

Hamiltonian lattice formulation of compact Maxwell-Chern-Simons theory

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In this paper, we derive a Hamiltonian lattice formulation of the $2 + 1$ D compact Maxwell-Chern-Simons theory within the zero instanton sector. We analytically solve the model and show that the mass gap in the continuum limit precisely reproduces the well-known continuum result. The formulation preserves key topological features, including the quantization of the Chern-Simons level, the degeneracy of energy eigenstates, the nontrivial properties of Wilson loops, and the mutual and self-statistics of anyons. By reconciling gauge compactness, locality, and canonical quantization within a unified framework, this work provides a foundation for conceivable future Hamiltonian-based simulations of lattice Maxwell-Chern-Simons theory. It aims to bridge topological field theory and quantum information science, with the goal of enabling Hamiltonian-based quantum simulations of Maxwell-Chern-Simons theory on both classical and quantum computers.

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I. INTRODUCTION

Chern-Simons theory is a topological quantum field theory with numerous applications in condensed matter and high-energy physics, including the study of anomalies, fermion/boson dualities, and the fractional quantum Hall effect [1]. Recently, a Lagrangian lattice formulation of pure Chern-Simons theory and its canonical quantization has been proposed [2,3]. However, pure lattice Chern-Simons theory faces a doubling problem, similar to the fermion doubling problem. The presence of a Maxwell term with Chern-Simons theory solves the doubling problem, in a way similar to the Wilson term that eliminates the fermion doubling problem [4]. The resulting theory is lattice Maxwell-Chern-Simons theory, which we consider in this paper. In the continuum, the Maxwell-Chern-Simons theory has been first introduced in Ref. [5], where it was

shown that in $2 + 1$ D, when a Chern-Simons term is added to the Maxwell Lagrangian, the photon becomes massive through a topological mechanism, without spontaneous symmetry breaking via the Higgs mechanism. This mechanism of topological mass generation is relevant for topological phases of matter (for a review see Ref. [6]), which are gapped systems with a new type of quantum order that is not due to symmetry breaking [7]. These phases of matter can have various exotic quantum features, like excitations with nontrivial statistics or ground-state degeneracy on manifolds with nontrivial topology.

The Lagrangian formulation of compact Maxwell-Chern-Simons theory for $2 + 1$ D Euclidean spacetime lattices has been first derived in Ref. [8] using the self-dual approximation introduced in Ref. [9] and later studied in other works (see, e.g., Refs. [10–12]). The Hamiltonian formulation has so far been derived only for noncompact gauge fields [13], but the corresponding formulation for compact gauge fields remains absent. Such a formulation would be particularly important for future Hamiltonian-based simulations of the theory on classical and quantum computers. The Hamiltonian formulation provides a promising approach to simulate sign-problem afflicted regimes

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in lattice field theory, including topological terms, chemical potentials, and out-of-equilibrium dynamics. Recently, there has been growing interest in the development of Hamiltonian-based simulation methods, including tensor network states [14,15], non-Gaussian states [16–19], machine-learning-based approaches [20–23], and quantum computing [24,25]. In particular, topological terms have been numerically investigated in $1 + 1\text{D}$ using tensor network states [26–30] and quantum algorithms [31,32], as well as in $3 + 1\text{D}$ using exact diagonalization [33]. Topological terms in $2 + 1\text{D}$ have not yet been simulated, due to the lack of a suitable Hamiltonian lattice formulation.

In this paper, we derive a Hamiltonian lattice formulation of compact Maxwell-Chern-Simons theory using the Villain approximation. We focus on the theory within the zero instanton sector, where we are able to construct a local and uniform (i.e., site-independent in form) Hamiltonian formulation. We analytically solve this theory and show that the mass gap in the continuum limit matches the well-known continuum formula. Moreover, topological features such as the quantization of the Chern-Simons level, the degeneracy of the energy eigenstates, and the nontrivial properties of Wilson loops are reproduced. When fermions are introduced into the theory, numerical methods become necessary. Our Hamiltonian formulation of Maxwell-Chern-Simons theory thus provides a foundation for conceivable future Hamiltonian-based simulations of the theory on classical and quantum computers. In particular, the compact and gauge-invariant structure of our Hamiltonian is a natural starting point for exploring quantum dynamics of topological gauge theory on near-term quantum devices. This formulation aims to bridge topological quantum field theory and quantum information science, with the goal of enabling quantum simulations of anyonic excitations and topological phases in $2 + 1$ dimensions, and may offer a possible route toward realizing topological lattice gauge theories within quantum computational frameworks.

