Collinear bootstrap for $\mathcal{N} = (1, 1)$ SYM theory

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We study the multicollinear behavior of tree amplitudes in the six-dimensional $\mathcal{N} = (1, 1)$ super Yang-Mills theory (sYM). A generalized dimensional reduction of the latter yields the four-dimensional $\mathcal{N} = 4$ sYM on the Coulomb branch, which is of interest for considerations of massive or off-shell scattering. To this end, we revisit the calculation of tree scattering in the former theory employing the collinear bootstrap and known massless limits. Assuming the universality of the double-collinear asymptotics, the result for six-leg superamplitudes differs from the one available in the literature. We further extract the triple-collinear splitting superamplitudes from these.

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I. INTRODUCTION

Scattering amplitudes develop singularities when adjacent external momenta become collinear [1–3]. This behavior is universal and depends solely on the attributes of coalescing particles.[1] Its understanding is of paramount importance for phenomenological applications, where cancellations between real and virtual divergences ensure finite physical cross sections, on the one hand. On the other, known factorization patterns furnish strong constraints on the structure of scattering amplitudes and provide an independent check on the correctness of their calculation.

Presently, we will address this question within the context of the six-dimensional $\mathcal{N} = (1, 1)$ super Yang-Mills theory (sYM). This theory is maximally supersymmetric and serves as a progenitor for $\mathcal{N} = 4$ sYM in four dimensions upon dimensional reduction. Depending on whether one keeps out-of-four-dimensional components of particles' six-dimensional null momenta vanishing or not, one can probe different branches of the theory. When they are set to zero, one obtains the conventional superconformal model, while, if they are not, one ends up on its Coulomb branch, where some of the states become massive. Our interest in the past few years was in a particular pattern of dimensional reduction that endows only external particles with nontrivial masses while keeping all states propagating in quantum loops massless. This can then be regarded as an off-shell generalization of massless on-shell scattering.

The on-shell scattering in the six-dimensional theory enjoys a superspace formulation [5], on the one hand, and an unconstrained spinor-helicity formalism [6], on the other. In addition, amplitudes possess covariant transformation properties under the dual conformal symmetry [7–10]. This allowed one to use these powerful techniques to study tree-level scattering amplitudes in a concise fashion [9,10]. The analysis in Ref. [10] went further in multiplicity than any other study. In particular, making use of a numerical implementation of Britto–Cachazo–Feng–Witten (BCFW) recursion relations [11,12], multiple forms of the six-leg amplitude were proposed there. Presently, we revisit its calculation by employing a different technique. Namely, constructing its ansatz in terms of dual conformally covariant building blocks, we then fixed unknown accompanying coefficients by relying on its (assumed) universal double-collinear behavior, with corresponding splitting amplitudes deduced recently in Ref. [13], and further constraints from four-dimensional massless limits. Our finding differs from [10]. Let us point out that the use of anticipated analytic properties of scattering amplitudes to constrain or fix their form is not new. It was used in the past to prove the form of next-to-maximal-helicity violating amplitudes on the conformal branch of the $\mathcal{N} = 4$ sYM [14]. Further, taking the triple-collinear limit of the found six-leg amplitude, we calculated the triple-collinear splitting superamplitude.

Our subsequent presentation is organized as follows. First, we give the basics of the $\mathcal{N} = (1, 1)$ on-shell superspace, spinor-helicity formalism, and dual conformal properties of scattering amplitudes. We next, provide a detailed derivation of the double-collinear splitting superamplitude in Sec. III starting from the five-leg amplitude. We assumed

[1]To be precise, this holds in the timelike kinematics only [4]. We will assume it throughout our current consideration.

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that it is universal for any number of legs and then used its form in Sec. IV B to find constraints on the six-leg amplitude. There is still some ambiguity left in the ansatz, so we projected out it on a component that involved only scalar external states. Since the latter depends only on Lorentz invariant six-dimensional products of particle momenta it can be obtained by an uplift of its four-dimensional counterpart. This projection then completely fixes the remaining freedom. Using so-obtained expressions as a starting point, we extracted from it a triple-collinear splitting amplitude. Finally, we provided an outlook for future use of our results. Some Appendices were added with details clarifying calculations in the main text.

II. SYMMETRIES OF $\mathcal{N}=(1,1)$ SUPERAMPLITUDES

The $\mathcal{N}=(1,1)$ sYM is a maximally supersymmetric gauge theory in six dimensions. All of its on-shell states are classified according to the little group $\mathrm{SU}(2)\times\mathrm{SU}(2)$ of its Lorentz group $\mathrm{SO}(5,1)\simeq\mathrm{SU}^*(4)$ [5,6]. The R-symmetry of the model is $\mathrm{SU}_R(2)\times\mathrm{SU}_R(2)$, however, it would require additional indices for the enumeration of physical degrees of freedom and result in their proliferation far beyond the number of available states. Thus, one sacrifices the R-group in favor of the little group such that only its $\mathrm{U}_R(1)\times\mathrm{U}_R(1)$ subgroup is left manifest, i.e., one imposes a truncation. The on-shell states of the model

$$\begin{aligned}&\text{gluons:}\quad && g^a{}_{\dot a},\\ &\text{scalars:}\quad && \phi,\phi',\phi'',\phi''',\\ &\text{gluinos:}\quad && \chi^a,\bar\chi_{\dot a},\psi^a,\bar\psi_{\dot a},\end{aligned}$$

can be packaged into a single CPT self-conjugate nonchiral superfield [5]

$$\begin{aligned}\Phi=\phi+\chi^a\eta_a+\bar\chi_{\dot a}\bar\eta^{\dot a}+\eta^2\phi'+\bar\eta^2\phi''+g^a{}_{\dot a}\eta_a\bar\eta^{\dot a}\\ +\psi^a\eta_a\bar\eta^2+\bar\psi_{\dot a}\bar\eta^{\dot a}\eta^2+\eta^2\bar\eta^2\phi''',\end{aligned}\tag{2.1}$$

as a terminating expansion in the complex, independent Grassmann variables η_a and $\bar\eta_{\dot a}$ that carry only the little group indices and possess positive and negative chirality

$$\eta\to e^{i\phi}\eta,\qquad \bar\eta\to e^{-i\phi}\bar\eta.\tag{2.2}$$

Scattering amplitudes in the theory are generated from the amputated vacuum expectation value of a product of superfields, schematically

$$\mathcal{A}_n=\langle\Phi_1\ldots\Phi_n\rangle.\tag{2.3}$$

They depend on n bosonic momenta P_i and $n+n$ of (anti) chiral charges Q_i and $\bar Q_i$, cumulatively called supermomenta. The theory benefits from a spinor-helicity formalism [6], which allows one to recast the super-Poincaré quantum numbers in terms of unconstrained Weyl spinors $\Lambda_i^{A,a}\equiv|i^a\rangle=\langle i^a|$ and $\bar\Lambda_{i,A,\dot a}\equiv|i_{\dot a}]=[i_{\dot a}|$,

$$\begin{aligned}P_i&=|i^a\rangle\langle i_a|, &\qquad \bar P_i&=|i_{\dot a}][i^{\dot a}|,\\ Q_i&=\langle i^a|\eta_{i,a}, &\qquad \bar Q_i&=[i_{\dot a}|\bar\eta_i^{\dot a}.\end{aligned}\tag{2.4}$$

Our conventions for these were thoroughly spelled out in Ref. [13] and will not be repeated here. Imposing the supermomentum conservation, we can extract it in terms of bosonic and fermionic delta functions

$$\mathcal{A}_n=i(2\pi)^6\delta^{(6)}\left(\sum_{i=1}^n P_i\right)\delta^{(4)}\left(\sum_{i=1}^n Q_i\right)\delta^{(4)}\left(\sum_{i=1}^n\bar Q_i\right)\hat{\mathcal{A}}_n,\tag{2.5}$$

and define reduced amplitudes $\hat{\mathcal{A}}_n$ which are homogeneous polynomials of order $n-4$ in Grassmann variables. The nonchiral nature of the theory imposes a more stringent constraint on these: they can be chosen to be polynomials of equal degrees $[n/2]-2$ both in the chiral and antichiral charges [10] yielding natural reduction properties to four dimensions.

