

Critical non-Abelian vortex string and 2D $\mathcal{N} = 2$ black hole

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It has been shown that the non-Abelian vortex string in 4D $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the $U(2)$ gauge group and $N_f = 4$ flavors becomes a critical superstring. Its 10D target space is a product of the flat 4D space and an internal noncompact Calabi-Yau threefold, namely, the conifold. It was also shown that the Coulomb branch of the associated string sigma model, which opens up at strong coupling, can be described by $\mathcal{N} = 2$ Liouville theory. We continue here the study of the recently proposed mass deformation of the $U(2)$ theory with $N_f = 4$, interpolating to SQCD with the $U(4)$ gauge group and $N_f = 8$ quarks, by analyzing the mass-deformed $\mathcal{N} = 2$ Liouville theory on the string world sheet, and show that it is always described by the trumpet geometry of the target space, which is T -dual to the 2D $\mathcal{N} = 2$ supersymmetric black hole. We use this correspondence to find the low-lying hadron spectrum in the deformed SQCD, and explain the expected increase in the number of hadronic states in the theory with more gauge fields and quarks by considering the near-Hagedorn behavior of the 2D black hole.

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I. INTRODUCTION

Non-Abelian vortices were first found in four-dimensional (4D) $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the gauge group $U(N)$ and $N_f \geq N$ flavors of quarks [1–4]. The non-Abelian vortex string is $1/2$ BPS (Bogomol’nyi-Prasad-Sommerfield) saturated and therefore has $\mathcal{N} = (2, 2)$ supersymmetry on its world sheet. In addition to four translational moduli, the non-Abelian string carries orientational moduli, as well as the size moduli if $N_f > N$ [1–4] (see [5–8] for reviews). The dynamics of the internal orientational and size moduli of the non-Abelian string are described by the effective two-dimensional (2D) sigma model on the string world sheet, the so-called $\mathcal{N} = (2, 2)$ supersymmetric weighted $\mathbb{CP}$ -model $\mathbb{WCP}(N, N_f - N)$.

It was shown in [9] that the non-Abelian solitonic string in $\mathcal{N} = 2$ SQCD with the $U(2)$ gauge group and $N_f = 2N = 4$ flavors of quark hypermultiplets behaves as a

critical superstring.¹ For $N_f = 2N$ the world sheet sigma model $\mathbb{WCP}(N, N)$ becomes conformal, and additionally for $N = 2$ the six orientational/size moduli can be combined with four translational moduli to form a ten-dimensional target space for a critical superstring [9,10]. In this case the target space of the world sheet theory is $\mathbb{R}^4 \times Y_6$, where Y_6 is a noncompact six-dimensional Calabi-Yau (CY) manifold, the conifold [11], see [12] for a review. The theory of the critical vortex string was identified as the type IIA superstring theory [10], and the spectrum of low-lying closed string excitations was found in [10,13].

Most of the massless and massive string modes have non-normalizable wave functions over the conifold Y_6 , i.e., they are not localized in 4D and cannot be interpreted as dynamical states in 4D theory, in particular, there are no massless 4D gravitons in the physical spectrum [10]. However, an excitation associated with the deformation of the complex structure modulus b of Y_6 has a

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¹By critical superstring we mean canonical superstring with $\mathcal{N} = 1$ local supersymmetry in the world-sheet (super)gravity sector. The $\mathcal{N} = (2, 2)$ supersymmetric world-sheet models, including supersymmetric $\mathcal{N} = 2$ Liouville belong to the matter sector of string theory.

(logarithmically) normalizable wave function and was interpreted as a massless baryon in the spectrum of hadrons of 4D $\mathcal{N} = 2$ SQCD [10].

To analyze the massive states, a different approach was chosen, similar to that used for little string theories (see [14] for a review). It is based on the equivalence [15] between the critical string on the conifold and a noncritical $c = 1$ string containing the Liouville field and a compact scalar at the self-dual radius (united into a complex scalar of $\mathcal{N} = 2$ Liouville theory [16,17]).² Later a similar correspondence was proposed (and treated as a holographic AdS/CFT-type duality) for the critical string on certain noncompact CY spaces with an isolated singularity in the so-called double scaling limit, and a noncritical $c = 1$ string with an additional Landau-Ginzburg $\mathcal{N} = 2$ superconformal system [18,19]. In the conifold case, this extra Ginzburg-Landau conformal field theory (CFT) is absent. The above equivalence was used in [13,20] to find the low-lying spectrum of hadrons in 4D $\mathcal{N} = 2$ SQCD with gauge group $U(2)$ and $N_f = 4$ quark flavors.

Recently, this equivalence was demonstrated in a more direct way. Namely, it was shown in [21] that Coulomb branches of world sheet $WCP(N, N)$ models on noncompact toric CY manifolds, which open up at strong coupling, can be described by $\mathcal{N} = 2$ Liouville theory with N -dependent background charge. Using the description of the theory of the critical non-Abelian string in terms of $\mathcal{N} = 2$ Liouville theory, the program of interpolation between SQCDs with different gauge groups and numbers of quark flavors was initiated in [22]. The motivation is to broaden the class of 4D $\mathcal{N} = 2$ SQCDs where the hadron spectrum can be described using the string theory of the non-Abelian string.

The idea was that, to study this interpolation, one can introduce quark masses in $\mathcal{N} = 2$ SQCD and by changing mass parameters decouple certain quark flavors. In particular, one can interpolate between $\mathcal{N} = 2$ SQCD with gauge group $U(2)$ and $N_f = 4$ (which supports a critical non-Abelian string) and $\mathcal{N} = 2$ SQCD with gauge group $U(4)$ and $N_f = 8$ quark flavors “integrating four quark flavors in” by reducing their mass parameters [22].

Of course, the quark masses break the conformal invariance on the world sheet, and the mass-deformed theory cannot be used directly for the string quantization. Instead, to find a true string vacuum, the effective supergravity equations of motion can be solved with appropriate initial conditions associated with the mass deformation [22].

In this paper, we continue these studies and show that actually the mass-deformed background found in [22] is T -dual to the $\mathcal{N} = 2$ supersymmetric 2D black hole with cigar geometry [23], which is the $\mathcal{N} = 2$ $SL(2, \mathbb{R})/U(1)$

coset Wess-Zumino-Novikov-Witten (WZNW) model [15,18,24,25]. In fact, the metric found in [22] can be reduced to the metric of a trumpet, first discussed in [26] and used in [27] for study of winding modes in the 2D black hole geometry of the cigar.

It is known that the original $\mathcal{N} = 2$ Liouville theory deformed with a Liouville superpotential has a mirror description in terms of a $\mathcal{N} = 2$ 2D black hole [28]. Using this, we argue that the combination of both superpotential and mass deformations leads to the black hole with the same cigar geometry, while the mass of the 2D black hole depends now on both deformation parameters. Namely, we conjecture that the total black hole mass is the sum of masses associated with each of the above deformations. This conjecture is one of the key points of our paper, however actually we use it only in the limit when the mass deformation dominates.

This allows us to find the low-lying hadron spectrum of the $\mathcal{N} = 2$ SQCD with the $U(4)$ gauge group and $N_f = 8$ quark flavors. In fact, it turns out that the form of the spectrum of string states (which determines the hadron spectrum in 4D SQCD) does not depend on the mass deformation, and only the number of states is changed. We also use the field theory arguments on the SQCD side to explain this surprising effect. We explain the expected increase in the number of hadronic states in SQCD, which has more quarks, by considering the near-Hagedorn behavior of the 2D black hole.

The paper is organized as follows. In Sec. II we briefly review the 4D $\mathcal{N} = 2$ supersymmetric SQCD and the world sheet theory of the non-Abelian string. In particular, we review results of [21] which show that the Coulomb branch of the world sheet $WCP(N, N)$ model is described by $\mathcal{N} = 2$ Liouville theory and describe the low-lying 4D spectrum in terms of string states in the world-sheet theory for $N = 2$. In Sec. III we consider the mass deformation and show that the mass-deformed solution, found in [22], is T -dual to the 2D black hole with cigar geometry. We also discuss the Liouville theory with both superpotential and mass deformations switched on, and formulate a conjecture how the deformation parameters are related to the background of the dual theory with cigar geometry. In Sec. IV we solve the Schrödinger equation for string excitations in the mass-deformed background, using the effective gravity approach, and find the discrete spectrum, the result confirms the statement about T -duality from Sec. III. In Sec. V we use the field theory arguments from SQCD to explain the surprising behavior of the 4D spectrum, predicted by string theory. In Sec. VI we finally study the near-Hagedorn behavior of the 2D black hole to explain the expected increase in hadron states in the 4D SQCD with more quark flavors. Sec. VII contains our conclusions, while the Appendix deals with an analytic continuation of the singular solution to the Schrödinger equation for string excitations.

²In [15] this equivalence was shown for topological versions of the string theories.

II. NON-ABELIAN VORTEX STRING AND $\mathcal{N} = 2$ LIOUVILLE THEORY

In this section we are going to review some basic facts about vortex strings in $\mathcal{N} = 2$ SQCD. The goal is to make the reader familiar with the world-sheet theory and to explain how the hadron spectrum emerges in this construction, which will be generalized later in this work.

A. Four-dimensional $\mathcal{N} = 2$ SQCD

Our starting point is the $\mathcal{N} = 2$ SQCD in 4D with eight supercharges and gauge group $U(1) \times SU(N)$; see, for example, [7] for a detailed review of this theory. We take the number of fundamental quark hypermultiplets to be $N_f = 2N$; with this choice, the β -function of the 4D SQCD is zero and the 4D coupling does not run. However, the conformal invariance of the 4D theory is explicitly broken by the Fayet-Iliopoulos (FI) term [29] with FI parameter ξ , which defines the vacuum expectation values (VEVs) of quarks. The FI parameter is not renormalized.

At weak coupling $g^2 \ll 1$, this theory is in the Higgs phase. In a vacuum where the first N quark flavors are massless at zero ξ , the matrix of adjoint scalars of the $\mathcal{N} = 2$ vector multiplet develops VEV of the form

$$\langle a \rangle = -\text{diag}(m_1, \dots, m_N), \quad (2.1)$$

where m_A ($A = 1, \dots, N_f$) are bare quark masses. Adjoint condensates (2.1) break $U(N)$ gauge group down to $U(1)^N$, with the masses of the off-diagonal gauge bosons given by $|m_k - m_l|$ ($k, l = 1, \dots, N$), while the quark masses of q^{kA} and $\tilde{q}_{Ak}$ (two complex scalars of the $\mathcal{N} = 2$ hypermultiplet) are equal to $|m_k - m_A|$.

At nonzero ξ , first N squarks also develop VEVs,

$$\langle q^{kA} \rangle = \sqrt{\xi} \delta^{kA}, \quad \langle \tilde{q}_{Ak} \rangle = 0 \quad k = 1, \dots, N, \quad A = 1, \dots, N_f. \quad (2.2)$$

These quarks' VEVs break the $U(N)$ gauge group, Higgsing all gauge bosons. The Higgsed gauge bosons combine with the screened quarks to form long $\mathcal{N} = 2$ multiplets with mass $\sim g\sqrt{\xi}$ in the limit of zero quark masses.

In this limit, the global flavor $SU(N_f)$ is also broken down by quark VEVs to the so-called color-flavor locked group. The resulting global symmetry is

$$SU(N)_{C+F} \times SU(N) \times U(1)_B, \quad (2.3)$$

see [7] for more details. The unbroken global $U(1)_B$ factor above is identified with a baryonic symmetry.

In the Higgs phase, quarks are screened, while monopoles are confined by non-Abelian strings. In fact, in $U(N)$ theories, confined monopoles are junctions of two distinct elementary non-Abelian strings, see [7] for a review. In

particular, baryons represent closed “necklace” configurations of monopoles on the string. At weak coupling, all these stringy hadrons are heavy and decay into perturbative states; however, at strong coupling, the theory enters the so-called instead-of-confinement phase [30,31]. In this phase quarks and gluons decay into monopole-antimonopole pairs and we are left with hadrons formed by monopoles confined by non-Abelian strings.

Below we assume that N is even, $N = 2K$, where K is integer and consider a special choice of quark masses,

$$\tilde{m}_A = m_A, \quad A = 1, \dots, N, \quad (2.4)$$

where $\tilde{m}_A \equiv m_{A+N}$, $A = 1, \dots, N$. This ensures that “extra” quarks with $A = (N+1), \dots, 2N$ have the same masses as the first N ones. Moreover, we assume that quark masses are taken to be

$$\{m_A\}_{A=1}^{N_f} = \underbrace{\{0, \dots, 0\}}_{N/2}, \underbrace{\{M, \dots, M\}}_{N/2}, \underbrace{\{0, \dots, 0\}}_{N/2}, \underbrace{\{M, \dots, M\}}_{N/2} \quad (2.5)$$

with M being the deformation parameter. In the large M limit, the original SQCD splits into two sectors with mutual interactions suppressed by the scale $1/M$,

$$\begin{aligned} &[SU(2K) + (N_f = 4K)] \\ &\xrightarrow{M \rightarrow \infty} [SU(K) + (N_f = 2K)] \times [SU(K) + (N_f = 2K)] \end{aligned} \quad (2.6)$$

(see [22] for details). In the opposite limit $M \rightarrow 0$, we recover the full $SU(2K)$ with $N_f = 4K$ fundamental flavors. Thus, varying the parameter M , we can interpolate between these two theories.

The case with $K = 2$, when both SQCDs in the rhs of (2.6) support critical non-Abelian strings, will be the main subject of this study.