The paper is organized as follows. Section II is a review section that sets the stage for the original results. In Sec. II A, we review the Lagrangian lattice formulation of compact Abelian Chern-Simons theory as proposed in Ref. [2]. In Sec. II B, we review the compact Maxwell lattice Hamiltonian and emphasize there are two versions of the Hamiltonian with instantons allowed or suppressed; this distinction is essential because the Chern-Simons term enforces the instanton-suppressed sector and leads to a qualitatively different Hamiltonian structure. Our original contributions begin in Sec. III, where we derive the Hamiltonian lattice formulation of compact Maxwell-Chern-Simons theory. Here, we discuss the quantization of the Chern-Simons level and the degeneracy of the states. We also discuss some physical properties of the lattice Maxwell-Chern-Simons theory, including the perimeter law of Wilson loops and their nontrivial linking, as well as the mutual and self-statistics of anyons. In Sec. IV A, we

derive the solution to the constant mode wave functions. In Sec. IV B, we analytically solve the quadratic Hamiltonian that we derived in Sec. III, and we plot the resulting band structure. In Sec. V, we summarize and discuss our results.

II. REVIEW OF RELEVANT BACKGROUND

In this section, we review the background material needed for our original results. We first review the Lagrangian lattice formulation of compact Chern-Simons theory in Sec. II A and then the Hamiltonian formulation of compact Maxwell theory in Sec. II B.

A. Compact Chern-Simons lattice action

In this section, we review the Lagrangian lattice formulation of compact Abelian Chern-Simons theory, based on Ref. [2]. This review establishes the lattice Chern-Simons action that serves as the starting point for our Hamiltonian derivation in Sec. III.

In the continuum, the Abelian Chern-Simons action is defined as

$$S_{\text{CS}}(A) = -\frac{ik}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \quad (1)$$

where A is the $U(1)$ gauge field configuration, and k is the quantized Chern-Simons level [34].

For the lattice discretization of this action, we consider a $2 + 1\text{D}$ cubic lattice with periodic boundary conditions in all directions. The $U(1)$ gauge field A lives on the links of the lattice. A naive lattice discretization of the continuous Chern-Simons action in Eq. (1) is

$$S_{\text{CS}}(A) = -\frac{ik}{4\pi} a^2 d\tau \sum_{x \in \text{sites}} \epsilon^{\mu\nu\rho} A_{x;\mu} \Delta_\nu A_{x+\hat{\mu};\rho}, \quad (2)$$

where a and $d\tau$ are the lattice spacings in the space and time directions, respectively, and $\hat{\mu}$ is a unit vector in the corresponding direction. Here, the finite (forward) difference operator of the A field on the lattice is defined as

$$\Delta_0 A_{x;\mu} = \frac{A_{x+\hat{0};\mu} - A_{x;\mu}}{d\tau}, \quad (3)$$

$$\Delta_i A_{x;\mu} = \frac{A_{x+\hat{i};\mu} - A_{x;\mu}}{a}, \quad (4)$$

where Δ_0 represents the time difference and $\Delta_{i \in \{1,2\}}$ represents the spatial difference.

We note that after integration by parts, Eq. (2) can be equivalently written in terms of the backward lattice difference operator,

$$S_{\text{CS}}(A) = -\frac{ik}{4\pi} a^2 d\tau \sum_{x \in \text{sites}} \epsilon^{\mu\nu\rho} A_{x;\mu} \hat{\Delta}_\nu A_{x-\hat{\nu};\rho}, \quad (5)$$

where

$$\hat{\Delta}_0 A_{x;\mu} = \frac{A_{x;\mu} - A_{x-\hat{0};\mu}}{d\tau}, \quad (6)$$

$$\hat{\Delta}_i A_{x;\mu} = \frac{A_{x;\mu} - A_{x-\hat{i};\mu}}{a}. \quad (7)$$

Thus, we can try to write the action symmetrically with both the forward and backward difference operators,

$$S_{\text{CS}}(A) = -\frac{ik}{4\pi} a^2 d\tau \sum_{x \in \text{sites}} \epsilon^{\mu\nu\rho} \frac{A_{x;\mu} \Delta_\nu A_{x+\hat{\mu};\rho} + A_{x;\mu} \hat{\Delta}_\nu A_{x-\hat{\mu};\rho}}{2}. \quad (8)$$