Since the six-dimensional Yang-Mills theory possesses dimensionful gauge coupling, it is not classically invariant under dilatations and special conformal boosts. This translates into the same properties for tree-level scattering. However, making use of the supersymmetric generalization of BCFW recursion relations, it was demonstrated in Refs. [7,8,10] that the reduced tree-level amplitudes $\hat{\mathcal{A}}_n^{(0)}$ transform covariantly under the dual conformal inversion $\mathcal{I}$ with the same conformal weight for all legs,

$$\mathcal{I}\hat{\mathcal{A}}_n^{(0)}=X_1^2\ldots X_n^2\hat{\mathcal{A}}_n^{(0)}.\tag{2.6}$$

Here, X_i are the region momenta, also known as dual coordinates, defined along with their supersymmetric counterparts Θ_i and $\bar\Theta_i$ as

$$P_i=X_{ii+1},\qquad Q_i=\Theta_{ii+1},\qquad \bar Q_i=\bar\Theta_{ii+1},\tag{2.7}$$

with the adopted convention $*_{ij}\equiv *_i-*_j$ throughout this paper. The discrete inversion is defined on these as

$$\mathcal{I}X^{AB}=\frac{\bar X_{AB}}{X^2},\quad \mathcal{I}\Theta^A=\Theta^B\frac{\bar X_{BA}}{X^2},\quad \mathcal{I}\bar\Theta_A=\frac{X^{AB}}{X^2}\bar\Theta_B,\tag{2.8}$$

where we displayed the $\mathrm{SU}^*(4)$ Lorentz indices explicitly for notational clarity. The inversion is an involution with $\mathcal{I}^2=1$. Though the proof of Eq. (2.6) via BCFW recursion relations requires, at intermediate stages, defining the operation of inversion on the spinor-helicity variables carrying the little group indices as well, we do not need to spell them out explicitly here since the reduced

amplitudes ultimately depend only on super-Poincaré charges P_i, Q_i, and $\bar{Q}_i$ in light of their nonchiral nature.

The fact that $\hat{\mathcal{A}}_n$ are dual translationally invariant functions of the dual variables $(X_i, \Theta_i, \bar{\Theta}_i)$ and covariant under the dual inversion provides severe constraints on their form. There is a unique way to construct primary building blocks by uplifting a four-dimensional construction from Ref. [15]. The required dual conformal covariants are [9,10,15]

$$\begin{aligned}\langle B_{i,jk}| &= \langle\Theta_{ij}|\bar{X}_{jk}X_{ki} + \langle\Theta_{ik}|\bar{X}_{kj}X_{ji},\\ |\bar{B}_{i,jk}] &= -\mathrm{cc}(\langle B_{i,jk}|),\end{aligned} \tag{2.9}$$

where cc stands for the chiral conjugate, i.e., changing all unbarred symbols with barred and vice versa. The (square) kets are defined as minus the chiral conjugate of (angle) bras, as in Ref. [10]. These invert as

$$\mathcal{I}\langle B_{i,jk}| = \frac{\langle B_{i,jk}|\bar{X}_i}{X_i^2X_j^2X_k^2}, \tag{2.10}$$

under Eq. (2.8). Though one may sandwich an odd number of momenta between bras and kets of the same chirality and define chiral conjugate pairs of conformal covariants; these lead to a decoupling of chiral and antichiral degrees of freedom and, therefore, can be eliminated from potential candidates of viable building blocks due to the nonchiral nature of the $\mathcal{N}=(1,1)$ sYM. Similarly, one could sandwich an even number of momenta between the brackets of opposite chirality as more generic ingredients, but the one with just a unit matrix in between them is a good, minimal starting point [10],

$$\Omega_{ijklm} = \frac{1}{2}\langle B_{i,jl}|\bar{B}_{i,km}] + \mathrm{cc}. \tag{2.11}$$

The inner product involved acquires the same conformal weights for all indices. These objects are of Grassmann degree $1+1$.

Color-ordered tree amplitudes exhibit simple analytic behavior when the invariant mass formed by adjacent momenta goes on its mass shell: they develop poles in corresponding Mandelstam invariants $S_{ii+1}\equiv(P_i+P_{i+1})^2$. In conjunction with the known Grassmann degree of n-leg amplitudes and their dual inversion properties, one can predict the four-leg amplitude unambiguously (up to an overall phase) [5,6,10] and put forward a compact ansatz for the five-leg case

$$\hat{\mathcal{A}}_4^{(0)} = \frac{1}{S_{12}S_{23}}, \qquad \hat{\mathcal{A}}_5^{(0)} = \frac{-\Omega_{12345}}{S_{12}S_{23}S_{34}S_{45}S_{51}}. \tag{2.12}$$

The latter was proposed and confirmed with the numerical implementation of BCFW recursion [10]. This representation is to be contrasted with a lengthy multiterm expression devised in [5], and quoted in Eq. (A1) for readers' convenience. The latter does not exhibit obvious dual conformal properties, however, with a little work, the two can be analytically shown to be equivalent, see Appendix A.

III. DOUBLE-COLLINEAR SPLITTING SUPERAMPLITUDE

To prepare the ground for attacking the six-leg amplitude $\hat{\mathcal{A}}_6^{(0)}$, we first extract the double-collinear splitting amplitude from the five-leg one. We will implement this limit on legs 4 and 5. We expect that $\hat{\mathcal{A}}_5^{(0)}$ admits a factorized form

$$\begin{aligned}\hat{\mathcal{A}}_5^{(0)}(P_1,P_2,P_3,P_4,P_5) &\xrightarrow{4\|5} \hat{\mathcal{A}}_4^{(0)}(P_1,P_2,P_3,P)\\ &\quad \mathcal{S}\mathrm{plit}^{(0)}(-P;P_4,P_5),\end{aligned} \tag{3.1}$$

where the leg $P=P_4+P_5$ in the super-splitting amplitude $\mathcal{S}$plit is slightly off-shell, i.e., $P^2\neq 0$. Thus, one of the Mandelstam invariants $S_{45}=P^2$ is small with the rest being large.

To study the above asymptotics in a self-consistent manner, we need to specify how the collinear configuration is approached. This can be done with a Sudakov parametrization of the involved momenta P_4 and P_5 as was done in Ref. [4] in four dimensions. Namely, these lightlike momenta are decomposed as

$$P_4 = zP + K_\perp - \frac{K_\perp^2}{2z(P\cdot N)}N, \quad P_5 = \bar{z}P - K_\perp - \frac{K_\perp^2}{2z(P\cdot N)}N, \tag{3.2}$$

where the two momenta $P_{4,5}$ approach a common lightlike direction P, $P^2=0$, with corresponding momentum fractions being z and $\bar{z}\equiv 1-z$, respectively. The vector N is another light-cone momentum, $N^2=0$, moving in the opposite direction to P and is required in order to define the transverse momentum $K_\perp$ that parametrizes how the common collinear direction is reached, $K_\perp\cdot P = K_\perp\cdot N = 0$. The inner product of the two momenta decays quadratically in $K_\perp\to 0$

$$P_4\cdot P_5 = -\frac{K_\perp^2}{2z\bar{z}}, \tag{3.3}$$

so that $\langle 4_a|5_{\dot{a}}] \sim \langle 5_a|4_{\dot{a}}] \sim O(K_\perp/\sqrt{z\bar{z}})$. Since the amplitude diverges as $1/S_{45}\sim 1/K_\perp^2$ in the collinear limit, the momentum algebra in the numerator of the amplitude has to be performed to linear order in $K_\perp$ to extract the splitting amplitude. To this accuracy, we will use

$$P_4\bar{P}_5 = -P_5\bar{P}_4 = K_\perp\bar{P} + O(K_\perp^2), \tag{3.4}$$

for the matrix products.

