Generalized collective coordinate quantization of solitons with topological terms

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For the solitons of the theory including topological terms, the standard collective coordinate quantization does not construct the proper low energy effective theory. In the presence of topological terms, zero modes are classified into three groups: dynamical zero modes with constant velocity, cyclotron zero modes in a circular orbit, and static constraint zero modes. According to the mode expansion, the topological term contributes to the moduli metric of the dynamical zero modes. In addition, constraint zero modes do not have the kinetic term and produce constraints that strongly restrict the spectrum instead. In this paper, we propose a generalized collective coordinate quantization method and apply it to the $SU(3)$ Skyrmion with the Wess-Zumino-Witten term. We find five constraints. These reproduce the results of Cherman *et al.* [The collective quantization of three-flavored skyrmions revisited, Phys. Rev. D **72**, 094015 (2005)], eliminating all the $SU(3)$ multiplets except the nucleon octet and Δ baryon decuplet from the collective coordinate part of the spectrum.

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I. INTRODUCTION

One of the basic methods to analyze the dynamics of solitons is the collective coordinate quantization (CCQ) [1]. Generally, solitons are parametrized by several collective coordinates, and the set of solitons with the same energy is called the moduli space. Collective coordinates are related to the zero modes of the fluctuations around the soliton and are important in the low energy dynamics. In the CCQ method, the low energy effective theory is constructed by introducing the time-dependent collective coordinates to the soliton and putting them into the action. Solitons are described as nonrelativistic particles slowly moving in the moduli space in this approach.

However, the analysis based on the CCQ might not be reliable for solitons with topological terms, especially the chiral soliton models in quantum chromodynamics (QCD). One example is the chiral soliton model with ρ and ω mesons with the Wess-Zumino-Witten (WZW) term in two flavors shown in [3]. Solitons are interpreted as baryons and have $SU(2)$ collective coordinates that correspond to the spin/isospin degrees of freedom [4]. In a classical soliton configuration, the space components of ω meson are in pure gauge. When promoting the time-dependent collective coordinates, however, the space components of the ω meson field strength become nonzero valued proportional to the angular velocity since the topological charge behaves as a $U(1)$ charge of the ω meson. Therefore, the momentum of inertia has the contribution from ω. The same phenomenon can be seen in holographic QCD [5].

Another example is the $SU(3)$ Skyrmion with the WZW term. According to the naive CCQ, a constraint arises from the WZW term and indicates that the baryon state with baryon number $N_B = 1$ must have the particle with hypercharge $Y = N_c/3$ in the representation [6–8]. In $N_c = 3$ case, the allowed multiplets are 8, 10, $\overline{10}$, $\cdots$. 8 and 10 include nucleons and Δ baryons, respectively, and $\overline{10}$ predicts exotic hadrons such as θ^+ pentaquark. θ^+ is the particle with spin $\frac{1}{2}$, isospin 0, and strangeness $+1$. Although many experiments were held to search θ^+, only a few reported the existence of this particle [9].

On the theoretical side, the existence of θ^+ depends on the models or methods used to analyze the baryon spectrum. In the bound state approach (BSA) for the Skyrme model, no bound states or resonances are found that can be interpreted as θ^+ [12–14]. Cherman *et al.* pointed out that some zero modes of the Skyrmion are not dynamical [15]. Distinguishing dynamical and static zero modes, they found the upper bound on the hypercharge of the baryons classically that prohibits all the exotic baryons including θ^+ from the collective excitations. Opposing them, Walliser and Weigel introduced the rotation-vibration approach (RVA), which is the combination of BSA and the naive CCQ, and calculated the mass and the width of θ^+ [16].

In this paper, we improve the CCQ method in the presence of the topological term by solving the equation

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of motion (EOM) of the fluctuation around the soliton. It turns out that zero modes are classified into three groups due to the form of the solution of the EOM. In addition, the zero mode action and the moduli metric constructed from the mode expansion do not coincide with those derived from the naive CCQ. Our approach extends [15] to be able to quantize properly.

The organization of the paper is as follows. First, we discuss the spectrum of the fluctuation around the soliton in Sec. II, and see that there are several types of zero modes. In Sec. III, we construct the zero mode action, which describes the low energy dynamics of solitons correctly, by the mode expansion. The generalization of the CCQ method is also introduced in this section. This is applied to the $SU(3)$ Skyrme-WZW model in Sec. IV, and we show that θ^+ do not arise as the collective excitation, which reproduces the result of [15]. We also discuss the relation and consistency between BSA, RVA, and the generalized CCQ in this section. Section V is devoted to the conclusion and discussion of possible future directions.

II. THE SPECTRUM OF THE FLUCTUATION

A. The quadratic Lagrangian

The basic idea of the collective coordinate quantization is the quantum theory of fluctuations around the soliton solution. Consider an action of bosonic variables

$$S[\phi] = \int \mathrm{d}t L(\phi(t), \dot{\phi}(t)), \qquad \phi = \{\phi^x\}. \tag{1}$$

Here, x is the index of ϕ. In quantum field theory (QFT), x denotes the spatial coordinates and indices of internal symmetries, for example. The EOM is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{\phi}} - \frac{\delta L}{\delta \phi} = 0. \tag{2}$$

Assume that this has the static solution $\phi = \phi_{\mathrm{sol}}$, $\dot{\phi}_{\mathrm{sol}} = 0$. The static EOM is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{\phi}} = \frac{\delta L}{\delta \phi} = 0. \tag{3}$$

Note that the solution of the static EOM (3) with given energy is not unique in general and has several parameters. In QFT, these parameters are collective coordinates and the space of collective coordinates is called the moduli space of solitons. For example, the collective coordinate and the moduli space of a kink on a $1+1$ dimensional spacetime is the position of the kink $\mathbb{R}$.

Let the fluctuation around the static solution be ϕ_{sol} as $q(t) \coloneqq \phi(t) - \phi_{\mathrm{sol}}$. Substituting into the action, we have

$$\begin{aligned} &S[\phi_{\mathrm{sol}} + q] - S[\phi_{\mathrm{sol}}] \\ &\simeq \int \mathrm{d}t \left[\frac{1}{2}(\dot{q}^T M \dot{q} - q^T K q + q^T (A+B)\dot{q}) + P_0^T \dot{q}\right], \end{aligned} \tag{4}$$

in the leading order of q. Here, M, K, A, B, and P_0 are the variations of the Lagrangian

$$M_{xy} \coloneqq \frac{\delta^2 L}{\delta\dot{\phi}^x \delta\dot{\phi}^y}(\phi_{\mathrm{sol}}, 0), \tag{5a}$$

$$K_{xy} \coloneqq -\frac{\delta^2 L}{\delta\phi^x \delta\phi^y}(\phi_{\mathrm{sol}}, 0), \tag{5b}$$

$$A_{xy} \coloneqq \frac{\delta^2 L}{\delta\phi^x \delta\dot{\phi}^y}(\phi_{\mathrm{sol}}, 0) + \frac{\delta^2 L}{\delta\dot{\phi}^x \delta\phi^y}(\phi_{\mathrm{sol}}, 0), \tag{5c}$$

$$B_{xy} \coloneqq \frac{\delta^2 L}{\delta\phi^x \delta\dot{\phi}^y}(\phi_{\mathrm{sol}}, 0) - \frac{\delta^2 L}{\delta\dot{\phi}^x \delta\phi^y}(\phi_{\mathrm{sol}}, 0), \tag{5d}$$

$$(P_0)_x \coloneqq \frac{\delta L}{\delta\dot{\phi}^x}(\phi_{\mathrm{sol}}, 0), \tag{5e}$$

respectively. By definition, M, A, and K are symmetric and B is an antisymmetric operator. To guarantee the stability of ϕ_{sol} and the nonexistence of gauge degrees of freedom or constraints, we assume that M is positive definite and K is positive semidefinite. In the standard soliton models such as kinks [17] or $SU(2)$ Skyrmions [4], there are only M and K terms, which are the kinetic term and harmonic potential term of q, respectively. A, B, and P_0 terms come from the term with the first order of time derivatives in the original theory (1). These terms behave as the constant magnetic field in the q target space. In QFT, they appear in the presence of metric-independent topological terms such as θ terms, WZW terms, and Chern-Simons terms.

The EOM from (4) is

$$M\ddot{q} - B\dot{q} + Kq = 0. \tag{6}$$

Note that P_0 and the symmetric A do not contribute to the EOM, since $P_0^T\dot{q}$ and $q^T A\dot{q}$ are actually total derivative terms. For simplicity, we will neglect these terms.

