

Weyl anomaly for chiral fermions

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We compute the parity-odd part of the Weyl anomaly for chiral fermions in a background gravitational field. We start from a manifestly real form of the Lagrangian (that is, not only real up to a total derivative), and we regularize it by means of Pauli-Villars fermions. All parity-odd terms in the anomaly cancel in the *integrand*, so that the result of the anomaly is necessarily parity even.

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I. INTRODUCTION

There has recently been some interest in the Weyl (conformal) anomaly of a chiral fermion on a background gravitational field with metric $g_{\mu\nu}$. This was motivated by a paper by Bonora, Giaccari, and Lima de Souza (cf. the first paper in [1]) claiming that there is a parity-odd term in the trace anomaly proportional to the Pontryagin index. Symbolically

$$\langle g^{\mu\nu} T_{\mu\nu} \rangle \sim i R_{\mu\nu\rho\sigma} \eta^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}, \quad (1.1)$$

with an *imaginary* coefficient. Here $R_{\mu\nu\rho\sigma}$ are the components of Riemann's tensor and $\eta_{\rho\sigma\alpha\beta} = |g|^{1/2} \epsilon_{\rho\sigma\alpha\beta}$, $\epsilon_{\rho\sigma\alpha\beta}$ being the Levi-Civita symbol.

In addition, in [1] it was claimed that this contribution was suggested earlier by Christensen and Duff [2]. Actually, in [2] the conformal anomaly corresponding to a Euclidean field transforming with the (A, B) representation of the $SO(4)$ group is determined in terms of the Schwinger-DeWitt coefficients as

$$T_\mu^\mu = \frac{1}{2} (-1)^{2A+2B} (b_4(A, B) + b_4(B, A)). \quad (1.2)$$

Although both $b_4(1/2, 0)$ and $b_4(0, 1/2)$ contain pieces proportional to Pontryagin's term, they cancel in the sum, which is indeed parity even.

This term would violate unitarity and point to a fatal inconsistency in the physics of such chiral fermions. The main reason is that the energy momentum tensor is the coefficient of the response of the action to an arbitrary (but real) variation of the metric

$$\delta S \equiv \int d(\text{vol}) T_{\mu\nu} \delta g^{\mu\nu}. \quad (1.3)$$

If the trace of the energy momentum tensor is not real, then the action itself is also not real, which violates unitarity among other things.

This result however disagrees with the computations of some authors [2–8], but some other works like [9] support it. A useful summary of the literature discussing the proposal can be found in the book [10].

To be precise, in Ref. [1] it is argued that the total conformal anomaly of a chiral fermion is given by

$$T_\mu^\mu = \frac{1}{180 \times 16\pi^2} \left(\frac{11}{4} E_4 - \frac{9}{2} W^2 + i \frac{15}{4} P_1 \right). \quad (1.4)$$

Here E_4 is the integrand of the Euler characteristic, a topological invariant given by

$$\begin{aligned} \chi(M) &\equiv \frac{1}{32\pi^2} \int \eta_{abcd} R^{ab} \wedge R^{cd} = \int d(\text{vol}) E_4 \\ &= \frac{1}{2} \frac{1}{(4\pi)^2} \int d(\text{vol}) R^* R^* \\ &= \frac{1}{32\pi^2} \int d(\text{vol}) (R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2) \end{aligned} \quad (1.5)$$

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and W^2 is the square of Weyl's tensor, which can be characterized by being invariant under conformal transformations

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad (1.6)$$

namely,

$$W^\mu{}_{\nu\rho\sigma} \rightarrow W^\mu{}_{\nu\rho\sigma}. \quad (1.7)$$

It follows that $\sqrt{|g|}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$ is a conformal singlet in $n = 4$ dimensions:

$$W^2 \equiv W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2 \quad (1.8)$$

and P_1 is the Pontryagin density which upon integration and normalization yields the Poincaré number, $P_1(M)$ (another topological invariant [11]):

$$P_1(M) \equiv -\frac{1}{8\pi^2} \int \text{tr} R \wedge R = \frac{1}{(4\pi)^2} \int d(\text{vol}) R_{\mu\nu\rho\sigma}^* R^{\mu\nu\rho\sigma}. \quad (1.9)$$

The dual Riemann tensor is given by

$$R_{\mu\nu\alpha\beta}^* \equiv \frac{1}{2} \eta_{\mu\nu\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta}. \quad (1.10)$$

Our aim in this paper is to compute this anomaly using a different, and in our opinion, safer procedure [12–14]. We will start from a real Lagrangian and regulate the theory using Pauli-Villars fermions in an attempt to avoid the ambiguities of dimensional regularization in the presence of γ_5 .

Our main interest will be in the parity-odd part of the anomaly (the coefficient of the Pontryagin invariant) which is where there is disagreement in the literature.

Let us now be more precise about the definition of the problem.

First, the vierbein $e_\mu^a(x)$ is defined as follows:

$$\eta_{ab} e_\mu^a(x) e_\nu^b(x) = g_{\mu\nu}(x).$$

The Latin indices are Lorentz indices whereas the Greek indices are curvilinear indices.

Let $\omega_{\mu ab}(x)$ be the Lorentz (spin) connection defined in terms of the Levi-Civita one by

$$\omega_{\mu ab}(x) = -e_b^\lambda(x) \partial_\mu e_{\lambda a}(x) + e_{\lambda a}(x) \Gamma_{\rho\mu}^\lambda e_b^\rho(x).$$

Then, the classical action of the model reads as

$$S = \frac{1}{2} \int d^4x e(x) e_\mu^a(x) [i \bar{\xi}_\alpha(x) \bar{\sigma}^{\dot{a}aa} (\mathcal{D}_\mu \xi)_\alpha(x) - i (\mathcal{D}_\mu \bar{\xi})_{\dot{\alpha}}(x) \bar{\sigma}^{\dot{a}aa} \xi_\alpha(x)],$$

where $e(x) = \det e_\mu^a(x)$ and

$$(\mathcal{D}_\mu \xi)_\alpha(x) = \partial_\mu \xi_\alpha(x) - \frac{i}{2} \omega_{\mu ab}(x) (\sigma^{ab} \xi)_\alpha(x),$$

$$(\mathcal{D}_\mu \bar{\xi})_{\dot{\alpha}}(x) = \partial_\mu \bar{\xi}_{\dot{\alpha}}(x) + \frac{i}{2} (\bar{\xi}(x) \bar{\sigma}^{ab})_{\dot{\alpha}} \omega_{\mu ab}(x).$$

Here, following [15], we define

$$\sigma^{ab} = \frac{i}{4} (\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a), \quad \bar{\sigma}^{ab} = \frac{i}{4} (\bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a).$$