With the above results in our hands, we start inspecting the five-leg amplitude in the form of Eq. (A4). It is expected, based on universal factorization properties of gauge amplitudes, that none of the fermionic structures encoding particles other than 4 and 5 contribute to the 4||5 collinear limit. This is obvious from the representation (A4) since the alien structures are accompanied by the vanishing Mandelstam invariant $S_{45} \to 0$, i.e., the first line in Eq. (A4) provides a regular contribution to the amplitude and is not relevant. Thus, we turn to the remaining ones next. Let us start with the diagonal terms, i.e., the first two terms in the second line. Using the relations $S_{51} + S_{41} + S_{45} = S_{23}$ and $S_{34} + S_{35} + S_{45} = S_{12}$ between the Mandelstam invariants, we can rewrite them as

$$S_{51} Q_4 \bar{P}_3 P_5 \bar{Q}_4 = S_{23} Q_4 \bar{P}_3 P_5 \bar{Q}_4 + Q_4 \bar{P}_1 K_\perp \bar{P} P_3 \bar{Q}_4 + O(K_\perp^2), \tag{3.5}$$

$$S_{34} Q_5 \bar{P}_4 P_1 \bar{Q}_5 = S_{12} Q_5 \bar{P}_4 P_1 \bar{Q}_5 + Q_5 \bar{P}_1 K_\perp \bar{P} P_3 \bar{Q}_5 + O(K_\perp^2). \tag{3.6}$$

Notice that the first term in both equations is of order $K_\perp$ since they have momenta and chiral charges of coalescing states adjacent to each other and they are of the same order as the second terms. All that is left to analyze in Eq. (A4) is the last term in the second line. We find for it

$$\begin{aligned} Q_4[S_{34}S_{51} - \bar{P}_3 P_5 \bar{P}_4 P_1]\bar{Q}_5 &= Q_4[\bar{P}_3 P_4 \bar{P}_5 P_1 - \bar{P}_3 P_5 \bar{P}_4 P_1]\bar{Q}_5 \\ &= 2 Q_4 \bar{P}_3 K_\perp \bar{P} P_1 \bar{Q}_5 + O(K_\perp^2) \\ &= 2 Q_4 \bar{P}_1 K_\perp \bar{P} P_3 \bar{Q}_5 + O(K_\perp^2). \end{aligned} \tag{3.7}$$

Adding our findings together, we conclude that the numerator of the five-leg amplitude reduces in the 4||5 double-collinear limit to

$$-\Omega_{51234} \xrightarrow{4||5} -\Omega^{||}_{51234} = S_{23} Q_4 \bar{P}_3 P_5 \bar{Q}_4 + S_{12} Q_5 \bar{P}_4 P_1 \bar{Q}_5 + (Q_4 + Q_5)\bar{P}_1 P_4 \bar{P}_5 P_3 (\bar{Q}_4 + \bar{Q}_5) + O(K_\perp^2). \tag{3.8}$$

Here in the second term, we restored the near-collinear momenta $P_{4,5}$ from $K_\perp$ and P by using Eq. (3.4) backward, i.e., from right to left.

Finally, we need to eliminate the alien momenta P_1 and P_3 from all Dirac strings. This is accomplished by using the spinor-helicity representation (2.4) for the super-momenta and factoring out small, i.e., $O(K_\perp)$, Weyl inner products first and then imposing the strict collinear kinematics, i.e., just the first term in Eq. (3.2), on the rest of the accompanying kinematical factors. For instance, for the first term in Eq. (3.8) we get

$$\begin{aligned} Q_4 \bar{P}_3 P_5 \bar{Q}_4 &\simeq \sqrt{z\bar{z}} \eta_4^a \langle P_a | \bar{P}_3 | P_b \rangle \langle 5^b | 4_{\dot{a}}] \bar{\eta}_4^{\dot{a}} \\ &\simeq -\sqrt{z\bar{z}} S_{12} \langle 5_a | 4_{\dot{a}}] \eta_4^a \bar{\eta}_4^{\dot{a}}. \end{aligned} \tag{3.9}$$

Here, after the first equality sign, we used the strict collinearity conditions at the level of the six-dimensional Weyl spinors (and their chiral conjugates)

$$\langle 4^a | = \sqrt{z} \langle P^a |, \qquad \langle 5^a | = \sqrt{\bar{z}} \langle P^a |. \tag{3.10}$$

While, after the second one, we employed the relation $\langle P_a | \bar{P}_3 | P_b \rangle = \varepsilon_{ab} S_{12}$ with $S_{3P} = S_{12}$ stemming from the momentum conservation condition in the accompanying four-leg amplitude $P_1 + P_2 + P_3 + P = 0$. Similarly, we proceed with the rest of the terms in Eq. (3.8). In the last, one has to commute $P_{4,5}$ to the left/right to pull out a large Mandelstam invariant from the Dirac string, first. Notice that to the $O(K_\perp^2)$ accuracy, the two inner products $\langle 4_a | 5_{\dot{a}}] \simeq -\langle 5_a | 4_{\dot{a}}]$ are the same, up to a phase, as can be easily concluded from the six-dimensional Clifford algebra of the corresponding near-collinear momenta.

Summarizing our findings and dividing out the reduced four-leg tree amplitude (2.12), we deduce the double-collinear splitting superamplitude

$$\begin{aligned} \mathcal{S}\text{plit}^{(0)}(-P; P_4, P_5) &= \frac{-\Omega^{||}_{51234}}{z\bar{z} S_{45} S_{12} S_{23}} \\ &= \frac{\langle 4_a | 5_{\dot{a}}]}{\sqrt{z\bar{z}} S_{45}} \Big[\eta_4^a \bar{\eta}_4^{\dot{a}} + \eta_5^a \bar{\eta}_5^{\dot{a}} \\ &\quad - (\sqrt{z}\eta_4^a + \sqrt{\bar{z}}\eta_5^a)(\sqrt{z}\bar{\eta}_4^{\dot{a}} + \sqrt{\bar{z}}\bar{\eta}_5^{\dot{a}}) \Big], \end{aligned} \tag{3.11}$$

in agreement with Ref. [13].

IV. SIX-LEG SUPERAMPLITUDE

In this section, we are turning to one of the main objectives of this study: the construction of the six-leg superamplitude using constraints from its anticipated behavior in the double-collinear limit. Namely, assuming the universality of the splitting amplitude $\mathcal{S}$plit derived in the previous section, the equation

$$\hat{\mathcal{A}}_6^{(0)}(P_1, P_2, P_3, P_4, P_5, P_6) \xrightarrow{5||6} \hat{\mathcal{A}}_5^{(0)}(P_1, P_2, P_3, P_4, P) \, \mathcal{S}\text{plit}^{(0)}(-P; P_5, P_6), \tag{4.1}$$

imposes rather rigid constraints on the form of $\hat{\mathcal{A}}_6^{(0)}$. Let us proceed in a stepwise fashion.

First, we have to come up with a suitable basis for its building blocks. At initialization, we can use the minimal elements (2.11) for this purpose. This was the working hypothesis of Ref. [10] as well. We chose the set of 6 inequivalent elements

$$\{\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6\}$$
$$\equiv \{\Omega_{12345}, \Omega_{23456}, \Omega_{34561}, \Omega_{45612}, \Omega_{56123}, \Omega_{61234}\}, \quad (4.2)$$

which we enumerated by the value of their header index. Since the six-leg reduced amplitude is of Grassmann degree $2+2$ so a natural ansatz for it, taking into account the anticipated pole structure in Mandelstam invariants of adjacent momenta, reads

$$\hat{\mathcal{A}}_6^{(0)}(P_1, P_2, P_3, P_4, P_5, P_6) = \frac{\mathcal{N}_6}{S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}} \quad (4.3)$$

with

$$\mathcal{N}_6 = \sum_{i,j=1}^{6} \alpha_{ij}\Omega_i\Omega_j, \quad (4.4)$$

Here, the α_{ij} coefficients are some functions of S_{ij}. It is our goal to constrain or fix their form.

A. Double-collinear behavior

We will postpone the discussion of dual inversion properties to later stages and start with the analysis of the $5||6$ double-collinear behavior of Ω_i that we will name $\Omega_i^{||}$. A consideration akin to the one performed in the previous section demonstrates that out of the six elements (4.2), half of them are large and the other half are small. Namely, the large ones are of order $O(K_\perp^0)$ in the collinear kinematics and reduce to the Ω-structure of the five-leg amplitude (2.12) with label $5 = P$,

$$\Omega_6^{||} = \bar{z}\Omega_{P1234} = \bar{z}\Omega, \qquad \Omega_1^{||} = \Omega_{1234P} = \Omega,$$
$$\Omega_2^{||} = z\Omega_{234P1} = z\Omega. \quad (4.5)$$

Here we exhibited the ordering of labels in the intermediate expressions before we ultimately used the cyclic symmetry of the Ω-covariants (A2) in the five-leg amplitude. The small ones, $\Omega_{3,4,5}^{||}$ are of order $O(K_\perp^1)$. A linear superposition of $\Omega_3^{||}$ and $\Omega_5^{||}$ is related to $\Omega_4^{||}$ via the equation

$$S_{61}\Omega_3^{||} + S_{45}\Omega_5^{||} = S_{345}\Omega_4^{||}, \quad (4.6)$$

where we introduced a three-particle Mandelstam invariant which can be related to the distance in dual coordinates as $S_{345} = X_{36}^2$. $\Omega_4^{||}$ generates the numerator structure of the splitting amplitude (3.8),

$$\mathcal{S}\text{plit}^{(0)}(-P; P_5, P_6) = \frac{-\Omega_4^{||}}{z\bar{z}S_{56}S_{4P}S_{P1}}, \quad (4.7)$$

as in Eq. (3.11).