As shown in Refs. [35,36], this symmetric form of the action respects Osterwalder-Schrader reflection positivity. Next, we introduce a pictorial illustration of the lattice action. For this, we first explicitly write out the epsilon tensor in Eq. (2), which results in

$$S_{\text{CS}}(A) = -\frac{ik}{4\pi} a^2 d\tau \sum_{x \in \text{sites}} [A_{x;0}(\Delta_1 A_{x+\hat{0};2} - \Delta_2 A_{x+\hat{0};1}) - A_{x;1}(\Delta_0 A_{x+\hat{1};2} - \Delta_2 A_{x+\hat{1};0}) + A_{x;2}(\Delta_0 A_{x+\hat{2};1} - \Delta_1 A_{x+\hat{2};0})]. \quad (9)$$

This action can be visualized as

$$S_{\text{CS}}(A) = -\frac{ik}{4\pi} a^2 d\tau \sum_{\text{cubes}} \left[\begin{array}{c} \text{Diagram 1} \times \frac{1}{a} \left(\text{Diagram 2} \right) \\ - \text{Diagram 3} \times \frac{1}{d\tau} \left(\text{Diagram 4} \right) \\ + \text{Diagram 5} \times \frac{1}{d\tau} \left(\text{Diagram 6} \right) \end{array} \right]. \quad (10)$$

The diagrams represent the terms in the action. Diagram 1 is a cube with a solid line on the front face and dashed lines on the other faces. Diagram 2 is a parallelogram with vertices labeled $-A_2$, $-A_1$, A_2 , and A_1 . Diagram 3 is a cube with a solid line on the front face and dashed lines on the other faces. Diagram 4 is a parallelogram with vertices labeled A_2 , A_1 , $-\frac{d\tau}{a} A_0$, and $-\frac{d\tau}{a} A_0$. Diagram 5 is a cube with a solid line on the front face and dashed lines on the other faces. Diagram 6 is a parallelogram with vertices labeled A_1 , A_2 , $-\frac{d\tau}{a} A_0$, and $-\frac{d\tau}{a} A_0$.

Here, the solid lines correspond to the actual terms in the action, while the dashed lines are for visual guidance only; the dashed lines denote the positions of the other gauge fields in the products. Moreover, A_0 , A_1 , and A_2 denote the gauge fields on the links in the 0, 1, and 2 directions, which correspond to the directions upwards, right, and into the plane, respectively. Each solid “plaquette” in Eq. (10) corresponds to the two terms in the round brackets of Eq. (9), which—according to the definition of the finite difference operators in Eqs. (3) and (4)—can be expressed in terms of four gauge fields multiplied by a prefactor of $1/d\tau$ or $1/a$, respectively.

This naive definition of the lattice action in Eq. (10) is invariant under local gauge transformations, $A \rightarrow A + d\lambda$. However, it is not invariant under the discrete shift of the compact gauge variable, $A_0 \rightarrow A_0 + \frac{2\pi}{d\tau}$ and $A_i \rightarrow A_i + \frac{2\pi}{a}$, $i = 1, 2$. This discrete shift invariance is necessary because we aim to formulate lattice Chern-Simons theory with a compact gauge group, i.e., $U(1)$ rather than $\mathbb{R}$. The compactness of the gauge group implies the existence of

quantized magnetic fluxes, $\int_\Sigma dA \in 2\pi\mathbb{Z}$, where Σ is a closed surface. This indicates the presence of a monopole configuration if the surface is contractible. Consequently, lattice discretizations of compact $U(1)$ gauge theory generically comprise dynamical lattice-scale monopole configurations. However, monopoles are known to violate gauge invariance in the presence of a Chern-Simons term, as shown in the continuum limit [37,38]. While this has been considered an obstacle for formulating a gauge-invariant Chern-Simons theory on the lattice, this issue can be circumvented using the modified Villain approach [39–45]. In the conventional Villain approach [39], the standard algebra-valued gauge fields $A \in \mathbb{R}$ are accompanied by discrete plaquette variables $n \in \mathbb{Z}$, which encode the quantized magnetic flux. These variables n can be interpreted as discrete gauge fields for the $\mathbb{Z}$ subsymmetry of the noncompact $\mathbb{R}$ gauge symmetry, which is precisely the discrete shift symmetry mentioned above, $A_0 \rightarrow A_0 + \frac{2\pi}{d\tau}$ and $A_i \rightarrow A_i + \frac{2\pi}{a}$, $i = 1, 2$. Upon gauging these discrete shifts, one effectively studies compact $U(1) = \mathbb{R}/2\pi\mathbb{Z}$.

gauge theory instead of noncompact $\mathbb{R}$ gauge theory. This so-called Villain formulation of compact U(1) gauge theory contains instantons (i.e., monopoles in 2 + 1D), which can be eliminated by a Lagrange multiplier φ that constrains the discrete gauge field to be flat, called the modified Villain approach [41,42]. The lattice action and Hamiltonian formulations of the double Chern-Simons theory have been studied using the modified Villain approach [42,44,45].

