

Massless SSV. To maintain the dimension to be 1, the amplitude of massless $S_1 S_2 V_3$ must be of the following form

$$\mathcal{M}_0(S_1 S_2 V_3) = (a_1 k_1 + a_2 k_2) \cdot \epsilon_3$$

where a_1 and a_2 are dimensionless parameters to be fixed. On-shell gauge symmetry Eq. (21) constrains a_1 and a_2 to be

$$a_1 = -a_2. \quad (22)$$

Thus the amplitude for massless $1^0 2^0 3^T$ is

$$\mathcal{M}_0(S_1 S_2 V_3) = (k_1 - k_2) \cdot \epsilon_3 \quad (23)$$

up to an overall factor.

Furthermore, boson symmetry for the amplitude requires the two scalars not to be identical. Instead, the two must have different quantum numbers, which we can label as a , b , so that the amplitude could be written as

$$\mathcal{M}_0(S_1^a S_2^b V_3) = T_{ab} \mathcal{M}_0(S_1 S_2 V_3) \quad (24)$$

with $T_{ab} = -T_{ba}$. For example, the scalars can be in a $SO(N)$ group, T then is one of its generators. There is also a special case of $N = 2$, giving $SO(2) \cong U(1)$. In this case, the two scalars are to be identified as the antiparticles of each other.

“Massless” SVV. The amplitude of SVV can be generally written as

$$\mathcal{M}_0(S_1 V_2 V_3) = m \epsilon_2 \cdot \epsilon_3. \quad (25)$$

The coefficient m has to be of dimension 1. Notice there is not anything constraining it to be a constant yet. Generally it should be a function of momenta: $m = m(p_1, p_2)$. However, we notice Eq. (25) does not satisfy massless on-shell gauge symmetry [Eq. (21)], which implies Eq. (25) is not really a massless amplitude. In fact, as we will see in Sec. IV B, Eq. (25) can be only understood as part of the massive SVV amplitude that satisfies massive on-shell gauge symmetry. This also means the amplitude should disappear in the massless limit. Thus following strong massive-massless continuation, we must have $m \rightarrow 0$ as either one of the masses $m_i \rightarrow 0$,

$$\lim_{m_i \rightarrow 0} m = 0. \quad (26)$$

This indicates the dependence of m on particle masses in some form, which further points to a common origin of m and particle masses.

Massless VVV. Similar to the derivation of SSV , the massless amplitude of VVV can be written generally as

$$\mathcal{M}_0(V_1 V_2 V_3) = (a_{13} k_1 + a_{23} k_2) \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_2 + \text{cyclic}.$$

a_{13} , a_{23} and etc. are dimensionless parameters to be fixed by on-shell gauge symmetry, which gives

$$a_{13} = -a_{23} = a_{32} = -a_{12} = a_{21} = -a_{31}. \quad (27)$$

$\mathcal{M}_0(V_1 V_2 V_3)$ becomes

$$\mathcal{M}_0(V_1 V_2 V_3) = (\epsilon_1 \cdot \epsilon_2)(k_1 - k_2) \cdot \epsilon_3 + \text{cyclic} \quad (28)$$

up to an overall factor.

Moreover, boson symmetry requires a totally antisymmetric factor for the three vector bosons. Labeling them as a, b, c , respectively, we arrive at the final form of $\mathcal{M}_0(V_1 V_2 V_3)$,

$$\mathcal{M}_0(V_1^a V_2^b V_3^c) = f^{abc} \mathcal{M}_0(V_1 V_2 V_3) = f^{abc} [(\epsilon_1 \cdot \epsilon_2)(k_1 - k_2) \cdot \epsilon_3 + \text{cyclic}]. \quad (29)$$

f^{abc} is totally antisymmetric over a, b, c .

B. Massive three-point amplitudes

Having finished constructing the massless amplitudes, we now turn to the massive amplitudes. Apart from the trivial SSS amplitude, the possible amplitudes include SSV , SVV , and VVV , respectively.

Massive SSV. Let us start with the SSV amplitude, denoted as $\mathcal{M}(S_1 S_2 V_3)$. In the massless limit, $\mathcal{M}(S_1 S_2 V_3)$ decomposes into $\mathcal{M}_0(S_1 S_2 V_3)$ and $\mathcal{M}_0(S_1 S_2 \varphi_3)$ that are Lorentz covariant separately. $\mathcal{M}_0(S_1 S_2 V_3)$ is given by Eq. (23). $\mathcal{M}(S_1 S_2 \varphi_3)$ is the amplitude for three scalars, which is trivial.

Up to an overall scaling, the general form of $\mathcal{M}(S_1 S_2 V_3)$ is

$$\mathcal{M}(S_1 S_2 V_3) = (k_1 - k_2) \cdot \epsilon_3 - \lambda_{SS\varphi} \epsilon_3^4. \quad (30)$$

$\lambda_{SS\varphi}$ is a parameter that can be fixed by massive on-shell gauge symmetry,

$$\mathcal{M}(S_1 S_2 V_3)|_{\epsilon_3^M \rightarrow k_3^M} = 0, \quad (31)$$

which gives

$$\lambda_{S_1 S_2 \varphi} = -i \frac{(m_1^2 - m_2^2)}{m_V}. \quad (32)$$

Thus we found the solution. However, the relations between masses can still be constrained by strong massive-massless continuation, which requires that $\lambda_{SS\varphi}$ stays finite as one of the mass goes to 0,

$$\lim_{m_i \rightarrow 0} \lambda_{SS\varphi}(m_1, m_2, m_3) \neq \infty. \quad (33)$$

From it we have for m_1 and m_2 ,

$$\begin{aligned} m_1^2 &= c^{(0)} + c_1^{(1)} m_V + \mathcal{O}(m_V^2) \\ m_2^2 &= c^{(0)} + c_2^{(1)} m_V + \mathcal{O}(m_V^2). \end{aligned} \quad (34)$$

This is the most general result we can obtain from the two consistent conditions, indicating at least parts of particles masses have the same physical origin.

Now let us discuss the physical meaning of the solution [Eqs. (32) and (34)] in different limits.

First of all, if $m_V = 0$, the vector boson is massless. The $SS\varphi$ amplitude decouples from the gauge part. Applying

the Ward identity (massless OGS) gives $m_1 = m_2$. So the two scalars must have equal masses, which further implies a $SO(2) \cong U(1)$ symmetry between S_1 and S_2 . In this case, the SSV amplitude simply corresponds to QED with a massive complex scalar field.

Second, if we impose $m_1 = m_2$ while keeping $m_V \neq 0$, we obtain $\lambda_{SS\varphi} = 0$. The equal mass condition is equivalent to imposing a $SO(2) \cong U(1)$ symmetry on (S_1, S_2) . In this case the Goldstone mode is still a component of the vector boson, but never contributes to any amplitudes. In other words, the ‘‘Goldstone’’ component does not partake in the interactions. That means we should treat the ‘‘Goldstone’’ component as unphysical that comes from an auxiliary field. This scenario corresponds to the Stueckelberg theory.

Third, noticing $\lambda_{SS\varphi}$ has a dimension of 1, it is natural to assume it is related to the masses of the particles. Under this assumption we can demand $\lambda_{SS\varphi} \rightarrow 0$ as $m_V \rightarrow 0$, $m_1^2 - m_2^2$ satisfies

$$m_1^2 - m_2^2 = c^{(2)} m_V^2 + \mathcal{O}(m_V^3). \quad (35)$$

The scalar masses (after subtracting the constant term) are proportional to the vector mass at the lowest order, but not limited to it.

Finally, the results here only concern three-point amplitudes. In order to confirm the physical analysis above, we need to complete four-point amplitudes.