The EOM (6) is a linear differential equation with constant coefficients. It is convenient to substitute $q = e^{-i\omega t}\xi$ into (6) and we have

$$(-\omega^2 M + i\omega B + K)\xi = 0. \tag{7}$$

For this equation to have a nontrivial "eigenvector" ξ, ω must be the root of the following characteristic equation

$$\Phi(\omega) \coloneqq \det(-\omega^2 M + i\omega B + K) = 0. \tag{8}$$

When $B = 0$, this is merely an eigenvalue problem of $M^{-1}K$. We will mention some properties of (7) and (8). First, $\Phi(\omega)$ is an even polynomial of ω. It is shown by

$$\begin{aligned}\Phi(\omega) &= \det(-\omega^2 M + i\omega B + K) \\ &= \det((-\omega^2 M + i\omega B + K)^T) \\ &= \det(-\omega^2 M - i\omega B + K) = \Phi(-\omega). \end{aligned} \tag{9}$$

In addition, the roots of $\Phi(\omega)$ are real. Let (ω, ξ) be the solution of (7) and (8). Taking the inner product of $\xi^\dagger$ and (7),

$$-\omega^2 \xi^\dagger M \xi + i\omega \xi^\dagger B \xi + \xi^\dagger K \xi = 0. \tag{10}$$

The discriminant of this equation is

$$-(\xi^\dagger B \xi)^2 + 4(\xi^\dagger M \xi)(\xi^\dagger K \xi).$$

Since M and K are positive semidefinite and B is antisymmetric, $\xi^\dagger M \xi$ and $\xi^\dagger K \xi$ are non-negative and $\xi^\dagger B \xi$ is pure imaginary. Thus, the discriminant is always non-negative and ω is real. Taking the complex conjugate of (7), we have

$$(-\omega^2 M - i\omega B + K)\xi^* = 0. \tag{11}$$

Therefore, $(-\omega, \xi^*)$ also satisfies (7) and (8).

Under the mode expansion, zero modes $\omega = 0$ play the central role in CCQ. Zero modes are characterized by $Kq = 0 \Leftrightarrow q = \ker K$. The constant fluctuations along the $\ker K$ direction do not change the energy of $\phi_{\rm sol}$. Thus, zero modes correspond to the infinitesimal transformation of collective coordinates of $\phi_{\rm sol}$, and $\dim \ker K$ is equal to the dimension of the moduli space. We denote the indices of the collective coordinates and the basis of $\ker K$ by $a, b, \cdots$, and $\{X^a\}$, $\{\xi_a\}$, respectively. We can take the basis $\{\xi_a\}$ as

$$\xi_a = \frac{\partial \phi_{\rm sol}}{\partial X^a}, \tag{12}$$

for example.

B. Contradiction of the naive collective coordinate quantization

Now we discuss the naive CCQ to show that this method has a critical contradiction. In a naive CCQ, the effective action is acquired by introducing the time-dependent collective coordinates $\phi = \phi_{\rm sol}(X^a(t))$ and putting them into the original action. This corresponds to assuming $q(t)$ takes the form

$$q(t) = \sum_a X^a(t) \frac{\partial \phi_{\rm sol}}{\partial X^a} = \sum_a X^a(t) \xi_a. \tag{13}$$

Putting it into (4), we can evaluate the naive CCQ action $S_{\rm naive}[X^a] := S[\phi_{\rm sol}(X^a(t))]$ as

$$S_{\rm naive}[X^a] \simeq \int {\rm d}t \frac{1}{2} \sum_{a,b} (M_{ab} \dot{X}^a \dot{X}^b + B_{ab} X^a \dot{X}^b), \tag{14}$$

where $M_{ab} := \xi_a^T M \xi_b$, $B_{ab} := \xi_a^T B \xi_b$. The EOM of $S_{\rm naive}$ is

$$M_{ab} \ddot{X}^b - B_{ab} \dot{X}^b = 0. \tag{15}$$

This is an EOM of a charged particle with a curved background metric M_{ab} and a constant magnetic field B_{ab}.

However, substituting (13) into (6), we have

$$\sum_a \ddot{X}^a M \xi_a - \dot{X}^a B \xi_a = 0. \tag{16}$$

The only solutions of (16) are $\dot{X}^a \equiv 0$, since $M\xi_a$ and $B\xi_a$ are linearly independent in general [15]. Thus, the time-dependent part of the solution of (15) cannot be the solution of (16). In addition, $\dot{X}^a \equiv 0$ claims that there are no dynamics in zero modes, which contradicts the dynamics that solitons can actually have. For example, solitons have the freedom of center of mass coordinates when the theory has Poincaré symmetry. Using the naive CCQ, (16) implies that solitons cannot move at finite velocity when $B \neq 0$. However, one can easily construct the exact soliton solution moving with constant velocity through a Lorentz transformation.

Another disadvantage of the naive CCQ appears when we consider other $\omega \neq 0$ modes along with zero modes. One might propose the following *Ansatz*:

$$q = \sum_a X^a(t) \xi_a + q', \xi_a^T M q' = 0, \tag{17}$$

and regard q' as $\omega \neq 0$ modes. The action (4) would be

$$\begin{aligned} S = S_{\rm naive} &+ \int {\rm d}t \frac{1}{2} (\dot{q}'^T M \dot{q}' + q'^T B \dot{q}' - q'^T K q') \\ &+ \int {\rm d}t \frac{1}{2} (q'^T B \xi_a \dot{X}^a - \dot{q}'^T B \xi_a X^a). \end{aligned} \tag{18}$$

The first line is $S_{\rm naive}$ defined in (14) and the action of q', respectively. The second line, on the other hand, seems to be the "mixing" term between q' and zero modes. Generally, the "backreaction" on q' cannot be ignored and this prevents zero modes from being analyzed independently from the nonzero modes.

Therefore, the naive CCQ $\phi_{\rm sol}(X^a(t))$ is not an appropriate approximation to discuss the low energy physics of solitons and needs some extensions in $B \neq 0$ case.

C. Classification of zero modes

In this section, we solve (6) more strictly. First $\omega = 0$ is a multiple root of $\Phi(\omega)$ in (8), since $\Phi(\omega)$ is a polynomial of ω^2 as mentioned in Sec. II A. According to the theory of ordinary differential equations, the solutions associated with $\omega = 0$ are some polynomials of t. So let us introduce the following polynomial *Ansatz*

$$q(t) = \sum_{k=0}^{N} \frac{t^k}{k!} \eta_{(k)}, \tag{19}$$

where N is finite and $\eta_{(k)}$ are arbitrary time-independent vectors. Substituting it into (6), we have

$$\sum_{k=0}^{N-2} \frac{t^k}{k!} (M\eta_{(k+2)} - B\eta_{(k+1)} + K\eta_{(k)}) - \frac{t^{N-1}}{(N-1)!}(B\eta_{(N)} - K\eta_{(N-1)}) + \frac{t^N}{N!} K\eta_{(N)} = 0. \tag{20}$$

For (20) to hold for arbitrary t, $\eta_{(k)}$ must satisfy

$$\begin{aligned} K\eta_{(N)} &= 0, \\ B\eta_{(N)} - K\eta_{(N-1)} &= 0, \\ M\eta_{(k+2)} - B\eta_{(k+1)} + K\eta_{(k)} &= 0, k \le N-2. \end{aligned} \tag{21}$$

If $N \ge 2$, we have

$$\begin{aligned} &\eta_{(N)}^T M\eta_{(N)} - \eta_{(N)}^T B\eta_{(N-1)} + \eta_{(N)}^T K\eta_{(N-2)} \\ &= \eta_{(N)}^T M\eta_{(N)} + \eta_{(N-1)}^T K\eta_{(N-1)} = 0. \end{aligned} \tag{22}$$

Due to the positivity of M and K, $\eta_{(N)} \equiv 0$ for $N \ge 2$. Then we have

$$K\eta_{(1)} = 0 \Rightarrow \eta_{(1)} \in \ker K, \tag{23}$$

$$B\eta_{(1)} = K\eta_{(0)}. \tag{24}$$

Therefore, the allowed polynomial solution is

$$q(t) = \sum_a [(X_0^a + V_0^a t)\xi_a + V_0^a \eta_a], \tag{25}$$

where we replaced $\eta_{(0)}$ and $\eta_{(1)}$ as

$$\eta_{(0)} \to \sum_a (X_0^a \xi_a + V_0^a \eta_a), \eta_{(1)} \to \sum_a V_0^a \xi_a, \tag{26}$$

and η_a is taken to satisfy

$$K\eta_a = B\xi_a. \tag{27}$$

The η_a term is known in many-body physics or chiral soliton models with vector mesons [3,18]. However, such a η_a does not always exist for all ξ_a. Since it is obvious that $B\xi_a \in \mathrm{Im}\ K$, η_a is well defined only for the zero modes ξ_a satisfying

$$\xi^T B\xi_i = 0 \text{ for arbitrary } \xi \in \ker K, \tag{28}$$

these modes are called dynamical zero modes in [15].

For zero modes that $B\xi_a$ are not orthogonal to $\ker K$, it turns out that the only polynomial solution of (6) is a time-independent constant. We call them nondynamical zero modes in this paper. In [15], these modes are called static zero modes along with the constant part of the dynamical zero modes. Without loss of generality, we take nondynamical modes orthogonal to dynamical zero modes as

$$\xi^T M\xi_i = 0, \tag{29}$$

for convenience of calculation.