Following the modified Villain approach, the authors of Ref. [2] have built a gauge-invariant Chern-Simons lattice action by introducing integer degrees of freedom on each plaquette, as well as angular variables on each lattice site. In the following, we use a different sign convention of the Chern-Simons action and therefore obtain a slightly different Chern-Simons lattice action [for the original Chern-Simons lattice action introduced in Ref. [2], see their Eq. (17)],

$$\begin{aligned}
 S_{CS}(A, n, \varphi) = & -\frac{ik}{4\pi} a^2 d\tau \sum_{\text{cubes}} \left[\left(\text{Diagram 1} \right) \times \frac{1}{a} \left(\text{Diagram 2} \right) \right. \\
 & - \text{Diagram 3} \times \frac{1}{d\tau} \left(\text{Diagram 4} \right) \\
 & + \text{Diagram 5} \times \frac{1}{d\tau} \left(\text{Diagram 6} \right) \\
 & + \frac{2\pi}{a^2} \left(\text{Diagram 7} \right) \times \left(\text{Diagram 8} \right) + \frac{2\pi}{a^2} \left(\text{Diagram 9} \right) \times \left(\text{Diagram 10} \right) \\
 & - \frac{2\pi}{ad\tau} \left(\text{Diagram 11} \right) \times \left(\text{Diagram 12} \right) - \frac{2\pi}{ad\tau} \left(\text{Diagram 13} \right) \times \left(\text{Diagram 14} \right) \\
 & + \frac{2\pi}{ad\tau} \left(\text{Diagram 15} \right) \times \left(\text{Diagram 16} \right) + \frac{2\pi}{ad\tau} \left(\text{Diagram 17} \right) \times \left(\text{Diagram 18} \right) \Big] \\
 & + i \sum_{\text{cubes}} \left(\text{Diagram 19} - \frac{1}{2} \cdot \frac{kaA_2}{kd\tau A_0} \left(\text{Diagram 20} \right) \right) \times \left(\text{Diagram 21} \right), \tag{11}
 \end{aligned}$$

where $n_{0,1}$ and $n_{0,2}$ are integer variables living on timelike plaquettes, and $n_{1,2}$ are integer variables living on spacelike plaquettes. These integer variables are drawn as smaller squares in each plaquette for visual clarity, and the negative n values in the last term are indicated by light grey squares. In the last term in Eq. (11), there is an angular variable $\varphi \in [0, 2\pi)$ on each lattice site as the Lagrange multiplier to impose the zero-instanton constraint for each unit cell. (The physical reason for this constraint is that, in the presence of a Chern-Simons term,

an instanton in 2 + 1D must carry electric charge, analogously to the Witten effect in 3 + 1D [46]; consequently, the pure gauge theory cannot accommodate dynamical instantons. We elaborate on this point in the conclusion of Sec. II B.) For any configuration where a unit cell contains nonzero total flux [i.e., instanton number, shown as the cube in Eq. (11)], integrating over the corresponding φ from 0 to 2π in the partition function makes the partition function vanish. Therefore, the φ variables impose hard constraints that eliminate all instantons. Later, we

show that φ is not necessary in our lattice Hamiltonian formulation in Eq. (23) because we impose those constraints directly by ignoring the instanton configurations in the counting of degrees of freedom (an outline of this procedure can be found in the Appendix E).

There are two types of gauge transformations associated with the action $S_{CS}(A, n, \varphi)$ [see Eq. (22) in Ref. [2] for the corresponding mathematical formula]. For $\forall \lambda \in \mathbb{R}, \forall k \in 2\mathbb{Z}$,

$$S_{CS} \left(\begin{array}{c} A_0 + \frac{\lambda}{d\tau} \\ \varphi - k\lambda \quad \swarrow \quad \searrow \\ A_1 - \frac{\lambda}{a} \quad \bullet \quad A_2 + \frac{\lambda}{a} \\ \swarrow \quad \searrow \\ A_2 - \frac{\lambda}{a} \quad A_0 - \frac{\lambda}{d\tau} \end{array} , * \right) = S_{CS} \left(\begin{array}{c} A_0 \\ \varphi \quad \swarrow \quad \searrow \\ A_1 \quad \bullet \quad A_2 \\ \swarrow \quad \searrow \\ A_2 \quad A_0 \end{array} , * \right) \quad (12)$$