B. World-sheet theory of the non-Abelian string

1. $WCP(N, N)$ model

The presence of the color-flavor locked group $SU(N)_{C+F}$ in 4D $\mathcal{N} = 2$ SQCD with gauge group $U(N)$ is the reason for the formation of non-Abelian vortex strings [1–4]. The most important feature of these vortices is the presence of the orientational zero modes described by complex fields n^i , $i = 1, \dots, N$ living on the 2D world sheet. In our case, the number of quark flavors exceeds the number of colors; the solitonic vortices become semilocal and acquire extra size moduli [32], described by $N_f - N = N$ complex fields ρ^j with $j = 1, \dots, N$; see [1,4,32–35]. In $\mathcal{N} = 2$ SQCD the flux-tube strings are 1/2 BPS saturated and preserve $\mathcal{N} = (2, 2)$ supersymmetry with four supercharges on the world sheet. Their tension is determined exactly by the FI parameter,

$$\tau = 2\pi\xi. \quad (2.7)$$

The effective theory on the string world sheet is 2D $\mathcal{N} = (2, 2)$ supersymmetric $\mathbb{WCP}(N, N)$ model, defined as a low-energy limit of the $U(1)$ gauge theory [36], with twisted-mass deformation,

$$\begin{aligned} S = \int d^2x \Big\{ & |\nabla_\alpha n^i|^2 + |\tilde{\nabla}_\alpha \rho^j|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 \\ & + \frac{1}{e_0^2} |\partial_\alpha \sigma|^2 + \frac{1}{2e_0^2} D^2 - \frac{\Theta}{2\pi} F_{01} \\ & - |\sqrt{2}\sigma + m_i|^2 |n^i|^2 - |\sqrt{2}\sigma + \tilde{m}_j|^2 |\rho^j|^2 \\ & + D(|n^i|^2 - |\rho^j|^2 - \text{Re } \beta) \Big\}, \\ \alpha, \beta = 1, \dots, 2, \quad i, j = 1, \dots, N, \end{aligned} \quad (2.8)$$

see review [7] for details. The fields n^i and ρ^j have charges $+1$ and -1 , respectively,

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \quad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha. \quad (2.9)$$

Twisted masses of the n^i and ρ^j fields coincide with the masses m_i and $\tilde{m}_j$ of the 4D quarks. The complex scalar σ is a superpartner of the $U(1)$ gauge field A_α and D is the auxiliary field in the vector supermultiplet. These fields can be written in terms of the twisted chiral superfield Σ [36],³

$$\Sigma = \sigma + \sqrt{2}\theta_R \bar{\lambda}_L - \sqrt{2}\bar{\theta}_L \lambda_R + \sqrt{2}\theta_R \bar{\theta}_L (D - iF_{01}). \quad (2.10)$$

The complexified inverse coupling in (2.8),

$$\beta = \text{Re } \beta + i \frac{\Theta}{2\pi}, \quad (2.11)$$

is defined via the 2D FI term (twisted superpotential)

$$-\frac{\beta}{2} \int d^2\tilde{\theta} \sqrt{2} \Sigma = -\frac{\beta}{2} (D - iF_{01}). \quad (2.12)$$

For our purposes, the following facts about this 2D theory will be most essential:

- (1) The beta function for this sigma model vanishes, and this 2D theory is conformal in the zero-mass limit. Therefore, its target space is Ricci-flat and [being Kähler due to $\mathcal{N} = (2, 2)$ supersymmetry] represents a (noncompact) Calabi-Yau manifold. As usual for such compactifications the $\mathcal{N} = (2, 2)$ superconformal symmetry of 2D world-sheet theory

³Here spinor indices are written as subscripts, say $\theta^L = \theta_R$, $\theta^R = -\theta_L$. We also define the twisted measure $d^2\tilde{\theta} = \frac{1}{2} d\bar{\theta}_L d\theta_R$ to ensure that $\int d^2\tilde{\theta} \tilde{\theta}^2 = \int d\bar{\theta}_L d\theta_R \theta_R \bar{\theta}_L = 1$.

ensures $\mathcal{N} = 2$ spacetime supersymmetry in 4D after the Gliozzi–Scherk–Olive (GSO) projection.

- (2) The global symmetry group of the $\mathbb{WCP}(N, N)$ sigma model coincides with the unbroken global group of the 4D SQCD (2.3); see [10]. The fields n^i and ρ^j transform in the representations,

$$\left(\mathbf{N}, \mathbf{1}, \frac{1}{2} \right), \quad \left(\mathbf{1}, \mathbf{N}, \frac{1}{2} \right), \quad (2.13)$$

respectively. Note that another $U(1)$ symmetry, which rotates the n and ρ fields with opposite charges, is gauged.

- (3) 2D-4D correspondence: the BPS spectrum of states in this 2D world sheet theory coincides with the BPS spectrum of 4D states in the quark vacuum (2.2) given by the exact Seiberg-Witten solution [37] at $\xi = 0$. This coincidence was observed in [38,39] and explained later in [3,4] using the picture of 4D monopoles confined by the non-Abelian string that are seen as kinks in the 2D world-sheet theory.

The Calabi-Yau space associated with the massless $\mathbb{WCP}(2, 2)$ model is the conifold. It is six-dimensional, and, combined with the flat Minkowski 4D space, forms a ten-dimensional space required for a superstring to be critical [9,10]. In fact, we are interested in considering the $\mathbb{WCP}(2, 2)$ model at strong coupling $\beta = 0$, the so-called “thin string conjecture” put forward in [9,10] implies that only at vanishing β (2.11) we expect that the world sheet $\mathbb{WCP}(2, 2)$ model defines the consistent string theory for the solitonic non-Abelian vortex in $\mathcal{N} = 2$ 4D SQCD.

At $\beta = 0$, the conifold singularity $\det w^{ij} = 0$, where $w^{ij} = n^i \rho^j$ is the set of $U(1)$ gauge-invariant “mesonic” variables, can be alternatively smoothed by the deformation of the complex structure. Following [12], this deformation can be described as

$$\det w^{ij} = b, \quad (2.14)$$

where b is a complex parameter. Promoted to a 4D field, the deformation constant b was interpreted as a scalar component of a massless baryonic hypermultiplet of 4D $\mathcal{N} = 2$ QCD in [10]. Its quantum numbers with respect to the global symmetry group (2.3) for $N = 2$ are determined by (2.14),

$$(\mathbf{1}, \mathbf{1}, 2), \quad (2.15)$$

it is a singlet with respect to both $SU(2)$ groups with $B(b) = 2$ baryonic charge [10].

The massless field b can form a condensate. Thus, we have a new Higgs branch in 4D $\mathcal{N} = 2$ SQCD which develops only at the critical value of the 4D coupling constant $\tau_{SW} = 1$ associated with $\beta = 0$ [40].

2. $\mathcal{N} = 2$ Liouville theory

As we mentioned in the Introduction, it was recently shown in [21] that the Coulomb branch of the 2D $\mathbb{WCP}(N, N)$ world sheet model which opens up at strong coupling $\beta = 0$ can be described by $\mathcal{N} = 2$ Liouville theory (see [41] for a review of Liouville theory in general). Its bosonic action reads

$$S_{\text{eff}} = \frac{1}{4\pi} \int d^2x \sqrt{h} \left(\frac{1}{2} h^{\alpha\beta} (\partial_\alpha \phi \partial_\beta \phi + \partial_\alpha Y \partial_\beta Y) - \frac{Q}{2} \phi R^{(2)} \right), \quad (2.16)$$

for the real Liouville scalar field ϕ , supplemented by the real compact scalar $Y \sim Y + 2\pi$. Here $R^{(2)}$ is the Ricci scalar for the world sheet metric $h_{\alpha\beta}$, and $h = \det(h_{\alpha\beta})$. The linear dilaton in (2.16),

$$\Phi = -\frac{Q}{2} \phi, \quad (2.17)$$

contains the background charge Q for the Liouville field ϕ , given by

$$Q(N) = \sqrt{2(N-1)}. \quad (2.18)$$

The action in (2.16) leads to the following holomorphic stress tensor of the bosonic part of the theory,

$$T = -\frac{1}{2} [(\partial_z \phi)^2 + Q \partial_z^2 \phi + (\partial_z Y)^2], \quad (2.19)$$

so that the central charge is $c_L = 1 + 3Q^2 + 2 = 3(1 + Q^2)$. For the $N = 2$ conifold case, (2.18) gives $Q = \sqrt{2}$, so the central charge of the internal part of the world sheet theory equals nine as required for criticality.

The $\mathcal{N} = 2$ Liouville interaction superpotential (see [41]) for the $N = 2$ case has the form,

$$L_{\text{int}} = b \int d^2\tilde{\theta} e^{-\frac{\phi + iY}{Q}}, \quad (2.20)$$

where we promote the scalars ϕ and Y to (twisted) chiral superfields, and b is the conifold complex structure deformation parameter (2.14). It comes from the 2D FI term (2.12) in the $\mathbb{WCP}(2, 2)$ model when we use the relation

$$\sigma = \gamma e^{-\frac{\phi + iY}{Q}}, \quad (2.21)$$

between the complex scalar σ from the $U(1)$ gauge multiplet of the $\mathbb{WCP}(2, 2)$ model and the fields ϕ and Y in the Liouville theory, where $\gamma = -\sqrt{2}b/\beta$. Note that we must take the constant γ singular in the limit $\beta \rightarrow 0$ in order to keep the Liouville superpotential (2.20) finite, see [21]

for details. This superpotential is a marginal deformation of the $\mathcal{N} = 2$ Liouville theory (2.16).

To conclude this subsection, we note that the dilaton has a linear dependence on the Liouville coordinate ϕ , see (2.17). Therefore, the string coupling constant $g_s = e^\Phi$ would become large at large negative ϕ . On the other hand, at nonzero b , the Liouville wall prevents the field ϕ from penetrating into the region of large negative values. In fact, the maximum value of the string coupling is $g_s \sim 1/|b|$ for $Q = \sqrt{2}$. In this paper, we keep b large to ensure that the string coupling is small and the string perturbation theory is reliable, see [18, 20]. In particular, one can use the tree-level approximation to obtain the string spectrum. In terms of 4D SQCD, keeping b large means moving along the Higgs branch far away from the origin.

C. Primary operators on the world sheet

In this subsection, we review primary operators in the $\mathcal{N} = 2$ Liouville theory. For the $N = 2$ [$Q = \sqrt{2(N-1)} = \sqrt{2}$] case, they describe physical string states interpreted as hadrons in 4D SQCD; see [13] for details.

The primary operators at large ϕ have the form [40, 42, 43]

$$T_{j, m_L, m_R} \simeq e^{iQ(m_L Y_L - m_R Y_R)} [e^{Qj\phi} + R(j, m_{L,R}; k) e^{-Q(j+1)\phi}], \quad (2.22)$$

where $Y_{L,R}$ correspond to the holomorphic and antiholomorphic parts of the compact scalar, with quantum numbers $m_{L,R}$,

$$m_L = \frac{1}{2}(n_1 + kn_2), \quad m_R = \frac{1}{2}(n_1 - kn_2), \quad k = \frac{2}{Q^2}, \quad (2.23)$$

related to integer winding (n_1) and momentum (n_2) numbers, respectively. These operators should also obey $m_R = \pm m_L$ to have equal left and right conformal dimensions,

$$\Delta_{j,m} = \frac{Q^2}{2} \{m^2 - j(j+1)\} = \frac{1}{k} \{m^2 - j(j+1)\}, \quad (2.24)$$

and unitarity requires $\Delta_{j,m} > 0$. For our 4D string applications, we set in (2.22) $m_L = -m_R \equiv m$.⁴ This corresponds to the local or momentum operator with $n_2 \neq 0$, $n_1 = 0$ (depending on $Y = Y_L + Y_R$) in the Liouville theory, and to the dual vortex or winding state in the mirror cigar picture.

The so-called reflection coefficient in (2.22), given by [42–44]

⁴This is the condition for the type IIA string, while for the type IIB string $m_R = m_L$ [20].

$$R(j, m_L, m_R; k) = \left[\frac{1}{\pi} \frac{\Gamma(1 + \frac{1}{k})}{\Gamma(1 - \frac{1}{k})} \right]^{2j+1} \frac{\Gamma(1 - \frac{2j+1}{k}) \Gamma(m_L + j + 1) \Gamma(m_R + j + 1) \Gamma(-2j - 1)}{\Gamma(1 + \frac{2j+1}{k}) \Gamma(m_L - j) \Gamma(m_R - j) \Gamma(2j + 1)}, \quad (2.25)$$

vanishes for values of j and m from the discrete spectrum

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, \dots, \quad m = \pm\{j, j-1, j-2, \dots\}, \quad (2.26)$$

for $k = 1$ ($Q = \sqrt{2}$), which kills the rising exponential in (2.22), so that the primary operator gives a normalizable wave function at $j \leq -1/2$; see (2.30) below.⁵

The spectrum (2.26) was exactly determined using the mirror description [28] of the theory as a $\mathcal{N} = 2$ $SL(2, R)/U(1)$ coset with a cigar geometry with the level,

$$k = \frac{2}{Q^2}, \quad (2.27)$$

of the supersymmetric version of the Kač-Moody algebra in [24,45–48], see [49] for a review.

To exclude the states with negative norm, one has to impose an extra restriction [45–49],

$$-\frac{k+1}{2} \leq j < 0. \quad (2.28)$$

These conditions are also seen from the reflection coefficient (2.25), namely, they are associated with $\Gamma(1 + \frac{2j+1}{k})$ and $\Gamma(-2j - 1)$, see [40] for more details.

Moreover, we look for string states with normalizable wave functions,

$$\Psi_{j;m_L,m_R}(\phi, Y) = e^{-\Phi} T_{j;m_L,m_R} \sim_{\phi \rightarrow \infty} e^{Q(j+\frac{1}{2})\phi + iQ(m_L Y_L - m_R Y_R)}, \quad (2.29)$$

to be interpreted as hadrons in 4D $\mathcal{N} = 2$ SQCD. This requires

$$j \leq -\frac{1}{2}, \quad (2.30)$$

where the borderline case $j = -\frac{1}{2}$ is also included. Thus, for our value $k = 1$, we are left with only two options $j = -1/2$ and $j = -1$.⁶

⁵Another option is with $\text{Re } j = -1/2$, when both exponentials are present in (2.22), but they have the same normalization properties. This corresponds to the principal continuous representation with $j = -\frac{1}{2} + i\mathbb{R}$.

⁶Note that $j = -1$ is included for $k = 1$ as indicated in (2.28) because the pole of $\Gamma(1 + \frac{2j+1}{k})$ is canceled by the pole of $\Gamma(1 - \frac{1}{k})$ from the prefactor.

D. 4D SQCD low-lying hadron spectrum

Dressing the tachyon vertex (2.22) with the 4D plane wave,

$$\mathcal{T}_{j,m} = e^{ip_\mu x^\mu} T_{j,m,-m}, \quad (2.31)$$

we impose the mass-shell condition

$$\frac{p_\mu p^\mu}{2} + \Delta_{j,m} = \frac{1}{2}. \quad (2.32)$$

Here we use dimensionless notation, imposing normalization $4\pi\tau = 1$ ($\alpha' = 2$, $\alpha' \equiv 1/(2\pi\tau)$) and Minkowski 4D metric $\text{diag}(-1, 1, 1, 1)$. The GSO projection restricts $2m$ for the operator (2.31) at $k = 1$ to be odd [17,18], i.e.,

$$m = \frac{1}{2} + \mathbb{Z}. \quad (2.33)$$

Then the only possibility [see (2.26) and (2.28)] is $j = -\frac{1}{2}$. This determines the masses of the 4D scalars⁷ to be [13]

$$\frac{M_T^2}{2} = -\frac{p_\mu p^\mu}{2} = m^2 - \frac{1}{4}, \quad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \dots \quad (2.34)$$

In particular, the operator (2.22) with $j = -1/2$ and $m_L = \pm 1/2$ has conformal dimension (2.24)

$$\Delta_{j=-\frac{1}{2}, m=\pm\frac{1}{2}} = \frac{1}{2}, \quad (2.35)$$

so it is marginal and describes a massless string state in 4D. As was noted in [13] this massless state corresponds to the complex structure modulus b for the string compactification on the conifold. Two possible values of $m = \pm 1/2$ correspond to two real degrees of freedom of the complex scalar field b . The associated string state has a logarithmically normalizable wave function in terms of the conifold radial coordinate [10,50]. On the Liouville side, this corresponds to the borderline normalization of the massless state (2.22) with $j = -\frac{1}{2}$, $m = \pm\frac{1}{2}$, see [13] for details.