In this paper, we further divide nondynamical zero modes into two subspaces. One is the space spanned by simultaneous eigenvectors of $M^{-1}B$ and K, denoted as $\{\xi_A\}$, i.e.,

$$M^{-1}B\xi_A = -ib_A\xi_A, \qquad K\xi_A = 0, \qquad b_A \in \mathbb{R}_{\neq 0}, \tag{30}$$

and its orthogonal complement. For the former subspace, not only constant solutions but also time-dependent solutions of the form $e^{-ib_A t}$ can appear. In this paper, we shall refer to these modes as cyclotron zero modes. For the latter subspace, on the other hand, no time-dependent solution is allowed. As discussed in the next section, these modes induce constraints. Thus, we name them constraint zero modes.

In summary, there are two groups of zero modes:

(1) Dynamical zero modes $\{\xi_i\}$. We will use $i, j, \cdots \subset a, b, \cdots$ indices for the dynamical zero modes. These modes satisfy $B\xi_i \in \mathrm{Im}\ K$, or equivalently (28) in order to possess the correlation vector η_i. The general solution of (6) for these modes is

$$q(t) = \sum_i [(X_0^i + V_0^i t)\xi_i + V_0^i \eta_i]. \tag{31}$$

Notice that η_i satisfying (27) for a given ξ_i is not unique. One can add an arbitrary vector of $\ker K$ to η_i,

$$\eta_i \to \eta_i + \sum_a \mathcal{V}_i{}^a \xi_a. \tag{32}$$

(2) Nondynamical zero modes. These zero modes do not satisfy (28). Nondynamical zero modes are separated into two subspaces based on the form of the solution of EOM (6).
   (a) Cyclotron zero modes $\{\xi_A\}$. The space of these modes is spanned by the simultaneous eigenvectors

of $M^{-1}B$ and K as (30). Note that ξ_A are complex vectors and ξ_A^* also belongs to cyclotron zero modes, i.e., $(-b_A, \xi_A^*)$ also satisfy (30). We label these modes with $A, B, \cdots$ indices to satisfy

$$(b_{-A}, \xi_{-A}) = (-b_A, \xi_A^*), \tag{33}$$

for convenience. The solution of (6) for these modes is

$$q(t) = \sum_A \left(\frac{1}{2} Z_0^A (1 + e^{-ib_A t}) + \frac{i}{b_A} W_0^A (e^{-ib_A t} - 1) \right) \xi_A, \tag{34}$$

where Z^A and W^A are complex constants that satisfy $(Z^{-A}, W^{-A}) = ((Z^A)^*, (W^A)^*)$.

(b) Constraint zero modes $\{\xi_\alpha\}$. $\alpha, \beta \cdots$ indices will be used for constraint zero modes. The space of these modes are defined as the orthogonal complement of cyclotron zero modes as

$$\xi_\alpha^T B \xi_A = -ib_A \xi_\alpha^T M \xi_A = 0. \tag{35}$$

The general solution of (6) for these modes are time independent

$$q(t) = \sum_\alpha Y_0^\alpha \xi_\alpha. \tag{36}$$

These modes lack dynamical degrees of freedom and instead generate constraints, as we will see later.

As mentioned at (29), nondynamical zero modes are taken to be orthogonal to dynamical zero modes.

$$\xi_i^T M \xi_A = \xi_i^T M \xi_\alpha = 0, \tag{37}$$

for all i, A, and α.

One can distinguish dynamical and nondynamical zero modes from $B_{ab} := \xi_a^\dagger B \xi_b$, the matrix element of B on $\ker K$. According to (28) and (35), we have

$$B_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{AB} = -ib_A \delta_{AB} & 0 \\ 0 & 0 & \mathcal{E}_{\alpha\beta} := \xi_\alpha^T B \xi_\beta \end{pmatrix}. \tag{38}$$

Thus, dynamical and nondynamical zero modes are associated with $\ker B_{ab}$ and $\mathrm{Im}\, B_{ab}$, respectively. The separation between cyclotron and constraint zero modes needs to consider the whole B, which is far more difficult.

For remaining $\omega \neq 0$ modes, we call them oscillating modes. Oscillating modes are related to soliton excitations. We label the oscillating modes with some nonzero indices $m, n, \cdots$ to satisfy

$$(\omega_{-m}, \xi_{-m}) := (-\omega_m, \xi_m^*), \tag{39}$$

as well as cyclotron zero modes. These indices may take discrete integer values or represent continuous wave numbers, depending on the context. We assume that there is no degeneracy for nonzero frequencies for simplicity. Then the oscillating mode part of the generic solution of (6) is

$$q(t) = \sum_{m \neq 0} \frac{\alpha_0^m}{\sqrt{2|\omega_m|}} e^{-i\omega_m t} \xi_m. \tag{40}$$

Here, α_0^m are complex constants that satisfy $\alpha_0^{-m} = (\alpha_0^m)^*$. Note that oscillating modes ξ_m are not orthogonal to zero modes $\xi_a^T M \xi_m \neq 0$ in general.

III. COLLECTIVE COORDINATE QUANTIZATION

A. Mode expansion and zero mode action

In this section, we derive the effective zero mode action based on the full mode expansion. As introduced in the previous section, the zero modes are divided into dynamical zero modes, cyclotron zero modes, and constraint zero modes when $B \neq 0$, and they have different types of classical solutions. In addition, they appear differently in the action.

Including zero modes, we have the full general solution of (6) and the momentum conjugate $p(t)$ as

$$\begin{aligned} q(t) &= (X_0^i + V_0^i t)\xi_i + V_0^i \eta_i \\ &\quad + \sum_A \left(\frac{Z_0^A}{2}(1 + e^{-ib_A t}) + \frac{iW_0^A}{b_A}(e^{-ib_A t} - 1) \right) \xi_A \\ &\quad + Y_0^\alpha \xi_\alpha + \sum_m \frac{\alpha_0^m}{\sqrt{2|\omega_m|}} e^{-i\omega_m t} \xi_m, \end{aligned} \tag{41}$$

$$\begin{aligned} p(t) &= \frac{\partial}{\partial \dot{q}} L = M\dot{q} - \frac{1}{2} Bq \\ &= V_0^i M \xi_i + \sum_A \left(W_0^A - \frac{ib_A}{2} Z_0^A \right) e^{-ib_A t} M \xi_A \\ &\quad - \sum_m i \frac{\omega_m \alpha_0^m}{\sqrt{2|\omega_m|}} e^{-i\omega_m t} M \xi_m - \frac{1}{2} Bq. \end{aligned} \tag{42}$$

Here, the summation over the zero mode indices i and α is taken. Now let us consider the following mode expansion analogous to (41) and (42).

$$\begin{aligned} q(t) &= X^i(t)\xi_i + V^i(t)\eta_i + \sum_A Z^A(t)\xi_A \\ &\quad + Y^\alpha(t)\xi_\alpha + \sum_m \frac{\alpha^m(t)}{\sqrt{2|\omega_m|}} \xi_m, \end{aligned} \tag{43}$$

$$p(t) = V^i(t) M\xi_i + \sum_A \left(W^A(t) - i\frac{b_A}{2} Z^A(t) \right) M\xi_A - \sum_m \frac{i\omega_m \alpha^m(t)}{\sqrt{2|\omega_m|}} M\xi_m - \frac{1}{2} Bq(t). \tag{44}$$

Taking the inner products of (41), (42), and ξ_i, ξ_α, ξ_m, or η_i, we have

$$g_{ij} X^j + (\xi_i^T M\eta_j - \eta_i^T M\xi_j + \eta_i^T B\eta_j) V^j + \eta_i^T B\xi_\alpha Y^\alpha - \eta_i^T M\xi_A \left(W^A + \frac{ib_A}{2} Z^A \right) = \xi_i^T Mq - \eta_i^T \left(p - \frac{1}{2} Bq \right), \tag{45}$$

$$g_{ij} V^j = \xi_i^T \left(p - \frac{1}{2} Bq \right), \tag{46}$$

$$Z^A + \frac{1}{\mu_A} \xi_A^\dagger M\eta_i V^i = \frac{1}{\mu_A} \xi_A^\dagger Mq \tag{47}$$

$$W^A + \frac{ib_A}{2\mu_A} \xi_A^\dagger M\eta_i V^i = \frac{1}{\mu_A} \xi_A^\dagger p \tag{48}$$

$$-\mathcal{E}_{\alpha\beta} Y^\beta - \xi_\alpha^T B\eta_i V^i = \xi_\alpha^T \left(p - \frac{1}{2} Bq \right), \tag{49}$$

$$\alpha^m = \frac{\sqrt{2|\omega_m|}}{2\mu_m} \xi_m^\dagger \left(Mq + \frac{i}{\omega_m} \left(p - \frac{1}{2} Bq \right) \right). \tag{50}$$