$$e^{-S_{CS}} \left(\begin{array}{c} A_0 + \frac{2\pi}{d\tau} n_{0,2} - 1 \\ n_{0,1} + 1 \quad \square \quad \square \quad n_{0,1} - 1 \\ \quad \quad \quad \updownarrow \quad \updownarrow \\ \quad \quad \quad \square \quad \square \\ n_{0,2} + 1 \end{array} , * \right) = e^{-S_{CS}} \left(\begin{array}{c} A_0 \quad n_{0,2} \\ n_{0,1} \quad \square \quad \square \quad n_{0,1} \\ \quad \quad \quad \updownarrow \quad \updownarrow \\ \quad \quad \quad \square \quad \square \\ n_{0,2} \end{array} , * \right) \quad (13)$$

$$e^{-S_{CS}} \left(\begin{array}{c} \square \quad n_{0,1} + 1 \\ \square \quad n_{1,2} - 1 \\ \square \quad \rightarrow \quad A_1 + \frac{2\pi}{a} \\ n_{1,2} + 1 \quad \square \\ n_{0,1} - 1 \quad \square \end{array} , * \right) = e^{-S_{CS}} \left(\begin{array}{c} \square \quad n_{0,1} \\ \square \quad n_{1,2} \\ \square \quad \rightarrow \quad A_1 \\ n_{1,2} \quad \square \\ n_{0,1} \quad \square \end{array} , * \right) \quad (14)$$

$$e^{-S_{CS}} \left(\begin{array}{c} n_{0,2} + 1 \quad \square \quad A_2 + \frac{2\pi}{a} \\ n_{1,2} - 1 \quad \square \quad \square \quad n_{1,2} + 1 \\ \quad \quad \quad \updownarrow \quad \updownarrow \\ \quad \quad \quad \square \quad \square \\ n_{0,2} - 1 \end{array} , * \right) = e^{-S_{CS}} \left(\begin{array}{c} n_{0,2} \quad \square \quad A_2 \\ n_{1,2} \quad \square \quad \square \quad n_{1,2} \\ \quad \quad \quad \updownarrow \quad \updownarrow \\ \quad \quad \quad \square \quad \square \\ n_{0,2} \end{array} , * \right) \quad (15)$$

The $*$ in the parentheses represents all other variables, which are the same on the left- and right-hand sides in the equations. Equation (12) shows a local gauge transformation, under which S_{CS} itself is invariant. Equations (13)–(15) are the discrete gauge transformations, under which the partition function $e^{-S_{CS}}$ is invariant. In fact, S_{CS} changes by $i\pi k\mathbb{Z}$ from the last term in the action Eq. (11) under discrete gauge transformations. When k is an odd integer, there can be an extra π phase in S_{CS} , and the partition function changes by an extra minus sign. This extra minus sign can be canceled by the anomaly inflow from an auxiliary 4D bulk [2] or by introducing additional fermionic degrees of freedom [3]. More precisely, the odd- k theory is a theory of fermions, where the Chern-Simons action depends on the choice of the spin structure of the manifold [47]. Here, we only consider the pure gauge theory in $2 + 1$ D, so we assume that k is an even integer in

the beginning. What may be surprising is that, in Sec. III, after we add the Maxwell term and derive the compact Maxwell-Chern-Simons theory in Hamiltonian formulation, the corresponding Hamiltonian works for any integer k . We show that the fermionic nature for odd k is still present and can be probed with our formulation.

The authors of Ref. [2] have shown that the lattice action in Eq. (11) reproduces various aspects of the continuum Chern-Simons theory, including the level quantization and the discrete $\mathbb{Z}$ 1-form symmetry. In a more recent work by the same authors [3], they have tried to canonically quantize the lattice action in Eq. (11). However, without the Maxwell term, the quantization of the pure Chern-Simons action unavoidably has nontrivial commutators between gauge field variables on different links (see Eq. (23) in Ref. [3]). The noncommuting gauge field variables make it hard to find an orthonormal basis of

the Hilbert space and thus lose the capability of performing numerical simulations with their formulation.

B. Compact Maxwell lattice Hamiltonian

Before we discuss Maxwell-Chern-Simons theory, we take a step back and review the pure Maxwell theory. In this section, we revisit the compact Maxwell lattice gauge theory in $2+1$ D, also known as the $2+1$ D lattice QED. There is no Chern-Simons term in this section. The purpose of this review is twofold: first, we introduce the Villain approximation and the transfer matrix method that we use to derive our original results in Sec. III; second, we highlight the role of instanton suppression, which leads to a qualitatively different Hamiltonian (quadratic rather than cosine). This distinction is not widely appreciated in the literature, yet it is crucial for the Maxwell-Chern-Simons theory, where the Chern-Simons term enforces the instanton-suppressed sector.