Massive: SVV. Next we go on to construct the scalar-vector-vector amplitude, denoted as $\mathcal{M}(S_3 V_1 V_2)$. In massless limit, it decomposes into $\mathcal{M}_0(S_3 V_1 V_2)$, $\mathcal{M}_0(S_3 \varphi_1 V_2)$, $\mathcal{M}_0(S_3 V_1 \varphi_2)$, and $\mathcal{M}_0(S_3 \varphi_1 \varphi_2)$. The first one is given by Eq. (25), the second and the third are given by Eq. (23), the final one is trivially a three scalar amplitude.

Thus we conclude $\mathcal{M}(S_3 V_1 V_2)$ has the following general form

$$\begin{aligned} \mathcal{M}(S_3 V_1 V_2) &= m(\epsilon_1 \cdot \epsilon_2) - a_1(k_1 - k_3) \cdot \epsilon_2 \epsilon_1^4 \\ &\quad - a_2(k_2 - k_3) \cdot \epsilon_1 \epsilon_2^4 + a_3 \epsilon_1^4 \epsilon_2^4. \end{aligned} \quad (36)$$

m, a_1, a_2, a_3 are coefficients that can be determined by on-shell gauge symmetry in terms of particle masses up to an overall scaling.

To simplify the calculation, we first apply on-shell gauge symmetry on V_1 by choosing $\epsilon_1^M \rightarrow k_1^M$ and setting $s_2 = T$, this fixes a_1 to be $a_1 = -\frac{im}{2m_1}$; then exchange the label of particle 1 and particle 2 and repeat the last step, we then get $a_2 = -\frac{im}{2m_2}$; finally, we still apply on-shell gauge symmetry on V_1 and set $s_2 = L$, then a_3 is also fixed: $a_3 = \frac{mm_3^2}{2m_1 m_2}$.

To summarize, a_1 , a_2 , a_3 are fixed by on-shell gauge symmetry to be

$$a_1 = -\frac{im}{2m_1}, \quad a_2 = -\frac{im}{2m_2}, \quad a_3 = \frac{mm_3}{2m_1m_2}. \quad (37)$$

We can further extract an overall factor $g_{VVS} \equiv \frac{m}{m_1}$, after which the amplitude of SVV becomes

$$\mathcal{M}(S_3V_1V_2) = m_1(\epsilon_1 \cdot \epsilon_2) - a_1(k_1 - k_3) \cdot \epsilon_2 \epsilon_1^4 - a_2(k_2 - k_3) \cdot \epsilon_1 \epsilon_2^4 + a_3 \epsilon_1^4 \epsilon_2^4 \quad (38)$$

with

$$a_1 = -\frac{i}{2}a_2 = -\frac{im_1}{2m_2}a_3 = \frac{m_3^2}{2m_2}. \quad (39)$$

Thus the massive amplitudes are uniquely fixed given the particle masses. Similar to SSV , massive-massless continuation gives constraints on the mass parameters

$$\begin{aligned} m &= c_{1m}m_1 + \mathcal{O}(m_1^2) & m_2 &= c_{12}m_1 + \mathcal{O}(m_1^2) \\ m_3 &= c_{31}m_1 + \mathcal{O}(m_1^2). \end{aligned} \quad (40)$$

Massive: VVV. Next we turn to the amplitude of three spin-1 particles. We denote it as $\mathcal{M}(V_1V_2V_3)$. In the massless limit, the amplitude $\mathcal{M}(V_1V_2V_3)$ decomposes into

$$\mathcal{M}(V_1V_2V_3) \rightarrow \begin{cases} \mathcal{M}_0(V_1^T V_2^T V_3^T) \\ \mathcal{M}_0(V_1^0 V_2^T V_3^T) + \text{cyclic} \\ \mathcal{M}_0(V_1^0 V_2^0 V_3^T) + \text{cyclic} \\ \mathcal{M}_0(V_1^0 V_2^0 V_3^0) \end{cases}. \quad (41)$$

As in the case of SSV and SVV , we can obtain the general form of massive amplitude $\mathcal{M}(V_1V_2V_3)$ by making

continuation from the massless limit,

$$\begin{aligned} \mathcal{M}(V_1V_2V_3) &= [(\epsilon_1 \cdot \epsilon_2 + a_{12}\epsilon_1^4\epsilon_2^4)(k_1 - k_2) \cdot \epsilon_3 + \text{cyclic}] \\ &\quad + [b_{12}m_3\epsilon_1 \cdot \epsilon_2\epsilon_3^4 + \text{cyclic}] + c_{123}\epsilon_1^4\epsilon_2^4\epsilon_3^4. \end{aligned} \quad (42)$$

c_{123} has dimension 1, a_{12} , a_{23} , a_{31} and b_{12} , b_{23} , b_{31} are dimensionless coefficients. m_1 , m_2 , m_3 are taken to be physical masses of the corresponding particles.

Again we will apply on-shell gauge symmetry to fix the parameters. We apply it on k_3 by the replacement: $\epsilon_3^M \rightarrow k_3^M$ and evaluate the amplitude for different helicity combinations of k_1 and k_2 .

First, choosing $s_1 = T$ and $s_2 = T$ gives

$$\begin{aligned} b_{12} &= -i\frac{m_1^2 - m_2^2}{m_3^2} \quad \text{if } m_3 \neq 0 \\ b_{12} &= 0 \quad \text{and } m_1 = m_2 \quad \text{if } m_3 = 0. \end{aligned} \quad (43)$$

Next, choosing $s_1 = T$, $s_2 = L$ gives

$$\begin{aligned} a_{23} &= -\frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3} \quad \text{if } m_2 \neq 0 \quad \text{and } m_3 \neq 0, \\ a_{23} &= 0 \quad \text{if } m_3 = 0. \end{aligned} \quad (44)$$

Finally, choosing $s_1 = L$, $s_2 = L$ fixes c_{123} to be

$$c_{123} = 0. \quad (45)$$

We have fixed b_{12} , a_{23} , c_{124} . Because of cyclic invariance, we can simply rotate the indices on mass parameters and fix all the remaining parameters, which are summarized as follows:

If $m_1 \neq 0$, $m_2 \neq 0$, $m_3 \neq 0$:

$$\begin{aligned} b_{12} &= -i\frac{m_1^2 - m_2^2}{m_3^2}, & b_{23} &= -i\frac{m_2^2 - m_3^2}{m_1^2}, & b_{31} &= -i\frac{m_3^2 - m_1^2}{m_2^2}, \\ a_{12} &= -\frac{m_1^2 + m_2^2 - m_3^2}{2m_1m_2}, & a_{23} &= -\frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3}, & a_{31} &= -\frac{m_3^2 + m_1^2 - m_2^2}{2m_3m_1}, \\ c_{123} &= 0. \end{aligned} \quad (46)$$

If $m_1 = 0$, $m_2 \neq 0$, $m_3 \neq 0$:

$$\begin{aligned} b_{12} &= i, & b_{23} &= 0, & b_{31} &= -i, \\ a_{12} &= 0, & a_{23} &= -1, & a_{31} &= 0, \\ c_{123} &= 0, \\ m_2 &= m_3. \end{aligned} \quad (47)$$

Notice the additional condition of $m_2 = m_3$ has not been derived in the literature before. It seems to imply a remnant symmetry between particle 2 and 3 in the case of particle 1 being massless.

If $m_1 = 0$, $m_2 = 0$, $m_3 \neq 0$, then there is no nontrivial solution. This is commonly known as Yang's theorem. Now we have derived the couplings for the VVV amplitude with three massive particles [Eq. (46)], two massive

particles [Eq. (47)], and one massive particle (no solution), respectively.