Here $g_{ij} \coloneqq M_{ij} + \eta_i^T K\eta_j$ and $\mathcal{E}_{\alpha\beta} = B_{\alpha\beta}$. μ_A and μ_m are the "norm" of ξ_A and ξ_m defined as

$$\mu_A \coloneqq \xi_A^\dagger M\xi_A, \qquad 2\mu_m \coloneqq \xi_m^\dagger M\xi_m + \frac{\xi_m^\dagger K\xi_m}{\omega_m^2}, \tag{51}$$

respectively. For the properties of the matrix elements of M, B, and K used in the calculations, see the Appendix. Note that g_{ij} and $\mathcal{E}_{\alpha\beta}$ are invertible matrices. This is easy to prove from the positivity of M and K and the definition of ξ_α. As mentioned above, η_i has the ambiguity shown in (32). Under this transformation,

$$\xi_A^\dagger M\eta_i \to \xi_A^\dagger M\eta_i + \mu_A \mathcal{V}_i{}^A \tag{52}$$

$$\xi_\alpha^T B\eta_i \to \xi_\alpha^T B\eta_i + \mathcal{E}_{\alpha\beta} \mathcal{V}_i{}^\beta, \tag{53}$$

$$\begin{aligned} &\eta_i^T B\eta_j + \xi_i^T M\eta_j - \xi_j^T M\eta_i \\ &\to \eta_i^T B\eta_j + \xi_i^T M\eta_j - \xi_j^T M\eta_i + g_{ik} \mathcal{V}_j{}^k - g_{jk} \mathcal{V}_i{}^k \\ &\quad - ib_A [\mathcal{V}_j{}^A \eta_i^T M\xi_A - \mathcal{V}_i{}^A \eta_j^T M\xi_A] \\ &\quad + \mathcal{V}_j{}^\alpha (\xi_i^T M\xi_\alpha + \eta_i^T B\xi_\alpha) + \mathcal{V}_i{}^\alpha (\xi_\alpha^T B\eta_j - \xi_j^T M\xi_\alpha) \\ &\quad + \mathcal{V}_i{}^\alpha \mathcal{V}_j{}^\beta \mathcal{E}_{\alpha\beta}. \end{aligned} \tag{54}$$

Thus, the terms on the left-hand sides can vanish by taking appropriate $\mathcal{V}_i{}^a$. As a result, one has

$$X^i = g^{ij} \left[\xi_j^T Mq - \eta_j^T \left(p - \frac{1}{2} Bq \right) \right], \tag{55}$$

$$V^i = g^{ij} \xi_j^T \left(p - \frac{1}{2} Bq \right), \tag{56}$$

$$Z^A = \frac{1}{\mu_A} \xi_A^\dagger Mq, \tag{57}$$

$$W^A = \frac{1}{\mu_A} \xi_A^\dagger p, \tag{58}$$

$$Y^\alpha = -\mathcal{E}^{\alpha\beta} \xi_\beta^T \left(p - \frac{1}{2} Bq \right), \tag{59}$$

$$\alpha^m = \frac{\sqrt{2|\omega_m|}}{2\mu_m} \xi_m^\dagger \left(Mq + \frac{i}{\omega_m} \left(p - \frac{1}{2} Bq \right) \right), \tag{60}$$

with η_i satisfying

$$K\eta_i = B\xi_i, \eta_i^T M\xi_A = 0, \xi_\alpha^T B\eta_i = 0, \quad \eta_i^T B\eta_j + \xi_i^T M\eta_j - \xi_j^T M\eta_i = 0. \tag{61}$$

Here, g^{ij} and $\mathcal{E}_{\alpha\beta}$ are the inverse matrices of g_{ij} and $\mathcal{E}_{\alpha\beta}$, respectively.

Now let us expand the action. Consider the action in Hamiltonian formalism,

$$S = \int \mathrm{d}t (p^T \dot{q} - H(q,p)). \tag{62}$$

Substituting the mode expansion (43) and (44) into (62), we have

$$\begin{aligned} H &= \frac{1}{2} \left[\left(p^T - \frac{1}{2} q^T B \right) M^{-1} \left(p + \frac{1}{2} Bq \right) + q^T Kq \right] \\ &= \frac{1}{2} g_{ij} V^i V^j + \sum_A \frac{\mu_A}{2} \left(W^{-A} + \frac{ib_A}{2} Z^{-A} \right) \left(W^A - \frac{ib_A}{2} Z^A \right) + \sum_{\omega_m > 0} \frac{\mu_m \omega_m}{2} (\alpha^m \alpha^{-m} + \alpha^{-m} \alpha^m), \end{aligned} \tag{63}$$

$$p^T\dot{q} = g_{ij}V^i\dot{X}^j + \sum_A \mu_A W^{-A}\dot{Z}^A + \frac{1}{2}\mathcal{E}_{\alpha\beta}Y^\alpha\dot{Y}^\beta + \sum_{\omega_m>0}\frac{i\mu_m}{2}(\alpha^{-m}\dot{\alpha}^m - \alpha^m\dot{\alpha}^{-m}) + \frac{\mathrm{d}}{\mathrm{d}t}(\cdots). \quad (64)$$

Here, $\cdots$ are some functions of X^i, V^i, Y^α, and α^m, and can be ignored. Thus, the action (62) can be decomposed into four sections,

$$\int \mathrm{d}t(p^T\dot{q} - H(q,p)) = S[\alpha^m] + S[X^i, V^i] + S[Z^A, W^A] + S[Y^\alpha] \quad (65)$$

$$S[\alpha^m] = \sum_{\omega_m>0}\int \mathrm{d}t\mu_m\left[\frac{i}{2}(\alpha^{-m}\dot{\alpha}^m - \alpha^m\dot{\alpha}^{-m}) - \frac{\omega_m}{2}(\alpha^{-m}\alpha^m + \alpha^m\alpha^{-m})\right] \quad (66)$$

$$S[X^i, V^i] = \int \mathrm{d}t\left(g_{ij}V^i\dot{X}^j - \frac{1}{2}g_{ij}V^iV^j\right) \quad (67)$$

$$S[Z^A, W^A] = \sum_A \int \mathrm{d}t\mu_A\left[W^{-A}\dot{Z}^A - \frac{1}{2}\left(W^{-A} + \frac{ib_A}{2}Z^{-A}\right)\left(W^A - \frac{ib_A}{2}Z^A\right)\right] \quad (68)$$

$$S[Y^\alpha] = \int \mathrm{d}t\frac{1}{2}\mathcal{E}_{\alpha\beta}Y^\alpha\dot{Y}^\beta. \quad (69)$$

Each term is the action of oscillating modes, dynamical zero modes, cyclotron zero modes, and constraint zero modes, respectively. Note that there are no interactions between the oscillating modes and zero modes. The "mixing" term in (18) is due to a naive CCQ *Ansatz* and does not mean that zero modes couple to other modes in the leading order. $S[\alpha^m]$ is a sum of harmonic oscillator (HO) actions with frequency $\omega_m > 0$ in terms of creation and annihilation operators α^m. We can rewrite this term in a standard HO action by introducing

$$\alpha^m =: \frac{\sqrt{2|\omega_m|}}{2}\left(x_{|m|} + \frac{i}{\omega_m}p_{|m|}\right). \quad (70)$$

$S[X^i, V^i]$ is the action of the nonrelativistic particle in the background metric g_{ij} in the Hamiltonian formalism. In Lagrangian formalism, this term is

$$S[X^i] = \int \mathrm{d}t\frac{1}{2}g_{ij}\dot{X}^i\dot{X}^j. \quad (71)$$

$S[Z^A, W^A]$ is the sum of the action of the particle on a plane with a constant magnetic field in terms of the complex coordinates. The momentum conjugate of Z^A is $\mu_A W^{-A}$, and we can rewrite the action as

$$S[Z^A] = \sum_{b_A>0}\int \mathrm{d}t\mu_A[|\dot{Z}^A|^2 + b_A\,\mathrm{Im}(Z^{-A}\dot{Z}^A)]. \quad (72)$$

$S[Y^\alpha]$ is quite different from the other terms. Since Y^αs are absent in the Hamiltonian, this Lagrangian gives the constraint

$$\phi_\alpha := P_\alpha + \frac{1}{2}\mathcal{E}_{\alpha\beta}Y^\beta = 0, \quad (73)$$

where P_α is the momentum conjugate of Y^α. Calculating the Poisson brackets, we have

$$\{\phi_\alpha, \phi_\beta\}_{\mathrm{P}} = \mathcal{E}_{\alpha\beta}. \quad (74)$$

Since $\mathcal{E}_{\alpha\beta}$ is regular, ϕ_α are second-class constraints and one can define the Dirac bracket as

$$\{Y^\alpha, Y^\beta\}_{\mathrm{D}} = \mathcal{E}^{\alpha\beta}. \quad (75)$$

As a result, the zero mode part of the action is

$$S_{\mathrm{zero}} := S[X^i] + S[Y^\alpha] + S[Z^A] = \int \mathrm{d}t\frac{1}{2}\left[g_{ij}\dot{X}^i\dot{X}^j + \mathcal{E}_{\alpha\beta}Y^\alpha\dot{Y}^\beta + \sum_{b_A>0}2\mu_A(|\dot{Z}^A|^2 + b_A\,\mathrm{Im}(Z^{-A}\dot{Z}^A))\right]. \quad (76)$$

This is one of the main result of this paper. Note that there are no "mixing" terms between the zero modes and oscillating modes. The moduli metric G_{ab} is obtained from the kinetic terms as

$$G_{ab} = \begin{pmatrix} g_{ij} & 0 & 0 \\ 0 & \mu_A\delta_{AB} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (77)$$

It is obvious that the moduli metric is not invertible.

B. The generalization of the collective coordinate quantization

We now present the generalization of the CCQ. Comparing the action from the naive CCQ (14) and (76), the naive CCQ does not reproduce the moduli metric in two aspects.