In $2+1$ D lattice QED, it is common to consider the following compact Maxwell action [48],

$$S_{\text{Maxwell}}(A) = -\frac{\beta_0}{d\tau} \sum_{x \in \text{sites}} [\cos(ad\tau(\Delta_0 A_{x;1} - \Delta_1 A_{x;0})) + \cos(ad\tau(\Delta_0 A_{x;2} - \Delta_2 A_{x;0}))] - \frac{\beta d\tau}{a^2} \sum_{x \in \text{sites}} [\cos(a^2(\Delta_1 A_{x;2} - \Delta_2 A_{x;1}))], \quad (16)$$

where β_0 and β are the temporal and spatial gauge coupling coefficients, respectively, $d\tau$ and a are the temporal and spatial lattice spacings, respectively, and A_0 and A_i with $i = 1, 2$ are the gauge field in the temporal and spatial directions, respectively. This lattice action can be visualized,

$$S_{\text{Maxwell}}(A) = -\frac{\beta_0}{d\tau} \sum_{\text{time-like plaquettes}} \left[\cos \left(a \cdot \frac{d\tau}{a} A_0 \begin{array}{c} \xrightarrow{A_1} \\ \square \\ \xleftarrow{-A_1} \end{array} - \frac{d\tau}{a} A_0 \right) + \cos \left(a \cdot \frac{d\tau}{a} A_0 \begin{array}{c} \xrightarrow{A_2} \\ \square \\ \xleftarrow{-A_2} \end{array} - \frac{d\tau}{a} A_0 \right) \right] - \frac{\beta d\tau}{a^2} \sum_{\text{space-like plaquettes}} \left[\cos \left(a \cdot -A_2 \begin{array}{c} \xleftarrow{-A_1} \\ \square \\ \xrightarrow{A_1} \end{array} A_2 \right) \right]. \quad (17)$$

In order to do the canonical quantization to obtain the Hamiltonian formulation, we need to use the Villain approximation to expand the cosine terms in the action into quadratic forms [39]. The Villain approximated action is

$$S_{\text{Maxwell}}(A, n) = \frac{\beta_0}{2d\tau} \sum_{\substack{x \in \text{sites} \\ i \in \{1,2\}}} (ad\tau(\Delta_0 A_{x;i} - \Delta_i A_{x;0}) + 2\pi n_{x;0,i})^2 + \frac{\beta d\tau}{2a^2} \sum_{x \in \text{sites}} (a^2(\Delta_1 A_{x;2} - \Delta_2 A_{x;1}) + 2\pi n_{x;1,2})^2, \quad (18)$$

where the integer degrees of freedom n live on each plaquette. This Villain approximated lattice action can be visualized as

$$S_{\text{Maxwell}}(A, n) = \frac{\beta_0}{2d\tau} \sum_{\text{time-like plaquettes}} \left[\left(a \cdot \frac{d\tau}{a} A_0 \begin{array}{c} \xrightarrow{A_1} \\ \square \\ \xleftarrow{-A_1} \end{array} - \frac{d\tau}{a} A_0 + 2\pi \begin{array}{|c|} \hline n_{0,1} \\ \hline \end{array} \right)^2 + \left(a \cdot \frac{d\tau}{a} A_0 \begin{array}{c} \xrightarrow{A_2} \\ \square \\ \xleftarrow{-A_2} \end{array} - \frac{d\tau}{a} A_0 + 2\pi \begin{array}{|c|} \hline n_{0,2} \\ \hline \end{array} \right)^2 \right] + \frac{\beta d\tau}{2a^2} \sum_{\text{space-like plaquettes}} \left[\left(a \cdot -A_2 \begin{array}{c} \xleftarrow{-A_1} \\ \square \\ \xrightarrow{A_1} \end{array} A_2 + 2\pi \begin{array}{|c|} \hline n_{1,2} \\ \hline \end{array} \right)^2 \right], \quad (19)$$

which converges to the nonapproximated action in Eq. (17) in the $d\tau \rightarrow 0$ limit [39].