Consider now the 4D spin-2 states, corresponding to the vertex operators

$$\psi_L^\mu \psi_R^\nu e^{ip_\mu x^\mu} T_{j;m,-m}, \quad (2.36)$$

where $\psi_{L,R}^\mu$ are the world-sheet superpartners of the 4D coordinates x^μ . The GSO projection selects $2m$ to be even,

⁷These states are, of course, not tachyonic in 4D, but we will use the standard terminology and refer to them as “tachyons”.

$|m| = 0, 1, 2, \dots$ [18], thus we are left with the only value $j = -1$ in (2.26) and (2.28). This leads to the following masses of spin-2 states [13]:

$$M_V^2 = 2m^2, \quad |m| = 1, 2, \dots \quad (2.37)$$

We call 4D states with masses (2.37) massive “gravitons”,⁸ note that $m = 0$, associated with the massless 4D graviton, is excluded. The momentum m in the compact dimension is related to the baryonic charge as [13,20]

$$Q_B = 4m, \quad (2.38)$$

and all closed string states are baryons.

III. MASS DEFORMATION

In this section, we introduce the mass deformation and first review the solutions of effective gravity equations for the mass-deformed superstring background found in [22]. Then, we show that the mass deformed $\mathcal{N} = 2$ Liouville theory is T -dual to the 2D supersymmetric black hole with cigar geometry and reduce its target space metric to the metric of the trumpet. Finally, we discuss the theory with both the mass deformation and the Liouville superpotential switched on.

A. Metric after mass deformation

The bosonic part of the action of the type-II supergravity in string frame for the metric and dilaton is given by

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G} e^{-2\Phi} \{R + 4G^{MN} \partial_M \Phi \partial_N \Phi + \dots\}, \quad (3.1)$$

where G_{MN} is the D -dimensional metric, $M, N = 1, \dots, D$. Here $2\kappa^2 = (2\pi)^{\frac{D-2}{2}} g_s^2 / \tau^{\frac{D}{2}-1}$.

Einstein’s equations following from (3.1) have the form,

$$R_{MN} + 2D_M D_N \Phi = 0, \quad (3.2)$$

while the equation for the dilaton reads,

$$4G^{MN} \partial_M \Phi \partial_N \Phi - 2G^{MN} D_M D_N \Phi + p = 0, \quad (3.3)$$

where $p = \frac{D-10}{2}$ (in dimensionless units) is included when $D \neq 10$.

⁸Physical states in 4D SQCD come in $\mathcal{N} = 2$ supermultiplets. This name underlines that the highest spin components of the corresponding supermultiplets have spin two, see [20].

We assume that our space-time is a direct product of the flat 4D Minkowski space and an internal space which has a nontrivial metric,

$$ds_{\text{int}}^2 = g(\phi)(d\phi^2 + dY^2), \quad (3.4)$$

associated with mass-deformed $\mathcal{N} = 2$ Liouville theory. Thus, $D = 6$ and the solution, found in [22], for the metric warp factor is

$$g(\phi) = \frac{1}{1 - \frac{|b|^2}{|\mathcal{M}|^2} e^{-Q\phi}} = \frac{1}{1 - e^{-Q(\phi-\phi_0)}}, \quad (3.5)$$

$$\phi_0 = -\frac{1}{Q} \log \frac{|\mathcal{M}|^2}{|b|^2},$$

and for the dilaton,

$$\begin{aligned} \Phi(\phi) &= -\frac{Q}{2} \phi + \frac{1}{2} \log g(\phi) \\ &= -\frac{Q}{2} \phi - \frac{1}{2} \log [1 - e^{-Q(\phi-\phi_0)}]. \end{aligned} \quad (3.6)$$

We see that the warp factor of the metric and the dilaton are functions of the Liouville field ϕ and do not depend on Y .

The solution (3.5) satisfies the initial condition,

$$g(\phi) \approx 1 + \frac{|b|^2}{|\mathcal{M}|^2} e^{-\frac{2\phi}{Q}} + \dots, \quad (3.7)$$

imposed by the mass deformation only in our case $Q = \sqrt{2}$; see [22] for details. Here $\mathcal{M} = -\beta M/2$ is the rescaled parameter of the mass deformation, which we keep finite in the limit $\beta \rightarrow 0$.⁹

Notice that in the limit $\mathcal{M} \rightarrow \infty$ solution (3.5), (3.6) turns into solution of (3.2), (3.3) with flat metric with $g(\phi) = 1$ and the linear dilaton (2.17). Einstein’s equations are then valid trivially, while Eq. (3.3) is satisfied only when $Q = \sqrt{2}$ is matched with $p = -2$.

Note also that the first nontrivial term in the expansion of the warp factor (3.5) at large ϕ gives rise to the following deformation:

$$(\partial_z \phi - i\partial_z Y)(\partial_{\bar{z}} \phi + i\partial_{\bar{z}} Y) e^{-Q\phi}. \quad (3.8)$$

of the free action. This operator (3.8) has $j = -1$, $m = 0$, and is marginal with conformal dimension $\Delta = (1, 1)$. It is the bosonic part of so-called nonchiral marginal deformation of $\mathcal{N} = 2$ Liouville theory; see [41] for a review. We see that (3.5) and (3.6) represent an exact solution for the mass deformation, which is infinitesimally associated with the nonchiral marginal operator (3.8).

⁹This rescaling is due to the presence of the coefficient γ in the relation (2.21) missed in [22].

Putting together all terms, we get for the bosonic part of the world sheet action of the mass-deformed Liouville theory [22],

$$S_{\text{ws}} = \frac{1}{4\pi} \int d^2x \sqrt{h} \left\{ \frac{1}{2} g(\phi) [(\partial_a \phi)^2 + (\partial_a Y)^2] + \Phi(\phi) R^{(2)} + L_{\text{int}} \right\}, \quad (3.9)$$

where the metric warp factor $g(\phi)$ and the dilaton $\Phi(\phi)$ are given by (3.5) and (3.6), while the Liouville superpotential still takes the form (2.20) since it is not modified by the mass deformation [22]. The action (3.9) defines a continuous family of world-sheet CFT's to be used for the string quantization. They all have the central charge $c_L = 3(1 + Q^2) = 9$ (for our choice $Q = \sqrt{2}$) and the family is parametrized by the mass parameter $\mathcal{M}$ of marginal deformation.

The world-sheet action (3.9) describes a nontrivial interacting world-sheet theory, and in order to extract information about the hadron spectrum of the 4D SQCD, we will rather use some indirect duality arguments.

B. T -duality

Now let us now show that the mass-deformed theory is actually T -dual to the 2D black hole. Consider the $\mathcal{N} = 2$ supersymmetric version of the two-dimensional black hole [23], which is the $\text{SL}(2, \mathbb{R})/\text{U}(1)$ coset WZNW theory [15, 18, 24, 25] with the level k of supersymmetric Kač-Moody algebra. The bosonic part of the action reads

$$S_{\text{BH}} = \frac{k}{4\pi} \int d^2x \sqrt{h} \{ (\partial_a \rho)^2 + \tanh^2 \rho (\partial_a \theta)^2 \} + \frac{1}{4\pi} \int d^2x \sqrt{h} \Phi(\rho) R^{(2)}, \quad (3.10)$$

with the dilaton given by

$$\Phi(\rho) = \Phi_0 - \log \cosh \rho, \quad (3.11)$$

so that the target space has the form of a semi-infinite cigar with radial coordinate $0 \leq \rho < \infty$ and angular coordinate $\theta \sim \theta + 2\pi$, see Fig. 1. Here k is related to Q via (2.27).

The first observation is that the action (3.10) can be rewritten as the following 2D sigma model with a complex¹⁰ metric [51]:

$$S = \frac{1}{8\pi} \int d^2x \sqrt{h} \{ (\partial_\mu \phi)^2 + (\partial_\mu X)^2 - Q R^{(2)} \phi + e^{-Q(\phi - \phi_0)} (\partial_\mu \phi - i \partial_\mu X)^2 \}. \quad (3.12)$$

¹⁰We are grateful to A. Litvinov for important comments on this point.

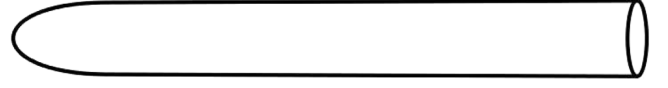


FIG. 1. Cigar geometry of 2D black hole.

Indeed, under the substitution

$$\phi = \frac{2}{Q} \log \cosh \rho + \phi_0, \quad X = \frac{2}{Q} (\theta + i \log \tanh \rho), \quad (3.13)$$

the action (3.12) turns into (3.10) if one takes

$$\Phi_0 = -\frac{Q}{2} \phi_0 \quad (3.14)$$

in (3.11). Let us now make a T -duality transformation of the action (3.12), rewriting it first as

$$\begin{aligned} S &= \frac{1}{8\pi} \int d^2z \sqrt{g} \{ (1 + e^{-Q\phi}) (\partial_\mu \phi)^2 + (1 - e^{-Q\phi}) (\partial_\mu X)^2 \\ &\quad - 2ie^{-Q\phi} \partial_\mu \phi \partial_\mu X - Q R^{(2)} \phi \} \\ &\equiv \frac{1}{4\pi} \int d^2z \sqrt{g} \left\{ \frac{1}{2} [G_{\phi\phi} (\partial_\mu \phi)^2 + G_{XX} (\partial_\mu X)^2 \right. \\ &\quad \left. + 2G_{\phi X} \partial_\mu \phi \partial_\mu X] - \frac{Q}{2} R^{(2)} \phi \right\}, \end{aligned} \quad (3.15)$$

with $\phi_0 = 0$ for simplicity. By a standard trick

$$\begin{aligned} \hat{S} &= \frac{1}{8\pi} \int d^2z \sqrt{g} \{ (\partial_\mu \phi)^2 + V_\mu^2 - Q R^{(2)} \phi \\ &\quad + e^{-Q\phi} (\partial_\mu \phi - i V_\mu)^2 + 2Y \epsilon_{\mu\nu} \partial_\mu V_\nu \}, \end{aligned} \quad (3.16)$$

with Lagrange multiplier Y so that $\tilde{S}|_{\delta \tilde{S}/\delta Y=0} = S$. However, solving the equations $\delta \tilde{S}/\delta V_\mu = 0$ one finds

$$\begin{aligned} V_\mu &= \frac{ie^{-Q\phi}}{1 - e^{-Q\phi}} \partial_\mu \phi - \frac{\epsilon_{\mu\lambda}}{1 - e^{-Q\phi}} \partial_\lambda Y \\ &= -\frac{G_{\phi X}}{G_{XX}} \partial_\mu \phi - \frac{\epsilon_{\mu\lambda}}{G_{XX}} \partial_\lambda Y. \end{aligned} \quad (3.17)$$

Substituting this back into (3.16) we come to the dual action,

$$\begin{aligned} S_{\text{dual}}[\phi, Y] &= \frac{1}{4\pi} \int d^2z \sqrt{g} \left\{ \frac{1}{2} [\tilde{G}_{\phi\phi} (\partial_\mu \phi)^2 + \tilde{G}_{YY} (\partial_\mu Y)^2 \right. \\ &\quad \left. + 2i\tilde{B}_{Y\phi} \epsilon_{\mu\nu} \partial_\mu Y \partial_\nu \phi] + R^{(2)} \Phi \right\}, \end{aligned} \quad (3.18)$$

with

$$\tilde{G}_{YY} = \frac{1}{G_{XX}}, \quad \tilde{G}_{\phi\phi} = G_{\phi\phi} - \frac{G_{\phi X}^2}{G_{XX}}, \quad \tilde{B}_{Y\phi} = \frac{G_{X\phi}}{G_{XX}} \quad (3.19)$$

and

$$\Phi = -\frac{Q}{2}\phi - \frac{1}{2}\log G_{XX} \quad (3.20)$$

coming from changing integration measure $DX \rightarrow DV$ (see e.g., [52]). We get, therefore, after the duality transformation,¹¹

$$\begin{aligned} \tilde{G}_{YY}(\phi) &= \tilde{G}_{\phi\phi}(\phi) = \frac{1}{1 - e^{-Q(\phi-\phi_0)}} = g(\phi), \\ \Phi(\phi) &= -\frac{Q}{2}\phi - \frac{1}{2}\log(1 - e^{-Q(\phi-\phi_0)}), \end{aligned} \quad (3.21)$$

coinciding exactly, after restoring the dependence on ϕ_0 in the warp factor, with solution (3.5) for the metric and the dilaton (3.6).

C. Trumpet geometry and fluctuations

Now let us make the world-sheet geometry of the mass-deformed $\mathcal{N} = 2$ theory

$$S_{\text{ws}} = \frac{1}{4\pi} \int d^2x \sqrt{h} \left\{ \frac{1}{2} g(\phi) [(\partial_\alpha \phi)^2 + (\partial_\alpha Y)^2] + \Phi(\phi) R^{(2)} \right\}, \quad (3.22)$$

with the metric warp factor (3.5) and the dilaton (3.6) more transparent. At $\phi = \phi_0$, this metric develops a naked singularity, so the geometry is defined at $\phi \geq \phi_0$. Making a change back to the radial coordinate,

$$e^{\frac{Q}{2}(\phi-\phi_0)} = \cosh \rho, \quad (3.23)$$

the action (3.22) turns into

$$\begin{aligned} S_{\text{ws}} &= \frac{k}{4\pi} \int d^2x \sqrt{h} ((\partial_\alpha \rho)^2 + \coth^2 \rho (\partial_\alpha \vartheta)^2) \\ &\quad + \frac{1}{4\pi} \int d^2x \sqrt{h} \Phi(\rho) R^{(2)}, \end{aligned} \quad (3.24)$$

with $\vartheta = \frac{Q}{2}Y$,

$$\vartheta \sim \vartheta + 2\pi/k, \quad (3.25)$$

and the dilaton is now given by

$$\Phi(\rho) = -\frac{Q}{2}\phi_0 - \log(\sinh \rho). \quad (3.26)$$

The target manifold of the dual theory (3.24) looks like a “trumpet”; see Fig. 2. This metric was first discussed

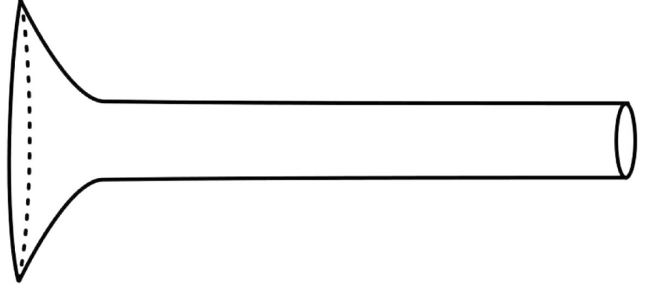


FIG. 2. Trumpet geometry. Asymptotically, at $\rho \rightarrow \infty$, it turns into a cylinder of radius Q .

in [26] and then in [27] for the black hole model (3.10). The point is that the effective theory for the winding modes of the string in the cigar metric (3.10) of the 2D black hole is given by fluctuations of “particles” (momentum modes) in the trumpet metric in (3.24) [27] (see also [53] for a review of the supersymmetric version). It is just a result of T -duality transformation, which here takes the form

$$\sqrt{2k} \tanh \rho \rightarrow \alpha' \frac{\coth \rho}{\sqrt{2k}}, \quad \alpha' = 2. \quad (3.27)$$

Taking the limit $\rho \rightarrow \infty$, we see that $k = 1$ corresponds to the self-dual point [this is also seen, of course, from (2.23)].