We shall denote spinorial indices by the first letters of the Greek alphabet ($\alpha, \dot{\alpha}, \beta, \dot{\beta}, \dots = 1, 2$) and reserve the middle letters of the Greek alphabet for four-dimensional vector indices ($\mu, \nu, \dots = 0, \dots, 3$). Our convention [15] for the Pauli matrices is as follows:

$$\begin{aligned} \sigma^0 &= \bar{\sigma}^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \sigma^1 &= -\bar{\sigma}^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma^2 &= -\bar{\sigma}^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \sigma^3 &= -\bar{\sigma}^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (1.11)$$

There is another notation frequently used representing chiral spinors as left and right ones, ψ_L or ψ_R . The relationship with ours is

$$\begin{aligned} \psi_L &\leftrightarrow \xi_\alpha, \\ \psi_R &\leftrightarrow \bar{\xi}_{\dot{\alpha}}. \end{aligned} \quad (1.12)$$

It is important to remark that the action is real without total derivatives. This fact will be relevant later on. The reader should bear in mind that the complex conjugation of the spinors is defined as in [15]:

$$(\xi_\alpha)^* = \bar{\xi}_{\dot{\alpha}}.$$

In constructing the quantum theory for the fermions in perturbation theory around Minkowski spacetime, we shall express the metric, $g_{\mu\nu}$, the vierbein e_μ^a , the inverse vierbein, e_a^μ , and the spin connection, $\omega_{a,bc}$ as follows (the perturbative parameter is related to Newton's constant by $\kappa^2 \equiv 8\pi G$):

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}, \\ e_\mu^a &= \delta_\mu^a + \kappa \mathcal{E}_\mu^{(1)a} + \kappa^2 \mathcal{E}_\mu^{(2)a} + \dots, \\ e_a^\mu &= \delta_a^\mu + \kappa \mathcal{E}_a^{(1)\mu} + \kappa^2 \mathcal{E}_a^{(2)\mu} + \dots, \\ \omega_{a,bc} &= e_a^\mu \omega_{\mu bc} = \kappa \Omega_{a,bc}^{(1)} + \kappa^2 \Omega_{a,bc}^{(2)} + \dots, \end{aligned}$$

in the above the terms involving the external gravitons become

$$\begin{aligned}\mathcal{E}_\mu^{(1)a} &= \frac{1}{2}h_a^\mu, & \mathcal{E}_\mu^{(2)a} &= -\frac{1}{8}h^{a\rho}h_{\rho\mu}, \\ \mathcal{E}_a^{(1)\mu} &= -\frac{1}{2}h_a^\mu, & \mathcal{E}_a^{(2)\mu} &= \frac{3}{8}h^{\mu\rho}h_{\rho a}, \\ \Omega_{a,bc}^{(1)} &= -\frac{1}{2}(\partial_b h_{ca} - \partial_c h_{ba}), \\ \Omega_{a,bc}^{(2)} &= \frac{1}{4}h_a^\mu(\partial_b h_{c\mu} - \partial_c h_{b\mu}) - \frac{1}{8}(h_b^\rho \partial_a h_{c\rho} - h_c^\rho \partial_a h_{b\rho}) \\ &\quad + \frac{1}{4}(h_b^\rho \partial_\rho h_{ca} - h_c^\rho \partial_\rho h_{ba}) - \frac{1}{4}(h_b^\rho \partial_c h_{\rho a} - h_c^\rho \partial_b h_{\rho a}).\end{aligned}$$

The linearized conformal transformation reads as

$$\delta_\theta h_{\mu\nu} = 2\left(\frac{\theta(x)}{\kappa}\eta_{\mu\nu} + h_{\mu\nu}\right). \quad (1.13)$$

The previous expansions in powers of κ , leads to the following expansion of the action S :

$$\begin{aligned}S &= S_0 + S_{\text{int}}, \\ S_0 &= \frac{1}{2}\int d^4x \left[i\bar{\xi}_\alpha(x)\bar{\sigma}^{\dot{\alpha}\alpha\mu}(\partial_\mu \xi)_\alpha(x) - i(\partial_\mu \bar{\xi})_{\dot{\alpha}}(x)\bar{\sigma}^{\dot{\alpha}\alpha\mu}\xi_\alpha(x) \right], \\ S_{\text{int}} &= \sum_n \kappa^n S^{(n)}.\end{aligned}$$

In both S_0 and S_{int} all the indices are flat. S_0 is the free action in Minkowski spacetime and S_{int} carry the interaction vertices between the graviton field $h_{\mu\nu}$ and the fermions. $S^{(n)}$ is quadratic on the fermion fields and involves n powers of $h_{\mu\nu}$ and/or its derivatives.

Let $\mathcal{Z}[h_{\mu\nu}]$ be the partition function of the model defined by perturbation theory around the Minkowski background:

$$\begin{aligned}\mathcal{Z}[h_{\mu\nu}] &= \frac{1}{\mathcal{N}} \int \mathcal{D}\xi_\alpha \mathcal{D}\bar{\xi}_{\dot{\alpha}} e^{iS} = \langle e^{iS_{\text{int}}} \rangle_0, \\ \mathcal{N} &= \int \mathcal{D}\xi_\alpha \mathcal{D}\bar{\xi}_{\dot{\alpha}} e^{iS_0}.\end{aligned}$$

$\langle \mathcal{O} \rangle_0$ denotes the average of $\mathcal{O}$ with regard to the free action S_0 :

$$\langle \mathcal{O} \rangle_0 = \frac{1}{\mathcal{N}} \int \mathcal{D}\xi_\alpha \mathcal{D}\bar{\xi}_{\dot{\alpha}} \mathcal{O} e^{iS_0}.$$

Then, the quantum action, $\mathcal{W}[h_{\mu\nu}]$, of the model is defined to be

$$\mathcal{W}[h_{\mu\nu}] = -i \text{Ln} \mathcal{Z}[h_{\mu\nu}] = -i \sum_n \frac{i^n}{n!} \langle (S_{\text{int}})^n \rangle_0^{(\text{connected})}. \quad (1.14)$$

The superscript “connected” signals that only connected contractions, as given by Wick’s theorem, are to be kept.

A. Pauli-Villars fields and the regularized quantum action

As it stands, $\mathcal{W}[h_{\mu\nu}]$, (1.14) is ill defined due to ultraviolet (UV) divergences. We shall associate to this ill-defined quantity a regularized—and, therefore, well-defined mathematically—expression. This will be done by introducing several massive Pauli-Villars (PV) bispinor fields: $\xi_\alpha^{(i)}$, $\bar{\xi}_{\dot{\alpha}}^{(i)}$, $i = 1 \dots f$, with classical actions

$$\begin{aligned}S^{(i)} &= \frac{1}{2} \int d^4x e(x) e_a^\mu(x) \left[i\bar{\xi}_{\dot{\alpha}}^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha a} (\mathcal{D}_\mu \xi_\alpha^{(i)})(x) \right. \\ &\quad \left. - i(\mathcal{D}_{\dot{\alpha}} \bar{\xi}^{(i)})(x) \bar{\sigma}^{\dot{\alpha}\alpha a} \xi_\alpha^{(i)}(x) \right] \\ &\quad - \frac{m_i}{2} \int d^4x e(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_{\dot{\alpha}}^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right],\end{aligned}$$

where $\xi^{(i)\alpha} = \epsilon^{\alpha\beta} \xi_\beta^{(i)}$ and $\bar{\xi}^{(i)\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}}^{(i)}$.