From the above Villain approximated lattice action, if we allow the presence of instantons, we can derive the familiar lattice QED Hamiltonian [49]. In the derivation, the presence of instantons allows an integer degree of freedom on every cubic unit cell in the lattice (i.e., the instanton number in every cube). Summing over these instanton degrees of freedom in the partition function provides an reversed Villain approximation and allows us to wrap the spacelike plaquettes into the familiar cosine term in the Hamiltonian again, while the timelike plaquettes become the E_i^2 term,

$$H_{\text{Maxwell with instantons}} = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{1}{e^2 a^2} \sum_{\text{plaquettes}} (1 - \cos a^2 B), \quad (20)$$

where E_i is the electric field in the i direction, and B is the magnetic field. The coefficients in Eq. (19) and the coupling constant are related by $\beta_0 = \beta = 1/e^2$. See Appendix E for an outline of the derivation.

Meanwhile, if we totally suppress the instantons, from the same Villain approximated lattice action in Eq. (19), we can derive a different version of the lattice Hamiltonian. In our derivation, we enforce the zero-instanton constraint by taking out the integer degrees of freedom which correspond to instanton numbers in cubes (this process is equivalent to adding the Lagrange multipliers to the action and then integrating them out in the partition function). Without summing over instanton degrees of freedom, the spacelike plaquette terms cannot be wrap into cosine terms and stay quadratic,¹

¹Because the space we study is a closed surface (2D torus due to periodic boundary conditions), there is one leftover integer degree of freedom—the total magnetic flux through the surface. Each integer n of total magnetic flux defines a sector of the theory, with the Hamiltonian,

$$H_n = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{a^2}{2e^2} \left[\left(B' + \frac{2\pi n}{a^2} \right)^2 + \sum'_{\text{plaq.}} B^2 \right],$$

where B' is the magnetic field on one special plaquette, and $\sum'_{\text{plaq.}}$ means summing all other plaquettes. The theory is gauge invariant after taking into account all the sectors with $n \in (-\infty, +\infty)$. In (21) for simplicity, we show H_0 as a demonstration. The appearance of integer operators in canonical quantization of Villain fields can also be found in other studies [3,50–52]. As a spoiler, in later sections, when we discuss the Maxwell-Chern-Simons theory, the gauge invariant condition from the Chern-Simons term enforces the total flux $n = 0$. Therefore, we do not see an $\hat{n}$ operator later in (23).

$$H_{\text{Maxwell without instantons}} = \frac{e^2}{2a^2} \sum_{\text{links}} E_i^2 + \frac{a^2}{2e^2} \sum_{\text{plaquettes}} B^2, \quad (21)$$

which comes together with two more constraints on its Hilbert space to guarantee the compactness of the theory. We emphasize that this Hamiltonian in Eq. (21) is a Hamiltonian of a **compact** Maxwell theory, and it is different from Eq. (20) because we enforce the zero-instanton constraint. Since we study a $2 + 1$ D cubic lattice with periodic boundary conditions in all directions, the 2D space is a torus, and thus, the zero-instanton constraint is nontrivial. An outline of the derivation of this Hamiltonian and the associated Hilbert space constraints can be found in the Appendix E.

To summarize, there are two types of Maxwell theory: instanton-allowed or instanton-suppressed. Sometimes the instanton-suppressed Maxwell theory is referred to as the “noncompact QED”. However, we would like to emphasize that the instanton-suppressed Maxwell theory is a compact theory due to the additional Hilbert space constraints. Later, when we discuss the Maxwell-Chern-Simons theory, we only talk about the instanton-suppressed Maxwell theory. This is because accompanied with the Chern-Simons action, instantons carry charges, which is similar to the Witten effect in $3 + 1$ D Maxwell theory with the θ term [46]. Therefore, in order to study the pure gauge theory without matter fields, here we have to suppress the instantons. We refer to Appendix A for more discussions on the instanton effects.

III. COMPACT MAXWELL-CHERN-SIMONS LATTICE HAMILTONIAN

Having reviewed the necessary ingredients—the lattice Chern-Simons action (Sec. II A) and the Maxwell Hamiltonian with instanton suppression (Sec. II B)—we now present the original results of this work. In the previous work of canonical quantization of lattice Chern-Simons theory [3], there is no proposal of adding the Maxwell term. In this section, we provide a formulation of the Maxwell-Chern-Simons lattice Hamiltonian, which can be implemented in classical and quantum simulations. We consider the sum of Eqs. (11) and (19) as the total action,

$$S(A, n, \varphi) = S_{\text{Maxwell}}(A, n) + S_{\text{CS}}(A, n, \varphi), \quad (22)$$

which is our starting point to obtain the lattice Hamiltonian for the Maxwell-Chern-Simons theory.