We can further obtain relations between the mass parameters by applying the strong massive-massless continuation on the couplings. This method is only useful in the case of all three being massive, in which the couplings depend on masses. Taking $m_1/m_2/m_3$ to 0 and requiring the couplings in Eq. (46) to be finite, we deduce that $m_1 \propto m_2 \propto m_3$ at the leading order. We can define them as functions of a common variable v and have

$$\begin{aligned} m_1 &= c_1 v + \mathcal{O}(v^2) & m_2 &= c_2 v + \mathcal{O}(v^2) \\ m_3 &= c_3 v + \mathcal{O}(v^3). \end{aligned} \quad (48)$$

Again, this is a signal that all particle masses have the same origin.

Before ending the VVV amplitude, we discuss two special cases of masses being equal.

First, let us take $m_1 = m_2 = m_3$, we then have $b_{12} = b_{23} = b_{31} = 0$ and $a_{12} = a_{23} = a_{31} = -\frac{1}{2}$. In this case the amplitude has $SO(3)$ symmetry for the three particles.

Second, we take two of the particles to have equal masses. Let us take $m_2 = m_3 \neq m_1$, we have

$$\begin{aligned} b_{12} &= i \left(1 - \frac{m_1^2}{m_3^2} \right), & b_{23} &= 0, & b_{31} &= -b_{12}, \\ a_{12} &= -\frac{m_1}{2m_3}, & a_{23} &= -\left(1 - \frac{m_1^2}{m_3^2} \right), & a_{31} &= -a_{12}. \end{aligned} \quad (49)$$

In addition we have $m_2 = m_3$ automatically for $m_1 = 0$ in Eq. (47).

For $m_1 > m_3$, the above two solutions correspond to WWZ and WWA vertices in the standard model (SM), respectively. This can be shown by defining the weak angle as $\cos \theta_W \equiv \frac{m_3}{m_1}$ and Eq. (49) as

$$\begin{aligned} b_{12} &= i \frac{\sin^2 \theta_W}{\cos^2 \theta_W}, & b_{23} &= 0, & b_{31} &= -i \frac{\sin^2 \theta_W}{\cos^2 \theta_W}, \\ a_{12} &= -\frac{1}{2 \cos \theta_W}, & a_{23} &= -\frac{\cos 2\theta_W}{2 \cos \theta_W}, & a_{31} &= -\frac{1}{2 \cos \theta_W}. \end{aligned} \quad (50)$$

This is indeed the couplings of WWZ , including $\varphi\varphi V$ and φVV vertices.

V. MASSIVE 4-POINT AMPLITUDES

As discussed before, three-point amplitudes are not generally viable by kinematics. The minimal number for n -point amplitudes to be kinematically allowed generically is $n = 4$. Thus in order to have consistent full theories, it is essential to construct up to four-point amplitudes at least, which is what we are doing in this section. The approach of constructing four-point amplitudes is explained in Sec. III, we mainly make use of consistent factorization in addition to the two conditions of on-shell gauge symmetry and massive-massless continuation.

According to the principle of quantum mechanics, to construct a four-point amplitude, we need to sum over all possible three-point amplitudes with different intermediate states and four-point contact terms. Moreover, bosonic symmetry (for the massless limit) constrains the number of types of particles (with different quantum numbers) for three-point amplitudes. For SSV , the two scalars are required to be of different types; for VVV , the three vector bosons also must be of different types. As a result, we can then classify different theories according to the number of vector bosons n_V and the number of scalars n_S , which fix the possible three-point amplitudes contributing to the final four-point amplitude:

- (1) $n_V < 3$
 - (a) $n_V = 1, n_S \geq 2$: SSV .
 - (b) $n_V \leq 2, n_S = 1$: VVS .
 - (c) $n_V \leq 2, n_S \geq 2$: VVS, SSV .
- (2) $n_V \geq 3$
 - (a) $n_V \geq 3, n_S = 0$: VVV .
 - (b) $n_V \geq 3, n_S = 1$: VVV, VVS .
 - (c) $n_V \geq 3, n_S \geq 2$: VVV, VVS , and SSV .

For every case with at least one scalar, we can also add SSS .

For $n_V < 2$, our strategy is to first construct the amplitudes from VVS , then add SSV . In this way we will be able to cover all three cases. For $n_V \geq 3$, we use a different strategy. We first try to construct four-point amplitudes from VVV alone, then expand to incorporate VVS . We will also neglect SSV and leave it to future work.

We will go into the massive case directly. Massless four-point amplitudes can be automatically included in the massive case, by taking masses to 0 and keeping the relevant degrees of freedom only. So they will not be discussed separately.

A. $n_V \leq 2$: Four-point amplitudes from VVS and SSV

Here we construct four-point amplitudes from VVS , SSV , and SSS . Except for the trivial $SSSS$, there are two possible four-point amplitudes $VVVV$ and $SSVV$, which we will construct one by one. After that, we will classify and discuss the underlying physics of the solutions.

I. VVVV from VVS

To construct the amplitude of VVVV we need only VVS. To simplify the analysis without losing generality, we focus on studying the case of $n_V = 1$, $n_S = 1$, so that there is only one independent VVS. Then we will generalize to two scalars in the end.

The total amplitude equals the sum of s , t , u channels and the contact term,

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c. \quad (51)$$

$\mathcal{M}_s$, $\mathcal{M}_t$, and $\mathcal{M}_u$ are s , t , u channels from three-point vertices. By consistent factorization we can obtain their forms when the intermediate states approach on shell. Plugging in the three-point amplitudes for VVS [Eqs. (38) and (39)] and SSV [Eqs. (30) and (32)], we get

$$\begin{aligned} \mathcal{M}_s(V_1 V_2 V_3 V_4) &\xrightarrow{p_{12}^2 \rightarrow m_S^2} \mathcal{M}(V_1 V_2 S_{12}) \frac{1}{p_{12}^2 - m_S^2} \mathcal{M}(-S_{12} V_3 V_4), \\ \mathcal{M}_t(V_1 V_2 V_3 V_4) &\xrightarrow{p_{13}^2 \rightarrow m_V^2} \sum_{s_{13}=\pm,0} \mathcal{M}(V_1 V_3 S_{13}) \frac{1}{p_{13}^2 - m_V^2} \mathcal{M}(-S_{13} V_2 V_4), \\ \mathcal{M}_u(V_1 V_2 V_3 V_4) &\xrightarrow{p_{14}^2 \rightarrow m_V^2} \sum_{s_{14}=\pm,0} \mathcal{M}(V_1 V_4 S_{14}) \frac{1}{p_{14}^2 - m_V^2} \mathcal{M}(-S_{14} V_2 V_3). \end{aligned} \quad (52)$$

Here $\mathcal{M}(VVS)$ should be understood to be multiplied by an overall coupling g_{VVS} . m_S and m_V are the masses of the scalar and the vector boson, respectively. The two vector bosons in VVS are identical, therefore have the same mass. The off-shell continuation for scalar is trivial, so amplitudes for s , t , u channels are simply

$$\begin{aligned} \mathcal{M}_s(V_1 V_2 V_3 V_4) &= \mathcal{M}(V_1 V_2 S_{12}) \frac{1}{p_{12}^2 - m_S^2} \mathcal{M}(-S_{12} V_3 V_4), \\ \mathcal{M}_t(V_1 V_2 V_3 V_4) &= \mathcal{M}(V_1 V_3 S_{13}) \frac{1}{p_{13}^2 - m_V^2} \mathcal{M}(-S_{13} V_2 V_4), \\ \mathcal{M}_u(V_1 V_2 V_3 V_4) &= \mathcal{M}(V_1 V_4 S_{14}) \frac{1}{p_{14}^2 - m_V^2} \mathcal{M}(-S_{14} V_2 V_3). \end{aligned} \quad (53)$$

To have the full amplitude we also need the contact term $\mathcal{M}_c$, which can be generally written as

$$\begin{aligned} \mathcal{M}_c &= a_1 \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_2 \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_3 \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 + b_{11} \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \cdot \epsilon_4^4 + b_{12} \epsilon_1^4 \cdot \epsilon_2^4 \epsilon_3 \cdot \epsilon_4 \\ &\quad + b_{21} \epsilon_1 \cdot \epsilon_3 \epsilon_2^4 \cdot \epsilon_4^4 + b_{22} \epsilon_1^4 \cdot \epsilon_3^4 \epsilon_2 \cdot \epsilon_4 + b_{31} \epsilon_1 \cdot \epsilon_4 \epsilon_2^4 \cdot \epsilon_3^4 + b_{32} \epsilon_1^4 \cdot \epsilon_4^4 \epsilon_2 \cdot \epsilon_3 + c \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4. \end{aligned} \quad (54)$$

$a_1, a_2, a_3, b_{11}, b_{12}, b_{21}, b_{22}, b_{31}, b_{32}, c$ are dimensionless coefficients that are to be fixed by on-shell gauge symmetry for the total amplitude. c is simply the self-coupling of Goldstone bosons, which we also label as $\lambda_{\varphi^4} \equiv c$. So up to an overall coupling g_{VVS}^2 , there are ten free parameters in total.