Collinear limits in the other five nearest-neighbor channels can be found from the above by a cyclic permutation.

B. Refined ansatz

Having established the collinear behavior of Ω's in one particular channel is sufficient to weed out redundant contributions in the previous crude ansatz (4.4). The factorized form of (4.1) suggests that the numerator of the six-leg amplitude when legs 5 and 6 coalesce can have products of large $\times$ small and small $\times$ small Ω's but not large $\times$ large as this would lead to way too strong collinear singularity, unless the latter is accompanied by a small Mandelstam invariant[2] in the amplitude. Thus, we choose a large Ω_i with the largest index $i(\mathrm{mod}\ 6)$ and multiply it by a linear combination of the small ones. This yields

$$\mathcal{N}_6 = \Omega_2(\alpha_1\Omega_3 + \alpha_2\Omega_4 + \alpha_3\Omega_5) + \text{cyclic permutations}, \quad (4.8)$$

with unknown functions α_i of the Mandelstam variables. In this manner, out of 36 terms in Eq. (4.4) only 18 survive, and there are only three truly unknown coefficients. Of course, terms obtained by cyclic permutations do contain the products of two large Ω's and seem to invalidate this form if corresponding α's are large. This does not happen, however, as we demonstrate next.

To constrain the form of α's, we turn next to the dual covariant properties of the amplitude (2.6). Let us inspect the dual inversion of the products of Ω's entering Eq. (4.8). In $\Omega_2\Omega_3$, $\Omega_2\Omega_4$, and $\Omega_2\Omega_5$, there are two labels that are not common to both and, thus, lead to mismatching powers of dual coordinates upon inversion. These arise for the labels $2\,\&\,1$, $3\,\&\,1$ and $4\,\&\,1$, respectively. To have the same weight for all contributing structures, we need to extract proper combinations of dual distances from α's. Notice, however, that the conformal weights of the two-particle Mandelstam invariants would compensate completely the ones from the thus-constructed numerator. So where will the required inversion weight on the right-hand side of Eq. (2.6) come from? It can only stem from the three-particle invariants, i.e., $S_{123} = S_{456} = X_{14}^2$, $S_{234} = S_{561} = X_{25}^2$ and $S_{345} = S_{612} = X_{36}^2$, introduced as a product in the denominator of the reduced amplitude. This does not violate the two-particle factorization of the amplitude. However, it brings in three-particle poles into the game. These can and do arise in six-leg amplitudes but they cannot simultaneously happen in partially overlapping channels in light of

[2]There is a remote possibility that some linear combination of large Ω's yields a small one as an $O(K_\perp^1)$ effect, but an inspection of higher terms in their expansion suggests that this is highly unlikely and thus can be disregarded in the first attempt to construct the numerator $\mathcal{N}_6$.

Steinmann relations[3] [16,17], which imposes rigid constraints on the structure of the numerator $\mathcal{N}_6$. Having this in mind, we rescale the α_i in Eq. (4.8) as

$$\alpha_i \to \frac{\alpha_i}{X_{14}^2 X_{25}^2 X_{36}^2}. \tag{4.9}$$

Now, we fulfill the goal of recovering the same conformal weight for individual terms in (4.8). Starting with $\Omega_2\Omega_3$, we notice that an obvious choice of a single power of X_{12}^2 is a no-starter since this distance is lightlike. Thus, the labels 1 and 2 should belong to two different squared distances $X_{1i}^2 X_{2j}^2/X_{ij}^2$ with i and j compensated by an appropriate denominator X_{ij}^2. Now, from the singularity structure of the double collinear limit, we know that we cannot have i and j to be next-to-nearest neighbors as this would result in a double pole in the corresponding Mandelstam variable forbidden by Eq. (4.1). Thus, the only option is to have $|i-j|=3$, which together with the fact that i and j cannot be adjacent to 1 and 2 uniquely fixes them to $i=3$ and $j=6$. We conclude

$$\alpha_1 = \frac{X_{13}^2 X_{26}^2}{X_{36}^2} a_1, \tag{4.10}$$

where a_1 is not a constant, rather, it is a possible function of dual conformal cross-ratios that one can introduce starting from six legs. These are

$$U_1 = \frac{X_{13}^2 X_{46}^2}{X_{14}^2 X_{36}^2}, \qquad U_2 = \frac{X_{24}^2 X_{15}^2}{X_{25}^2 X_{14}^2}, \qquad U_3 = \frac{X_{35}^2 X_{26}^2}{X_{36}^2 X_{25}^2}. \tag{4.11}$$

Notice that they transform into each other under the cyclic permutation $i \to i+1$,

$$U_i \to U_{i+1(\mathrm{mod}\,3)}. \tag{4.12}$$

Next, we turn to $\Omega_2\Omega_3$ where $3\,\&\,1$ are not common so that we restore the same conformal weight for all labels with

$$\alpha_2 = X_{13}^2 a_2, \tag{4.13}$$

with a_2 being potentially a function of U_i.

Finally, we consider $\Omega_2\Omega_3$ where the mismatched labels 4 and 1 are far-distant in a six-leg amplitude, therefore,

$$\alpha_3 = X_{14}^2 a_3. \tag{4.14}$$

Now, considering the $5||6$ collinear limit, we observe that to have any chance of getting the double-collinear amplitude from the linear combination of $\Omega_3^{||}$ and $\Omega_5^{||}$, a_3 should be a linear function of the conformal cross ratio U_1 since it contains the proper Mandelstam invariant S_{45} that accompanies $\Omega_5^{||}$ in Eq. (4.6). Linear dependence on U_3 is ruled out because it is a large variable not possessing proper dependence on two-particle invariants. However, U_2 is not as it vanishes linearly as a power of S_{56} and is thus a subleading $O(K_\perp^2)$ effect in the $5||6$ kinematics. Therefore, we further decompose a_3 as

$$a_3 = U_1 a_{3,1} + U_2 a_{3,2}, \tag{4.15}$$

where $a_{3,i}$ can depend on the cross ratios (4.11).

Substitution of the constraint equations (4.9), (4.10), and (4.13)–(4.15) into Eq. (4.8) is the first stage of what we dub as the *collinear bootstrap*. An inspection of the cyclic permutations of the first line in (4.8) immediately demonstrates that all contributions from products of two large Ω's are necessarily accompanied by the vanishing Mandelstam invariant S_{56}. Thus the dual-covariance properties of the amplitude ruled out a potential predicament in the ansatz from such terms. Finally, since the extracted denominator in Eq. (4.9) is symmetric under $i \to i+1(\,\mathrm{mod}\,3)$, it is only natural to assume that $a = \{a_1, a_2, a_{3,i}\}$ are symmetric functions of the conformal cross ratios (4.11),

$$a = a(U_{\sigma(1)}, U_{\sigma(2)}, U_{\sigma(3)}), \tag{4.16}$$

with $\sigma \in S_3$.

Now, we are in a position to carefully match our ansatz in the $5||6$ limit to the right-hand side of Eq. (4.1). Let us first collect all terms proportional to the large $\Omega_2^{||}$. Stripping the latter along with the denominator in Eq. (4.9) we find for them

$$\frac{X_{13}^2}{X_{36}^2}\left[a_1 X_{26}^2 \Omega_3^{||} + 2a_{3,1} X_{46}^2 \Omega_5^{||}\right] + X_{13}^2 a_2 \Omega_4^{||}. \tag{4.17}$$

Comparing the square bracket to Eq. (4.6), we conclude that

$$a_{3,1} = \frac{1}{2} a_1, \tag{4.18}$$

since then, the numerator of the double-splitting amplitude factorizes out producing

$$X_{13}^2 (a_1 + a_2) \Omega_4^{||}. \tag{4.19}$$

Similarly, we analyze the collinear behavior of the terms involving large $\Omega_1^{||}$ and $\Omega_6^{||}$. To form and factor out the splitting amplitude, we need to impose a condition similar to (4.18) on $a_{3,2}$ as well,

[3]At the tree level we are working with, this is obvious since there can be only one pole from $3 \to 1$ transition in a given Feynman graph.

$$a_{3,2} = \frac{1}{2} a_1. \tag{4.20}$$

Adding everything together we obtain

$$\mathcal{N}_6^{\|} = \frac{\Omega_4^{\|}}{X_{14}^2 X_{25}^2 X_{36}^2} \Big[(a_1 + a_2) X_{13}^2 \Omega_2^{\|} + (a_1 + a_2) X_{35}^2 \Omega_6^{\|} + X_{36}^2 [a_2 + a_1 (U_1 + U_3)] \Omega_1^{\|} \Big]. \tag{4.21}$$

Finally, recalling that $X_{36}^2 \longrightarrow {}^{5\|6} z X_{13}^2 + \bar{z} X_{35}^2$ along with $U_1 + U_3 \longrightarrow {}^{5\|6} 1$ in the collinear limit, we find

$$\mathcal{N}_6^{\|} = \frac{\Omega_1^{\|} \Omega_4^{\|}}{S_{4P} S_{P1}}, \tag{4.22}$$

making use of Eq. (4.5) and the collinear constraint on the sum of the remaining two unknowns

$$2(a_1 + a_2) \xrightarrow{5\|6} 1. \tag{4.23}$$

Equation (4.22) is of the form of the right-hand side of Eq. (4.1). Our double-collinear bootstrap is unable, however, to fix the functions a_1 and a_2 individually.