The usual PV regulators (as explained, for example in Faddeev and Slavnov’s book [16] page 131, or in Itzykson and Zuber’s [17] page 411) consist in replacing some fermionic Gaussian integral

$$\det \not{D} \quad (1.15)$$

by

$$\det \not{D} \prod_{j=1}^f (\det \not{D}_j)^{c_j}, \quad (1.16)$$

where the massive covariant derivatives are defined as

$$\not{D}_j \equiv \not{D} - m_j, \quad (1.17)$$

where the set of masses m_j and constants c_j are determined by some equations that will be made explicit in what follows. In our case the Gaussian integral does not really define a determinant, but the regularization proceeds by replacing everywhere in the above formulas the determinant in them by our Gaussian integral, i.e., by replacing the Gaussian path integrals which formally give the determinants in (1.16) with our Gaussian path integrals.

By using the PV regulators, we define

$$\begin{aligned}\mathcal{W}^{(i)}[h_{\mu\nu}] &= -i \text{Ln} \mathcal{Z}_i[h_{\mu\nu}] \\ &= -i \sum_n \frac{i^n}{n!} \langle (S_{\text{int}}^{(i)})^n \rangle_0^{(\text{connected})}, \quad i = 1 \dots f.\end{aligned} \quad (1.18)$$

Finally, we regularize $\mathcal{W}[h_{\mu\nu}]$ by replacing it with $\mathcal{W}^{(PV)}[h_{\mu\nu}]$, the latter defined as follows:

$$\mathcal{W}^{(PV)}[h_{\mu\nu}] = \sum_{i=0}^f c_i \mathcal{W}^{(i)}[h_{\mu\nu}], \quad (1.19)$$

with the understanding that $i = 0$ labels the physical field: $\xi_\alpha^{(0)} = \xi_\alpha$, $\bar{\xi}_{\dot{\alpha}}^{(0)} = \bar{\xi}_{\dot{\alpha}}$, $m_0 = 0$ and $c_0 = 1$. The coefficients c_i are real numbers.

The c_i 's above are the PV parameters and, along with Pauli-Villars masses, are to be chosen so that the loop integrals are UV finite by power counting at nonexceptional momenta. This regularization breaks the fermion number owing to the PV mass terms, but respects diffeomorphism invariance. Any dependence on the PV masses will disappear in the physical theory.

It is well known that when dealing with anomalies it is of paramount importance not to carry out any kind of simplification in the integrands of the nonregularized UV divergent integrals. So, here, we shall be careful and choose the c_i 's and the Pauli-Villars masses so that all the loop integrals are regularized before carrying out any manipulations in the integrands of the loop integrals.

It is not difficult to come to the conclusion that the most UV divergent loop integrals which occur in $\mathcal{W}[h_{\mu\nu}]$ are

$$\int \frac{d^4 q}{(2\pi)^4} \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2n}}}{(q + P_1)^2 (q + P_2)^2 \cdots (q + P_n)^2},$$

where the P_i 's are (nonexceptional) linear combinations of the external momenta. The Pauli-Villars regularization of the previous integral amounts to its replacement with

$$\int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=0}^f c_i \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2n}}}{((q + P_1)^2 - m_i^2)((q + P_2)^2 - m_i^2) \cdots ((q + P_n)^2 - m_i^2)} \right]. \quad (1.20)$$

In (1.17) we have added to the action fields with similar couplings to gravity as the original fields but with equal or opposite statistics; namely, the coefficients c_i can be negative. The masses m_j and coefficients c_i are chosen in such a way that the ratio is UV finite.

In fact all these integrals will be UV finite by power counting if the c_i 's and m_i 's satisfy the Pauli-Villars conditions, which, in the case at hand, become

$$\begin{aligned} c_0 &= 1, & \sum_{i=0}^f c_i &= 0, \\ \sum_{i=1}^f c_i m_i^2 &= 0 & \text{and} & \sum_{i=1}^f c_i m_i^4 = 0. \end{aligned}$$

The condition on m_i^4 has to do with the fact that the integrals in (1.20) diverge as the fourth power of the loop momentum. An explicit set that satisfies the above conditions is given by

$$\begin{aligned} m_1 &= M, & m_2 &= M\sqrt{2}, & m_3 &= M\sqrt{3}, \\ c_1 &= -3, & c_2 &= 3, & c_3 &= -1. \end{aligned} \quad (1.21)$$

But this is not all, for the mass terms of the Pauli-Villars fields also couple to gravity so that we shall have interaction vertices linear in the m_i 's. These vertices do not occur in the massless physical theory. Further, unlike in the massless case, the free propagators

$$\langle \xi_\alpha^{(i)}(x) \xi_\beta^{(i)}(y) \rangle_0 = \int \frac{d^4 p}{(2\pi)^4} m_i \epsilon_{\beta\alpha} G_i(p) e^{-ip(x-y)},$$

$$\langle \bar{\xi}_{\dot{\alpha}}^{(i)}(x) \bar{\xi}_{\dot{\beta}}^{(i)}(y) \rangle_0 = \int \frac{d^4 p}{(2\pi)^4} m_i \epsilon_{\dot{\alpha}\dot{\beta}} G_i(p) e^{-ip(x-y)},$$

$$G_i(p) = \frac{i}{p^2 - m_i^2} \quad (1.22)$$

have numerators linear in the m_i 's.

Now, the integrals in (1.20) come from contributions where only propagators of the type $\langle \xi_\alpha^{(i)}(x) \bar{\xi}_{\dot{\alpha}}^{(i)}(y) \rangle_0$ and only vertices which read as

$$\int d^4 x H_\nu^\mu[h](x) (\bar{\xi} \bar{\sigma}^\nu \partial_\mu \xi - \partial_\mu \bar{\xi} \bar{\sigma}^\nu \xi)(x), \quad (1.23)$$

$H_\nu^\mu[h](x)$ being a function of $h_{\mu\nu}$, occur. Hence, each time a propagator of the type $\xi \bar{\xi}$ is replaced, in an integral such as (1.20), by a propagator of the kind displayed in (1.22), the UV degree of divergence of the corresponding integral decreases by one unit. The same decrease by one unit takes place when a vertex of type (1.23) is replaced by a vertex linear in m_i . We thus conclude that only integrals with fewer than 5 powers of m_i can be UV divergent: recall that the UV degree of divergence of the integral in (1.20) is 4. Let us point out that since the Pauli-Villars mass terms break the fermion number symmetry, only even powers of the m_i 's occur in the Feynman integrals. To conclude, we will meet UV divergent integrals of the type

$$\int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=1}^f c_i m_i^{2n'} \frac{q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2(n-n')}}}{((q+P_1)^2 - m_i^2)((q+P_2)^2 - m_i^2) \cdots ((q+P_n)^2 - m_i^2)} \right],$$

with $n' = 1, 2$. Making these integrals UV finite does not require new Pauli-Villars conditions.