Let us now say a few words about the expected spectrum for the string in trumpet geometry. The primary operators of the coset theory, corresponding to the string in the 2D black hole, are actually given at large ρ by the formula similar to the one in (2.22) (written for Liouville theory)

$$T_{j,m_L,m_R} \simeq e^{2i(m_L\theta_L + m_R\theta_R)} [e^{2j\rho} + R(j, m_{L,R}; k) e^{-2(j+1)\rho} + \dots], \quad (3.28)$$

where now the Liouville momenta and windings change roles and are interpreted as windings and momenta of the string in the cigar geometry, n_2 and n_1 respectively; see (2.23). Hence, once we are interested in the operators (3.28) with $n_2 \neq 0$ ($m_L = -m_R$), they correspond to the windings on the black hole side, and (in the minisuperspace approximation) it is natural to look for them as solutions of the target-space equations of motion in the trumpet geometry [27]. This target-space approximation works for large $k \rightarrow \infty$, in contrast to direct Liouville quasiclassics, which works at large $Q^2 = 2/k$. One can even be convinced that such reasoning is valid in the theory when both marginal deformations—the Liouville superpotential and the trumpet warp factor—are switched on independently.

D. Switching on both superpotential and mass deformations

As is commonly believed, $\mathcal{N} = 2$ Liouville theory with Liouville superpotential has a mirror description [28] in

¹¹An extra B -field $\tilde{B}_{Y\phi} = i \frac{e^{-Q(\phi-\phi_0)}}{1-e^{-Q(\phi-\phi_0)}}$ in the dual theory is almost inessential in two dimensions and we neglect it below.

terms of the $\mathcal{N} = 2$ supersymmetric version of the two-dimensional black hole (3.10) with the level $k = \frac{2}{Q^2}$. The constant Φ_0 entering the expression for the dilaton (3.11) in this background is given by

$$\Phi_0 = \Phi_0^{(b)} = -\frac{1}{k} \ln |b| \quad (3.29)$$

in terms of the coefficient b (modulus of the conifold complex structure) in front of the Liouville superpotential by the standard argument with the shift $\phi \rightarrow \phi - Q \log |b| \equiv \phi - \phi_{\text{wall}}$, where ϕ_{wall} can be interpreted as a “position” of the “Liouville wall” (superpotential).

As was shown in [23], the value of the dilaton at the horizon Φ_0 determines the ADM mass of the black hole,

$$M_{\text{BH}} = \frac{Q}{2} e^{-2\Phi_0} = \frac{1}{\sqrt{2k}} e^{-2\Phi_0}, \quad (3.30)$$

and therefore the entropy

$$S_{\text{BH}} = \frac{M_{\text{BH}}}{T} = 2\pi e^{-2\Phi_0} \quad (3.31)$$

of the 2D black hole with the Hawking temperature $T = \frac{1}{2\pi R} = \frac{Q}{4\pi}$. Hence, we see that the black hole entropy (and therefore the mass at fixed radius $R = \sqrt{2k}$) is completely determined by a single free parameter Φ_0 of the solution (3.11).

For the supersymmetric black hole associated with the Liouville theory, this parameter is totally determined (3.29) by the constant in front of the Liouville superpotential. Thus, for the black hole mass associated with the Liouville theory with superpotential deformation we have

$$M_{\text{BH}}^{(b)} = \frac{Q}{2} |b|^{2/k}. \quad (3.32)$$

On the other hand, as was shown in the previous section, the mass-deformed Liouville theory (3.22) is T -dual to the same theory (3.10) with the cigar geometry, where now the constant part of the dilaton,

$$\Phi_0^{(\mathcal{M})} = -\frac{Q}{2} \phi_0 = -\frac{1}{2} \ln \frac{|b|^2}{|\mathcal{M}|^2}, \quad (3.33)$$

which is given by the location of the singularity at $\phi = \phi_0$ of the warp factor (3.5) of the metric, i.e., is actually determined by the mass deformation parameter $\mathcal{M}$. The associated black hole mass is

$$M_{\text{BH}}^{(\mathcal{M})} = \frac{Q}{2} \frac{|b|^2}{|\mathcal{M}|^2}. \quad (3.34)$$

Let us now make a physically plausible conjecture. If both marginal deformations are switched on [see (3.9)], one still

gets the same dual black hole theory (3.10) with cigar geometry, and we only have to fix the value of “total” Φ_0 or total mass of the black hole [they are related by (3.30)] in terms of the initial data in the theory. We conjecture that the total mass of the black hole upon switching on both deformations is the sum of masses (3.32) and (3.34),

$$M_{\text{BH}}^{\text{total}} = M_{\text{BH}}^{(b)} + M_{\text{BH}}^{(\mathcal{M})}. \quad (3.35)$$

This gives for the total mass and entropy of the black hole, corresponding to the theory with both deformations,

$$\frac{M_{\text{BH}}^{\text{total}}}{Q/2} = \frac{S_{\text{BH}}^{\text{total}}}{2\pi} = |b|^{2/k} + \frac{|b|^2}{|\mathcal{M}|^2}. \quad (3.36)$$

Let us now try to justify the conjecture (3.35), (3.36):

- (i) Each term in the rhs (3.35) dominates at large negative values of (3.29) or (3.33), when $e^{-2\Phi_0} = e^{-2\Phi_0^{(b)}} + e^{-2\Phi_0^{(\mathcal{M})}}$ can be rewritten as

$$e^{2|\Phi_0|} = e^{2|\Phi_0^{(b)}|} + e^{2|\Phi_0^{(\mathcal{M})}|} \quad (3.37)$$

or, in the limit, as a “tropical sum”,

$$|\Phi_0| = |\Phi_0^{(b)}| \oplus |\Phi_0^{(\mathcal{M})}| \\ = \max(|\Phi_0^{(b)}|, |\Phi_0^{(\mathcal{M})}|). \quad (3.38)$$

- (ii) The asymptotic condition (3.38) does not uniquely fix the exact form of the function in (3.35), but it is reasonable to have a linear addition law for the entropy and mass in (3.36).
- (iii) At large $\mathcal{M} \rightarrow \infty$ the black hole mass does not feel the mass deformation. However, as we reduce the mass parameter $\mathcal{M}$, both terms in (3.36) become of the same order and the system goes through a crossover. At small $\mathcal{M}$ the second term in (3.36) dominates.

Formula (3.35) is one of the key points of our paper. It comes out of the fact that both (undeformed and mass-deformed) theory are formulated in the same string background, and the only parameter which can depend on the mass deformation is the dilaton constant Φ_0 . The exact form of this formula can be verified only by rather complicated analysis of the world-sheet theory (3.9), so we proposed instead its most straightforward form motivated by the idea that the full mass or entropy of the black hole can be naturally thought to be the sum of two independent contributions associated with the Liouville superpotential and the mass deformation. It obviously reproduces both “boundary” cases when either the Liouville superpotential deformation or the nontrivial metric warp factor, arising due to the mass deformation, dominates. These two limits of cigar theory are effectively described either by its mirror or T -dual formulations, which

relate (3.9) to the black hole (3.10) with Φ_0 determined either by (3.29) or by (3.33) respectively.

IV. STRING SPECTRUM AFTER MASS DEFORMATION

In this section, we use an effective gravity approach to study the string spectrum associated with the mass-deformed $\mathcal{N} = 2$ Liouville world sheet theory. First, we review the setup for this problem developed in [22], then we study the associated Schrödinger equation and find its discrete spectrum. We show that this spectrum coincides with the spectrum of a 2D black hole with cigar geometry, confirming the result of the previous section that these theories are related by T -duality.

A. Tachyon equations

Primary tachyon vertex operators (2.22) can be described as scalar fields in the effective supergravity (3.1). To take them into account, we add the tachyonic (with the “wrong” sign in the D -dimensional space-time) term,

$$S_{\text{tachyon}} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G} e^{-2\Phi} \times \{-G^{MN} \partial_M \bar{T}_{j,m} \partial_N T_{j,m} + |\mathcal{T}_{j,m}|^2\}, \quad (4.1)$$

to the gravity action (3.1), cf. [54]. This gives the equation for the tachyon field,

$$D_N D^N T_{j,m} - 2\partial_N \Phi \partial^N T_{j,m} + T_{j,m} = 0, \quad (4.2)$$

and we neglect the back reaction of tachyons on the background (3.5), (3.6) for the metric and the dilaton.

Dressing the tachyon with the dependence on the 4D coordinates as in (2.31) we rewrite the tachyon equation (4.2) in the form,

$$D_n D^n T_{j,m} - 2\partial_n \Phi \partial^n T_{j,m} + (1 + M_T^2) T_{j,m} = 0, \quad (4.3)$$

where derivatives are now taken only with respect to two internal coordinates, $\{\phi, Y\}$. In (4.3) we have used the relation,

$$M_T^2 = 2\Delta_{j,m} - 1, \quad (4.4)$$

which coincides with Eq. (2.32), obtained by world sheet methods, see [13] for details. Here the conformal dimension $\Delta_{j,m}$ is given by (2.24). The relation (4.4) is also easily verified by substitution,

$$T_{j,m} \approx e^{Q[j\phi + imY]}, \quad (4.5)$$

in the flat background with the linear dilaton (2.17).

Since the warp factor (3.5) does not depend on Y , one can look for solutions of (4.3) using the ansatz,

$$T_{j,m}(\phi, Y) = e^\Phi e^{iQmY} \Psi_{j,m}(\phi), \quad (4.6)$$

where the factor e^Φ kills first derivatives in (4.3). This substitution gives the Schrödinger equation for the wave function $\Psi_{j,m}(\phi)$ [22],

$$-\partial_\phi^2 \Psi_{j,m} + V_{\text{eff}}(\phi) \Psi_{j,m} = E_j \Psi_{j,m}, \quad (4.7)$$

where the potential is given by

$$V_{\text{eff}}(\phi) = -\frac{Q^2}{4} \frac{1}{(e^{Q(\phi-\phi_0)} - 1)^2} - \frac{Q^2(m^2 - j(j+1))}{e^{Q(\phi-\phi_0)} - 1} \\ \underset{\phi \rightarrow \phi_0}{\sim} -\frac{1}{4(\phi - \phi_0)^2} + \dots, \quad (4.8)$$

while energy levels are determined by j ,

$$E_j = -Q^2 \left(j + \frac{1}{2}\right)^2. \quad (4.9)$$

The potential (4.8) is attractive for $m^2 - j(j+1) > 0$ and tends to zero at $\phi \rightarrow \infty$. Therefore, one expects the continuous spectrum¹² with $j = -\frac{1}{2} + i\mathbb{R}$ with positive E_j and a discrete spectrum with negative E_j .

Equation (4.3) for tachyons can also be written in the trumpet background (3.21). Introducing the wave function via

$$T_{j,m} = \frac{e^{2im\theta}}{\sqrt{\sinh 2\rho}} \Psi_{j,m}^{(t)}, \quad (4.10)$$

[cf. with (4.6)] one gets a similar to (4.7) Schrödinger equation,

$$-\partial_\rho^2 \Psi_{j,m}^{(t)} + V_{\text{eff}}(\rho) \Psi_{j,m}^{(t)} = E_j^{(t)} \Psi_{j,m}^{(t)}, \quad (4.11)$$

where the effective potential is now given by

$$V_{\text{eff}}(\rho) = -\frac{1}{4 \sinh^2 \rho} + \frac{1 - 16m^2}{4 \cosh^2 \rho} \quad (4.12)$$

and the energy levels are just

$$E_j^{(t)} = -(2j + 1)^2. \quad (4.13)$$

¹²Note that we are looking for the spectrum of normalizable and borderline-normalizable states. The continuous spectrum mentioned here corresponds to the *principal* continuous spectrum of the cigar theory [24,45–48], see [49] for a review.

The wave function $\Psi_{j,m}^{(t)}$ at large ρ , where the potential goes to zero has the form

$$\Psi_{j,m}^{(t)}(\rho) \approx e^{2(j+\frac{1}{2})\rho}. \quad (4.14)$$

The potential (4.12) is also attractive and, of course, gives the same spectrum as the one from (4.8). Solutions of Eqs. (4.11) and (4.7) are related by

$$\Psi_{j,m}^{(t)}(\rho) = \Psi_{j,m}(\phi) \sqrt{\coth \rho} \quad (4.15)$$

together with (3.23).

Equation (4.11) describes the momentum modes with $n_2 \neq 0$ [see (2.23)] in the trumpet geometry (3.24). The nonsupersymmetric version of this equation was considered first in [27] for winding modes (also with $n_2 \neq 0$) for the dual black hole theory with the cigar geometry.

B. Equations for massive “gravitons”

Consider now string states with integer m and j associated with massive “gravitons”; see (2.36). The gravity action for the scalar components $V_{j,m}$ of the corresponding spin-2 supermultiplets has the form

$$S_{\text{graviton}} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G} e^{-2\Phi} \{ -G^{MN} \partial_M \bar{V}_{j,m} \partial_N V_{j,m} \}, \quad (4.16)$$

where the mass term is absent in contrast to (4.1). Much in the same way as before for the tachyons, one gets the equations of motion

$$D_n D^n V_{j,m} - 2\partial_n \Phi \partial^n V_{j,m} + M_V^2 V_{j,m} = 0, \quad (4.17)$$

where $V_{j,m}$ depends only on ϕ and Y . Taking the limit of large ϕ , where $g(\phi) \rightarrow 1$, and looking for a solution of (4.17) with the ansatz

$$V_{j,m} \approx e^{Q[j\phi + imY]}, \quad (4.18)$$

we get

$$M_V^2 = 2\Delta_{j,m}. \quad (4.19)$$

This coincides with (2.37) for $j = -1$. Substituting this into (4.17), we get

$$D_n D^n V_{j,m} - 2\partial_n \Phi \partial^n V_{j,m} + 2\Delta_{j,m} V_{j,m} = 0, \quad (4.20)$$

This equation has the same form as the equation of motion (4.3) for tachyons; thus we conclude that the wave function for massive “gravitons” defined via

$$V_{j,m}(\phi, Y) = e^\Phi e^{iQmY} \Psi_{j,m}(\phi), \quad (4.21)$$

satisfies the same Schrödinger equation (4.7) or (4.11), and we need only select solutions with different (integer) values of m and j .

C. Exact spectrum and profiles

Luckily enough, the second-order differential equation (4.7) can be solved exactly, and the solution profiles are even expressible in terms of elementary functions. The conditions for the spectrum that arise in the process will also give us an important hint.