C. Scalar projection

To alleviate the complication of the collinear bootstrap, we will fix the remaining freedom with a projection on a known component. Upon dimensional reduction down to four space-time dimensions, (2.3) generates all massless amplitudes when the extradimensional components of particles' momenta are set to zero. It produces the maximal helicity-violating scattering as well as all the rest, i.e., the nonmaximal helicity-violating amplitudes. In this limit, $g^2{}_{\dot{1}}$ and $g^1{}_{\dot{2}}$ correspond to the positive and negative helicity gluons, while $g^1{}_{\dot{1}}$ and $g^2{}_{\dot{2}}$ are the four-dimensional scalars, which along with ϕ, ϕ', ϕ'', and ϕ''' restore the $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ internal symmetry of the spinless sector. Without loss of generality, it is convenient to identify ϕ and ϕ''' with their four-dimensional counterparts ϕ^{34} and ϕ^{12} of the sextet ϕ^{AB}.

Since the scattering amplitude involving only scalars depends solely on Mandelstam invariants, knowing its massless form in four dimensions will easily allow us to find it in six by an elementary uplift

$$s_{ij\ldots k} \to S_{ij\ldots k}, \tag{4.24}$$

where $s_{ij\ldots k}$ is a multiparticle Lorentz invariant in four dimensions. Thus, we consider

$$\begin{aligned} \mathcal{A}_6^{4\mathrm{D}} &= \langle \phi_1^{12} \phi_2^{12} \phi_3^{34} \phi_4^{34} \phi_5^{34} \phi_6^{12} \rangle \\ &= \frac{\langle 45 \rangle [12] \langle 6 | p_4 + p_5 | 3]}{s_{456} \langle 56 \rangle [23] \langle 4 | p_5 + p_6 | 1]} \\ &\quad - \frac{\langle 34 \rangle [61] \langle 2 | p_3 + p_4 | 5]}{s_{561} \langle 23 \rangle [56] \langle 4 | p_5 + p_6 | 1]}, \end{aligned} \tag{4.25}$$

which is a next-to-maximal helicity-violating amplitude that was extracted from, e.g., Ref. [15] in the spinor-helicity formalism. It can be converted to a form involving only Mandelstam invariants by multiplying the numerator and denominator with appropriate angle/square brackets and forming strings of Dirac matrices from these. In this manner, we get

$$\begin{aligned} \mathcal{A}_6^{4\mathrm{D}} &= \frac{1}{s_{56} s_{23}} \left(\frac{s_{12} s_{45}}{s_{456}} + \frac{s_{16} s_{34}}{s_{561}} - s_{612} \right) \\ &= \frac{x_{36}^2}{x_{15}^2 x_{24}^2} (u_1 + u_3 - 1), \end{aligned} \tag{4.26}$$

where we displayed after the first equality sign the form that exhibits the singularity structure of the amplitude when groups of adjacent momenta become collinear, while after the second one, we showed it in terms of four-dimensional analogs of conformal cross ratios.

To extract this component from our ansatz (4.8), we set the supercharges corresponding to legs 3, 4 and 5 to zero,

$$\{ Q_3, Q_4, Q_5, \bar{Q}_3, \bar{Q}_4, \bar{Q}_5 \} = 0, \tag{4.27}$$

solve Q_1, Q_2 as well as their chiral conjugates from the supermomentum-conserving delta functions in terms of Q_6

$$Q_1 = -\frac{Q_6 \overline{P_2} P_1}{S_{12}}, \qquad Q_2 = -\frac{Q_6 \overline{P_1} P_2}{S_{12}}. \tag{4.28}$$

We list the thus-reduced nonvanishing Ω's, which we label Ω_i^{sc}, in Appendix B. Q_6 and $\bar{Q}_6$ have now to be integrated out to project on the component in question. Introducing the shorthand notation

$$\langle \langle \ldots \rangle \rangle_6 \equiv \int d^2\eta_6 d^2\bar{\eta}_6 \ldots, \tag{4.29}$$

the only relation one has to use is

$$\langle \langle Q_6^A Q_6^B \bar{Q}_{6,C} \bar{Q}_{6,D} \rangle \rangle_6 = P_6^{AB} \bar{P}_{6,CD}, \tag{4.30}$$

in the product of two Ω's. This converts the latter into lengthy Dirac traces which can be evaluated with, for example, FeynCalc [18,19] and yield

$$\frac{\langle \langle \Omega_1^{\mathrm{sc}} \Omega_4^{\mathrm{sc}} \rangle \rangle_6}{X_{14}^3 X_{25}^2 X_{36}^2} = \frac{X_{26}^2 X_{35}^2 X_{46}^2}{X_{13}^2} (1 - U_1 + U_2 - U_3), \tag{4.31}$$

$$\frac{\langle\langle \Omega_1^{\rm sc}\Omega_5^{\rm sc}\rangle\rangle_6}{X_{14}^3X_{25}^2X_{36}^2}=\frac{X_{26}^2X_{35}^2X_{36}^2}{X_{13}^2}(1-U_1+U_2-U_3), \tag{4.32}$$

$$\frac{\langle\langle \Omega_1^{\rm sc}\Omega_6^{\rm sc}\rangle\rangle_6}{X_{14}^3X_{25}^2X_{36}^2}=2\frac{X_{24}^2X_{26}^2X_{35}^2X_{36}^2}{X_{13}^2X_{25}^2}(1-U_1), \tag{4.33}$$

$$\frac{\langle\langle \Omega_4^{\rm sc}\Omega_5^{\rm sc}\rangle\rangle_6}{X_{14}^3X_{25}^2X_{36}^2}=2\frac{X_{15}^2X_{26}^2X_{36}^2X_{46}^2}{X_{13}^2X_{14}^2}(1-U_3), \tag{4.34}$$

$$\frac{\langle\langle \Omega_4^{\rm sc}\Omega_6^{\rm sc}\rangle\rangle_6}{X_{14}^3X_{25}^2X_{36}^2}=\frac{X_{26}^2X_{36}^2X_{46}^2}{X_{13}^2}(1-U_1+U_2-U_3), \tag{4.35}$$

$$\frac{\langle\langle \Omega_5^{\rm sc}\Omega_6^{\rm sc}\rangle\rangle_6}{X_{14}^3X_{25}^2X_{36}^2}=\frac{X_{26}^2X_{36}^4}{X_{13}^2}(1-U_1+U_2-U_3). \tag{4.36}$$

Substituting these expressions into our ansatz and equating it to the uplifted form of the six-scalar amplitude (4.26), we find that the only solution for the last two unknown functions a_1 and a_2 is

$$a_1=-\frac{1}{2}a_2=\frac{1}{U_1+U_2+U_3-3}. \tag{4.37}$$

These correctly reproduce the expected collinear behavior (4.23). We notice, however, that the six-leg amplitude develops a spurious pole at $U_1+U_2+U_3=3$, which should cancel upon extraction of individual components.