(i) The dynamical zero mode component of the moduli metric is g_{ij} in (77). However, the evaluated metric from the naive CCQ is M_{ij}, which does not have the term $\eta_i^T K\eta_j$.

(ii) The naive CCQ does not distinguish zero modes and (14) has the kinetic term of the constraint zero modes $\frac{1}{2}M_{\alpha\beta}\dot{Y}^\alpha\dot{Y}^\beta$, which actually does not exist in (76).

To solve the first problem (i), we add the η_i term to the naive *Ansatz* as

$$q(t) = \sum_i (X^i(t)\xi_i + \dot{X}^i(t)\eta_i) + \sum_\alpha Y^\alpha(t)\xi_\alpha + \sum_A Z^A\xi_A. \tag{78}$$

Substituting to (4) and ignoring the time derivative term with order higher than three, we have

$$\begin{aligned} S &= \int \mathrm{d}t \frac{1}{2}[M_{ij}\dot{X}^i\dot{X}^j + M_{\alpha\beta}\dot{Y}^\alpha\dot{Y}^\beta + \mathcal{E}_{\alpha\beta}Y^\alpha\dot{Y}^\beta \\ &\quad + \eta_i^T(B\xi_j - K\eta_j)\dot{X}^i\dot{X}^j + \xi_i^T B\eta_j X^i\ddot{X}^j \\ &\quad + \eta_i^T B\xi_\alpha\dot{X}^i\dot{Y}^\alpha + \xi_\alpha^T B\eta_i\ddot{X}^i Y^\alpha] \\ &= \int \mathrm{d}t \frac{1}{2}[(M_{ij} + \eta_i^T(2B\xi_j - K\eta_j))\dot{X}^i\dot{X}^j \\ &\quad + M_{\alpha\beta}\dot{Y}^\alpha\dot{Y}^\beta + \mathcal{E}_{\alpha\beta}Y^\alpha\dot{Y}^\beta] \\ &= \int \mathrm{d}t \frac{1}{2}[g_{ij}\dot{X}^i\dot{X}^j + M_{\alpha\beta}\dot{Y}^\alpha\dot{Y}^\beta + \mathcal{E}_{\alpha\beta}Y^\alpha\dot{Y}^\beta]. \end{aligned} \tag{79}$$

Here, we used (61) and the cyclotron zero mode part is omitted.

The second problem (ii) can be solved by removing the kinetic term of constraint zero modes $\frac{1}{2}M_{\alpha\beta}\dot{Y}^\alpha\dot{Y}^\beta$ by hand.

In summary, the procedure of the generalized CCQ method is as follows.

(1) Find all the zero modes (or equivalently the collective coordinates) of the solutions.
(2) Calculate $\xi_a^T B\xi_b$, $M\xi_a$, and $B\xi_a$ to classify the zero modes into dynamical zero modes, cyclotron zero modes, and constraint zero modes.
(3) For dynamical zero modes, find the η_i that satisfy (61).
(4) Add the η_i term to the naive CCQ *Ansatz* $\phi(t) = \phi_{\mathrm{sol}}(X^a(t))$. The coefficient of the η_i term is the first order of $\dot{X}^a$ and determined by substituting the *Ansatz* into the EOM.
(5) Put the *Ansatz* to the original action.
(6) Ignore the time derivative terms with order higher than three.
(7) Ignore the constraint zero mode components of the metric $M_{\alpha\beta}$.
(8) The effective action takes the form of

$$S = \int \mathrm{d}t\left(\frac{1}{2}\mathcal{G}_{ab}(X)\dot{X}^a\dot{X}^b + \mathcal{A}_a(X)\dot{X}^a\right). \tag{80}$$

Note that $\mathcal{G}_{ab}(X)$ is not full rank in general. Because of this, the effective action involves second-class constraints.

In practical analysis, the most difficult part is the distinction between the cyclotron and the constraint zero modes. To determine the dynamical zero modes, only B_{ab} is needed, and this is relatively easy to evaluate. The difference between the cyclotron and constraint zero modes, on the other hand, depends on whether the eigenvector of $M^{-1}B$ exists in $\ker K$ or not. Because of this, it is necessary to construct the exact M, B, and ξ_a from the exact soliton configuration in general.

IV. APPLICATION TO THE SKYRMION

The $SU(3)$ Skyrme model with the WZW term is a good model for applying the generalized CCQ method discussed above. In this section, we use the following indices.

$$\begin{aligned} &4\,\text{dim spacetime}\ \mu,\nu,\cdots = 0,1,2,3, \\ &5\,\text{dim spacetime}\ M,N,\cdots = 0,1,2,3,4, \\ &SU(3)\ a,b,\cdots = 1,2,\ldots,8, \\ &\text{spatial and SU(2)}\ i,j,\cdots = 1,2,3, \\ &SU(3)/(SU(2)\times U(1))\alpha,\beta,\cdots = 4,5,6,7. \end{aligned} \tag{81}$$

In addition, we use λ_a as the Gell-Mann matrices.

A. Skyrmion with WZW term and the naive CCQ

First, we review the $SU(3)$ Skyrmion with the WZW term [19–21] and the baryon spectrum from the naive CCQ. For details, see [6–8]. The action of the model is

$$S[U] = S_{\mathrm{Skyrme}} + S_{\mathrm{mass}} + S_{\mathrm{WZW}}, \tag{82}$$

$$S_{\mathrm{Skyrme}} = \int \mathrm{d}^4x\,\mathrm{tr}\left[\frac{f_\pi^2}{16}L_\mu^2 + \frac{1}{32e^2}[L_\mu, L_\nu]^2\right], \tag{83}$$

$$S_{\mathrm{mass}} = \int \mathrm{d}^4x\,\mathrm{tr}\left[\frac{f_\pi^2}{16}\mathcal{M}(U + U^\dagger - 2)\right] \tag{84}$$

$$S_{\mathrm{WZW}} = -i\frac{N_c}{240\pi^2}\int_{M_5}\mathrm{d}^5x\epsilon^{KLMNO}\mathrm{tr}(L_K L_L L_M L_N L_O). \tag{85}$$

Here, $U(x) \in SU(3)$ is the pion field and $L_\mu := \partial_\mu U U^\dagger$. M_5 is the five-dimensional manifold with the boundary ∂M_5 being the four-dim spacetime. ϵ^{KLMNO} is the Levi-Civita symbol with $\epsilon_{01234} = +1$. $\mathcal{M} = \mathrm{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$ is the mass matrix of pions and kaons. The parameters of the theory are the pion decay constant f_π, the Skyrme parameter e, and the meson mass $\mathcal{M}$. In this section, we first discuss in the chiral limit $\mathcal{M} = 0$. We take the $ef_\pi/2 = 1$ unit. The EOM from (82) is

$$\frac{1}{e^2}\partial_\mu\left(L^\mu-\frac{1}{4}[L_\nu,[L^\nu,L^\mu]]\right)+i\frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma 4}L_\mu L_\nu L_\rho L_\sigma=0. \tag{86}$$

The one-baryon configuration is written in the spherically symmetric *Ansatz*,

$$U(x)=U_{\rm sol}(x)\coloneqq\begin{pmatrix}\exp(if(r)n^i\lambda_i) & 0\\ 0 & 1\end{pmatrix}, \tag{87}$$

where $r\coloneqq\sqrt{|x^i|^2}$ and $n^i\coloneqq x^i/r$. This *Ansatz* is invariant under $SO(3)$ "rotation"

$$U(x)=V_R U(R^{-1}x)V_R^\dagger, V_R=\begin{pmatrix}V_R^{SU(2)} & 0\\ 0 & 1\end{pmatrix},\quad R\in SO(3). \tag{88}$$

Here, $V_R^{SU(2)}$ is the $SU(2)$ representation of R. $f(r)$ satisfies the EOM

$$(r^2+2\sin^2 f)f''+2rf'+\sin 2f\left(f'^2-1-\frac{\sin^2 f}{r^2}\right)=0, \tag{89}$$

where $f'(r)=\frac{{\rm d}f(r)}{{\rm d}r}$ and $U_{\rm sol}(r\to\infty)\to 1\Leftrightarrow f(\infty)=0$. The baryon number is

$$N_B=\int {\rm d}^3\boldsymbol{x}\rho_{N_B}(\boldsymbol{x})\coloneqq\frac{1}{24\pi^2}\epsilon^{0ijk4}\int {\rm d}^3\boldsymbol{x}\,{\rm tr}(L_iL_jL_k)=-\frac{2}{\pi}\int_0^\infty {\rm d}r\sin^2 f f'=\frac{1}{\pi}f(0)-\frac{1}{2\pi}\sin 2f(0). \tag{90}$$

By requiring $N_B=1$, we obtain the boundary condition $f(0)=\pi$.