II. NO PARITY-ODD TERMS IN THE CONFORMAL ANOMALY

A simple way of understanding the Weyl anomaly stems from a Weyl transformation in the path integral [18]. This will be enough to show that no parity-odd terms appear in the anomaly. Recall that

$$\begin{aligned} S[g, \xi, \bar{\xi}] &= \frac{1}{2} \int d^4 x e(x) e_a^\mu(x) \left[i \bar{\xi}_\alpha(x) \bar{\sigma}^{\dot{\alpha}aa} (\mathcal{D}_\mu \xi)_a(x) - i (\mathcal{D}_\mu \bar{\xi})_{\dot{\alpha}}(x) \bar{\sigma}^{\dot{\alpha}aa} \xi_a(x) \right], \\ S^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}] &= \frac{1}{2} \int d^4 x e(x) e_a^\mu(x) \left[i \bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}aa} (\mathcal{D}_\mu \xi)_a^{(i)}(x) - i (\mathcal{D}_\mu \bar{\xi}^{(i)})_{\dot{\alpha}}(x) \bar{\sigma}^{\dot{\alpha}aa} \xi_a^{(i)}(x) \right], \\ S_m^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}] &= -\frac{m_i}{2} \int d^4 x e(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right], \end{aligned}$$

and that in (1.19) we defined the regularized Pauli-Villars effective action $\mathcal{W}^{(PV)}[g_{\mu\nu}]$ by

$$\mathcal{W}^{(PV)}[g_{\mu\nu}] = -i \sum_{i=0}^f c_i \ln \mathcal{Z}_i[g_{\mu\nu}],$$

where

$$\mathcal{Z}_i[g_{\mu\nu}] = \frac{1}{\mathcal{N}_i} \int \mathcal{D}\xi_\alpha^{(i)} \mathcal{D}\bar{\xi}_\alpha^{(i)} e^{iS^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}] + iS_m^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}]}.$$

Then by performing the Weyl transformations

$$g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \quad \xi \rightarrow \Omega^{-3/2} \xi, \quad \bar{\xi} \rightarrow \Omega^{-3/2} \bar{\xi}, \quad \xi^{(i)} \rightarrow \Omega^{-3/2} \xi^{(i)}, \quad \bar{\xi}^{(i)} \rightarrow \Omega^{-3/2} \bar{\xi}^{(i)}, \quad (2.1)$$

we get

$$\mathcal{W}^{(PV)}[\Omega^2 g_{\mu\nu}] = -i \sum_{i=0}^f c_i \ln \mathcal{Z}_i[\Omega^2 g_{\mu\nu}] = -i \sum_{i=0}^f c_i \ln \frac{1}{\mathcal{N}_i} \int \mathcal{D}\xi_\alpha^{(i)} \mathcal{D}\bar{\xi}_\alpha^{(i)} e^{iS^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}] + iS_m^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}; \Omega]}, \quad (2.2)$$

where the transformed PV mass terms read as

$$S_m^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}; \Omega] = -\frac{m_i}{2} \int d^4 x \Omega(x) e(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right].$$

The previous equation shows that the Pauli-Villars mass terms explicitly break Weyl invariance.

Let us linearize around the unit transformation

$$\Omega(x) = 1 + \theta(x), \quad 0 \leq \theta(x) \ll 1.$$

Now, the infinitesimal version of (2.2) reads as

$$\delta_\theta \mathcal{W}^{(PV)}[g_{\mu\nu}] = \sum_{i=1}^f c_i \frac{1}{\mathcal{Z}_i[g] \mathcal{N}_i} \int \mathcal{D}\xi_\alpha^{(i)} \mathcal{D}\bar{\xi}_\alpha^{(i)} \left(\delta_\theta S_m^{(i)} \right) e^{iS^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}] + iS_m^{(i)}[g, \xi^{(i)}, \bar{\xi}^{(i)}]}, \quad (2.3)$$

where

$$\delta_\theta S_m^{(i)} = -\frac{1}{2} m_i \int d^4x \theta(x) e(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right], \quad i \geq 1.$$

The right-hand side of (2.3) yields the anomaly, and at order 2 in the number of graviton fields $h_{\mu\nu}$ it is given by

$$\begin{aligned} \mathcal{A}_2 = & -\frac{1}{2} \sum_{i=1}^f c_i \left\langle \delta_\theta^{(0)} S_m^{(i)} S_{\text{kin}}^{(i),1} S_{\text{kin}}^{(i),1} \right\rangle_0^{(c)} + i \sum_{i=1}^f c_i \left\langle \delta_\theta^{(0)} S_m^{(i)} S_{\text{kin}}^{(i),2} \right\rangle_0^{(c)} + i \sum_{i=1}^f c_i \left\langle \delta_\theta^{(0)} S_m^{(i)} S_{\text{spin}}^{(i),2} \right\rangle_0^{(c)} \\ & + i \sum_{i=1}^f c_i \left\langle \delta_\theta^{(1)} S_m^{(i)} S_{\text{kin}}^{(i),1} \right\rangle_0^{(c)} + \sum_{i=1}^f c_i \left\langle \delta_\theta^{(2)} S_m^{(i)} \right\rangle_0^{(c)}. \end{aligned} \quad (2.4)$$

Let us recall that the superscript (c) stands for the connected contributions, the subscript 0 denotes the vacuum expectation value when $h_{\mu\nu}$ is set to zero, and the Weyl variation of the PV mass terms becomes

$$\begin{aligned} \delta_\theta^{(0)} S_m^{(i)} &= -\frac{1}{2} m_i \int d^4x \theta(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right], \\ \delta_\theta^{(1)} S_m^{(i)} &= -\frac{1}{4} m_i \int d^4x \theta(x) h_\mu^\mu(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right], \\ \delta_\theta^{(2)} S_m^{(i)} &= -\frac{1}{4} m_i \int d^4x \theta(x) H_2(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right], \end{aligned}$$