We start with a substitution,

$$\Psi(x) \equiv w(x) \frac{\sqrt{e^{Qx} - 1}}{e^{Qmx}}, \quad (4.22)$$

and change the variable $x \equiv \phi - \phi_0$ to $z \equiv 1 - e^{Qx}$. Then $w(z)$ satisfies the hypergeometric equation

$$w'' + \frac{1 - 2(1-m)z}{z(1-z)} w' + \frac{(j+m)(j+1-m)}{z(1-z)} w = 0, \quad (4.23)$$

and the solution regular at $z = 0$ (i.e., at $\phi = \phi_0$) is given by

$$w_{\text{reg}} = F(\alpha, \beta, 1, z), \quad (4.24)$$

a standard hypergeometric function with

$$\alpha = -j - m, \quad \beta = 1 + j - m. \quad (4.25)$$

The second (linearly independent) solution is singular at $z = 0$, nonsymmetric under $m \leftrightarrow -m$, and therefore should be discarded (see Appendix).

To continue analytically the solution (4.24) to the region $z \rightarrow -\infty$ along the negative real axis (corresponding to $\phi \rightarrow \infty$) one can use (9.132) from [55] and get

$$\begin{aligned} w_{\text{reg}}|_{z \rightarrow -\infty} &= F(\alpha, \beta, 1; z)|_{z \rightarrow -\infty} \\ &= \frac{\Gamma(1+2j)}{\Gamma(1+j-m)\Gamma(1+j+m)} (-z)^m \\ &\quad \times [(-z)^j + R_{\text{reg}}(-z)^{-1-j} + \dots]. \end{aligned} \quad (4.26)$$

At $\phi \rightarrow +\infty$ ($z \rightarrow -\infty$), for the function $\Psi_{j,m} \sim (-z)^{-m+1/2} w_{\text{reg}}(z)$ so the first term in the brackets corresponds to the falling exponent $e^{Q(j+1/2)\phi}$, while $(-z)^{-1-j}$ gives the rising $e^{-Q(j+1/2)\phi}$ in the wave function $\Psi_{j,m}$ corresponding to (2.22) for $j \leq -1/2$. The reflection coefficient in (4.26) is given by

$$R_{\text{reg}} = \frac{\Gamma(-1-2j)\Gamma(1+j-m)\Gamma(1+j+m)}{\Gamma(1+2j)\Gamma(-j-m)\Gamma(-j+m)}. \quad (4.27)$$

The discrete spectrum is determined by the zeros of the reflection coefficient, see Sec. II C, which ensures that the rising exponent is absent at

$$-j - |m| = -n, \quad n = 0, 1, \dots, \quad (4.28)$$

The reflection coefficient (4.27) coincides with the k -independent part of the exact formula (2.25), obtained using the $SL(2, R)/U(1)$ coset (super)conformal theory, and matches with the statement of the previous section that the mass-deformed $\mathcal{N} = 2$ Liouville theory is T -dual to the supersymmetric black hole with cigar geometry (so these theories have the same spectrum of string states). Indeed, we see that Eqs. (4.23) and (4.11) do not depend on k , so our result (4.27) from the effective gravity approach reproduces the exact expression (2.25) only in the limit of large k (or the leading order in α'), while missing the $\alpha' \sim 1/k$ world-sheet corrections¹³ (even in the supersymmetric case). We point out again here, that although the gravity approximation in the mirror to Liouville “cigar theory” works only at large $k \rightarrow \infty$, still the k -independent conclusions can be applied to the most interesting critical $k = 1$ case. In particular, the reflection coefficient (4.27) reproduces the correct discrete spectrum (2.26) however misses the lower unitarity bound for allowed values of j in (2.28).

Under conditions (4.28) for integer and half-integer j and m the parameters (4.25) of the hypergeometric equation (4.23) become integers, and it can be solved in terms of rational functions. For the branch of solutions to (4.28) with $m \geq |j| > 0$ one gets the polynomial solutions,

$$w_{j,m>0}^{(+)}(z) = F(-j-m, 1+j-m, 1; z) = \frac{(1-z)^{2m}}{n!} \frac{d^n}{dz^n} (z^n (1-z)^{j-m}) \Big|_{n=j+m}. \quad (4.29)$$

For negative values of $m \leq j < 0$, using a Kummer identity one can write

$$\begin{aligned} w_{j,m<0}^{(-)}(z) &= F(-j-m, 1+j-m, 1; z) \\ &= (1-z)^{2m} F(-j+m, 1+j+m, 1; z) \\ &= (1-z)^{-2|m|} F(-j-|m|, 1+j-|m|, 1; z) \\ &= \frac{1}{n!} \frac{d^n}{dz^n} (z^n (1-z)^{j+m}) \Big|_{n=j-m}, \end{aligned} \quad (4.30)$$

¹³Although the target space geometry (3.9) itself in the supersymmetric case (in contrast to the bosonic 2D black hole) does not receive the $\alpha' \sim 1/k$ corrections.

where the bottom line follows from (4.29). We see that $w_{j,m<0}^{(-)}(z) = w_{j,|m|}^{(+)}(z)(1-z)^{-2|m|}$, and therefore both degenerate hypergeometric solutions with $m = \pm|m|$ by (4.22) give rise to the same wave function $\Psi_{j,\pm|m|}$. To conclude this section, we list the wave functions $\Psi(x)$ for the low-lying string states with allowed values of $j = -1/2$ and $j = -1$, see the table below,

j	m	α	β	$w_{\text{reg}}(z)$	$\Psi(x)$	Asympt. $x \rightarrow \infty$
$-\frac{1}{2}$	$-\frac{1}{2}$	1	1	$\frac{1}{1-z}$	$e^{-\frac{Qx}{2}} \sqrt{e^{Qx} - 1}$	1
	$\frac{1}{2}$	0	0	1		
	$-\frac{3}{2}$	2	2	$\frac{1+z}{(1-z)^3}$	$e^{-\frac{3Qx}{2}} (2 - e^{Qx}) \sqrt{e^{Qx} - 1}$	-1
	$\frac{3}{2}$	-1	-1	$1+z$		
-1	-1	2	1	$\frac{1}{(1-z)^2}$	$e^{-Qx} \sqrt{e^{Qx} - 1}$	$e^{-\frac{Q}{2}x}$
	1	0	1	1		
	-2	3	2	$\frac{1+2z}{(1-z)^4}$	$e^{-2Qx} (3 - 2e^{Qx}) \sqrt{e^{Qx} - 1}$	$-2e^{-\frac{Q}{2}x}$
	2	-1	-2	$1+2z$		

and for the “border case” $j = -\frac{k}{2}$, $m = \pm\frac{k}{2}$ these functions take the form

$$\begin{aligned} \Psi_{-k/2, \pm k/2}^{(i)}(\rho) &= \sqrt{\tanh \rho} \cosh^{1-k} \rho, \\ \Psi_{-k/2, \pm k/2}(\phi) &= e^{-\frac{\phi-\phi_0}{Q}} \sqrt{e^{Q(\phi-\phi_0)} - 1}. \end{aligned} \quad (4.31)$$

V. SQCD INTERPRETATION

The results of Sec. III D show that when both the superpotential and the mass deformation are switched on in the $\mathcal{N} = 2$ Liouville theory, one gets the same cigar geometry in the dual black hole picture, and the only effect of the mass deformation is an increase of the black hole mass, see (3.36). This means that as we reduce the mass parameter $\mathcal{M}$ the mass spectrum of the string states does not change. In particular, the low-lying hadron spectrum of the mass-deformed 4D SQCD is still given by Eqs. (2.34) and (2.37) from Sec. II D.

This rather surprising result from the string theory side seems to run into a contradiction with the intuitive expectations on the field theory side. One would expect the emergence of new hadronic states in SQCD with the masses determined by $\mathcal{M}$, which become lighter as we reduce $\mathcal{M}$. Now we are going to demonstrate that in $\mathcal{N} = 2$ supersymmetric QCD this intuitive expectation is actually incorrect under rather mild assumptions.

Of course, one cannot calculate the hadron masses directly from SQCD, since the theory is at an extremely strong-coupling regime $g^2 \rightarrow \infty$ (associated with $\beta = 0$), and therefore our approach is to use the effective string theory of the critical non-Abelian string to find the hadron spectrum in SQCD. However, we can make certain plausible assumptions on the hadron mass dependence on quark mass parameters, typical for $\mathcal{N} = 2$ SQCD, and on the

quantum numbers of allowed states. In particular, we assume that the squared hadron masses, depending on ξ and on quark masses, are given by the sum of the (ξ -dependent) non-BPS part and the squared central charge $Z_{BPS}(m_i, \tilde{m}_j)$ of the $\mathcal{N} = 2$ supersymmetric algebra for a given state,

$$m_H^2 = m_{\text{non-BPS}}^2(\xi) + |Z_{BPS}(m_i, \tilde{m}_j)|^2. \quad (5.1)$$

The BPS central charges are given by exact formulas, depending on the global symmetry charges of a given state, cf. [37].

To illustrate (5.1), consider the masses of the perturbative states (quarks and gluons and their superpartners) in the Higgs phase at weak coupling. The non-BPS part of the mass, say, of the squark q^{kA} , $A = 1, \dots, N_f$, arising due to squark condensation (2.2), is determined by the D -term in the SQCD scalar potential ($|q^A|^2 - |\tilde{q}_A|^2 - N\xi$)², see [7] for details, and gives $m_{\text{non-BPS}}^2 \sim g^2 \xi$. On the other hand, the BPS part of the mass of q^{kA} arises from the F -term in the SQCD scalar potential, $|(a + m_A)q^A|^2$ which turns into $|m_k - m_A|^2$ when VEVs of adjoint scalars (2.1) are taken into account.

To discuss the BPS parts of the masses of 4D SQCD states, it is convenient to use the 2D-4D correspondence mentioned in Sec. II B, which claims that the BPS spectrum of 2D world sheet theory coincides with the BPS spectrum of 4D SQCD in the quark vacuum (2.2) at $\xi = 0$. For example, the quark q^{kA} with $A = N + j$ has the same mass $|m_k - \tilde{m}_j|$ as the gauge-invariant “meson” $w^{kj} = n^k \rho^j$ in the $\mathbb{WCP}(N, N)$ model.

A. BPS central charges

We have already discussed the global symmetry group (2.3) of the $\mathbb{WCP}(N, N)$ model in Sec. II B, so that the n^i and ρ^j fields belong to its representations (2.13). The exact formula for their BPS central charges has the form,

$$Z_{BPS} = i\vec{m}_n \vec{q}_n - i\vec{m}_\rho \vec{q}_\rho, \quad (5.2)$$

where $\vec{m}_n = \{m_1, \dots, m_N\}$ and $\vec{m}_\rho = \{\tilde{m}_1, \dots, \tilde{m}_N\}$ are the masses of n and ρ fields (they coincide with the masses of $2N$ flavors of quarks, see Sec. II B). Here we use that $\mathfrak{u}_N = \mathfrak{su}_N \oplus \mathfrak{u}_1$ and consider the $\mathfrak{gl}_N \supset \mathfrak{u}_N$ charges $\vec{q}_n$ and $\vec{q}_\rho$ (instead of $\mathfrak{sl}_N \supset \mathfrak{su}_N$ charges), since both n^i and ρ^j fields transform under the fundamental representation $\mathbf{N}$ of $\mathfrak{gl}_N$.

In the simplest “canonical” basis,

$$(\vec{e}_i)_k \equiv \delta_{ik}, \quad i, k = 1, \dots, N \quad (5.3)$$

the Cartan $\mathfrak{gl}_N$ -generators

$$(H_i)_{jk} = \delta_{ij} \delta_{ik} \quad (5.4)$$

determine the charges $\vec{q}_{n^i}$ by

$$H_k \vec{e}_i = (q)_k \vec{e}_i \quad (5.5)$$

for both $\vec{q}_{n^i}$ and $\vec{q}_{\rho^j}$, and therefore the “perturbative states” n^i and ρ^j have charges,

$$\vec{q}_{n^i} = \vec{e}_i, \quad \vec{q}_{\rho^j} = \vec{e}_j, \quad (5.6)$$

so that they form the basis of the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$. Substituting (5.6) into the BPS formula (5.2), we immediately obtain, for an arbitrary “mesonic” $n^i \rho^j$ state,

$$Z_{n^i \rho^j} = i(m_i - \tilde{m}_j), \quad (5.7)$$

as expected. Any charge vector can be decomposed as

$$\vec{q} = \frac{1}{N} \vec{e} B + \vec{q}_\perp, \quad (5.8)$$

where $\vec{e} = \sum_j \vec{e}_j$ is the diagonal vector, $B = \sum_j q_j$ is the corresponding baryonic charge, while $\vec{q}_\perp$ is the vector of the $\mathfrak{sl}_N$ charges, satisfying $\vec{q}_\perp \cdot \vec{e} = 0$. In the $N-1$ -dimensional hyperplane orthogonal to $\vec{e}$, the basis vectors $\{\vec{e}_i\}$ project to $\{\vec{v}_i = \vec{e}_i - \frac{1}{N} \vec{e}\}$ and satisfy

$$\vec{v}_i \vec{v}_j = \delta_{ij} - \frac{1}{N}, \quad \sum_j \vec{v}_j = 0. \quad (5.9)$$

For the set of $\mathfrak{sl}_N$ charges, let us take the root basis $\vec{\alpha}_{ij} = \vec{v}_i - \vec{v}_j = \vec{e}_i - \vec{e}_j$ for the simple roots

$$\vec{\alpha}_a = \vec{\alpha}_{a,a+1}, \quad a = 1, \dots, N-1, \quad \vec{\alpha}_a^2 = 2, \quad (5.10)$$

which have the charges $h_1 = \text{diag}(1, -1, 0, \dots, 0)$, $h_2 = \text{diag}(0, 1, -1, 0, \dots, 0)$, ..., $h_{N-1} = \text{diag}(0, \dots, 0, 1, -1, \dots)$ with respect to $\mathfrak{sl}_N$ Cartan generators

$$h_a = H_a - H_{a+1}, \quad a = 1, \dots, N-1 \quad (5.11)$$

also satisfying

$$\begin{aligned} [h_a, e_b] &= C_{ab} e_b, & [h_a, f_b] &= -C_{ab} f_b, \\ [e_a, f_b] &= \delta_{ab} h_a, & a, b &= 1, \dots, N-1, \end{aligned} \quad (5.12)$$

with $C_{ab} = \vec{\alpha}_a \cdot \vec{\alpha}_b = 2\delta_{ab} - \delta_{a+1,b} - \delta_{a,b+1}$ being the Cartan matrix of $\mathfrak{sl}_N$, while $e_a = E_{a,a+1}$ and $f_a = E_{a+1,a}$ are the matrices of simple roots. The corresponding charges in the fundamental representation $\mathbf{N}$,

$$h_a(\vec{v}_j) = \vec{\alpha}_a \cdot \vec{v}_j = \delta_{aj} - \delta_{a+1,j}, \quad (5.13)$$

are always integers. One therefore gets

$$\vec{q}_\perp = \sum_{i=1}^N q_i \vec{v}_i = \sum_{a=1}^{N-1} q^a \vec{\alpha}_a, \quad (5.14)$$

so that

$$\sum_{b=1}^{N-1} C_{ab} \mathbf{q}^b = \sum_{i=1}^N q_i \vec{\alpha}_a \cdot \vec{v}_i = q_a - q_{a+1} \equiv Q_a, \quad a = 1 \dots N-1 \quad (5.15)$$

is the charge with respect to the a th $\mathfrak{sl}_2$ subalgebra (if $Q_a = 0$ for a certain a , the state is a singlet with respect to this $\mathfrak{sl}_2$ subalgebra).