In this paper, we write the fluctuation u around $U_{\rm sol}$ as

$$U(x)=(1+u(x))U_{\rm sol}(x)\Leftrightarrow u=UU_{\rm sol}^\dagger-1. \tag{91}$$

The zero modes of the fluctuation are $u=L_i^{\rm sol}$ corresponding to the spatial translation, and $u=\xi_a\coloneqq\frac{i}{2}(\lambda_a-U_{\rm sol}\lambda_{a\neq 8}U_{\rm sol}^\dagger)$ for $a\neq 8$ corresponding to the $SU(3)_V/U(1)$ symmetry. Therefore, the moduli space of the $B=1$ Skyrmion is $\mathbb{R}^3\times SU(3)_V/U(1)$. We focus on the latter throughout this analysis.

The *Ansatz* of the naive CCQ is

$$U(x)=V(t)U_{\rm sol}(x^i)V^\dagger(t),\quad V(t)=\exp\left(iv^a(t)\frac{\lambda_a}{2}\right)\in SU(3). \tag{92}$$

Note that $V(t)$ has one gauge degree of freedom that does not change $U(x)$.

$$V(t)\to V(t)\exp\left(-iv_8'(t)\frac{\lambda_8}{2}\right). \tag{93}$$

Under this *Ansatz*, the action (82) reduces to

$$S-S[U_{\rm sol}]=\int {\rm d}tL_{\rm naive}[V]=\int {\rm d}t\left[\frac{I}{2}(\Omega_R^i)^2+\frac{I'}{2}(\Omega_R^\alpha)^2-\frac{N_c}{2\sqrt{3}}\Omega_R^8\right]. \tag{94}$$

Here, I and I' are

$$I=\int {\rm d}^3\boldsymbol{x}\rho_I(r)\coloneqq\frac{2}{3}\int {\rm d}^3\boldsymbol{x}\frac{\sin^2 f}{e^2}\left(1+f'^2+\frac{\sin^2 f}{r^2}\right), \tag{95}$$

$$I'=\int {\rm d}^3\boldsymbol{x}\rho_{I'}(r)\coloneqq\int {\rm d}^3\boldsymbol{x}\frac{1}{2e^2}(1-\cos f)\left(1+\frac{f'^2}{4}+\frac{\sin^2 f}{2r^2}\right), \tag{96}$$

and Ω_R^a are defined as

$$-iV^\dagger\dot V=\Omega_R^a\frac{\lambda_a}{2}. \tag{97}$$

The momentum conjugates to v^a are

$$P_a=\frac{\partial L_{\rm naive}}{\partial\dot v^a}=e_a{}^i I\Omega_i^R+e_a{}^\alpha I'\Omega_\alpha{}^8-\frac{N_c}{2\sqrt{3}}e_a{}^8,\quad e_a{}^b(v)\coloneqq\frac{\partial\Omega_R^b}{\partial\dot v^a}. \tag{98}$$

It is convenient to introduce $Q_a^R\coloneqq(e^{-1})_a{}^bP_b$. These are the generators of the transformation,

$$V\to VV_R^\dagger,\qquad V_R\in SU(3)_R, \tag{99}$$

because the Poisson brackets of Q_a^Rs are

$$\{Q_a^R V\}_{\rm P}=-iV\frac{\lambda_a}{2}, \tag{100}$$

$$\{Q_a^RQ_b^R\}_{\rm P}=f_{abc}Q_c^R. \tag{101}$$

Thus, (98) can be rewritten as

$$Q_a^R = \begin{cases} I\Omega_R^i & a = i \\ I'\Omega_R^\alpha & a = \alpha \\ -\frac{N_c}{2\sqrt{3}} & a = 8. \end{cases} \tag{102}$$

The effective theory (94) has one constraint $\phi_8 := Q_8^R + \frac{N_c}{2\sqrt{3}} \equiv 0$. This constraint reflects the gauge transformation (93).

The effective action (94) has the following symmetry.

$$V(t) \to V_L V(t), \qquad V_L \in SU(3). \tag{103}$$

$$V(t) \to V(t) V_R^\dagger, \qquad V_R = \begin{pmatrix} V_R^{SU(2)} & 0 \\ 0 & 1 \end{pmatrix}, \tag{104}$$

Note that the whole $SU(3)_R$ transformation (99) is not a symmetry, and only the $SU(2)_R$ subgroup is preserved as a symmetry of this theory.

V_L and V_R transformations correspond to $SU(3)_V$ transformations and spatial $SO(3)$ rotations (88), respectively. The conserved charges of $SU(3)_V$ are

$$Q_a^L := -D_{ab} Q_b^R, \qquad D_{ab} := \frac{1}{2}\mathrm{tr}(V^\dagger \lambda_a V \lambda_b). \tag{105}$$

Here D_{ab} is the adjoint representation matrix of V. Thus, Q_L^a and Q_R^a are related to each other through a $SU(3)$ transformation. Isospin and hyperchage operators are identified as $I_i := Q_i^L$ and $Y := \frac{2}{\sqrt{3}} Q_8^L$, respectively. The conserved charges of $SO(3)$ rotation are $J_i := Q_i^R$, which are equivalent to the spin.

Due to the relation (105), physical states belong to the (r, r) representation of $SU(3)_V \times SU(3)_R$ and can be labeled as

$$|\mathrm{phys}\rangle = |r : I, I^3, Y; J, J^3, Y^R\rangle. \tag{106}$$

Note that (I, I^3, Y) and (J, J^3, Y^R) are indices in r multiplet of $SU(3)_V$ and $SU(3)_R$, respectively. The Hamiltonian from (94) is

$$H = \frac{(Q_i^R)^2}{2I} + \frac{(Q_a^R)^2}{2I'}, \tag{107}$$

and the energy spectrum for the $SU(3)$ representation r is

$$E = \frac{J(J+1)}{2I} + \frac{1}{2I'}\left(C_2(r) - J(J+1) - \frac{N_c^2}{12}\right). \tag{108}$$

Here, J is the spin $(Q_i^R)^2 = J(J+1)$ and $C_2(r) = (Q_a^R)^2$ is the Casimir operator.

Physical states must satisfy the first-class constraint condition,

$$\phi_8 = 0 \Leftrightarrow Y^R = \frac{N_c}{3}, \tag{109}$$

where $Y^R := -\frac{2}{\sqrt{3}} Q_8^R$ is the generator of the gauge transformation (93) analogous to the hypercharge Y. Equation (109) restricts the representation and the spin of the multiplets. In order to satisfy (109), physical states must belong to the representations that contain $Y = \frac{N_c}{3}$ states. Moreover, the number of $Y = \frac{N_c}{3}$ states in the representation is $2J + 1$, where J is the spin of the multiplet. For $N_c = 3$, the allowed multiplets and their spin are

$$\begin{aligned} 8 &= [1,1], & J &= \frac{1}{2}, \\ 10 &= [3,0], & J &= \frac{3}{2}, \\ \bar{10} &= [0,3], & J &= \frac{1}{2}, \\ &\vdots \end{aligned} \tag{110}$$

The octet represents the nucleon multiplet N. One of the decuplets with spin $\frac{3}{2}$ corresponds to the Δ baryons. The spin $\frac{1}{2}$ antidecuplet contains spin $\frac{1}{2}$, isospin 0, and hypercharge $+2$ particle, which is called the θ^+ pentaquark.

B. The generalized CCQ

Now let us adopt our CCQ method developed in Sec. III B to the Skyrme model. Substituting (91) to (82) or (86), one can read off M and B as

$$M(u, u') := -\frac{1}{e^2}\int \mathrm{d}^3x\, \mathrm{tr}\left(uu' + \frac{1}{4}[L_{\mathrm{sol}}^i, u][L_{\mathrm{sol}}^i, u']\right), \tag{111}$$

$$B(u, u') := -\frac{iN_c}{24\pi^2}\int \mathrm{d}x^3 \epsilon^{0ijk}\mathrm{tr}(u\{[u', L_{\mathrm{sol}}^i], L_{\mathrm{sol}}^j L_{\mathrm{sol}}^k\}). \tag{112}$$

Substituting $u = \xi_a = \frac{i}{2}(\lambda_a - U_{\mathrm{sol}}\lambda_a U_{\mathrm{sol}}^\dagger)$ into (112), we have

$$B_{ab} := B(\xi_a, \xi_b) = -\frac{N_c}{2\sqrt{3}} f_{ab8}. \tag{113}$$

The nonzero components of f_{ab8} are $f_{458} = f_{678} = \frac{\sqrt{3}}{2}$. Therefore, $\{\xi_i\}$ are dynamical zero modes and $\{\xi_\alpha\}$ are nondynamical zero modes.