and the kinetic and spin parts are defined as

$$\begin{aligned} S_{\text{kin}}^{(i),1} &= \int d^4x \left\{ \frac{i}{2} H_{1,\nu}^\mu(x) \left[\bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha\nu} \partial_\mu \xi_\alpha^{(i)}(x) - \partial_\mu \bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha\nu} \xi_\alpha^{(i)}(x) \right] \right. \\ &\quad \left. - \frac{m_i}{4} h_\mu^\mu(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right] \right\}, \\ S_{\text{kin}}^{(i),2} &= \int d^4x \left\{ \frac{i}{2} H_{2,\nu}^\mu(x) \left[\bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha\nu} \partial_\mu \xi_\alpha^{(i)}(x) - \partial_\mu \bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha\nu} \xi_\alpha^{(i)}(x) \right] \right. \\ &\quad \left. - \frac{m_i}{4} H_2(x) \left[\xi^{(i)\alpha}(x) \xi_\alpha^{(i)}(x) + \bar{\xi}_\alpha^{(i)}(x) \bar{\xi}^{(i)\dot{\alpha}}(x) \right] \right\}, \\ S_{\text{spin}}^{(i),2} &= \int d^4x \frac{1}{16} \epsilon^{\mu\nu\rho\sigma} h_\mu^\lambda(x) \partial_\nu h_{\rho\lambda}(x) \bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}_\sigma^{\dot{\alpha}\alpha} \xi_\alpha^{(i)}(x), \\ H_{1,\nu}^\mu(x) &= \frac{1}{2} \left(h_\rho^\rho(x) \delta_\nu^\mu - h_\nu^\mu(x) \right), \\ H_{2,\nu}^\mu(x) &= -\frac{1}{4} h_\rho^\rho(x) h_\nu^\mu(x) + \frac{1}{8} (h_\rho^\rho(x))^2 \delta_\nu^\mu - \frac{1}{4} h^{\rho\sigma}(x) h_{\rho\sigma}(x) + \frac{3}{8} h^{\mu\rho}(x) h_{\rho\nu}(x), \\ H_2(x) &= \frac{1}{4} (h_\rho^\rho(x))^2 - \frac{1}{2} h^{\rho\sigma}(x) h_{\rho\sigma}(x). \end{aligned}$$

Before we show that there is no parity-odd contribution to $\mathcal{A}_2$ in (2.4), let us discuss how this result bears on the properties of the renormalized effective action obtained from our regularized effective action $\mathcal{W}^{(PV)}[h_{\mu\nu}]$ in (1.19). Let us define the renormalized effective action by minimal subtraction, i.e., by subtracting from $\mathcal{W}^{(PV)}[h_{\mu\nu}]$ its minimal UV divergent contribution. Then we conclude that this subtraction will not give rise to any anomalous parity-odd Weyl contribution to the three point part of renormalized effective action, for it turns out that $\mathcal{A}_2$ carries no parity-odd contribution for any value of the regulator masses and parameters. This in turn implies that no parity-odd Weyl anomaly occurs in the renormalized theory under consideration since, as shown in [19], the only parity-odd term which can arise reads as

$$r_2 \int d^4x \sqrt{|g|(x)} \theta(x) R_{\mu\nu\rho\sigma}^*(x) R^{\mu\nu\rho\sigma}(x).$$

The coefficient r_2 is just given by the term order 2 in the number of $h_{\mu\nu}$ in the previous expression. Of course, the value of r_2 can be obtained from $\mathcal{A}_2$.

A. No parity-odd anomaly from triangles

There are terms including a variation of a Pauli-Villars mass and two terms from the kinetic terms that would give rise, upon Wick contractions, to *triangle* type Feynman diagrams. Let us examine them.

First of all,

$$-\frac{1}{2} \sum_{i=1}^f c_i \left\langle \delta_\theta^{(0)} S_m^{(i)} S_{\text{kin}}^{(i),1} S_{\text{kin}}^{(i),1} \right\rangle_0^{(c)} = -\frac{1}{2} (\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3).$$

We shall show below that this contribution to (2.4) yields no parity-odd anomalous term. Indeed, in the previous equation, $\mathcal{T}_m$, $m = 1, 2, 3$, is given by

$$\begin{aligned} \mathcal{T}_1 = & \frac{1}{8} \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \\ & \times \left\{ \sum_{i=1}^f c_i m_i \left\langle \left[\xi^{(i)\alpha} \xi_\alpha^{(i)}(x_1) + \bar{\xi}_\alpha^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_1) \right] \left[j_{R\mu_2}^{(i)\nu_2}(x_2) - j_{L\mu_2}^{(i)\nu_2}(x_2) \right] \left[j_{R\mu_3}^{(i)\nu_3}(x_3) - j_{L\mu_3}^{(i)\nu_3}(x_3) \right] \right\rangle_0^{(c)} \right\}. \end{aligned} \quad (2.5)$$

Here the left and right currents are defined as

$$\begin{aligned} j_R^{(i)\nu}{}_\mu(x) &\equiv \bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha\nu} \partial_\mu \xi_\alpha^{(i)}(x), \\ j_L^{(i)\nu}{}_\mu(x) &\equiv \partial_\mu \bar{\xi}_\alpha^{(i)}(x) \bar{\sigma}^{\dot{\alpha}\alpha\nu} \xi_\alpha^{(i)}(x). \end{aligned} \quad (2.6)$$

$$\begin{aligned} \mathcal{T}_2 = & \frac{1}{8} i \int dx_1 dx_2 dx_3 \theta(x_1) h_{\mu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \\ & \times \left\{ \sum_{i=1}^f c_i m_i^2 \left\langle \left[\xi^{(i)\alpha} \xi_\alpha^{(i)}(x_1) + \bar{\xi}_\alpha^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_1) \right] \left[\xi^{(i)\alpha} \xi_\alpha^{(i)}(x_2) + \bar{\xi}_\alpha^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_2) \right] \left[j_{R\mu_3}^{(i)\nu_3}(x_3) - j_{L\mu_3}^{(i)\nu_3}(x_3) \right] \right\rangle_0^{(c)} \right\} \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \mathcal{T}_3 = & -\frac{1}{32} \int dx_1 dx_2 dx_3 \theta(x_1) h_{\mu_2}^{\mu_2}(x_2) h_{\mu_3}^{\mu_3}(x_3) \\ & \times \left\{ \sum_{i=1}^f c_i m_i^3 \left[\xi^{(i)\alpha} \xi_\alpha^{(i)}(x_1) + \bar{\xi}_\alpha^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_1) \right] \left[\xi^{(i)\alpha} \xi_\alpha^{(i)}(x_2) + \bar{\xi}_\alpha^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_2) \right] \left[\xi^{(i)\alpha} \xi_\alpha^{(i)}(x_3) + \bar{\xi}_\alpha^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_3) \right] \right\}. \end{aligned} \quad (2.8)$$