To fix the coefficients we apply on-shell gauge symmetry on ϵ_1^M . For s , t , u channels we get

$$\begin{aligned} \mathcal{M}_s^{\epsilon_1^M \rightarrow k_1^M} &= \frac{i}{2} g_{VVS}^2 \epsilon_2^4 \left(m_V \epsilon_3 \cdot \epsilon_4 + \frac{i}{2} (2p_3 + p_4) \cdot \epsilon_4 \epsilon_3^4 + \frac{i}{2} (2p_4 + p_2) \cdot \epsilon_3 \epsilon_4^4 + \frac{m_S^2}{2m_V} \epsilon_3^4 \epsilon_4^4 \right), \\ \mathcal{M}_t^{\epsilon_1^M \rightarrow k_1^M} &= \frac{i}{2} g_{VVS}^2 \epsilon_3^4 \left(m_V \epsilon_2 \cdot \epsilon_4 + \frac{i}{2} (2p_2 + p_4) \cdot \epsilon_4 \epsilon_2^4 + \frac{i}{2} (2p_4 + p_3) \cdot \epsilon_2 \epsilon_4^4 + \frac{m_S^2}{2m_V} \epsilon_2^4 \epsilon_4^4 \right), \\ \mathcal{M}_u^{\epsilon_1^M \rightarrow k_1^M} &= \frac{i}{2} g_{VVS}^2 \epsilon_4^4 \left(m_V \epsilon_2 \cdot \epsilon_3 + \frac{i}{2} (2p_2 + p_3) \cdot \epsilon_3 \epsilon_2^4 + \frac{i}{2} (2p_3 + p_4) \cdot \epsilon_2 \epsilon_3^4 + \frac{m_S^2}{2m_V} \epsilon_2^4 \epsilon_3^4 \right). \end{aligned} \quad (55)$$

For the contact channel we get

$$\begin{aligned} \mathcal{M}_c^{\epsilon_1^M \rightarrow k_1^M} &= a_1 k_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_2 \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_3 \epsilon_1 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_2 + b_{11} \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \cdot \epsilon_4^4 + b_{12} \epsilon_1^4 \cdot \epsilon_2^4 \epsilon_3 \cdot \epsilon_4 \\ &\quad + b_{21} \epsilon_1 \cdot \epsilon_3 \epsilon_2^4 \cdot \epsilon_4^4 + b_{22} \epsilon_1^4 \cdot \epsilon_3^4 \epsilon_2 \cdot \epsilon_4 + b_{31} \epsilon_1 \cdot \epsilon_4 \epsilon_2^4 \cdot \epsilon_3^4 + b_{32} \epsilon_1^4 \cdot \epsilon_4^4 \epsilon_2 \cdot \epsilon_3 + c \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4. \end{aligned} \quad (56)$$

Adding them up we obtain the following results:

$$a_1 = a_2 = a_3 = 0,$$

$$g_{\phi\phi VV} = b_{11} = b_{12} = b_{21} = b_{22} = b_{31} = b_{32} = \frac{1}{2}g_{VVS}^2,$$

$$c \equiv \lambda_{\phi^4} = \frac{3}{4}g_{VVS}^2 \frac{m_S^2}{m_V^2}. \quad (57)$$

From Eq. (57), $g_{\phi\phi VV}$ (i.e., b_{ij}) and λ_{ϕ^4} are related by $\lambda_{\phi^4} = \frac{3}{2} \frac{m_S^2}{m_V^2} g_{\phi\phi VV}$. Thus all four-point couplings are fixed by g_{VVS} , an overall coupling.

If we have more than one scalar, the results can be generalized naturally. For example, let us assume we have two scalars S^a and S^b that couple with V , the solution of on-shell gauge symmetry becomes

$$a_1 = a_2 = a_3 = 0,$$

$$b_{11} = b_{12} = b_{21} = b_{22} = b_{31} = b_{32} = \frac{1}{2}(g_{VVS^a}^2 + g_{VVS^b}^2),$$

$$c \equiv \lambda_{\phi^4} = \frac{3}{4m_V^2}(g_{VVS^a}^2 m_a^2 + g_{VVS^b}^2 m_b^2). \quad (58)$$

In order to completely fix the couplings, we need two parameters as input, which we can choose as g_{VVS^a} and the ratio of $r_1 \equiv \frac{g_{VVS^a}}{g_{VVS^b}}$.

2. SSVV from SSV and VVS

Here we construct SSVV, which is from either VVS or SSV. SSV also already implies the existence of VVS and SSS. Our strategy is to first construct SSVV with SSV, VVS, and SSS, then take $g_{VVS} = 0$ and $g_{SSV} = 0$, respectively, to see the minimal structure for the amplitude.

Again, to avoid unnecessary complication, we focus on the case with minimal number of particles, i.e., with only one type of vector boson and two types of scalars. This implies one SSV amplitude $\mathcal{M}(S^a S^b V)$; as well as two VVS amplitudes of $\mathcal{M}(VVS^a)$ and $\mathcal{M}(VVS^b)$. We focus on analyzing the example of $\mathcal{M}(S_a^a S_2^a V_3 V_4)$, as there is no essential difference in other four-point amplitudes.

The total amplitude of $\mathcal{M}(S_1^a S_2^a V_3 V_4)$ can be constructed from s, t, u channels and the contact term,

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c \quad (59)$$

with

$$\begin{aligned} \mathcal{M}_s(S_1^a S_2^a V_3 V_4) &= \sum_{i=a,b} \mathcal{M}(S_1^a S_2^a S_{12}^i) \frac{1}{p_{12}^2 - m_i^2} \mathcal{M}(S_{12}^i V_3 V_4), \\ \mathcal{M}_t(S_1^a S_2^a V_3 V_4) &= \mathcal{M}(S_1^a V_3 S_{13}^b) \frac{1}{p_{13}^2 - m_b^2} \mathcal{M}(S_{13}^b S_2^a V_4) + \mathcal{M}(S_1^a V_3 V_{13})^{M_{13}} \frac{-g_{M_{13} N_{13}}}{p_{13}^2 - m_V^2} \mathcal{M}^{N_{13}}(V_{13} S_2^a V_4^{s_4}), \\ \mathcal{M}_u(S_1^a S_2^a V_3 V_4) &= \mathcal{M}(S_1^a V_4 S_{14}^b) \frac{1}{p_{14}^2 - m_i^2} \mathcal{M}(S_{14}^b S_2^a V_3) + \mathcal{M}(S_1^a V_4 V_{14})^{M_{14}} \frac{-g_{M_{14} N_{14}}}{p_{14}^2 - m_V^2} \mathcal{M}^{N_{14}}(V_{14} S_2^a V_3^{s_3}). \end{aligned} \quad (60)$$

Notice we have to sum over all possible particles in the intermediate particles for every channel.