D. Final expression

Having completely fixed all unknown functions accompanying the dual covariants Ω_i, we can finally present the six-leg amplitude. It is

$$\hat{\mathcal{A}}_6^{(0)}=\frac{\mathcal{N}_6}{X_{13}^2X_{24}^2X_{35}^2X_{46}^2X_{51}^2X_{62}^2X_{14}^2X_{25}^2X_{36}^2(U_1+U_2+U_3-3)}, \tag{4.38}$$

with

$$\mathcal{N}_6=\frac{X_{13}^2X_{26}^2}{X_{36}^2}\Omega_2\Omega_3-2X_{13}^2\Omega_2\Omega_4+\frac{1}{2}X_{14}^2(U_1+U_2)\Omega_2\Omega_5 + \text{cyclic permutations}. \tag{4.39}$$

This expression disagrees with the one advocated in Ref. [10] from the numerical implementation of six-dimensional BCFW recursion in the coefficient of the last term in the numerator, it is half of the one in that work. We also inspected another form of the alleged six-leg amplitude, given by Eq. (3.78) there. As it stands, it possesses the correct double-collinear behavior in the $5||6$ channel. However, it does not yield the expected form when projected on the scalar component (4.26), which we extracted in the previous section. Namely, the term $\Omega_4\Omega_5$ in Eq. (3.78) is an obstruction for an overall factorization of the spurious zero from the (U_1+U_3-1) factor in (4.26), provided one changes the relative sign in the factor accompanying the $\Omega_3\Omega_6$ structure. Moreover, simply neglecting this term changes the location of the spurious pole to $U_1-U_2+U_3=1$, rather than $U_2+U_3-U_1=1$ reported there. These naive changes alter the proper collinear limit, however. It is unclear if one can salvage that formula.

V. TRIPLE-COLLINEAR SPLITTING SUPERAMPLITUDE

Having constructed the six-leg amplitude, we can now consider its triple-collinear limit and extract the corresponding triple-collinear splitting superamplitude

$$\hat{\mathcal{A}}_6^{(0)}(P_1,P_2,P_3,P_4,P_5,P_6)\xrightarrow{4||5||6}\hat{\mathcal{A}}_4^{(0)}(P_1,P_2,P_3,P)\,\mathcal{S}\text{plit}^{(0)}(-P;P_4,P_5,P_6). \tag{5.1}$$

As in Sec. III, we introduce the Sudakov decomposition for the near-collinear momenta

$$P_i=z_iP+K_{i,\perp}-\frac{K_{i,\perp}^2}{2z_i(P\cdot N)}N,\quad i=4,5,6 \tag{5.2}$$

with

$$\sum_i z_i=1,\qquad \sum_i K_{i,\perp}=0, \tag{5.3}$$

to have a clear identification of the $K_\perp$-scaling for various terms in this kinematics.

Since the reduced four-leg amplitude is just an overall scalar factor (2.12), we strip it down and define the triple-splitting superamplitude as

$$\mathcal{S}\text{plit}^{(0)}(-P;P_4,P_5,P_6)=\frac{\mathcal{N}_6^{|||}/(S_{12}S_{23})^2}{z_4z_6S_{45}S_{56}[\bar{z}_4S_{45}+\bar{z}_6S_{56}+(z_4z_6-3\bar{z}_4\bar{z}_6)S_{456}]}. \tag{5.4}$$

The unfortunate feature of this representation is that the splitting superamplitude inherits the spurious pole of the six-leg amplitude. Accepting this feature as unavoidable, it suffices to analyze the triple-collinear behavior of individual dual covariants Ω_i that we denote as $\Omega_i^{|||}$. Since the triple splitting has to have $1/S$-type singularity in the small Mandelstam invariants $S=\{S_{45},S_{56},S_{456}\}$, the products $\Omega_i^{|||}\Omega_j^{|||}$ should scale at least as $O(K_\perp^2)$. The large kinematical denominator $(S_{12}S_{23})^2$ then has to cancel out with the same factors emerging from $\Omega_i^{|||}$. Last but not least, $\Omega_i^{|||}$ have to depend only on the (anti)chiral changes $Q_i/\bar{Q}_i$ with $i=4$, 5, 6 and be oblivious to the rest.

Analyzing the scaling of $\Omega_i^{|||}$ with $K_\perp$, we find that the six dual covariants form two groups, one containing small $\Omega_i^{|||}\sim O(K_\perp)$ with $i=1,2,5,6$ and another one with two ultrasmall ones $\Omega_i^{|||}\sim O(K_\perp^3)$ for $i=3,4$. Dropping terms of orders higher than corresponding leading terms, the independence of $\Omega_i^{|||}$ on the (anti)chiral charges $Q_i/\bar{Q}_i$ with $i=1,2,3$ becomes evident. In this manner, as an intermediate result, we find their leading asymptotics as $4||5||6$,

$$\begin{aligned}
\Omega_1^{|||}&=\frac{1}{2}(Q_4+Q_5+Q_6)(\bar{P}_4+\bar{P}_5+\bar{P}_6)P_1[(\bar{P}_5+\bar{P}_6)P_3\bar{Q}_4-S_{34}(\bar{Q}_5+\bar{Q}_6)]+\text{cc},\\
\Omega_2^{|||}&=\frac{1}{2}[S_{23}(Q_5+Q_6)-S_{234}(Q_4+Q_5+Q_6)]\bar{P}_3(P_4+P_5)(\bar{Q}_4+\bar{Q}_5)+\text{cc},\\
\Omega_3^{|||}&=\frac{1}{2}(Q_4+Q_5)(\bar{P}_4+\bar{P}_5)P_3[S_{56}\bar{Q}_4-\bar{P}_4(P_5+P_6)(\bar{Q}_5+\bar{Q}_6)]+\text{cc},\\
\Omega_4^{|||}&=\frac{1}{2}[S_{56}Q_4-(Q_5+Q_6)(\bar{P}_5+\bar{P}_6)P_4][\bar{P}_5P_1\bar{Q}_6-S_{16}\bar{Q}_5]+\text{cc},\\
\Omega_5^{|||}&=\frac{1}{2}[Q_6\bar{P}_1P_5-S_{16}Q_5][S_{12}\bar{Q}_6-\bar{P}_6P_3(\bar{Q}_4+\bar{Q}_5+\bar{Q}_5)]+\text{cc},\\
\Omega_6^{|||}&=\frac{1}{2}[S_{12}Q_6-(Q_4+Q_5+Q_6)\bar{P}_3P_6]\bar{P}_1(P_4+P_5+P_6)(\bar{Q}_4+\bar{Q}_5+\bar{Q}_5)+\text{cc}.
\end{aligned}$$

In this form, Ω's superficially possess residual dependence on the alien momentum labels $i=1,2,3$. It is eliminated upon extraction of small contractions $\langle i|j]$, $i,j=4,5,6$, and further imposing the strict triple-collinear limit in all factors accompanying them, i.e., setting $P_i=z_iP$. In this manner, we deduce the final expressions for $\Omega^{|||}$'s,

$$\begin{aligned}
\Omega_1^{|||}&=\frac{1}{2}S_{12}S_{23}[\langle 4_a|5_{\dot a}](\sqrt{z_4}\eta_5^a-\sqrt{z_5}\eta_4^a)+\langle 4_a|6_{\dot a}](\sqrt{z_4}\eta_6^a-\sqrt{z_6}\eta_4^a)+\langle 5_a|6_{\dot a}](\sqrt{z_5}\eta_6^a-\sqrt{z_6}\eta_5^a)]\\
&\quad\times[\bar{z}_4\sqrt{z_4}\eta_4^a-z_4\sqrt{z_5}\eta_5^a-z_4\sqrt{z_6}\eta_6^a]+\text{cc},
\end{aligned}\tag{5.5}$$

$$\Omega_2^{|||}=\frac{1}{2}S_{12}S_{23}[\bar{z}_4\sqrt{z_4}\eta_4^a-z_4\sqrt{z_5}\eta_5^a-z_4\sqrt{z_6}\eta_6^a]\langle 4_a|5_{\dot a}](\sqrt{z_4}\bar{\eta}_5^{\dot a}-\sqrt{z_5}\bar{\eta}_4^{\dot a})+\text{cc},\tag{5.6}$$

$$\Omega_3^{|||}=\frac{1}{2}S_{12}(z_4\eta_5^a-\sqrt{z_4z_5}\eta_4^a)\langle 4_a|5_{\dot a}][S_{56}\bar{\eta}_4^{\dot a}+(\bar{Q}_5P_6+\bar{Q}_6P_5)|4^{\dot a}]]+\text{cc},\tag{5.7}$$

$$\Omega_4^{|||}=\frac{1}{2}S_{23}[S_{56}\eta_4^a+(Q_5\bar{P}_6+Q_6\bar{P}_5)|4^a\rangle]\langle 4_a|5_{\dot a}](z_6\bar{\eta}_5^{\dot a}-\sqrt{z_5z_6}\bar{\eta}_6^{\dot a})+\text{cc},\tag{5.8}$$

$$\Omega_5^{|||}=\frac{1}{2}S_{12}S_{23}(\sqrt{z_5}\eta_6^a-\sqrt{z_6}\eta_5^a)\langle 5_a|6_{\dot a}][z_6\sqrt{z_4}\bar{\eta}_4^{\dot a}+z_6\sqrt{z_5}\bar{\eta}_5^{\dot a}-\bar{z}_6\sqrt{z_6}\bar{\eta}_6^{\dot a}]+\text{cc},\tag{5.9}$$

$$\begin{aligned}
\Omega_6^{|||}&=\frac{1}{2}S_{12}S_{23}[z_6\sqrt{z_4}\eta_4^a+z_6\sqrt{z_5}\eta_5^a-\bar{z}_6\sqrt{z_6}\eta_6^a]\\
&\quad\times[\langle 4_a|5_{\dot a}](\sqrt{z_4}\bar{\eta}_5^{\dot a}-\sqrt{z_5}\bar{\eta}_4^{\dot a})+\langle 4_a|6_{\dot a}](\sqrt{z_4}\bar{\eta}_6^{\dot a}-\sqrt{z_6}\bar{\eta}_4^{\dot a})+\langle 5_a|6_{\dot a}](\sqrt{z_5}\bar{\eta}_6^{\dot a}-\sqrt{z_6}\bar{\eta}_5^{\dot a})]+\text{cc}.
\end{aligned}\tag{5.10}$$