1. No parity-odd Weyl anomaly from $\mathcal{T}_1$

Let us express $\mathcal{T}_1$ in (2.5) in terms of the three different products of the left and right currents and their symmetrizations

$$\mathcal{T}_1 = \frac{1}{8} \sum_{m=1}^6 \mathcal{T}_{1m},$$

where

$$\begin{aligned}
 \mathcal{T}_{11} &= \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \left\{ \sum_{i=1}^f c_i m_i \left\langle \xi^{(i)\alpha} \xi_{\alpha}^{(i)}(x_1) j_{R\mu_2}^{(i)\nu_2}(x_2) j_{R\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right\}, \\
 \mathcal{T}_{12} &= \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \left\{ \sum_{i=1}^f c_i m_i \left\langle \bar{\xi}_{\dot{\alpha}}^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_1) j_{R\mu_2}^{(i)\nu_2}(x_2) j_{R\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right\}, \\
 \mathcal{T}_{13} &= \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \left\{ \sum_{i=1}^f c_i m_i \left\langle \xi^{(i)\alpha} \xi_{\alpha}^{(i)}(x_1) j_{L\mu_2}^{(i)\nu_2}(x_2) j_{L\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right\}, \\
 \mathcal{T}_{14} &= \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \left\{ \sum_{i=1}^f c_i m_i \left\langle \bar{\xi}_{\dot{\alpha}}^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_1) j_{L\mu_2}^{(i)\nu_2}(x_2) j_{L\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right\}, \\
 \mathcal{T}_{15} &= -2 \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \left\{ \sum_{i=1}^f c_i m_i \left\langle \xi^{(i)\alpha} \xi_{\alpha}^{(i)}(x_1) j_{R\mu_2}^{(i)\nu_2}(x_2) j_{L\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right\}, \\
 \mathcal{T}_{16} &= -2 \int dx_1 dx_2 dx_3 \theta(x_1) H_{1,\nu_2}^{\mu_2}(x_2) H_{1,\nu_3}^{\mu_3}(x_3) \left\{ \sum_{i=1}^f c_i m_i \left\langle \bar{\xi}_{\dot{\alpha}}^{(i)} \bar{\xi}^{(i)\dot{\alpha}}(x_1) j_{L\mu_2}^{(i)\nu_2}(x_2) j_{R\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right\}.
 \end{aligned}$$

Recall that the integrand of $\mathcal{T}_1$ is symmetric under the exchange of (μ_2, ν_2, x_2) and (μ_3, ν_3, x_3) .

In the sequel the following traces of Pauli matrices—see Ref. [15]—will be used to identify the parity-odd contributions to the connected Green functions we shall be dealing with:

$$\text{tr}[\sigma^{\mu} \bar{\sigma}^{\nu} \sigma^{\rho} \bar{\sigma}^{\delta}] = 2(\eta^{\mu\nu} \eta^{\rho\delta} - \eta^{\mu\rho} \eta^{\nu\delta} + \eta^{\mu\delta} \eta^{\nu\rho} + i\epsilon^{\mu\nu\rho\delta}). \quad (2.9)$$

The application of Wick's theorem and (2.9) readily leads to the following results for the parity-odd contributions of $\mathcal{T}_{1m}$:

$$\begin{aligned}
 \mathcal{T}_{11}[\text{odd}] &= 4 \int \prod_{j=1}^3 \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + p_3) \theta(p_1) H_{1,\nu_2}^{\mu_2}(p_2) H_{1,\nu_3}^{\mu_3}(p_3) \\
 &\quad \times \left\{ \int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=1}^f c_i m_i^2 \frac{\epsilon^{\nu_4 \nu_5 \nu_2 \nu_3} [(q + p_3)_{\mu_2} (q + p_3)_{\mu_3} (p_2 + p_3)_{\nu_4} q_{\nu_5} - 2(q + p_2 + p_3)_{\mu_2} (q + p_3)_{\mu_3} p_{3\nu_4} q_{\nu_5}] }{(q^2 - m_i^2)((q + p_3)^2 - m_i^2)((q + p_2 + p_3)^2 - m_i^2)} \right] \right\}, \\
 \mathcal{T}_{12}[\text{odd}] &= 4 \int \prod_{j=1}^3 \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + p_3) \theta(p_1) H_{1,\nu_2}^{\mu_2}(p_2) H_{1,\nu_3}^{\mu_3}(p_3) \\
 &\quad \times \left\{ \int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=1}^f c_i m_i^2 \frac{\epsilon^{\nu_4 \nu_5 \nu_2 \nu_3} [2(q + p_2 + p_3)_{\mu_2} (q + p_3)_{\mu_3} p_{2\nu_4} q_{\nu_5} - (q + p_2 + p_3)_{\mu_2} q_{\mu_3} (p_2 + p_3)_{\nu_4} q_{\nu_5}] }{(q^2 - m_i^2)((q + p_3)^2 - m_i^2)((q + p_2 + p_3)^2 - m_i^2)} \right] \right\}, \\
 \mathcal{T}_{13}[\text{odd}] &= -\mathcal{T}_{12}[\text{odd}], \\
 \mathcal{T}_{14}[\text{odd}] &= -\mathcal{T}_{11}[\text{odd}], \\
 \mathcal{T}_{15}[\text{odd}] &= -8 \int \prod_{j=1}^3 \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + p_3) \theta(p_1) H_{1,\nu_2}^{\mu_2}(p_2) H_{1,\nu_3}^{\mu_3}(p_3) \\
 &\quad \times \left\{ \int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=1}^f c_i m_i^2 \frac{\epsilon^{\nu_4 \nu_5 \nu_2 \nu_3} [p_{2\nu_4} p_{3\nu_5} (q + p_3)_{\mu_2} (q + p_3)_{\mu_3} + p_{2\nu_4} q_{\nu_5} (q + p_3)_{\mu_2} p_{3\mu_3} + p_{3\nu_4} q_{\nu_5} p_{2\mu_2} q_{\mu_3}] }{(q^2 - m_i^2)((q + p_3)^2 - m_i^2)((q + p_2 + p_3)^2 - m_i^2)} \right] \right\}, \\
 \mathcal{T}_{16}[\text{odd}] &= -\mathcal{T}_{15}[\text{odd}].
 \end{aligned}$$

It follows that

$$\mathcal{T}_{11}[\text{odd}] + \mathcal{T}_{14}[\text{odd}] = 0, \quad \mathcal{T}_{12}[\text{odd}] + \mathcal{T}_{13}[\text{odd}] = 0, \quad \mathcal{T}_{15}[\text{odd}] + \mathcal{T}_{16}[\text{odd}] = 0,$$

so that no parity-odd contribution to the Weyl anomaly comes from $\mathcal{T}_1$.