Next we have to derive η_i that satisfy (61) for ξ_i and distinguish whether $\{\xi_\alpha\}$ are cyclotron or constraint zero modes. Putting (91) with $u = iv^a(t)\xi_a$ into (86), we have

$$\frac{3}{2}\rho_I(\delta_{ij}-n_in_j)\ddot{v}^j=0, \tag{114}$$

$$\rho_{I'}\ddot{v}^\alpha+\frac{N_c}{2\sqrt{3}}\rho_{N_B}(1-\cos f)f_{\alpha\beta 8}\dot{v}^\beta=0. \tag{115}$$

This is the explicit form of (16) in this model. The first equation indicates that $B\xi_i=0$ and it is obvious that $\eta_i\equiv 0$. The second equation shows that $\{\xi_\alpha\}$ are constraint zero modes. For (114) to have time-dependent solutions, $\rho_I'\propto\rho_{N_B}(1-\cos f)$ is needed. However, this is not the case. This follows from the asymptotic behavior of $f(r)\sim r^{-2}$ [22,23]: $\rho_{I'}(r)\sim r^{-4}$ and $\rho_{N_B}(r)\sim r^{-9}$. Therefore, the effective Lagrangian should not have the kinetic term proportional to I'.

$$S_{\rm eff}=\int \mathrm{d}t\, L_{\rm eff}=\int \mathrm{d}t\left[\frac{I}{2}(\Omega_i^R)^2-\frac{N_c}{2\sqrt{3}}\Omega_8^R\right]. \tag{116}$$

The Hamiltonian from (116) and its eigenvalue are

$$H_{\rm eff}=\frac{(Q_i^R)^2}{2I}, E_{\rm eff}=\frac{J(J+1)}{2I}. \tag{117}$$

Since I' is lost in (116), Q_a^R are now

$$Q_a^R=\begin{cases} I\Omega_R^i, & a=i\\ 0, & a=\alpha\\ -\frac{N_c}{2\sqrt{3}}. & a=8\end{cases}. \tag{118}$$

This shows that the effective theory (116) has five constraints: $\phi_8=Q_8^R+\frac{N_c}{2\sqrt{3}}\equiv 0$, which also appears from (94), and $\phi_\alpha:=Q_\alpha^R\equiv 0$, which are overlooked in the naive CCQ. The Poisson brackets between the constraints are

$$\{\phi_\alpha,\phi_\beta\}_{\rm P}\approx f_{\alpha\beta i}Q_i^R-\frac{N_c}{2\sqrt{3}}f_{\alpha\beta 8},\{\phi_\alpha,\phi_8\}_{\rm P}\approx 0, \tag{119}$$

where $\approx$ is the weak equality. This implies that ϕ_8 is first class, while four ϕ_αs are second class.

In the naive CCQ, $\phi_8=0$ is the only constraint and every state that satisfies (109) can be physical. On the other hand, however, there ar e also second-class constraints ϕ_α in our method, and they give further restrictions. To deal with the second-class constraints, a naive approach is to reduce the degrees of freedom by introducing the Dirac bracket. However, the $SU(3)$ symmetry becomes implicit, and the commutation rules become complicated. To deal with the second-class constraints in a covariant way, we adopt the Gupta-Bleuler quantization method, originally developed for quantum electrodynamics [24–26].

First, let us introduce the following operators

$$e_1^\pm:=\frac{1}{\sqrt{2}}(\phi_4\pm i\phi_5), e_2^\pm:=\frac{1}{\sqrt{2}}(\phi_6\pm i\phi_7). \tag{120}$$

The second class constraints become $e_1^\pm=e_2^\pm=0$. The commutators between $e_1^\pm$ and $e_2^\pm$ are

$$[e_1^+,e_1^-]=\frac{1}{2}Q_3^R+\frac{\sqrt{3}}{2}Q_8^R, \tag{121}$$

$$[e_2^+,e_2^-]=-\frac{1}{2}Q_3^R+\frac{\sqrt{3}}{2}Q_8^R, \tag{122}$$

$$[e_1^\pm,e_2^\pm]=0, \tag{123}$$

$$[e_1^\pm,e_2^\mp]=\pm\frac{1}{2}(Q_1^R\pm iQ_2^R). \tag{124}$$

According to (122), either a pair (e_1^+,e_2^+) or (e_1^-,e_2^-) can be regarded as the first-class constraints, and the other as the gauge fixing condition. Then the physical state can be defined as

$$\begin{aligned} e_1^+|\text{phys}\rangle=e_2^+|\text{phys}\rangle=0,\\ \text{or}\quad e_1^-|\text{phys}\rangle=e_2^-|\text{phys}\rangle=0.\end{aligned} \tag{125}$$

We understand the meaning of these conditions as follows. $e_1^\pm$ and $e_2^\pm$ are the operators that shift the weight (J_3,Q_8^R) on the $SU(3)_R$ weight diagram by simple roots as

$$[J_3,e_1^\pm]=\pm\frac{1}{2}e_1^\pm, \tag{126}$$

$$[J_3,e_2^\pm]=\mp\frac{1}{2}e_2^\pm, \tag{127}$$

$$[Q_8^R,e_1^\pm]=\pm\frac{\sqrt{3}}{2}e_1^\pm, \tag{128}$$

$$[Q_8^R,e_2^\pm]=\pm\frac{\sqrt{3}}{2}e_2^\pm. \tag{129}$$

According to (128) and (129), (e_1^+,e_2^+) are the raising operators and (e_1^-,e_2^-) are the lowering operators of Q_8^R, respectively. Hence (125) requires physical states to be the highest or lowest eigenstates of $Y^R\propto Q_8^R$ in the representation. Since the $SU(3)_V$ and $SU(3)_R$ representations coincide, (125) yields

$$Y^R=Y_{\max},\quad \text{or}\quad Y_{\min}, \tag{130}$$

where $Y_{\max}$ and $Y_{\min}$ are the maximum and minimum values of the hypercharge in the representation, respectively. By combining (130) with (109), we obtain

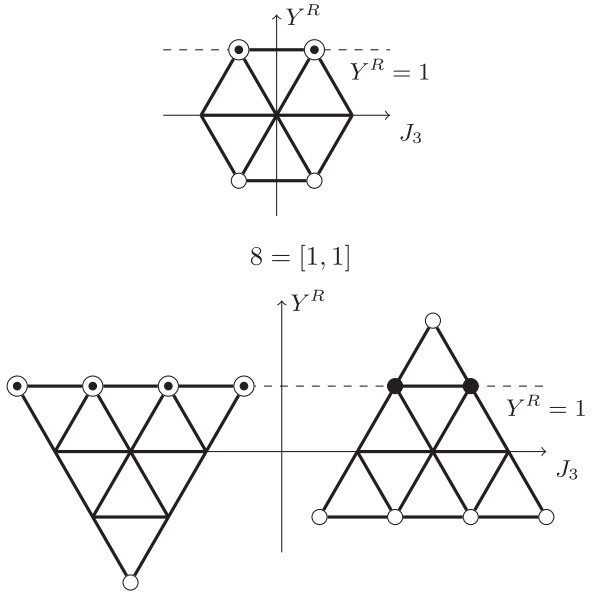


FIG. 1. Weight diagram of 8, 10, and $\overline{10}$ in a $SU(3)_R$ weight lattice. Black and white circles are the weights that satisfy (109) and (125), respectively. Physical states that satisfies both (109) and (125) are marked with $\odot$, which black and white circles match. $8 = [1,1]$ and $10 = [3,0]$ have 2 and 4 $\odot$ s, respectively, while $\overline{10} = [0,3]$ do not. This means that the antidecuplet $\overline{10}$ is not included in the physical state.

$$Y_{\max} = \frac{N_c}{3}. \tag{131}$$

This condition eliminates more representations than (109). Equation (131) forbids the representations that includes $Y > \frac{N_c}{3}$ states. For $N_c = 3$, only the nucleon octet $8 = [1,1]$ and the decuplet $10 = [3,0]$ related to Δ satisfy (131). On the other hand, all the other multiplets that appeared in the naive CCQ, including the antidecuplet $\overline{10} = [0,3]$ with the θ^+ pentaquark, contains $Y \geq 2$ states and are not allowed in our method. This is visualized in Fig. 1. As a result, the infamous θ^+ pentaquark is a mere artifact of the naive CCQ and never appears in the collective coordinate sector.

The classification of zero modes into dynamical and nondynamical zero modes was first presented in [15] for the Skyrme-WZW model. According to this distinction, the upper bound of the hypercharge $Y \leq \frac{N_c}{3}$ is derived. However, this bound was evaluated classically and was unclear whether this holds even in the quantum level. The generalized CCQ in our approach solve this problem by removing the kinetic term of constraint zero modes and treating $\phi_\alpha \equiv 0$ as constraints quantum mechanically.

C. The meson mass term

We briefly discuss the effects of the masses of pions and kaons (84) to the result of the generalized CCQ in this section. It is convenient to introduce $\delta m^2 \coloneqq m_K^2 - m_\pi^2$ to see the effects of meson masses.

The mass term affects the baryon spectrum in two ways. First, it deforms the classical soliton configuration through the EOM. In the spherically symmetric solution (87), $f(r)$ depends on m_π^2. This modifies the numerical values of the classical soliton mass $M_{\rm sol}$ an the moment of inertia I. Second, the mass term breaks the $SU(3)_V$ symmetry and the constraint zero modes $\{\xi_\alpha\}$ are not genuine zero modes for the $\delta m^2 \neq 0$ case.