Their substitution into Eq. (4.39) with the simultaneous replacement of all dual distances with their strict collinear limits (except for the vanishing invariants),

$$\{X_{13}^2,X_{24}^2,X_{35}^2,X_{46}^2,X_{51}^2,X_{62}^2,X_{14}^2,X_{25}^2,X_{36}^2\}\xrightarrow{4||5||6}\{S_{12},S_{23},z_4S_{12},S_{45},S_{56},z_6S_{23},S_{456},\bar{z}_4S_{23},\bar{z}_6S_{12}\},\tag{5.11}$$

yields the final expression for the triple-splitting amplitude. Since this expression is merely a sum of its parts and is extremely lengthy, we do not present it here to save space. As an ultimate check on these expressions, we considered the projection of $\mathcal{S}\text{plit}^{(0)}(-P;P_4,P_5,P_6)$ on the purely scalar sector to observe the required cancellation of the spurious pole and confirm agreement with the expected form of the amplitude (4.26). This is demonstrated in Appendix C.

VI. CONCLUSIONS

In this paper, we studied six-dimensional amplitudes in $\mathcal{N}=(1,1)$ sYM theory in the multicollinear limits. We first used known double-collinear splitting amplitudes to constrain the form of the six-leg case, which was previously extracted from the numerical BCFW recursion relations. The collinear bootstrap, while strongly constraining its form, leaves some freedom in the ansatz. The latter was fixed by considering a simple component of this superamplitude. Relying on this result, we extracted from it the triple-collinear splitting superamplitude.

The result of this consideration is of interest for multiple reasons. First, the use of the double- and triple-collinear behavior will undoubtedly help to construct the seven-leg amplitude in this theory and then find the quadruple-splitting amplitude from it. The process can then be repeated for higher multiplicity. The use of the known analyticity structure will certainly benefit from numerical BCFW recursions for the amplitude determination and these should be used in tandem.

The triple-splitting amplitude can now be used for the construction of integrands of the double-splitting amplitude. Previously, this question was addressed at one-loop order [13], where the tree-level double splitting was sufficient to find the integrand employing the unitarity-cut sewing technique [20–22]. At two loops, in addition to iterated double two-particle cuts, one has to include a three-particle cut of the five-leg superamplitude and the triple-collinear superamplitude found in this work.

These questions are currently under study and their results will be announced elsewhere.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: EQUIVALENCE OF FIVE-LEG SUPERAMPLITUDES

In this appendix, we provide a simple analytical demonstration of the equivalence between the compact representation of the five-leg reduced superamplitude (2.12) to the original lengthier form [5]

$$\hat{\mathcal{A}}_5^{(0)}=\frac{1}{\prod_{i=1}^5 S_{i,i+1}}\Big(Q_1\bar{P}_2P_3\bar{P}_4P_5\bar{Q}_1+\text{cyclic}+\frac{1}{2}(Q_1\bar{P}_2[P_3,\bar{P}_4,P_5]\bar{Q}_2+Q_3\bar{P}_4[P_5,\bar{P}_1,P_2]\bar{Q}_4+(Q_3+Q_4)\bar{P}_5[P_1,\bar{P}_2,P_3]\bar{Q}_5+\text{cc})\Big),\tag{A1}$$

where $[P_i,\bar{P}_j,P_k]\equiv P_i\bar{P}_jP_k-P_k\bar{P}_jP_i$. The Ω covariants in Eq. (2.12) are cyclically symmetric on the support of the super-momentum conserving delta functions,

$$\Omega\equiv\Omega_{12345}=\Omega_{23451}=\Omega_{34512}=\Omega_{45123}=\Omega_{51234}.\tag{A2}$$

We will focus on the Ω_{51234} ordering since it will be particularly useful for our follow-up study of the $4||5$ collinear asymptotics adopted in Sec. III of the body of the paper. Making use of the relations between super-region momenta and super-Poincaré ones (2.7), we can recast the bras and kets (2.9) entering Ω_{51234} as

$$\langle B_{5,13}|=-S_{34}Q_5+(Q_3+Q_5)(\bar{P}_3+\bar{P}_4)P_5,\ |\bar{B}_{5,24}]=-S_{51}\bar{Q}_4+\bar{P}_4(P_1+P_5)(\bar{Q}_5+\bar{Q}_1).\tag{A3}$$

The inner products $\langle B|\bar{B}]$ can be further simplified, to give

$$-2\Omega_{51234}=S_{45}[S_{34}Q_5\bar{Q}_1-Q_4\bar{P}_3P_5\bar{Q}_1+S_{51}Q_3\bar{Q}_4-Q_3\bar{P}_4P_1\bar{Q}_5-Q_3\bar{P}_4P_5\bar{Q}_1]+S_{51}Q_4\bar{P}_3P_5\bar{Q}_4+S_{34}Q_5\bar{P}_4P_1\bar{Q}_5+Q_4[S_{34}S_{51}-\bar{P}_3P_5\bar{P}_4P_1]\bar{Q}_5+\text{cc}.\tag{A4}$$

Now, to show its equivalence to (A1), it is necessary to use super-momentum conservation conditions in Eq. (A1), such that only the structures emerging in the dual conformal representation show up. It is not an easy fit, as *a priori* one could end up with multiple extra contributions that will vanish only upon the use of cyclicity and

momentum conservation. One would want to avoid these unnecessary complications. The form of the invariant Ω_{5123} gives us a hint on how to proceed. Namely, the first term in it depends on the supercharges $(Q_3, Q_4, Q_5) \times (\bar{Q}_1, \bar{Q}_4, \bar{Q}_5)$. Thus, we will eliminate the rest from the amplitude (A1). This is not completely unambiguous since one can trade these supercharges between the direct and chiral conjugate contributions. Therefore, it necessitates a trial-and-error procedure. We can demonstrate the sought-after equivalence with the following steps. First, we use the relation by applying (anti)chiral charge conservation in the first term of the second line of Eq. (A1)

$$\begin{aligned} Q_1\bar{P}_2[P_3,\bar{P}_4,P_5]\bar{Q}_2 = &-Q_1\bar{P}_2P_3\bar{P}_4P_5\bar{Q}_1 - (Q_3+Q_4+Q_5)\bar{P}_2P_3\bar{P}_4P_5\bar{Q}_1 \\ &- Q_1\bar{P}_2[P_3,\bar{P}_4,P_5](\bar{Q}_3+\bar{Q}_4+\bar{Q}_5). \end{aligned} \tag{A5}$$