In addition, the contact term gives,

$$\mathcal{M}_c(S_1^a S_2^a V_3 V_4) = g_{S^a S^a VV} \epsilon_3 \cdot \epsilon_4 + \lambda_{SS\phi^a \phi^a} \epsilon_3^4 \epsilon_4^4. \quad (61)$$

Thus the full amplitude of $S_1^a S_2^a V_3 V_4$ has seven free parameters: $\lambda_{S^a S^a S^a}, \lambda_{S^a S^a S^b}, g_{VVS^a}, g_{VVS^b}, g_{S^a S^b V}, g_{S^a S^a VV}$,

and $\lambda_{SS\phi^a \phi^a}$. Those parameters are subject to additional constraints from on-shell gauge symmetry. Replacing ϵ_3^M with p_3^M , we have,

$$\mathcal{M}_{\text{tot}}^{\epsilon_3^M \rightarrow p_3^M} = 0, \quad (62)$$

which gives the following solution:

$$\begin{aligned} g_{S^a S^a VV} &= -2 \left(g_{S^a S^b V}^2 - \frac{g_{VVS^a}^2}{4} \right), \\ \frac{1}{2} (\lambda_{S^a S^a S^a} g_{VVS^a} + \lambda_{S^a S^a S^b} g_{VVS^b}) - \lambda_{S^a S^a \phi\phi} m_V &= g_{S^a S^b V}^2 \frac{m_a^2 - m_b^2}{m_V} - \frac{m_a^2}{2m_V} g_{VVS^a}^2. \end{aligned} \quad (63)$$

The first line comes from taking $s_4 = T$, the second line comes from taking $s_4 = L$. Exchanging a and b , we further get

$$g_{S^b S^b VV} = -2 \left(g_{S^a S^b V}^2 - \frac{g_{VVS^b}^2}{4} \right)$$

$$\frac{1}{2} (\lambda_{S^a S^a S^a} g_{VVS^a} + \lambda_{S^a S^a S^b} g_{VVS^b}) - \lambda_{S^a S^a \varphi\varphi} m_V = g_{S^a S^b V}^2 \frac{m_a^2 - m_b^2}{m_V} - \frac{m_a^2}{2m_V} g_{VVS^a}^2. \quad (64)$$

Now let us analyze the solution in Eq. (63) by taking various limits.

First, we reduce the number of scalars to one, which means $g_{S^a S^b V} = 0$ and $\lambda_{S^a S^a S^b} = 0$. Setting $S^a = S$, Eq. (63) becomes

$$g_{SSVV} = \frac{1}{2} g_{VVS}^2,$$

$$\lambda_{\varphi\varphi SS} = g_{VVS} \frac{\lambda_{SSS}}{m_V} + \frac{m_S^2}{2m_V^2} g_{VVS}^2. \quad (65)$$

In this case, the three-point amplitudes that contribute to the full amplitude are VVS and SSS . There is only one free parameter λ_{SSS} after setting the overall coupling $g_{VVS} = 1$. Considering that λ_{SSSS} also has no constraints on it, all couplings can be fixed by particle masses except for the scalar self-couplings (λ_{SSS} and λ_{SSSS}). However, strong massive-massless continuation on Eq. (65) (second line) constrains the behavior of λ_{SSS} at the limit of $m_V \rightarrow 0$,

$$\lim_{m_V \rightarrow 0} \lambda_{SSS} = 0. \quad (66)$$

So λ_{SSS} does not exist in the massless limit. Also it indicates that λ_{SSS} has the same physical origin as the particle masses.

Next, we take $g_{VVS^a} = g_{VVS^b} = 0$ and $m_a = m_b$. Equation (63) then reduces to

$$g_{S^a S^a VV} = -2g_{S^a S^b V}^2 \lambda_{S^a S^a \varphi\varphi} = 0. \quad (67)$$

In this case, φ does not contribute to the amplitude. Physically it means that we should treat it as an auxiliary mode, instead of a physical, dynamic degree of freedom. Moreover, scalar couplings $\lambda_{S^a S^a S^a/S^b}$ decouple from the amplitude.

Finally, we analyze the general case in Eq. (63). The three conditions [Eq. (63) and the freedom of overall scaling] reduce the number of free parameters from 7 to 4, only enough to eliminate the two four-point couplings and one three-point coupling. In addition we treat $\lambda_{S^a S^a S^a}$ and $\lambda_{S^a S^a S^b}$ as input. We then have five relevant parameters constrained by three conditions. In summary, we still need $5 - 3 = 2$ free parameters. To fix the remaining two parameters, we can set the ratios,

$$r_1 \equiv \frac{g_{VVS^a}}{g_{VVS^b}}, \quad r_2 \equiv \frac{g_{VVS^a}}{g_{S^a S^b V}} \quad (68)$$

as input, in addition to $\lambda_{S^a S^a S^a}$ and $\lambda_{S^a S^a S^b}$.

3. Underlying theories and models

Now having finished constructing four-point amplitudes for $n_V < 3$, we then set out to discuss the underlying physical theories and models for those constructions.

From one vector boson and one scalar, we can construct only one three-point amplitude of VVS and three four-point amplitudes of $SSSS$, $SSVV$, and $VVVV$. In constructing those amplitudes, we found all couplings are fixed by the masses, except for the scalar couplings of SSS and $SSSS$. Moreover, by applying strong massive-massless continuation on both three-point and four-point amplitudes, we found that when one of the masses goes to 0, other particles' masses and g_{VVS} also go to 0. This implies a common origin for the all particle masses and couplings of mass dimension, a mark of SSB. However, the relations between the masses are generally nonlinear. Physically, the nonlinearity between masses and scalar couplings being free parameters indicate the Higgs potential is not limited to the terms of $\dim \leq 4$. In order to fix $\dim \geq 4$ terms for Higgs potential, we need to go beyond four point in constructing amplitudes, this is not within the scope of this paper. For example, we can add dim-6 SMEFT operators, or consider Coleman-Weinberg potential. In conclusion, the underlying constructible theory with one massive vector boson and one scalar, is massive QED with SSB, with the corresponding Lagrangian being composed of a vector field and a complex scalar field. Moreover, the specific shape of the Higgs potential cannot be determined.

From one vector boson and two scalars, we can construct three-point amplitudes of SVV and SSV , as well as four-point amplitudes of $SSSS$, $SSVV$, and $VVVV$. In constructing those amplitudes, we need two more parameters in Eq. (68) as input, apart from masses, to completely fix the couplings except for scalar self-couplings. The other aspects are the same as the case of one vector boson and one scalar. Thus the underlying theory is still massive gauge theory with SSB, but with more than one Higgs field. For a specific model, we can take the $U(1)$ version of two-Higgs-doublet model as an example. The two parameters in Eq. (68) fix the angle between the two vacuum expectation values (β) and the mixing angle of the two neutral and CP -even scalars (α). This case can also be a basis for

analyzing more general cases, such as the non-Abelian cases with mixing between Higgs fields, etc.

Finally, there is a special limit for one vector boson and two scalars, in which any couplings involving the Goldstone φ are zero. In the cases we studied, there are two necessary conditions. The first is that the two scalars have equal masses. The second is all VVS couplings are 0. The underlying theory of this scenario is Stueckelberg theory, in which the vector boson has mass without SSB. φ is only a pure gauge that does not contribute to any amplitude.

B. $n_V \geq 3$: Four-point amplitudes from VVV, VVS, and SSV

For $n_V \geq 3$, we have possible three-point amplitudes of VVV, VVS, and SSV. The possible four-point amplitudes constructed are VVVV, VVSS, and SSSS. SSSS is not constrained by on-shell gauge symmetry, so we will skip it. Similarly, we also will not construct VVSS, which has been constructed in the case of $n_V = 1$. There are no new physical insights for $n_V \geq 3$. So in summary we will only construct VVVV with VVV and VVS. Our strategy is to

construct VVVV from the three-point amplitude VVV alone first (by assuming there is no scalar), to see if it is possible to be compatible with consistent conditions. If it does not work, we will then add VVS.