Hence $\mathbf{q}^a = \sum_{b=1}^{N-1} C_{ab}^{-1} Q_b$ are “dual charges” [not necessarily integers, in contrast to $\{Q_b\}$, due to $C_{ab}^{-1} = \min(a, b) - \frac{ab}{N}$], and therefore

$$\begin{aligned} Z_{BPS} &= i \frac{B}{N} (\vec{e} \cdot \vec{m}_n - \vec{e} \cdot \vec{m}_\rho) + i \vec{m}_n \vec{q}_n^\perp - i \vec{m}_\rho \vec{q}_\rho^\perp \\ &= i \frac{B}{N} \left(\sum_{j=1}^N m_j - \sum_{j=1}^N \tilde{m}_j \right) + i \sum_{a=1}^{N-1} ((m_a - m_{a+1}) \mathbf{q}^a - (\tilde{m}_a - \tilde{m}_{a+1}) \tilde{\mathbf{q}}^a), \\ &= i \frac{B}{N} \left(\sum_{j=1}^N m_j - \sum_{j=1}^N \tilde{m}_j \right) + i \sum_{a,b=1}^{N-1} ((m_a - m_{a+1}) C_{ab}^{-1} Q_b - (\tilde{m}_a - \tilde{m}_{a+1}) C_{ab}^{-1} \tilde{Q}_b). \end{aligned} \quad (5.16)$$

1. $SU(2)$ case

Now let us consider the starting point of our interpolation procedure at $\mathcal{M} \rightarrow \infty$, namely the $\mathbb{WCP}(2, 2)$ model. The global symmetry group (2.3) of this model is

$$SU(2) \times SU(2) \times U(1)_B, \quad (5.17)$$

and each $SU(2)$ factor here is broken down to $U(1)$ by the quark mass difference.

Following (5.4), (5.5), and (5.6), one gets

$$H_1 = \text{diag}(1, 0), \quad H_2 = \text{diag}(0, 1), \quad (5.18)$$

and

$$\vec{q}_{n^1} = \vec{q}_{\rho^1} = (1, 0), \quad \vec{q}_{n^2} = \vec{q}_{\rho^2} = (0, 1) \quad (5.19)$$

while, in the $\mathfrak{su}_2 \oplus \mathfrak{u}_1$ terms, for $h = H_1 - H_2$ and the unit $H_1 + H_2$, we get

$$\begin{aligned} (Q, B)_{n^1} &= (\tilde{Q}, B)_{\rho^1} = (1, 1), \\ (Q, B)_{n^2} &= (\tilde{Q}, B)_{\rho^2} = (-1, 1), \end{aligned} \quad (5.20)$$

for the projections of isospin and baryonic charge.

The global symmetry group of the $\mathbb{WCP}(2, 2)$ model is (5.17), so the formula (5.2) reads in this case,

$$\begin{aligned} Z_{BPS} &= i \vec{m}_n \vec{q}_n - i \vec{m}_\rho \vec{q}_\rho \\ &= i \frac{B}{2} \left(\sum m - \sum \tilde{m} \right) + i \vec{m}_n \vec{q}_n^\perp - i \vec{m}_\rho \vec{q}_\rho^\perp. \end{aligned} \quad (5.21)$$

Each $\vec{q}_\perp$ corresponds to an $\mathfrak{sl}_2$ charge via

$$\vec{q}_\perp = \sum_{i=1,2} q_i \vec{v}_i = \frac{1}{2} (q_1 - q_2) \vec{\alpha} = \frac{Q}{2} \vec{\alpha} = \mathbf{q} \vec{\alpha}, \quad (5.22)$$

where

$$\vec{v}_1 = \vec{e}_1 - \frac{1}{2} \vec{e} = \frac{1}{2} (\vec{e}_1 - \vec{e}_2) = \frac{1}{2} \vec{e} - \vec{e}_2 = -\vec{v}_2 \quad (5.23)$$

and $\vec{\alpha} = \vec{e}_1 - \vec{e}_2$. Formula (5.21) therefore gives

$$\begin{aligned} Z_{BPS} &= i \frac{B}{2} (m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2) + i (m_1 - m_2) \mathbf{q} \\ &\quad - i (\tilde{m}_1 - \tilde{m}_2) \tilde{\mathbf{q}}, \end{aligned} \quad (5.24)$$

for instance, $Z_{n^1 \rho^2} = i(m_1 - \tilde{m}_2)$ [cf. with (5.7)].

Now let us turn to the nonperturbative states; the lightest is the massless stringy baryon, associated with the conifold complex structure parameter b . Eq. (2.14) shows that the b -baryon transforms with respect to (5.17) as $\det w^{ij}$, namely, it is a singlet with respect to both $SU(2)$ factors in (5.17) and has $B = 2$. This state is characterized by the $\mathfrak{u}_2$ charges (5.19), i.e.,

$$\vec{q}_b = (1, 0) + (0, 1) = (1, 1) \quad (5.25)$$

or

$$Q = \tilde{Q} = (\vec{q}_b)_1 - (\vec{q}_b)_2 = 0, \quad B = (\vec{q}_b)_1 + (\vec{q}_b)_2 = 2. \quad (5.26)$$

With these B and Q , Eq. (5.24) gives

$$Z_b = i(m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2). \quad (5.27)$$

This result was obtained in [31]. Note that within our interpolation procedure the mass of this state in the

$\mathbb{WCP}(2,2)$ model vanishes even at finite $\mathcal{M}$ because $\tilde{m}_i = m_i$; moreover, $m_1 = m_2 = 0$.

All massive string states in the theory with $N = 2$ are also singlets with respect to both $SU(2)$ factors in (5.17), see Sec. II D. They differ only by the value of their baryonic charge B . Therefore, their central charges are just multiples of the one in (5.27). Thus, the BPS part of the mass is always zero for these states. They become massive due to the first non-BPS term in (5.1), which depends only on the string tension, $\tau = 2\pi\xi$.

2. $SU(4)$ case

Now consider our interpolation procedure at finite $\mathcal{M}$. In particular, at $\mathcal{M} = 0$ the global symmetry group reads

$$SU(4)_n \times SU(4)_\rho \times U(1)_B, \quad (5.28)$$

where each $SU(4)$ factor is associated with n^i and ρ^j fields, respectively, $i, j = 1, \dots, 4$. At nonzero $\mathcal{M}$ each $SU(4)$ factor is broken down to

$$SU(4) \rightarrow SU(2)_{1,2} \times SU(2)_{3,4} \times U(1), \quad (5.29)$$

see (2.5). The groups $SU(2)_{1,2}$ and $SU(2)_{3,4}$ rotate the fields $n^{1,2}$ ($\rho^{1,2}$) and $n^{3,4}$ ($\rho^{3,4}$) respectively, and therefore correspond to the $a = 1$ and $a = 3$ subgroups.

To consider the nonperturbative string states, we have to decide first which quantum numbers with respect to the global group (5.28) are allowed. In the $N = 2$ case all string states are singlets with respect to both $SU(2)$ factors in (5.17), but all have nonzero baryonic charges B . Moreover, from the point of view of their transformation properties these states should behave as powers of $\det w$ to be nontrivial on the deformed conifold, see (2.14); this latter requirement (assuming, in particular, equality $Q = \tilde{Q}$ of the n - and ρ -charges) we call a “conifold rule”. These arguments suggest that we propose the following generalization of these rules¹⁴ for the $\mathbb{WCP}(4,4)$ model:

- (i) First, we assume that all allowed stringy states at nonzero $\mathcal{M}$ are singlets with respect to both $SU(2)_{1,2}$ and $SU(2)_{3,4}$ subgroups in (5.29) for both $SU(4)$ groups associated with n and ρ fields, see (5.28). It means that

$$Q_1 = Q_3 = \tilde{Q}_1 = \tilde{Q}_3 = 0. \quad (5.30)$$

In contrast to the $N = 2$ case, the b -baryon now belongs to a nontrivial (long fundamental) representation $\mathbf{6}$ in $\mathbf{4} \times \mathbf{4} = \mathbf{10} + \mathbf{6}$ of $SU(4)$ with the highest weight $\vec{\mu}_2 = \vec{\nu}_1 + \vec{\nu}_2$. Hence, a stronger requirement for all states to be singlets of the whole

$SU(4)$ would kill the b -baryon state, which is expected, however, to survive the transition to a theory with a larger gauge group;

- (ii) Second, we still assume a generalized “conifold rule”, requiring that the charges of a stringy state with respect to the $SU(4)_\rho$ and $SU(4)_n$ groups coincide, i.e., in addition to (5.30), impose

$$Q_2 = \tilde{Q}_2 = Q, \quad (5.31)$$

being the charge of the diagonal $SU(2)_{2,3} \subset SU(4)_d \subset SU(4)_n \times SU(4)_\rho$.

Formula (5.16) then gives for the baryons,

$$\begin{aligned} Z_{BPS} &= i \frac{B}{4} \left(\sum_{j=1}^4 m_j - \sum_{j=1}^4 \tilde{m}_j \right) \\ &+ i \sum_{a,b=1}^3 ((m_a - m_{a+1} - \tilde{m}_a + \tilde{m}_{a+1}) C_{a2}^{-1} Q) \\ &= i(m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2) \left(\frac{B}{4} + \frac{Q}{2} \right) \\ &+ i(m_3 + m_4 - \tilde{m}_3 - \tilde{m}_4) \left(\frac{B}{4} - \frac{Q}{2} \right). \end{aligned} \quad (5.32)$$

Consider first the $b_{1,2}$ -baryon, which is a particular component ($w^{11}w^{22} - w^{12}w^{21}$) of $\mathbf{6}_n \times \mathbf{6}_\rho$ with $u(4)$ charges,

$$\begin{aligned} \vec{q}_{b_{1,2}} &= \vec{q}_1 + \vec{q}_2 = (1, 0, 0, 0) + (0, 1, 0, 0) \\ &= (1, 1, 0, 0). \end{aligned} \quad (5.33)$$

For $\{Q_a\}$ and the baryonic charge one gets

$$\begin{aligned} Q_1 &= (\vec{q}_{b_{1,2}})_1 - (\vec{q}_{b_{1,2}})_2 = 1 - 1 = 0, \\ Q_3 &= (\vec{q}_{b_{1,2}})_3 - (\vec{q}_{b_{1,2}})_4 = 0 - 0 = 0, \\ Q_2 &\equiv Q = (\vec{q}_{b_{1,2}})_2 - (\vec{q}_{b_{1,2}})_3 = 1 - 0 = 1, \\ B &= \sum_j (\vec{q}_{b_{1,2}})_j = 1 + 1 + 0 + 0 = 2. \end{aligned} \quad (5.34)$$

Similarly, for the $b_{3,4}$ state, which is another component ($w^{33}w^{44} - w^{34}w^{43}$) of $\mathbf{6}_n \times \mathbf{6}_\rho$, we have

$$\begin{aligned} Q_1 &= (\vec{q}_{b_{3,4}})_1 - (\vec{q}_{b_{3,4}})_2 = 0 - 0 = 0, \\ Q_3 &= (\vec{q}_{b_{3,4}})_3 - (\vec{q}_{b_{3,4}})_4 = 1 - 1 = 0, \\ Q_2 &\equiv Q = (\vec{q}_{b_{3,4}})_2 - (\vec{q}_{b_{3,4}})_3 = 0 - 1 = -1, \\ B &= \sum_j (\vec{q}_{b_{3,4}})_j = 0 + 0 + 1 + 1 = 2. \end{aligned} \quad (5.35)$$

Hence, one gets here two baryons ($b_{1,2}$ and $b_{3,4}$) with charges $Q = \pm 1$ with respect to the $\tilde{a}_2$ -subgroup $SU(2)_{2,3} \subset SU(4)$, respectively. Their masses are

$$Z_{b_{1,2}} = i(m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2) \quad (5.36)$$

¹⁴Below in this section we present only some qualitative features of the hadron states in $SU(4)$ gauge theory. We postpone a more detailed analysis of this picture in a generic $SU(N)$ case for a separate publication.

and

$$Z_{b_{3,4}} = i(m_3 + m_4 - \tilde{m}_3 - \tilde{m}_4) \quad (5.37)$$

as expected.

Geometrically, all states with $\mathbf{u}_4 = (\mathfrak{s}\mathbf{u}_4)_d \oplus (\mathbf{u}_1)_B$ charges form an integer lattice $\mathbb{Z}^4 \subset \mathbb{R}^4$. The $(\mathfrak{s}\mathbf{u}_2)_{1,2} \times (\mathfrak{s}\mathbf{u}_2)_{3,4}$ singlet states we are looking for form a sublattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, generated by the (orthogonal) vectors $(\vec{e}_1 + \vec{e}_2, \vec{e}_3 + \vec{e}_4)$, i.e.,

$$\vec{q} = l\vec{q}_{b_{1,2}} + k\vec{q}_{b_{3,4}} = l(\vec{e}_1 + \vec{e}_2) + k(\vec{e}_3 + \vec{e}_4), \quad (5.38)$$

where l and k are non-negative integers, see Fig. 3. The baryonic charge of these states is measured by

$$B = \vec{e} \cdot \vec{q} = 2(k + l) \quad (5.39)$$

and the BPS masses of these stringy states are given by

$$\begin{aligned} Z_{\text{BPS}} &= i(\vec{m}_n - \vec{m}_\rho)\vec{q} \\ &= i[l(m_1 + m_2 - \tilde{m}_1 - \tilde{m}_2) + k(m_3 + m_4 - \tilde{m}_3 - \tilde{m}_4)], \\ l + k &= \frac{B}{2}. \end{aligned} \quad (5.40)$$

B. Number of states

Actually, the main observation we need from the previous Sec. VA is that under our mass deformation we always have $\tilde{m}_i = m_i$, and therefore the BPS parts of the masses of all these states in (5.1) vanish for the states satisfying the “conifold rule”. This is a simple consequence of the general formula (5.2), and one can easily see this effect in the considered above examples (5.36), (5.37) and for the generic case (5.40).