Together with its chiral conjugate, it eliminates the first term in the first line. The last term above, i.e., involving Q_1 is moved to the cc contribution but its conjugate is extracted from it: this term matches the pattern we are looking for. The second and third terms in the first line in (A1) can be brought to the form

$$Q_2\bar{P}_3P_4\bar{P}_5P_1\bar{Q}_2 = -\frac{1}{2}(Q_3+Q_4+Q_5)\bar{P}_1P_5\bar{P}_4P_3(\bar{Q}_1+\bar{Q}_4+\bar{Q}_5) + \text{cc}, \tag{A6}$$

$$Q_3\bar{P}_4P_5\bar{P}_1P_2\bar{Q}_3 = -\frac{1}{2}Q_3\bar{P}_4P_5\bar{P}_1P_2(\bar{Q}_1+\bar{Q}_4+\bar{Q}_5) + \text{cc}, \tag{A7}$$

again using the super-momentum conservation. The rest of the contributions in Eq. (A1) already possess the required structure and so they are just left intact, however, the remaining two self-conjugate terms, proportional to $Q_i\ldots\bar{Q}_i$, with $i=4,5$, are then split evenly between the direct contribution and its cc image. Summarizing this short consideration, we find the numerator of the five-leg amplitude (A1) to be

$$\begin{aligned} 2\text{Numerator}\big[\hat{\mathcal{A}}_5^{(0)}\big] = &-(Q_3+Q_4+Q_5)\bar{P}_1P_5\bar{P}_4P_3(\bar{Q}_1+\bar{Q}_4+\bar{Q}_5) \\ &- Q_3\bar{P}_4P_5\bar{P}_1P_2(\bar{Q}_1+\bar{Q}_4+\bar{Q}_5) - (Q_3+Q_4+Q_5)\bar{P}_2P_5\bar{P}_4P_3\bar{Q}_1 \\ &+ (Q_3+Q_4+Q_5)[\bar{P}_5,P_4,\bar{P}_3]P_2\bar{Q}_1 + Q_3\bar{P}_4[P_5,\bar{P}_1,P_2]\bar{Q}_4 \\ &+ (Q_3+Q_4)\bar{P}_5[P_1,\bar{P}_2,P_3]\bar{Q}_5 + Q_4\bar{P}_5P_1\bar{P}_2P_3\bar{Q}_4 + Q_5\bar{P}_1P_2\bar{P}_3P_4\bar{Q}_5 + \text{cc}. \end{aligned} \tag{A8}$$

Although this representation looks lengthy, its equivalence to $-2\Omega_{51234}$ is now easy to demonstrate. It boils down to the use of momentum conservation and repeated application of Dirac algebra. Let us show it for just one term as an example. We find after a chain of transformations

$$2\text{Numerator}\big[\hat{\mathcal{A}}_5^{(0)}\big]_{Q_4\bar{Q}_4} = -Q_4\bar{P}_1P_5\bar{P}_4P_3\bar{Q}_4 + Q_4\bar{P}_5P_1\bar{P}_2P_3\bar{Q}_4 + \text{cc} \tag{A9}$$

$$\begin{aligned} &= -S_{34}Q_4\bar{P}_1P_5\bar{Q}_4 - Q_4\bar{P}_5P_1(\bar{P}_4+\bar{P}_5)P_3\bar{Q}_4 + \text{cc} \\ &= -2S_{34}Q_4[\bar{P}_1P_5+\bar{P}_5P_1]\bar{Q}_4 - 2S_{51}Q_4\bar{P}_5P_3\bar{Q}_4 \end{aligned} \tag{A10}$$

where in the second line we anticommuted the last two Dirac matrices in the first term, imposing the on-shell condition $P_4\bar{Q}_4=0$ along the way. While in the second term, we used the on-shellness of lines $P_i\bar{P}_i=0$ and the momentum conservation. Next, in the second term of the second line, we anticommuted $\bar{P}_4$ to the right and $\bar{P}_5$ to get the anticipated expression in Eq. (A4) since the first term of the third lines vanishes by means of the Dirac algebra and $Q_4\bar{Q}_4=0$. The rest can be done accordingly term by term. We thus prove the equality

$$\text{Numerator}[\hat{\mathcal{A}}_5^{(0)}] = -\Omega_{51234}. \tag{A11}$$

APPENDIX B: SCALAR PROJECTIONS OF Ω'S

We list here the scalar projections of the dual covariants Ω_i. Imposing conditions (4.27) and (4.28), they are

$$2S_{12}\Omega_1^{\text{sc}} = S_{34}Q_6(\bar{P}_4+\bar{P}_5)P_3\bar{P}_2P_1\bar{Q}_6 + \text{cc}, \tag{B1}$$

$$\Omega_2^{\rm sc} = \Omega_3^{\rm sc} = 0, \tag{B2}$$

$$2S_{12}\Omega_4^{\rm sc} = -S_{45}Q_6\bar{P}_5(P_6+P_1)P_3\bar{P}_2P_1\bar{Q}_6 + {\rm cc}, \tag{B3}$$

$$2S_{12}\Omega_5^{\rm sc} = S_{345}Q_6\bar{P}_1P_2(\bar{P}_6+\bar{P}_1)P_5\bar{Q}_6 + {\rm cc}, \tag{B4}$$

$$2S_{12}\Omega_6^{\rm sc} = S_{345}Q_6(\bar{P}_4+\bar{P}_5)P_3\bar{P}_2P_1\bar{Q}_6 + {\rm cc}. \tag{B5}$$

APPENDIX C: SCALAR COMPONENT OF TRIPLE SPLITTING

In this appendix, we offer a check on the correctness of Eqs. (5.5)–(5.10). We chose the scalar component, which is found by setting $Q_4 = Q_5 = \bar{Q}_4 = \bar{Q}_5 = 0$ and integrating out Q_6 and $\bar{Q}_6$ as in Eq. (4.30). In this manner, we find

$$\langle\langle \Omega_1^{|||}\Omega_4^{|||}\rangle\rangle_6 = z_4z_6S_{12}S_{23}^2S_{45}[(\bar{z}_6-z_4)S_{456} - \bar{z}_4S_{45} + \bar{z}_6S_{56}], \tag{C1}$$

$$\langle\langle \Omega_1^{|||}\Omega_5^{|||}\rangle\rangle_6 = z_4z_6\bar{z}_6S_{12}^2S_{23}^2[(\bar{z}_6-z_4)S_{456} - \bar{z}_4S_{45} + \bar{z}_6S_{56}], \tag{C2}$$

$$\langle\langle \Omega_1^{|||}\Omega_6^{|||}\rangle\rangle_6 = 2z_4z_6\bar{z}_6S_{12}^2S_{23}^2[\bar{z}_6S_{456} - S_{45}], \tag{C3}$$

$$\langle\langle \Omega_4^{|||}\Omega_5^{|||}\rangle\rangle_6 = 2z_6\bar{z}_6(\bar{z}_6-z_4)S_{12}S_{23}^2S_{45}S_{56}, \tag{C4}$$

$$\langle\langle \Omega_4^{|||}\Omega_6^{|||}\rangle\rangle_6 = z_6\bar{z}_6S_{12}S_{23}^2S_{45}[(\bar{z}_6-z_4)S_{456} - \bar{z}_4S_{45} + \bar{z}_6S_{56}], \tag{C5}$$

$$\langle\langle \Omega_5^{|||}\Omega_6^{|||}\rangle\rangle_6 = z_6\bar{z}_6^2S_{12}^2S_{23}^2[(\bar{z}_6-z_4)S_{456} - \bar{z}_4S_{45} + \bar{z}_6S_{56}], \tag{C6}$$

with all other projections vanishing. Substituting these expressions into (4.39) and using the triple-collinear asymptotics of the Mandelstam invariants (5.11), we find

$$\begin{aligned}\langle\langle \mathcal{N}_6\rangle\rangle_6 = {}& S_{12}\left(\frac{z_4z_6}{\bar{z}_4} + \frac{S_{45}}{S_{456}}\right)\langle\langle \Omega_1^{|||}\Omega_4^{|||}\rangle\rangle_6 - 2S_{45}\langle\langle \Omega_1^{|||}\Omega_5^{|||}\rangle\rangle_6 + \frac{S_{45}S_{56}}{S_{456}}\langle\langle \Omega_1^{|||}\Omega_6^{|||}\rangle\rangle_6 \\ & + \frac{z_4}{\bar{z}_4}S_{12}\langle\langle \Omega_4^{|||}\Omega_5^{|||}\rangle\rangle_6 - 2z_4S_{12}\langle\langle \Omega_4^{|||}\Omega_6^{|||}\rangle\rangle_6 + \frac{z_4}{\bar{z}_6}S_{45}\langle\langle \Omega_5^{|||}\Omega_6^{|||}\rangle\rangle_6,\end{aligned} \tag{C7}$$

which after a little algebra gives the factorized form of the splitting amplitude's numerator

$$\langle\langle \mathcal{N}_6\rangle\rangle_6/(S_{12}S_{23})^2 = [\bar{z}_4S_{45} + \bar{z}_6S_{56} + (z_4z_6 - 3\bar{z}_4\bar{z}_6)S_{456}]\frac{z_4z_6S_{45}}{\bar{z}_4S_{456}}[(\bar{z}_6-z_4)S_{456} - \bar{z}_4S_{45}]. \tag{C8}$$

Here, the first factor cancels the spurious pole in Eq. (5.4). The rest agree with the collinear limit of the six-dimensional uplift of the scalar amplitude (4.26).

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