1. VVVV from VVV

Here we set to construct VVVV from VVV and VVS. We start with $n_S = 0$, meaning no VVS first. We also put the “color” explicitly for the amplitudes. For VVV we have $\mathcal{M}(V_1^a V_2^b V_3^c) = f^{abc} \mathcal{M}(V_1 V_2 V_3)$, with f^{abc} to be the antisymmetric tensor. For simplicity, we set all vector bosons to have the same mass.

The condition of consistent factorization, combined by a natural off-shell continuation, ensures the total four-point amplitude $\mathcal{M}(V^a V^b V^c V^d)$ equals the sum of s, t, u channels and the contact channel,

$$\mathcal{M}_{\text{tot}}(V_1^a V_2^b V_3^c V_4^d) = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c \quad (69)$$

with $\mathcal{M}_s, \mathcal{M}_t, \mathcal{M}_u$ being

$$\begin{aligned} \mathcal{M}_s(V_1^a V_2^b V_3^c V_4^d) &= \sum_e \mathcal{M}(V_1^a V_2^b V_{12}^e) M_{12} \frac{-g_{12} N_{12}}{p_{12}^2 - m_V^2} \mathcal{M}^{N_{12}}(V_{12}^e V_3^c V_4^d), \\ \mathcal{M}_t(V_1^a V_2^b V_3^c V_4^d) &= \sum_e \mathcal{M}(V_1^a V_3^c V_{13}^e) M_{13} \frac{-g_{13} N_{13}}{p_{13}^2 - m_V^2} \mathcal{M}^{N_{13}}(V_{13}^e V_2^b V_4^d), \end{aligned} \quad (70)$$

$$\mathcal{M}_u(V_1^a V_2^b V_3^c V_4^d) = \sum_e \mathcal{M}(V_1^a V_4^d V_{14}^e) M_{14} \frac{-g_{14} N_{14}}{p_{14}^2 - m_V^2} \mathcal{M}^{N_{14}}(V_{14}^e V_2^b V_3^c). \quad (71)$$

The contact term $\mathcal{M}_c$ has no pole, therefore it should be the linear combination of all possible terms from Lorentz contraction between $\epsilon_i^\mu, \epsilon_i^\mu$ with $i = 1, 2, 3, 4$.

$$\begin{aligned} \mathcal{M}_c &= a_{12,34} \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_{13,24} \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_{14,23} \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 + b_{12,34} \epsilon_1 \cdot \epsilon_2 \epsilon_3^4 \epsilon_4^4 + b_{13,24} \epsilon_1 \cdot \epsilon_3 \epsilon_2^4 \epsilon_4^4 \\ &+ b_{14,23} \epsilon_1 \cdot \epsilon_4 \epsilon_2^4 \epsilon_3^4 + b_{34,12} \epsilon_3 \cdot \epsilon_4 \epsilon_1^4 \epsilon_2^4 + b_{24,13} \epsilon_2 \cdot \epsilon_4 \epsilon_1^4 \epsilon_3^4 + b_{23,14} \epsilon_2 \cdot \epsilon_3 \epsilon_1^4 \epsilon_4^4 + \lambda_{\varphi^4} \epsilon_1^4 \epsilon_2^4 \epsilon_3^4 \epsilon_4^4. \end{aligned} \quad (72)$$

The coefficients are to be determined by the additional constraint of on-shell gauge symmetry.

We first apply on-shell gauge symmetry on $\mathcal{M}(V_1 V_2 V_{12})$ with one of the other legs, say $p_{12} = p_1 + p_2$, being off shell. Choosing the particle for replacing ϵ^M with p^M to be p_1 , we then have,

$$\mathcal{M}_{\epsilon_1^M \rightarrow p_1^M}^{p_{12}^2 \neq m_V^2}(V_1 V_2 V_{12}) = \left(\epsilon_2 \cdot \epsilon_3 - \frac{1}{2} \epsilon_2^4 \epsilon_3^4 \right) (p_{12}^2 - m_V^2) + p_{12}^M \cdot \epsilon_M^{12*} \left(-\frac{i}{2} m_V \epsilon_2^4 - p_3 \cdot \epsilon_2 \right). \quad (73)$$

Making use of it, we can obtain the solution for $\mathcal{M}_{\text{tot}}(V_1^a V_2^b V_3^c V_4^d)_{\epsilon_1^M \rightarrow p_1^M} = 0$. However, because the solution is a little complicated, we simplify it by solving on-shell gauge symmetry with s_2, s_3, s_4 being transverse (T) and longitudinal (L), respectively.

First, taking $s_2 = T, s_3 = T, s_4 = T$, in this case only terms such as $\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4$ contribute to $\mathcal{M}_c$, which can be written as

$$\mathcal{M}_c^{\epsilon_1 \rightarrow p_1}(s_2 = T, s_3 = T, s_4 = T) = a_{12,34} p_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 + a_{13,24} p_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 + a_{14,23} p_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3. \quad (74)$$

We evaluate $(\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u)_{\epsilon_1^M \rightarrow p_1^M}$ first, which gives

$$\begin{aligned} & (\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u)_{\epsilon_1^M \rightarrow p_1^M} \\ &= -(f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce})[\epsilon_2 \cdot \epsilon_3 (p_1 + 2p_2) \cdot \epsilon_4 - 2\epsilon_2 \cdot \epsilon_4 p_2 \cdot p_3 + 2\epsilon_3 \cdot \epsilon_4 (-p_1 - p_4) \cdot \epsilon_2] \\ &+ f^{abe} f^{cde}[p_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - p_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3] + f^{ace} f^{dbe}[p_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 - p_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4] \\ &+ f^{ade} f^{bce}[p_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - p_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4]. \end{aligned} \quad (75)$$

Comparing with Eq. (74), we find that there is no solution for on-shell gauge symmetry, unless there is an additional condition,

$$f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce} = 0 \quad (76)$$

in which case we have the solution for coefficients,

$$a_{12,34} = f^{ace} f^{bde} - f^{ade} f^{bce}; \quad a_{13,24} = f^{ade} f^{bce} - f^{abe} f^{cde}; \quad a_{14,23} = f^{abe} f^{cde} - f^{ace} f^{dbe} \quad (77)$$

we then obtain $\mathcal{M}_c$ with all particles being transverse,

$$\begin{aligned} \mathcal{M}_c &= -f^{abe} f^{cde}[\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3] \\ &- f^{ace} f^{dbe}[\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 - \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4] \\ &- f^{ade} f^{bce}[\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4]. \end{aligned} \quad (78)$$

Equation (76) is simply Jacobi identity, indicating that the involved vector bosons with different quantum numbers belong to the same Lie group. Notice all transverse cases reduce to massless vector scattering directly when $m_V \rightarrow 0$, thus we arrive at the conclusion that the only consistent massless theory with vector boson scattering is Yang-Mills theory.

Then we continue to apply on-shell gauge symmetry with one or several particles being longitudinal. We found, however, there is no nontrivial solution for on-shell gauge symmetry. In particular, the solutions for $b_{ij,kl}$ by applying OGS on $s_2 = L, s_3 = T, s_4 = T$ and $s_2 = L, s_3 = L, s_4 = T$ contradict with each other.