It actually means that the form of the spectrum (5.1) does *not* change as we reduce the mass deformation parameter $\mathcal{M}$. However, this is not so trivial, since the *number* of physical states changes for each fixed baryon charge B . Namely, at $\mathcal{M} \rightarrow \infty$, with the infinite mass barrier, certain quarks totally decouple from the theory. In fact, in this limit the theory factorizes into two noninteracting SQCDs as shown in (2.6) for $K = 2$, and there is no way to form a state with mixed $b_{1,2}$ and $b_{3,4}$ global charges. Hence, at $\mathcal{M} \rightarrow \infty$ we have only two states with central charge given by (5.40), corresponding to $l = B/2, k = 0$ and/or $l = 0, k = B/2$, while all other states are separated by infinite barriers. Instead, for finite $\mathcal{M}$ at each value of B more states are allowed, actually, at small $\mathcal{M}$, all of them with $l + k = B/2$, so that the whole “cone” in Fig. 3 with internal points, marked with red, appears in the spectrum, increasing the number of states with the same mass.

We also expect that, as we reduce $\mathcal{M}$, the system goes through a certain number of crossovers, similar to walls of marginal stability for BPS states, where the number of

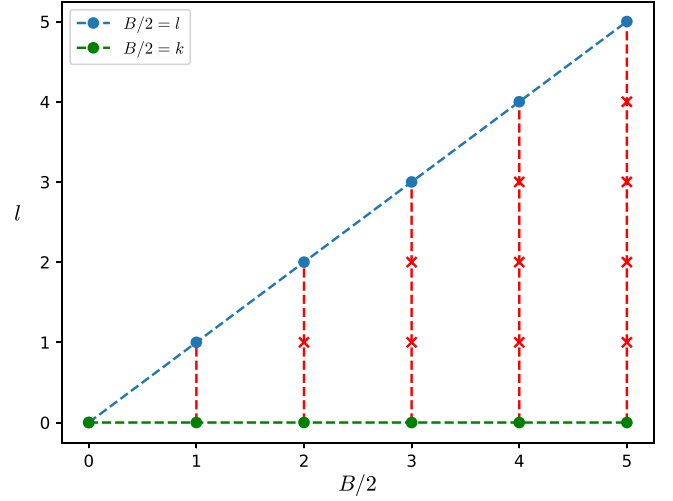


FIG. 3. States with different baryonic charges. The blue dashed line corresponds to $l = \frac{B}{2}, k = 0$, the green one to $l = 0, k = \frac{B}{2}$. Red crosses are new states that emerge as we reduce $\mathcal{M}$.

states jumps. In the next section we will see that this behavior is in qualitative agreement with results from string theory.

VI. NUMBER OF STATES FROM BLACK HOLE

A. Hagedorn behavior

Now let us return to the string theory described by $\mathcal{N} = 2$ Liouville world sheet model (3.9) with both Liouville superpotential and mass deformation switched on. As we discussed in Sec. III D, this theory is dual to the 2D black hole (3.10) with a cigar geometry. We also conjectured that the mass of the black hole is given by Eq. (3.36) and depends on both deformation parameters b and $\mathcal{M}$. We will show in this section that, although the string spectrum does not change as we reduce the mass deformation parameter $\mathcal{M}$ (which has a reasonable explanation on the field theory side, see the previous section), the number of states on each energy level increases.

One of the ways to find the number of states is to calculate the entropy of the system. In Euclidean formulation, the compact dimension of the target space can be interpreted as a temperature $T = (2\pi R)^{-1}$, where $R = \sqrt{2k}$ is the asymptotic radius of the cigar, see (3.10). We have to stress here that in our theory real time is one of the 4D Minkowski coordinates and has nothing to do with the compact dimension around the cigar, but we use the trick with the Euclidean 2D black hole to estimate the number of states in 4D theory.

This idea, however, immediately faces a problem. In string theory there is a limiting Hagedorn temperature T_H [56] beyond which the higher energy levels are no longer suppressed by the factor $\exp(-E/T)$ due to the exponential growth of the density of the string states

$$\omega(E) \sim \exp\left(\frac{E}{T_H}\right) \quad (6.1)$$

and the partition function

$$Z = \int_0^\infty dE \omega(E) e^{-\frac{E}{T}} \quad (6.2)$$

becomes divergent at $T \geq T_H$ (see [57] and references therein). Similarly, the exponential growth (6.1) leads to another unusual fact [58], that string theory belongs to a nonlocalizable class of theories where the Green functions can have singularities in a finite space-time domain (of the order of the string length).

In the Euclidean formulation, the Hagedorn behavior is described by the so-called thermal scalar, which is a winding string mode around the thermal circle [57]. As temperature approaches T_H this mode becomes massless and presumably becomes tachyonic at $T > T_H$, leading to an instability.

For the black holes we are interested in here, the Hagedorn behavior is related to the black hole/excited strings transition [59,60]. At low temperatures, we have a well-defined black hole geometry with small α' corrections. As the temperature grows above some critical value, the string's size exceeds its Schwarzschild radius and the black hole turns into an excited string [60].

Similar behavior was found for the 2D black hole with the linear dilaton (3.10) in [61]. As we reduce k (increase temperature), the theory enters a strong coupling regime where $\alpha' \sim 1/k$ corrections grow, and below some critical value k_c , the black hole, as a geometric object, no longer exists. Moreover, it is argued in [61] that the critical value $k_c = 1$ is equal to unity, i.e., exactly the value we are interested in for the critical non-Abelian string in 4D SQCD. At this value, the thermal winding scalar becomes massless [61,62], see also [63] for the nonsupersymmetric version. Below, in this section, we consider k above the critical value $k = 1$ to regularize the theory. We use the target-space effective action to calculate the entropy, associated with the thermal scalar, and find its dependence on the mass deformation parameter $\mathcal{M}$.

B. Thermal scalar

The thermal scalar for the 2D black hole (3.10) is our (massless in 4D) b -baryon, associated with the conifold complex structure parameter b . It corresponds to the minimal winding $n_2 = \pm 1$ state, see (2.23) [57], i.e., its quantum numbers are

$$m = \pm \frac{k}{2}, \quad j = -\frac{k}{2}. \quad (6.3)$$

This state belongs to the discrete spectrum, see Sec. II C; for $k > 1$ this state is normalizable and for $k = 1$ it is on the

borderline between normalizable and non-normalizable modes. It is seen from the asymptotic behavior,¹⁵

$$\Psi_b^{(t)}(\rho) \underset{\rho \rightarrow \infty}{\approx} e^{2(j+\frac{1}{2})\rho} = e^{-(k-1)\rho}, \quad (6.4)$$

of the wave functions $\Psi_b^{(t)}(\rho) = \Psi_{-k/2, \pm k/2}^{(t)}(\rho)$, see (4.14). It shows that at $k \rightarrow 1$ the thermal scalar becomes massless (with respect to the ρ dimension), but never becomes tachyonic, $-E_j^{(t)} \geq 0$, see (4.13).

Its effective theory can be described by the tachyonic action (4.1) in the supergravity background (3.1). For the tachyon (2.31) with quantum numbers (6.3) on mass-shell, when the conformal dimension (2.24) satisfies

$$2\Delta_{j,m}(T_b) = 2\Delta_{-k/2, \pm k/2} = 1, \quad (6.5)$$

the action (4.1) can be reduced to 2D Euclidean theory,

$$S_b^E = \frac{1}{2\tilde{\kappa}^2} \int d\rho d\vartheta \sqrt{G} e^{-2\Phi} \{g^{nn'} \partial_n \bar{T}_b \partial_{n'} T_b - |T_b|^2\}, \quad (6.6)$$

where $\frac{1}{2\tilde{\kappa}^2} = \frac{V_{4D}}{2\kappa^2} = \frac{V_{4D}}{(2\pi)^3 \alpha'^2}$ is proportional to the volume of the 4D Minkowski space, while the string coupling $g_s^2 = \exp(2\Phi_0)$ is included into $\exp(-2\Phi)$.

Since our thermal scalar (b -baryon) is a winding mode in cigar theory, (6.6) is written in the T -dual trumpet metric (3.24),

$$ds^2 = 2k(d\rho^2 + \coth^2 \rho d\vartheta^2), \quad (6.7)$$

with the dilaton (3.26), see [26,27].

To calculate the entropy of the black hole we use the formula for the (dominant) part of the black hole entropy, associated with the thermal scalar

$$s = \left(1 - R \frac{\partial}{\partial R}\right) \log Z \approx \left(R \frac{\partial}{\partial R} - 1\right) S_b^E, \quad (6.8)$$

in the quasiclassical limit (see review [65] for a recent discussion). Here $R = \sqrt{2k}$ is the radius of the cigar at $\rho \rightarrow \infty$, proportional to the inverse Hawking temperature. However, one should use the formula (6.8) with great care, since changing the level k not only deforms the radius of the “temperature circle” but also many properties of the whole theory, in particular it rescales the radial direction in the background (6.7), see [64] for the discussion of this peculiarity for the case of the nonsupersymmetric 2D black hole.

In order to avoid this problem, we first take the action (6.6) after the substitution

¹⁵This behavior was found in [62], see also [63,64] for the nonsupersymmetric version.

$$T_b(\rho, \theta) = e^{2im\theta} \mathbb{T}(\rho) = e^{\pm i \frac{R^2}{2} \theta} \mathbb{T}(\rho), \quad (6.9)$$

together with (k -dependent!) metric (6.7) and dilaton (3.26). After the integration over the compact direction $\vartheta \sim \vartheta + \frac{4\pi}{R^2}$ [we separate the radius and the level k for a moment, cf. with (3.25)] one gets

$$S_b^E = \frac{2\pi e^{-2\Phi_0} k}{\tilde{\kappa}^2 R^2} \int_0^\infty d\rho \sinh(2\rho) \times \left\{ \frac{1}{2k} (\partial_\rho \mathbb{T})^2 + \left(\frac{R^4}{8k} \tanh^2 \rho - 1 \right) \mathbb{T}^2 \right\}. \quad (6.10)$$

To find the entropy one substitutes (6.10) into (6.8), where the derivative is taken only of the explicit R -dependence, since the possible extra terms, coming from the derivatives of the fields, vanish, being proportional to the equations of motion,¹⁶ and we keep k constant when taking R -derivatives, putting $R = \sqrt{2k}$ only at the end. This leads to

$$s = \left(R \frac{\partial}{\partial R} - 1 \right) S_b^E \Big|_{R=\sqrt{2k}} = \frac{\pi e^{-2\Phi_0}}{\tilde{\kappa}^2} \int_0^\infty d\rho \sinh(2\rho) \times \left\{ -\frac{3}{2k} (\partial_\rho \mathbb{T})^2 + \left(\frac{k}{2} \tanh^2 \rho + 3 \right) \mathbb{T}^2 \right\}. \quad (6.11)$$

Substituting here $\mathbb{T}(\rho) = \frac{\Psi_{-k/2, \pm k/2}(\rho)}{\sqrt{\sinh 2\rho}} = \frac{1}{\sqrt{2} \cosh^k \rho}$ and computing the integral, one finds

$$\begin{aligned} \frac{(2\pi\alpha')^2}{V_{4D}} s &= \frac{e^{-2\Phi_0}}{k-1} = \frac{1}{k-1} (e^{-2\Phi_0^{(b)}} + e^{-2\Phi_0^{(\mathcal{M})}}) \\ &= \frac{\sqrt{2k}}{k-1} M_{\text{BH}}^{\text{total}}, \end{aligned} \quad (6.12)$$

where in the last two equalities we have used (3.35) and (3.36). Hence, we see that the entropy (6.12) diverges at $k \rightarrow 1$, a similar result for the entropy of the nonsupersymmetric 2D black hole was obtained in [64]. This divergence comes from the infrared region $\rho \rightarrow \infty$, due to asymptotic behavior (6.4), leading to explosion of the norm $\int d\rho \sinh(2\rho) |\mathbb{T}|^2 = \int d\rho |\Psi_b|^2$ of the thermal scalar wave function at $k \rightarrow 1$.

The rest of the contributions into the black hole entropy (not associated with the thermal scalar) are also proportional to $M_{\text{BH}}^{\text{total}} = \frac{S_{\text{BH}}^{\text{total}}}{2\pi\sqrt{2k}}$, but give subleading terms in

¹⁶For the thermal scalar $\mathbb{T}$ equation of motion is equivalent to (4.11) at $j = -k/2$, $m = \pm k/2$ under $\mathbb{T}(\rho) = \frac{\Psi_{-k/2, \pm k/2}(\rho)}{\sqrt{\sinh 2\rho}}$, while in the equations of motion for the background (6.7) and (3.26) we neglect the contribution of the field $\bar{\mathbb{T}}$.

the limit $k \rightarrow 1$. As argued in [61,62] the critical value $k_H = 1$ is related to the black hole/excited strings phase transition.

Note that the Hagedorn temperature in flat space in the supersymmetric case corresponds to $k_H^{\text{flat}} = 2$, see [53] for a review. Still, in the cigar geometry one can increase the temperature till k reaches the critical value $k_H = 1$ from above, similar behavior for the nonsupersymmetric case was discussed in [64].

Observe now that due to (3.36) at the initial stage of the mass deformation at $\mathcal{M} \rightarrow \infty$ the entropy (6.12) does not really feel the mass parameter, being just $s \sim |b|^2 (k-1)^{-1}$. However, at small $\mathcal{M} \rightarrow 0$ the black hole mass increases due to the second term in (3.36), giving a “second divergence” to the entropy, becoming

$$\frac{(2\pi\alpha')^2}{V_{4D}} s \sim \frac{1}{k-1} \frac{|b|^2}{\mathcal{M}^2}. \quad (6.13)$$

Hence, we always meet the Hagedorn behavior of excited strings, which implies that all high energy levels are filled now in the system. However, we also see that the number of states (finite at each level with fixed energy or baryonic charge B) increases dramatically as we reduce $\mathcal{M}$. This qualitatively confirms our expectations on the field theory side. Our 4D $\mathcal{N} = 2$ supersymmetric theory interpolates from SQCD with the gauge group $U(2)$ and $N_f = 4$ to SQCD with the gauge group $U(4)$ and $N_f = 8$ as we reduce the mass deformation parameter $\mathcal{M}$, and one naturally expects more hadronic states to appear in a theory with more quark matter multiplets.

VII. CONCLUSIONS

In this work we examined a particular mass deformation of $\mathcal{N} = 2$ SQCD with gauge group $U(4)$ and $N_f = 8$ hypermultiplets of fundamental quarks, introducing a bare mass $\mathcal{M}$ for half of the quark flavors, while keeping the other half without a mass term. At large $\mathcal{M}$, this theory splits into two non-interacting sectors, each representing an SQCD with two colors and four flavors. As reviewed in Sec. IID, the hadron spectrum of the latter theory is known from the string theory of the critical non-Abelian string [13,20].

Perhaps our most surprising result is that the spectrum of string states of the mass-deformed theory does *not* change when we reduce the mass parameter $\mathcal{M}$ interpolating to SQCD with more quark flavors. In particular, the low-lying spectrum of hadrons in 4D SQCD is still given by (2.34) and (2.37) and is actually independent of $\mathcal{M}$.