In order to apply the generalized CCQ to $\{\xi_\alpha\}$, δm^2 must be small compared with the WZW term. This is because the origin of the constraints $Q_\alpha^R = 0$ is the WZW term. The typical scale of the WZW term contribution can be estimated from $M^{-1}B$ of (111) and (112) as

$$m_{\rm WZW} = \frac{N_c e^2}{24\pi^2} \times 3! \times 4 = \frac{N_c e^2}{\pi^2}, \tag{132}$$

in the $\frac{e f_\pi}{2} = 1$ unit. Note that $m_{\rm WZW}$ is $\mathcal{O}(N_c^0)$. The factor $3! \times 4$ is due to the summation over the indices of ϵ^{0ijk4} and commutators. Substituting the numerical values $(e, f_\pi) = (5.45, 127\ \text{MeV})$ for example [4], we have $m_{\rm WZW} = 3.17$ GeV. The physical kaon mass $m_K = 495$ MeV [27] is about 16% of $m_{\rm WZW}$, and it is expected that the constraint $Y \leq \frac{N_c}{3}$ approximately holds.

Since $\{\xi_\alpha\}$ are not genuine zero modes, it is also possible to treat them as oscillating modes. Hence, other methods that analyze nonzero modes would give similar results.

D. Comparison with other approaches

In this section, we discuss the relation and consistency between our generalized CCQ and other methods that include nonzero modes: BSA [12–14], RVA [16].

BSA and RVA are similar in *Ansatz*, but give opposite results for the θ^+ pentaquark. The *Ansätze* of BSA and RVA are

$$U_{\rm BSA}(\eta^a(x)) = A_2(t) U_{\rm sol}^{\frac{1}{2}}(\boldsymbol{x}) \exp\left(\frac{i}{f_\pi}\eta^a(x)\lambda_a\right) U_{\rm sol}^{\frac{1}{2}}(\boldsymbol{x}) A_2^\dagger(t), \tag{133}$$

$$U_{\rm RVA}(\tilde{\eta}^a(x)) = V(t) U_{\rm sol}^{\frac{1}{2}}(\boldsymbol{x}) \exp\left(\frac{i}{f_\pi}\tilde{\eta}^a(x)\lambda_a\right) U_{\rm sol}^{\frac{1}{2}}(\boldsymbol{x}) V^\dagger(t). \tag{134}$$

Note that collective coordinate of BSA is $A_2(t) \in SU(2)$. This is because baryons with $S \neq 0$ are expressed as nonzero modes.

η^a and $\tilde{\eta}^a$ are introduced to describe meson excitations around the baryon (η^a is denoted as K_a in [12,13]). The fundamental difference between the BSA and RVA is the definition of these fluctuations.

In BSA, η^a are defined to satisfy the Skyrme model EOM in the leading order of N_c. This is equivalent to (6) or its mode expansion (7). Hence, η^a correspond to the oscillating modes, which represents meson excitations, bound states, and resonances properly. As discussed in Sec. III A, the zero mode part and oscillating mode part of the action can be separated as (65). As a result, the generalized CCQ and BSA are consistent and should give the same result. In fact, it is already known that there are no bound states and resonances that can be interpreted as θ^+ in BSA for $\delta m^2 \lesssim 1\ \mathrm{GeV}^2$ region, where m_K is small compared with m_{WZW}. This indeed agrees with the result of the generalized CCQ. Thus, we conclude that θ^+ does not appear in the Skyrme model.

In contrast, $\tilde{\eta}^a$ in RVA are defined to be orthogonal to zero modes $\xi_{a\neq 8}$. This is associated with q' in (17) with the orthogonal condition $\xi_a^T M q' = 0$. In [16], $\tilde{\eta}^a$ are interpreted as "background" excitations that not related to the exotic baryons. The existence of θ^+ is based on decomposing the phase shift of η^a in BSA into the "resonance" part and the "background" part from $\tilde{\eta}^a$. However, the definition of $\tilde{\eta}^a$ causes several problems for the analysis. First, $\tilde{\eta}^a$ cannot represent any of meson excitations, bound states, or resonances that η^a do. Due to the orthogonal condition $\xi_a^T M q' = 0$, the ξ_a components are subtracted from $\tilde{\eta}^a$. This prevents $\tilde{\eta}^a$ from satisfying the ξ_a part of the Skyrme model EOM. For this reason, it is questionable to interpret $\tilde{\eta}^a$ as a background part, as pointed out in [28]. Second, the "mixing" terms appear in the action. The RVA *Ansatz* corresponds to (17) and the action possesses the "mixing" term of $V(t)$ and $\tilde{\eta}^a$ due to the orthogonal condition, which is treated perturbatively. However, this "mixing" term is $\mathcal{O}(N_c^0)$, which is of the same order as other terms of the Hamiltonian in large N_c, and perturbation is in fact inappropriate. As mentioned in Sec. II B, this prevents us from applying CCQ to $V(t)$ independent from $\tilde{\eta}^a$. Concerning these problems, the existence of θ^+ in RVA might be a mere artifact as well as in the naive CCQ. A proper discussion of the θ^+ in RVA would require redefining the "background" phase shift to be consistent with the Skyrme model EOM or taking the effect of "mixing" term into account completely.

V. CONCLUSION

We discussed the general collective coordinate quantization of solitons in the presence of topological terms. The basic idea of the method is the classification of zero modes. The moduli metric constructed from our method is noninvertible, and the effective action contains constraints in general. We applied our method to the Skyrmion with the WZW term and identified four additional constraints that were not accounted for in the naive approach. Using the Gupta-Bleuler quantization and group theoretical arguments, we demonstrated that only the nucleon and Δ baryon multiplets are permitted as physical states. As a result, we conclude that the θ^+ state does not arise as a physical state in the Skyrme model, but rather is an artifact of the naive CCQ scheme, which agrees with the previous analysis in [15] and [12–14].

Our analysis is based on the nongauge systems and requires further developments to be applicable to solitons in gauge theories. For gauge systems, the fluctuation around the soliton also has the gauge transformation, and the mode expansion depends on the gauge fixing condition. Thus, we need to discuss the gauge invariance of the effective action derived in our method.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: INNER PRODUCTS OF THE "EIGENVECTORS"

We define the matrix elements of M as

$$M_{ab} \coloneqq \xi_a^\dagger M \xi_b, \qquad M_{am} \coloneqq \xi_a^\dagger M \xi_m,$$
$$M_{ma} \coloneqq \xi_m^\dagger M \xi_a, \qquad M_{mn} \coloneqq \xi_m^\dagger M \xi_n. \tag{A1}$$

The matrix elements of B and K are defined as well. From the symmetry of M, B, and K, the matrix elements satisfy the following symmetry relations:

$$M_{ab} = M_{ba}, \quad M_{ma} = M_{a-m}, \quad M_{mn} = M_{-n-m}, \tag{A2}$$

$$B_{ab} = -B_{ba}, \quad B_{ma} = -B_{a-m}, \quad B_{mn} = -B_{-n-m}, \tag{A3}$$

$$K_{ab} = K_{am} = K_{ma} = 0, \qquad K_{mn} = K_{-n-m}. \tag{A4}$$

Taking the inner product with (7), we have

$$-\omega_m^2 M_{am} + i\omega_m B_{am} = 0,$$
$$\Leftrightarrow B_{am} = -i\omega_m M_{am}, \tag{A5}$$

$$-\omega_n^2 M_{mn} + i\omega_n B_{mn} + K_{mn} = 0. \tag{A6}$$

When $m \neq n$,

$$\begin{aligned}\omega_n^2 M_{mn} - i\omega_n B_{mn} &= K_{mn} = K_{-n-m} \\ &= \omega_{-m}^2 M_{-n-m} - i\omega_{-m} B_{-n-m} \\ &= \omega_m^2 M_{mn} - i\omega_m B_{mn}\end{aligned} \tag{A7}$$

is satisfied from the symmetry (A2)–(A4). Therefore,

$$K_{mn} = -\omega_m\omega_n M_{mn}, \qquad B_{mn} = -i(\omega_m + \omega_n)M_{mn}, \quad \text{(A8)}$$

for any $m \neq n$. For $m = n$, (A7) is trivial and (A8) is not valid because of the positivity. Instead, we can generalize (A8) as

$$K_{mn} = -\omega_m\omega_n M_{mn} + 2\mu_m\omega_m\omega_n\delta_{mn}, \quad \text{(A9)}$$

$$B_{mn} = -i(\omega_m + \omega_n)M_{mn} + 2i\mu_m\omega_m\delta_{mn}. \quad \text{(A10)}$$

Here, μ_m is the "norm" of ξ_m introduced in (51).

The cyclotron zero mode components are quite simple. First, it is easy to see that

$$B_{AB} = -ib_B M_{AB} = -ib_A M_{AB}$$
$$\Rightarrow M_{AB} = \mu_A\delta_{AB}, \quad \text{(A11)}$$

with the norm μ_A. From the condition (28), we have

$$0 = B_{iA} = -ib_A M_{iA}. \quad \text{(A12)}$$

For the inner products with the oscillating modes, we have

$$M_{Am} = \frac{i}{\omega_m}B_{Am} = \frac{b_A}{\omega_m}M_{Am}$$
$$\Rightarrow M_{Am} = 0, B_{Am} = 0. \quad \text{(A13)}$$

Here, we assume that $b_A \neq \omega_m$. In the case $b_A = \omega_m$, one can redefine ξ_m to be orthogonal to ξ_A. Since the constraint zero modes and η_i are defined to satisfy (35) and (61), respectively, the cyclotron zero modes are orthogonal to both. Hence, they are orthogonal to all other modes.

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