Choosing $s_2 = L, s_3 = T$, and $s_4 = T$, on-shell gauge symmetry is satisfied if the coefficients in Eq. (72) satisfy the following conditions:

$$b_{34,12} = 0; b_{24,13} = 0; b_{23,14} = 0, \quad (79)$$

which fixes the $VV\phi\phi$ vertices. By permutation symmetry we also obtain all $b_{ij,kl}$ coefficients,

$$b_{12,34} = b_{34,12} = 0; \quad b_{13,24} = b_{24,13} = 0; \quad b_{14,23} = b_{23,14} = 0. \quad (80)$$

On the other hand, the solution of $b_{ij,kl}$ for OGS by choosing $s_2 = L, s_3 = L, s_4 = T$ is

$$b_{14,23} = -f^{abe} f^{cde} = f^{ace} f^{dbe} \quad b_{34,12} = 0 \quad b_{24,13} = 0. \quad (81)$$

It is impossible for Eqs. (80) and (81) to be correct at the same time, except for the trivial solution of $f^{abe} f^{cde} = f^{ace} f^{dbe} = 0$. Therefore, it is impossible to satisfy on-shell gauge symmetry for $\mathcal{M}(V^a V^b V^c V^d)$ with vector bosons only.

For the case of elementary particles, the only solution is to add additional degrees of freedom in the theory. For $\mathcal{M}(V^a V^b V^c V^d)$, the only possibility is to add additional scalars, and consequently the three-point amplitude $\mathcal{M}(VVS)$. For simplicity, we only add one scalar to the theory. This gives three additional s, t, u channels contributing to $\mathcal{M}(V^a V^b V^c V^d)$ with the scalar to be the

intermediate particle. The total amplitude now becomes

$$\begin{aligned} \mathcal{M}_{\text{tot}}(V_1^a V_2^b V_3^c V_4^d) &= \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_s^S + \mathcal{M}_t^S \\ &+ \mathcal{M}_u^S + \mathcal{M}_c. \end{aligned} \quad (82)$$

The scalar channels' contribution to on-shell gauge symmetry was obtained in Eq. (55).

Applying on-shell gauge symmetry again for $s_2 = L, s_3 = T, s_4 = T$, the related coefficients are modified to

$$b_{34,12} = \frac{1}{2} g_{V^a V^b S} g_{V^c V^d S}. \quad (83)$$

Using permutation, we can obtain all $b_{ij,kl}$ coefficients,

$$\begin{aligned} b_{12,34} &= b_{34,12} = \frac{1}{2} g_{V^a V^b S} g_{V^c V^d S}, \\ b_{13,24} &= b_{24,13} = \frac{1}{2} g_{V^a V^c S} g_{V^b V^d S}, \\ b_{14,23} &= b_{23,14} = \frac{1}{2} g_{V^a V^d S} g_{V^b V^c S}. \end{aligned} \quad (84)$$

We then apply on-shell gauge symmetry to $s_2 = L, s_3 = L, s_4 = T$, which gives the following constraint on VVS vertices:

$$\begin{aligned} 0 &= -\frac{1}{4} f^{abe} f^{cde} \epsilon_2^A \epsilon_3^A (2p_2 + p_4) \cdot \epsilon_4 + \frac{1}{4} f^{ace} f^{dbe} \epsilon_2^A \epsilon_3^A (2p_3 + p_4) \cdot \epsilon_4 + b_{34,12} \epsilon_3 \cdot \epsilon_4 (-im_V) \epsilon_2^4 \\ &\quad + b_{24,13} \epsilon_3 \cdot \epsilon_4 (-im_V) \epsilon_3^4 + b_{14,23} p_1 \cdot \epsilon_4 \epsilon_2^4 \epsilon_3^4 + g_{V^a V^b S} g_{V^c V^d S} \left[\frac{i}{2} m_V \epsilon_2^4 (\epsilon_3 \cdot \epsilon_4) + \frac{1}{4} (2p_3 + p_4) \cdot \epsilon_4 \epsilon_2^4 \epsilon_3^4 \right] \\ &\quad + g_{V^a V^c S} g_{V^d V^b S} \left[\frac{i}{2} m_V \epsilon_3^4 (\epsilon_2 \cdot \epsilon_4) + \frac{1}{4} (p_2 + p_4) \cdot \epsilon_4 \epsilon_2^4 \epsilon_3^4 \right]. \end{aligned} \quad (85)$$

Plugging in Eq. (84), we obtain the following relations between g_{VVS} and f^{abc} :

$$\begin{aligned} f^{abe} f^{cde} &= g_{V^a V^c S} g_{V^b V^d S} - g_{V^a V^d S} g_{V^b V^c S}, \\ f^{ace} f^{dbe} &= -g_{V^a V^b S} g_{V^c V^d S} + g_{V^a V^d S} g_{V^b V^c S}. \end{aligned} \quad (86)$$

$g_{V^i V^j S}$ is symmetric with $i \leftrightarrow j$ for equal vector boson masses, e.g., $g_{V^b V^a S} = g_{V^a V^b S}$. We can also apply OGS on $s_2 = L, s_3 = T, s_4 = L$ and $s_2 = T, s_3 = L, s_4 = L$, but there are no new independent results. Equation (86) gives us the only constraints between the group structure coefficients f^{abc} and the VVS couplings.

Finally, we choose $s_2 = L, s_3 = L, s_4 = L$ and apply on-shell gauge symmetry on p_1 , which fixes λ_{φ^4} to be

$$\lambda_{\varphi^4} = \frac{1}{4} \frac{m_S^2}{m_V^2} (g_{V^a V^b S} g_{V^c V^d S} + g_{V^a V^c S} g_{V^d V^b S} + g_{V^a V^d S} g_{V^b V^c S}). \quad (87)$$

So we have successfully constructed massive VVVV from VVV, with all vector boson masses being equal. We proved a scalar is needed for the construction to be consistent. We believe this conclusion holds for general cases of vector boson masses. All couplings are constrained by particle masses and the group structure constants f^{abc} in Eqs. (84), (86), and (87).

Those solutions can be further simplified if we introduce a mild and reasonable assumption. Because all vector bosons have the same mass and there is only one scalar, it is reasonable to assume all VVS couplings are equal, i.e., $g \equiv g_{V^a V^b S} = g_{V^c V^d S} = g_{V^a V^c S} = g_{V^b V^d S} = g_{V^a V^d S} = g_{V^b V^c S}$. However, in this case, Eq. (86) becomes $f^{abe} f^{cde} = f^{ace} f^{dbe} = f^{ade} f^{bce} = 0$, reducing the solution to the

Abelian case. Nevertheless, there is a way to save the situation by making one of the VVS couplings equal to 0. For example, we can take

$$\begin{aligned} g_{V^a V^b S} &= 0 \quad \text{or} \quad g_{V^c V^d S} = 0, \\ g_{VVS} &\equiv g_{V^a V^c S} = g_{V^b V^d S} = g_{V^a V^d S} = g_{V^b V^c S}. \end{aligned} \quad (88)$$

The condition eliminates one scalar mediated channel. Equation (86) then reduces to

$$f^{abe} f^{cde} = 0 \quad g_{VVS}^2 = f^{ace} f^{dbe} = f^{ade} f^{bce}. \quad (89)$$

We see that one of the vector boson mediated channels is also eliminated. As a result, the VVVV amplitude has only s and t channels that can be organized as planar diagrams. This can be checked in the simplest case of the $SU(2)$ group. In the $W^\pm$ basis, the diagrams correspond to exactly Eq. (89) with $f^{abc} = \epsilon^{abc}$ and $a, b, c = \pm, 3$. It is also interesting to point out that Eq. (89) is also the solution for

$$\begin{aligned} g_{V^a V^b S} g_{V^c V^d S} &= g_{V^a V^d S} g_{V^b V^c S} = 0, \\ g_{VVS} &\equiv g_{V^a V^c S} = g_{V^b V^d S}. \end{aligned} \quad (90)$$

In other words, there is only one scalar mediated channel while all relevant VVS couplings are equal.