On the string theory side, this fact is established with the help of the deformed $\mathcal{N} = 2$ Liouville theory (3.9) on the world sheet of the non-Abelian strings supported in $\mathcal{N} = 2$ SQCD. First, we proved that the mass-deformed theory with the Liouville superpotential dropped in (3.9) is in fact

T -dual to the $\mathcal{N} = 2$ black hole with the same cigar geometry. Then, combining mirror and T -duality arguments, we concluded that the theory (3.9) with both deformations switched on is actually dual to the 2D black hole with increasing mass, determined by the deformation parameters. We also used field theory arguments on the SQCD side to explain this surprising behavior of the string spectrum.

Nevertheless, as a result of the deformation, the multiplicities of the hadron SQCD states with given baryonic charges increase as we reduce $\mathcal{M}$ and interpolate to SQCD with more quarks. We estimated the entropy of the resulting black hole near the Hagedorn transition, showing that the string theory results qualitatively confirm our field theory expectations.

We have also derived the string spectrum, associated with the mass-deformed Liouville theory, explicitly solving the equations for vertex operators' wave functions in the gravity background. Our result for the reflection coefficient (which fixes the discrete spectrum) coincides with the known exact result for the 2D black hole in the large k limit. This matches our conclusion that the two theories are just related by T -duality. It also shows that the effective gravity approach for the reflection coefficient misses the $\alpha' \sim 1/k$ correction even in the supersymmetric case, where it reproduces the metric and dilaton exactly.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: THE ANALYTIC CONTINUATION OF THE SINGULAR SOLUTION AT $z=0$ TO $z=-\infty$

The second, singular at $z = 0$ solution to Eq. (4.23) takes the form

$$w_{\sin g} = F(\alpha, \beta, 1, z) \log z + \sum_{k=1}^{\infty} (z)^k \frac{(\alpha)_k (\beta)_k}{(k!)^2} \{ \psi(\alpha + k) - \psi(\alpha) + \psi(\beta + k) - \psi(\beta) - 2\psi(k+1) + 2\psi(1) \}, \quad (\text{A1})$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol.

To make an analytic continuation of this solution to infinity, consider Eq. (9.131.2) from [55] and apply it to our case $\gamma - \alpha - \beta = \epsilon$ with $\epsilon \rightarrow 0$,

$$\begin{aligned} F(\alpha, \beta, \gamma; 1-z) &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha, \beta, \alpha+\beta-\gamma+1; z) + z^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1; z) \\ &= \lim_{\epsilon \rightarrow 0} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} (\Gamma(\epsilon) F(\alpha, \beta, 1-\epsilon; z) + z^\epsilon \Gamma(-\epsilon) F(\beta, \alpha, 1+\epsilon; z)) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} F(\alpha, \beta, 1; z) \lim_{\epsilon \rightarrow 0} (\Gamma(\epsilon) + z^\epsilon \Gamma(-\epsilon)) \\ &= -\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} F(\alpha, \beta, 1; z) (2\gamma_e + \ln z), \end{aligned} \quad (\text{A2})$$

where α and β are given by (4.25), $\gamma = 1-2m$, while γ_e is the Euler's constant. In (A2), when moving to line four we use the symmetry of the hypergeometric function under interchange of its first two parameters (α and β), and in the last line we use the following expansion:

$$\Gamma(\epsilon) + (z)^\epsilon \Gamma(-\epsilon) \stackrel{\epsilon \rightarrow 0}{\approx} - (2\gamma_e + \ln z) + O(\epsilon). \quad (\text{A3})$$

Hence, neglecting the $\sim \gamma_e$ term, from (A2) we get

$$F(\alpha, \beta, 1; z) \ln z \approx -\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma; 1-z). \quad (\text{A4})$$

This means that the singular solution at $z = 0$ behaves as the regular one at $z = 1$. Using this property, we make an analytic continuation to infinity,

$$w_{\text{sing}} \approx -\frac{\Gamma(-j-m)\Gamma(1+2j)}{\Gamma(1+j-m)} (z-1)^m \times \left[(z-1)^j F\left(-j-m, -j+m, -2j; \frac{1}{1-z}\right) + R_{\text{sing}} (z-1)^{-1-j} F\left(1+j-m, 1+j+m, 2+2j; \frac{1}{1-z}\right) \right], \quad (\text{A5})$$

where

$$R_{\text{sing}} = \frac{\Gamma(-1-2j)\Gamma^2(1+j-m)}{\Gamma(1+2j)\Gamma^2(-j-m)}. \quad (\text{A6})$$

Obviously, R_{sing} has no symmetry under the replacement $m \rightarrow -m$, which contradicts our condition $m \equiv m_L = -m_R$, see footnote 4 on page 18.

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- [1] A. Hanany and D. Tong, Vortices, instantons and branes, *J. High Energy Phys.* **07** (2003) 037.
 - [2] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Non-Abelian superconductors: Vortices and confinement in $\mathcal{N} = 2$ SQCD, *Nucl. Phys.* **B673**, 187 (2003).
 - [3] M. Shifman and A. Yung, Non-Abelian string junctions as confined monopoles, *Phys. Rev. D* **70**, 045004 (2004).
 - [4] A. Hanany and D. Tong, Vortex strings and four-dimensional gauge dynamics, *J. High Energy Phys.* **04** (2004) 066.
 - [5] D. Tong, TASI lectures on solitons, *arXiv:hep-th/0509216*.
 - [6] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Solitons in the Higgs phase: The moduli matrix approach, *J. Phys. A* **39**, R315 (2006).
 - [7] M. Shifman and A. Yung, Supersymmetric solitons and how they help us understand non-Abelian gauge theories, *Rev. Mod. Phys.* **79**, 1139 (2007). for an expanded version see *Supersymmetric Solitons*, (Cambridge University Press, Cambridge, England, 2009).
 - [8] D. Tong, Quantum vortex strings: A review, *Ann. Phys. (Amsterdam)* **324**, 30 (2009).
 - [9] M. Shifman and A. Yung, Critical string from non-Abelian vortex in four dimensions, *Phys. Lett. B* **750**, 416 (2015).
 - [10] P. Koroteev, M. Shifman, and A. Yung, Non-Abelian vortex in four dimensions as a critical string on a conifold, *Phys. Rev. D* **94**, 065002 (2016).
 - [11] P. Candelas and X. C. de la Ossa, Comments on conifolds, *Nucl. Phys.* **B342**, 246 (1990).
 - [12] A. Neitzke and C. Vafa, Topological strings and their physical applications, *arXiv:hep-th/0410178*.
 - [13] M. Shifman and A. Yung, Critical non-Abelian vortex in four dimensions and little string theory, *Phys. Rev. D* **96**, 046009 (2017).
 - [14] D. Kutasov, Introduction to Little String Theory, in *Superstrings and Related Matters 2001, Proceedings of the ICTP Spring School of Physics*, Edited by C. Bachas, K. S. Narain, and S. Randjbar-Daemi (2002), pp. 165–209.
 - [15] D. Ghoshal and C. Vafa, $c = 1$ string as the topological theory of the conifold, *Nucl. Phys.* **B453**, 121 (1995).
 - [16] E. Ivanov and S. Krivonos, U(1) supersymmetric extension of the Liouville equation, *Lett. Math. Phys.* **7**, 523 (1983).
 - [17] D. Kutasov and N. Seiberg, Noncritical superstrings, *Phys. Lett. B* **251**, 67 (1990).
 - [18] A. Giveon and D. Kutasov, Little string theory in a double scaling limit, *J. High Energy Phys.* **10** (1999) 034.
 - [19] A. Giveon, D. Kutasov, and O. Pelc, Holography for noncritical superstrings, *J. High Energy Phys.* **10** (1999) 035.
 - [20] M. Shifman and A. Yung, Hadrons of $\mathcal{N} = 2$ supersymmetric QCD in four dimensions from little string theory, *Phys. Rev. D* **98**, 085013 (2018).
 - [21] P. Gavrylenko, E. Ievlev, A. Marshakov, I. Monastyrskii, and A. Yung, 2D sigma models on non-compact Calabi-Yau and $N = 2$ Liouville theory, *Phys. Rev. D* **111**, 106003 (2025).
 - [22] A. Yung, Flowing between string vacua for the critical non-Abelian vortex with a deformation of $N = 2$ Liouville theory, *Phys. Rev. D* **110**, 025004 (2024).
 - [23] E. Witten, String theory and black holes, *Phys. Rev. D* **44**, 314 (1991).
 - [24] S. Mukhi and C. Vafa, Two-dimensional black hole as a topological coset model of $c = 1$ string theory, *Nucl. Phys.* **B407**, 667 (1993).
 - [25] H. Ooguri and C. Vafa, Two-dimensional black hole and singularities of CY manifolds, *Nucl. Phys.* **B463**, 55 (1996).

- [26] A. Giveon, Target space duality and stringy black holes, *Mod. Phys. Lett. A* **06**, 2843 (1991).
- [27] R. Dijkgraaf, H. Verlinde, and E. Verlinde, String propagation in a black hole geometry, *Nucl. Phys.* **B371**, 269 (1992).
- [28] K. Hori and A. Kapustin, Duality of the fermionic 2-D black hole and $N = 2$ Liouville theory as mirror symmetry, *J. High Energy Phys.* **08** (2001) 045.
- [29] P. Fayet and J. Iliopoulos, Spontaneously broken supergauge symmetries and goldstone spinors, *Phys. Lett.* **51B**, 461 (1974).
- [30] M. Shifman and A. Yung, Lessons from supersymmetry: “Instead-of-Confinement” mechanism, *Int. J. Mod. Phys. A* **29**, 1430064 (2014).
- [31] E. Ievlev, M. Shifman, and A. Yung, String baryon in four-dimensional $N = 2$ supersymmetric QCD from the 2D-4D correspondence, *Phys. Rev. D* **102**, 054026 (2020).
- [32] For a review see e.g. A. Achucarro and T. Vachaspati, Semilocal and electroweak strings, *Phys. Rep.* **327**, 347 (2000).
- [33] M. Shifman and A. Yung, Non-Abelian semilocal strings in $\mathcal{N} = 2$ supersymmetric QCD, *Phys. Rev. D* **73**, 125012 (2006).
- [34] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, On the moduli space of semilocal strings and lumps, *Phys. Rev. D* **76**, 105002 (2007).
- [35] M. Shifman, W. Vinci, and A. Yung, Effective world-sheet theory for non-Abelian semilocal strings in $\mathcal{N} = 2$ supersymmetric QCD, *Phys. Rev. D* **83**, 125017 (2011).
- [36] E. Witten, Phases of $N = 2$ theories in two dimensions, *Nucl. Phys.* **B403**, 159 (1993).
- [37] N. Seiberg and E. Witten, Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD, *Nucl. Phys.* **B431**, 484 (1994).
- [38] N. Dorey, The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms, *J. High Energy Phys.* **11** (1998) 005.
- [39] N. Dorey, T. J. Hollowood, and D. Tong, The BPS spectra of gauge theories in two and four dimensions, *J. High Energy Phys.* **05** (1999) 006.
- [40] E. Ievlev and A. Yung, Critical non-Abelian vortex and holography for little string theory, *Phys. Rev. D* **104**, 114033 (2021).
- [41] Y. Nakayama, Liouville field theory: A decade after the revolution, *Int. J. Mod. Phys. A* **19**, 2771 (2004).
- [42] J. Teschner, Operator product expansion and factorization in the $H + (3)$ WZNW model, *Nucl. Phys.* **B571**, 555 (2000).
- [43] O. Aharony, A. Giveon, and D. Kutasov, LSZ in LST, *Nucl. Phys.* **B691**, 3 (2004).
- [44] P. Baseilhac and V. Fateev, Expectation values of local fields for a two-parameter family of integrable models and related perturbed conformal field theories, *Nucl. Phys.* **B532**, 567 (1998).
- [45] L. J. Dixon, M. E. Peskin, and J. D. Lykken, $N = 2$ superconformal symmetry and $SO(2,1)$ current algebra, *Nucl. Phys.* **B325**, 329 (1989).
- [46] P. M. S. Petropoulos, Comments on $SU(1,1)$ string theory, *Phys. Lett. B* **236**, 151 (1990).
- [47] S. Hwang, Cosets as gauge slices in $SU(1,1)$ strings, *Phys. Lett. B* **276**, 451 (1992).
- [48] J. M. Evans, M. R. Gaberdiel, and M. J. Perry, The no ghost theorem for $AdS(3)$ and the stringy exclusion principle, *Nucl. Phys.* **B535**, 152 (1998).
- [49] J. M. Evans, M. R. Gaberdiel, and M. J. Perry, The no-ghost theorem and strings on AdS_3 , in *Proceedings of the 1998 ICTP Spring School of Physics, Nonperturbative Aspects of Strings, Branes and Supersymmetry*, Edited by M. J. Duff *et al.* (1998), pp. 435–444.
- [50] A. Strominger, Massless black holes and conifolds in string theory, *Nucl. Phys.* **B451**, 96 (1995).
- [51] T. Eguchi and Y. Sugawara, D-branes in singular Calabi-Yau manifold and $N = 2$ Liouville theory, *Nucl. Phys.* **B598**, 467 (2001).
- [52] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, An introduction to T duality in string theory, *Nucl. Phys. B, Proc. Suppl.* **41**, 1 (1995).
- [53] T. G. Mertens, Hagedorn string thermodynamics in curved spacetimes and near black hole horizons, Ph.D. thesis, Gent University, 2015.
- [54] J. Polchinski, Critical behavior of random surfaces in one dimension, *Nucl. Phys.* **B346**, 253 (1990).
- [55] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, translated from Russian (Elsevier/Academic Press, Amsterdam, 2007), 7th ed.
- [56] R. Hagedorn, Statistical thermodynamics of strong interactions at high-energies, *Nuovo Cimento Suppl.* **3**, 147 (1965).
- [57] J. J. Atick and E. Witten, The Hagedorn transition and the number of degrees of freedom of string theory, *Nucl. Phys.* **B310**, 291 (1988).
- [58] V. Fainberg and A. Marshakov, A propagator for the fermionic string, *Phys. Lett. B* **211**, 81 (1988).
- [59] L. Susskind, Some speculations about black hole entropy in string theory, [arXiv:hep-th/9309145](https://arxiv.org/abs/hep-th/9309145).
- [60] G. T. Horowitz and J. Polchinski, A correspondence principle for black holes and strings, *Phys. Rev. D* **55**, 6189 (1997).
- [61] A. Giveon, D. Kutasov, E. Rabinovici, and A. Sever, Phases of quantum gravity in AdS_3 and linear dilaton backgrounds, *Nucl. Phys.* **B719**, 3 (2005).
- [62] D. Kutasov, Accelerating branes and the string/black hole transition, [arXiv:hep-th/0509170](https://arxiv.org/abs/hep-th/0509170).
- [63] V. Kazakov, I. K. Kostov, and D. Kutasov, A matrix model for the two-dimensional black hole, *Nucl. Phys.* **B622**, 141 (2002).
- [64] Y. Chen and J. Maldacena, String scale black holes at large D , *J. High Energy Phys.* **01** (2022) 095.
- [65] E. Witten, Introduction to black hole thermodynamics, [arXiv:2412.16725](https://arxiv.org/abs/2412.16725).