2. No parity-odd Weyl anomaly from $\mathcal{T}_2$

The only contributions to $\mathcal{T}_2$ in (2.7) which may give rise to a trace over four Pauli matrices, and thus an parity-odd contribution, are

$$\begin{aligned} \mathcal{T}_{21} &= \left(\frac{1}{8}i\right) \int dx_1 dx_2 dx_3 \left[\theta(x_1) h_{\mu_2}^{\mu_2}(x_2) + \theta(x_2) h_{\mu_1}^{\mu_1}(x_1) \right] H_{1,\nu_3}^{\mu_3}(x_3) \\ &\quad \times \left\{ \sum_{i=1}^f c_i m_i^2 \left[\left\langle \xi^{(i)\alpha_1}(x_1) \xi_{\alpha_1}^{(i)}(x_1) \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2) \bar{\xi}^{(i)\dot{\alpha}_2}(x_2) j_{R\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right] \right\}, \\ \mathcal{T}_{22} &= \left(-\frac{1}{8}i\right) \int dx_1 dx_2 dx_3 \left[\theta(x_1) h_{\mu_2}^{\mu_2}(x_2) + \theta(x_2) h_{\mu_1}^{\mu_1}(x_1) \right] H_{1,\nu_3}^{\mu_3}(x_3) \\ &\quad \times \left\{ \sum_{i=1}^f c_i m_i^2 \left[\left\langle \bar{\xi}_{\dot{\alpha}_1}^{(i)}(x_1) \bar{\xi}^{(i)\dot{\alpha}_1}(x_1) \xi^{(i)\alpha_2}(x_2) \xi_{\alpha_2}^{(i)}(x_2) j_{L\mu_3}^{(i)\nu_3}(x_3) \right\rangle_0^{(c)} \right] \right\}. \end{aligned} \quad (2.10)$$

A tedious computation shows that the parity-odd contribution to $\mathcal{T}_{21}$ reads as

$$\begin{aligned} \mathcal{T}_{21}[\text{odd}] &= - \int \prod_{j=1}^3 \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2 + p_3) [\theta(p_1) h_{\mu_2}^{\mu_2}(p_2) + \theta(p_2) h_{\mu_1}^{\mu_1}(p_1)] H_{1,\nu_3}^{\mu_3}(p_3) \\ &\quad \times \left\{ \int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=1}^f c_i m_i^2 \frac{\epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} p_{2\nu_1} p_{3\nu_2} (q + p_3)_{\mu_3} q_{\nu_4}}{(q^2 - m_i^2)((q + p_3)^2 - m_i^2)((q + p_2 + p_3)^2 - m_i^2)} \right] \right\}, \end{aligned} \quad (2.11)$$

whereas the parity-odd contribution to $\mathcal{T}_{22}[\text{odd}]$ is

$$\mathcal{T}_{22}[\text{odd}] = -\mathcal{T}_{21}[\text{odd}]. \quad (2.12)$$

Putting it all together we conclude that $\mathcal{T}_2$ carries no parity-odd contributions.

3. No parity-odd anomaly from $\mathcal{T}_3$

$\mathcal{T}_3$ in (2.8) never yields parity-odd contributions since the maximum number of Pauli matrices in them to trace over is two.

B. No parity-odd anomaly from bubble-type contributions

There are also terms that would give rise under contractions to *bubble* type Feynman diagrams. These terms carry one variation of the PV mass term and a kinetic term to second order. Here, we shall show that no parity-odd contribution to the Weyl anomaly will arise from either

$$\mathcal{B}_1 = \sum_{i=1}^f c_i \left\langle \delta_\theta^{(0)} S_m^{(i)} S_{\text{kin}}^{(i),2} \right\rangle_0^{(c)} \quad (2.13)$$

or else

$$\mathcal{B}_2 = \sum_{i=1}^f c_i \left\langle \delta_\theta^{(0)} S_m^{(i)} S_{\text{spin}}^{(i),2} \right\rangle_0^{(c)} \quad (2.14)$$

or again,

$$\mathcal{B}_3 = \sum_{i=1}^f c_i \left\langle \delta_\theta^{(1)} S_m^{(i)} S_{\text{kin}}^{(i),1} \right\rangle_0^{(c)} \quad (2.15)$$

in (2.4).

Neither $\mathcal{B}_1$ nor $\mathcal{B}_3$ can give rise to parity-odd anomalous terms since the maximum number of Pauli matrices to trace over is two. Let us analyze next what the situation is with regard to $\mathcal{B}_2$.

Let us express $\mathcal{B}_2$ as follows:

$$\mathcal{B}_2 = \mathcal{B}_{21} + \mathcal{B}_{22},$$

where

$$\begin{aligned} \mathcal{B}_{21} &= -\frac{1}{2} \int dx_1 dx_2 \theta(x_1) H_{\mu\nu\rho}(x_2) \left\{ \epsilon^{\mu\nu\rho\nu_2} \sum_{i=0}^f c_i m_i \left\langle \xi^{(i)\alpha_1}(x_1) \xi_{\alpha_1}^{(i)}(x_1) \bar{\xi}_{\alpha_2}^{(i)}(x_2) \bar{\sigma}_{\nu_2}^{\alpha_2} \xi_{\alpha_2}^{(i)}(x_2) \right\rangle_0^{(c)} \right\}, \\ \mathcal{B}_{22} &= -\frac{1}{2} \int dx_1 dx_2 \theta(x_1) H_{\mu\nu\rho}(x_2) \left\{ \epsilon^{\mu\nu\rho\nu_2} \sum_{i=0}^f c_i m_i \left\langle \bar{\xi}_{\alpha_1}^{(i)}(x_1) \bar{\xi}^{(i)\alpha_1}(x_1) \bar{\xi}_{\alpha_2}^{(i)}(x_2) \bar{\sigma}_{\nu_2}^{\alpha_2} \xi_{\alpha_2}^{(i)}(x_2) \right\rangle_0^{(c)} \right\}, \\ H_{\mu\nu\rho}(x) &= \frac{1}{16} h_\mu^\lambda(x) \partial_\nu h_{\rho\lambda}(x). \end{aligned} \quad (2.16)$$

A little computation yields

$$\begin{aligned} \mathcal{B}_{21} &= -2 \int \prod_{j=1}^2 \frac{d^4 p_j}{(2\pi)^4} (2\pi)^4 \delta(p_1 + p_2) \theta(p_1) H_{\mu\nu\rho}(p_2) \left\{ \int \frac{d^4 q}{(2\pi)^4} \left[\sum_{i=1}^f c_i m_i^2 \frac{\epsilon^{\mu\nu\rho\lambda} (q + p_1)_\lambda}{(q^2 - m_i^2)((q + p_1)^2 - m_i^2)} \right] \right\}, \\ \mathcal{B}_{22} &= -\mathcal{B}_{21}, \end{aligned}$$

so that

$$\mathcal{B}_{21} + \mathcal{B}_{22} = 0.$$

Hence, we conclude that $\mathcal{B}_2$ in (2.14) gives rise to no parity-odd anomalous term.

C. No parity-odd anomaly from tadpole-type contributions

The terms in the second order variation of the PV mass terms would give rise to tadpoles upon Wicks contractions. The tadpole-type contributions to the anomaly, if any, come from

$$\sum_{i=1}^f c_i \left\langle \delta_\theta^{(2)} S_m^{(i)} \right\rangle_0^{(c)}.$$

Since this contribution involves no Pauli matrices, there is no parity-odd term coming from it.