2. The underlying theory with amplitudes from VVV

For the number of vector bosons equal to or larger than 3, we have the three-point VVV, from which we try to construct VVVV. For all particles being transverse, it leads to Jacobi identity for the couplings of VVV, thus establishing the classic result of massless Yang-Mills theory. For other polarizations, we found that an additional scalar is

needed for self-consistency of the amplitude as required for on-shell gauge symmetry. After adding the scalar and the subsequent three-point amplitude VVS , we are able to construct $VVVV$ with all four-point couplings and VVS couplings fixed by masses and group structure constants [Eqs. (84), (86), and (87)]. Those results, combined with the results of VVV in Sec. IV B, are enough for us to conclude that, for $VVVV$ with the number of vector bosons larger than or equal to three and all particles elementary, the only possible underlying theory is Yang-Mills theory with SSB. In particular, we can conclude that Stueckelberg theory cannot be applied on theory with the vertex of VVV , i.e., non-Abelian gauge theory. The reason is that the Goldstone mode ϕ always contributes nontrivially to amplitudes in this case. Therefore, it is impossible to have Stueckelberg theory, in which ϕ is simply pure gauge.

Furthermore, we also discuss the solutions under the reasonable assumption of all VVS couplings being equal. We found that, in order to obtain nontrivial solutions, all amplitudes in one of the s, t, u channels must vanish, whether they are mediated by scalars or vector bosons. The couplings are fixed as in Eqs. (88) and (89). The corresponding amplitudes then have only planar diagrams. A specific example is the $SU(2)$ group in the basis of $W^\pm$.

VI. CONCLUSION

In this paper we set out to construct massive amplitudes with elementary particles. We focus on amplitudes with vector bosons and scalars up to four point. For this purpose, we propose two consistent conditions for the amplitudes: on-shell gauge symmetry and strong massive-massless continuation [Item (I)]. The latter condition is to ensure the particles are elementary by imposing on the analytic property of amplitudes as functions of masses. The former condition is simply a massive Ward identity that becomes the Goldstone equivalence theorem in high energy limit. However, we argued in the paper that it can be seen as coming from Lorentz symmetry and unitarity, combined with the principle of manifest massive-massless continuation. By mixing the vector boson with an auxiliary scalar, constrained by the mixing condition in Eq. (1), we eliminated the state with negative norm. Both the mixing condition and the corresponding polarization vectors have smooth massless limits. Most importantly, we naturally obtain on-shell gauge symmetry.

After proposing the two consistent conditions, we continue to construct three-point and four-point amplitudes. To construct three point, we apply the principle of massive-massless continuation on amplitudes, concluding that massive amplitudes can be written as linear combinations of the massless counterparts [Eq. (15)]. So the basic strategy is to construct the massless amplitudes first, then apply the two consistent conditions to construct massive amplitudes, which fix the coefficients and their relations. With this strategy, we successfully construct all possible

three-point massive amplitudes: SSV , SVV , and VVV . All couplings, up to an overall scaling, are fixed in terms of masses. For VVV we also discuss the solution with different mass combinations and reconstruct the VVV amplitudes in the SM. Finally, when one of the masses goes to 0, other masses and dimensional couplings also must go to 0, indicating common physical origin.

To construct four-point amplitudes, we make use of consistent factorization that the four-point amplitude factorizes into products of three-point amplitudes when either of s, t, u channels goes on shell for one of the particles. Starting from four-point amplitudes when the intermediate states are on shell, we then extend the propagator off shell continuously to obtain the amplitudes at general kinematics [Eq. (19)]. We classify our construction according to the number of vector bosons n_V : for $n_V < 3$, we construct $VVSS$ and $VVVV$. With only one scalar we found all couplings are fixed in terms of masses, except for scalar self-couplings. The underlying theory is massive scalar QED with SSB. With two scalars, we also need the ratio of $\frac{g_{VVSS^a}}{g_{VVS^b}}$ and $\frac{g_{VVSS^a}}{g_{SS^a S^b V}}$. Physically, the underlying theory for one vector boson and two scalars is the $U(1)$ version of two-Higgs-Doublet-Model (2HDM). The two ratios are the mixing angles between two VEVs and between two neutral, CP -even scalars. There is also a special case when the scalars have equal masses. In this case the ‘‘Goldstone’’ mode ϕ does not contribute to the amplitude. This corresponds to Stueckelberg theory, in which there is no need for symmetry breaking for a vector boson to have mass.

For $n_V \geq 3$, we construct $VVVV$ from VVV . We found that in order to satisfy on-shell gauge symmetry, additional scalars must be added. After adding an additional scalar and vertex VVS , the amplitude of $VVVV$ is successfully constructed. All four-point couplings except scalar self-couplings are then fixed. VVS couplings are further constrained by the group structure constants in Eq. (86). We therefore conclude the only possible underlying theory for $VVVV$ from VVS is Yang-Mills theory with SSB, when all particles are elementary. We also discuss the solutions in some special conditions such as the case of all VVS couplings being equal, the $SU(2)$ group, etc.

In comparison with other similar works in the literature, our approach of constructing amplitudes has a few advantages. First of all, our method has manifest gauge symmetry at the level of amplitudes by making Goldstone bosons part of the physical spectrum. Second, we essentially derive on-shell gauge symmetry from Lorentz symmetry and some reasonable assumptions of mixing between vector boson and scalar, thus putting our approach on a firm theoretical foundation. Third, the condition of strong massive-massless continuation allows us to distinguish theories with elementary particles from composite particles through analytic properties of amplitudes. Finally, our results are also more complete compared with other approaches, as demonstrated by the conclusions of Stueckelberg theory,

particles masses have the same physical origin, Higgs self-couplings are modified by $\dim > 4$ terms of the Higgs potential, etc.

Our results open the door to many new directions, while also leaving a few questions to be answered. To name a few topics for future research: four-point amplitudes with fermions remain to be constructed; for $VVVV$ from VVV , we have only studied the case of equal masses. The amplitude with general masses is still yet to be fully constructed. Furthermore, it is also interesting to go beyond four-point amplitudes to see what can be learned.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: DERIVATION OF VECTOR-SCALAR KINEMATIC LAGRANGIAN

We will derive the kinematic Lagrangian terms for $V - \varphi$ mixing, from the assumption that both V and φ have mass m_V , and physical condition $\partial^\mu V_\mu = m_V \varphi$.

The kinematic Lagrangian related to the vector boson is

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2}m_V^2 V_\mu V^\mu + \mathcal{L}_{V-\varphi}.$$

$\mathcal{L}_{V-\varphi}$ is unknown. The Euler-Lagrangian equation gives

$$-(\partial^2 + m_V^2)V_\mu + \partial_\mu(\partial \cdot V) + (V - \varphi \text{ terms}) = 0.$$

Applying the physical condition $\partial^\mu V_\mu = m_V \varphi$ fixes the $V - \varphi$ term above to be $-m_V \partial_\mu \varphi$, meaning $\mathcal{L}_{V-\varphi}$ is

$$\mathcal{L}_{V-\varphi} = m_V V_\mu \partial^\mu \varphi.$$

In similar way we can derive the Lagrangian for φ , of which the general form is

$$\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m_\varphi^2 \varphi^2.$$

The equation of motion for φ is then

$$(\partial^2 + m_\varphi^2)\varphi + m_V \partial^\mu V_\mu = 0.$$

Again plugging in the physical condition eliminates V_μ and gives

$$(\partial^2 + m_\varphi^2 - m_V^2)\varphi = 0.$$

This fixes $m_\varphi = 0$.

To sum up, the kinematic Lagrangian for V and φ with mixing is

$$\begin{aligned} \mathcal{L}_{V^2} &= -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{1}{2}m_V^2 V_\mu^2, \\ \mathcal{L}_{V-\varphi} &= m_V V_\mu \partial^\mu \varphi, \\ \mathcal{L}_{\varphi^2} &= \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi. \end{aligned} \tag{A1}$$

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