III. CONCLUSIONS

When computing the Weyl anomaly in perturbation theory one has to deal with the following type of vacuum expectation values:

$$\begin{aligned} &\langle 0 | \delta S_{\text{int}} S_{\text{int}} \cdots S_{\text{int}} | 0 \rangle \\ &= \int dx dx_1 \cdots dx_n \langle 0 | \delta \mathcal{L}_{\text{int}}(x) \mathcal{L}_{\text{int}}(x_1) \cdots \mathcal{L}_{\text{int}}(x_n) | 0 \rangle, \end{aligned} \quad (3.1)$$

where $|0\rangle$ is the Fock vacuum of the theory, and S_{int} the part of the action which contains the interaction, and δS_{int} is its symmetry variation. Obviously, we assume that S_{int} is Hermitian and so is δS_{int} . Formally, the vacuum expectation value in question is real; but this is not a mathematically rigorous statement, for, in general, the correlation functions on the right-hand side of (3.1) contain UV divergences, which must be regularized. If there is a regularization method that explicitly preserves the formal reality of correlation functions under scrutiny, it is clear that no parity-odd contribution to the Weyl anomaly should show up. In this paper we have put forward such a regularization method—a Pauli-Villars regularization—based on the use of a real Lagrangian (not just a real Lagrangian modulo a total derivative) for each Pauli-Villars field. Notice, however, that the regularized correlation functions are well defined once they are represented in momentum space, and once the loop contributions coming from the physical fields and the Pauli-Villars fields have been collected as a linear combination inside the loop integrations. Hence, to show that the correlation functions which may give rise to a parity-odd contribution to the anomaly do cancel, we have checked that for each contribution of this type there is always a contribution with the opposite sign. This leads to the conclusion that the renormalized, by minimal subtraction, gravitational effective action carries no parity-odd Weyl anomaly—see discussion right above Sec. II A. Finally, we believe that the formal reality of the correlation functions should be preserved in the regularized theory since it comes from the reality of the classical

action and should be taken as a fundamental property of the quantum theory.

Let us end with a comment. There is no known topological interpretation of the Weyl anomaly; in particular, there is no known relationship with any index theorem. In this respect it is interesting to highlight the work [20] where the exact form of the one-loop beta function of QCD is recovered from an index theorem on twistor space.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: SOME COMPUTATIONAL DETAILS

As an example of the standard techniques used to obtain the results displayed in this paper, we shall give here some details of the computations to be done to reproduce the results in Sec. II A 2.

To calculate $\mathcal{T}_{21}$ in (2.10), one needs to work out the correlation function

$$\left\langle \xi_{\alpha_1}^{(i)}(x_1) \xi_{\alpha_1}^{(i)}(x_1) \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2) \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2) \bar{\xi}_{\dot{\alpha}_3}^{(i)}(x_3) \bar{\xi}_{\dot{\alpha}_3}^{(i)}(x_3) \partial_{\mu_3} \xi_{\alpha_3}^{(i)}(x_3) \right\rangle_0^{(c)}. \quad (\text{A1})$$

A trivial application of Wick's theorem leads to the conclusion that the previous Green function is equal to

$$4e^{\alpha_1\beta_1}\epsilon^{\dot{\alpha}_2\dot{\beta}_2}\bar{\sigma}^{\dot{\alpha}_3\alpha_3\nu_3}\left[\left\langle \xi_{\alpha_1}^{(i)}(x_1) \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2) \right\rangle_0 \left\langle \bar{\xi}_{\dot{\beta}_2}^{(i)}(x_2) \partial_{\mu_3} \xi_{\alpha_3}^{(i)}(x_3) \right\rangle_0 \left\langle \bar{\xi}_{\dot{\alpha}_3}^{(i)}(x_3) \xi_{\beta_1}^{(i)}(x_1) \right\rangle_0 - \left\langle \xi_{\alpha_1}^{(i)}(x_1) \bar{\xi}_{\dot{\alpha}_2}^{(i)}(x_2) \right\rangle_0 \left\langle \bar{\xi}_{\dot{\beta}_2}^{(i)}(x_2) \bar{\xi}_{\dot{\alpha}_3}^{(i)}(x_3) \right\rangle_0 \left\langle \partial_{\mu_3} \xi_{\alpha_3}^{(i)}(x_3) \xi_{\beta_1}^{(i)}(x_1) \right\rangle_0\right]. \quad (\text{A2})$$

Now, the free propagators—see Ref. [15]—in the previous equation read as

$$\begin{aligned} \left\langle \xi_{\alpha}^{(i)}(x) \bar{\xi}_{\dot{\alpha}}^{(i)}(y) \right\rangle_0 &= \int \frac{d^4 p}{(2\pi)^4} p \sigma_{\alpha\dot{\alpha}} G_i(p) e^{-ip(x-y)}, \\ \left\langle \bar{\xi}_{\dot{\alpha}}^{(i)}(x) \partial_{\mu} \xi_{\alpha}^{(i)}(y) \right\rangle_0 &= \int \frac{d^4 p}{(2\pi)^4} (ip_{\mu}) p \sigma_{\alpha\dot{\alpha}} G_i(p) e^{-ip(x-y)}, \\ \left\langle \xi_{\alpha}^{(i)}(x) \xi_{\beta}^{(i)}(y) \right\rangle_0 &= \int \frac{d^4 p}{(2\pi)^4} m_i \epsilon_{\beta\alpha} G_i(p) e^{-ip(x-y)}, \\ \left\langle \bar{\xi}_{\dot{\alpha}}^{(i)}(x) \bar{\xi}_{\dot{\beta}}^{(i)}(y) \right\rangle_0 &= \int \frac{d^4 p}{(2\pi)^4} m_i \epsilon_{\dot{\alpha}\dot{\beta}} G_i(p) e^{-ip(x-y)}, \\ \left\langle \partial_{\mu} \xi_{\alpha}^{(i)}(x) \xi_{\beta}^{(i)}(y) \right\rangle_0 &= \int \frac{d^4 p}{(2\pi)^4} m_i (-ip_{\mu}) \epsilon_{\beta\alpha} G_i(p) e^{-ip(x-y)}, \\ G_i(p) &= \frac{i}{p^2 - m_i^2}. \end{aligned} \quad (\text{A3})$$

The substitution of the free propagators in (A3) in Eq. (A2), and some algebra, yields the correlation function in (A1) in terms of its Fourier transform. This expression for the Green function in (A3) substituted in (2.10) yields (2.11). Notice that, on the one hand, the product of three free propagators in the second line of (A2) does not contribute to the parity-odd part of $\mathcal{T}_{21}$, for it leads to the trace of the product of only two Pauli matrices. On the other hand, the computation of the contribution to $\mathcal{T}_{21}$ coming from the first line of (A2) demands to work out the trace over the product of four Pauli matrices, which yields a term involving the $\epsilon^{\mu\nu\rho\sigma}$ pseudo-tensor.

Finally, Eq. (2.12) can be obtained in a completely analogous way.

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