The Chern-Simons term, being topological, does not contribute to the Hamiltonian. Therefore, we would expect to see a Maxwell-Chern-Simons lattice Hamiltonian that looks similar to the Hamiltonian of compact QED with instantons suppressed as shown in

Eq. (21). We would also expect to obtain two additional Hilbert space constraints. However, the expression of the electric field E appearing in the Hamiltonian, the Gauss' law, and the Hilbert space constraints are modified due to the Chern-Simons term.

We use the transfer matrix method [53], suppress the instantons by ignoring their configurations in the counting of degrees of freedom, and construct a lattice Hamiltonian based on the action in Eq. (22) (see also Appendix E for an outline of the derivation),

$$\begin{aligned} \hat{H} = & + \frac{e^2}{2a^2} \sum_{\text{plaquettes}} \left[\left(\left[\overrightarrow{\hat{p}_1} - \left(\frac{ka^2}{4\pi} \right) \overrightarrow{\hat{A}_2} \right]^2 + \left(\left[\overrightarrow{\hat{p}_2} + \left(\frac{ka^2}{4\pi} \right) \overrightarrow{\hat{A}_1} \right]^2 \right) \right. \right. \\ & \left. \left. + \frac{1}{2e^2} \sum_{\text{plaquettes}} \left(\begin{array}{c} \text{---} \hat{A}_1 \text{---} \\ \left| \begin{array}{c} \text{---} \hat{A}_2 \text{---} \\ \text{---} \hat{A}_1 \text{---} \end{array} \right| \\ \text{---} \hat{A}_2 \text{---} \end{array} \right)^2 \right. \right. \\ & \left. \left. = \sum_{x \in \text{sites}} \frac{e^2}{2a^2} \left[\left(\hat{p}_{x;1} - \frac{ka^2}{4\pi} \hat{A}_{x-\hat{2};2} \right)^2 + \left(\hat{p}_{x;2} + \frac{ka^2}{4\pi} \hat{A}_{x-\hat{1};1} \right)^2 \right] + \frac{1}{2e^2} \left(\square \hat{A}_{x;1,2} \right)^2, \right. \end{aligned} \quad (23)$$

where e^2 is the coupling constant, a is the lattice spacing, k is the Chern-Simons level, $\hat{A}_i$ is the gauge field operator in the direction $i = 1$ or 2 , and $\hat{p}_i$ is the corresponding conjugate momentum operator. They have the standard commutation relation $[\hat{A}_{x,i}, \hat{p}_{y,j}] = i\delta_{x,y}\delta_{i,j}$. We also define $\square \hat{A}_{x;1,2} \equiv \hat{A}_{x;1} + \hat{A}_{x+\hat{1};2} - \hat{A}_{x+\hat{2};1} - \hat{A}_{x;2}$. The action coefficients and the coupling constant are related by $\beta_0 = \beta = \frac{1}{e^2}$. This lattice Hamiltonian lives on a constraint Hilbert space which we detail below.

In the pure Maxwell lattice theory as we discussed in the previous section, there is an extra integer degree of freedom denoting the total flux through the closed space surface. However, after adding the Chern-Simons term, the total flux is enforced to be zero due to gauge invariant constraints (as shown in the Appendix E). Therefore, we do not have an integer operator in (23).

A. Hilbert space and constraints

We construct the Hilbert space on a 2D time slice, which is a 2D square lattice. The Hilbert space is constructed with a set of basis $\{|A\rangle : A \in \mathbb{R}^{2N_1N_2}\}$, where A denotes a configuration of the gauge fields, N_1 and N_2 are the lattice sizes in the two spatial directions, respectively, and $2N_1N_2$ is the total number of links; i.e., each basis vector is labeled by a configuration in which every link has a variable ranging in $(-\infty, +\infty)$. We note that our formulation provides a natural basis in terms of the gauge field configurations for classical and quantum simulations.

The compactness of the theory is reflected by several constraints on the Hilbert space. Any physical state $|\psi\rangle$ in the Hilbert space needs to satisfy the following constraints, where $\hat{A}$ is the gauge field operator, $\hat{p}$ is the conjugate

momentum operator, k is the Chern-Simons level, and a is the lattice spacing,

(1)

$$\hat{G}|\psi\rangle = 0, \quad (24)$$

where

$$\begin{aligned} \hat{G} = & \begin{array}{c} \begin{array}{c} \overrightarrow{\hat{p}_2} \\ \left| \begin{array}{c} \overleftarrow{\hat{p}_1} \\ \overrightarrow{\hat{p}_1} \\ \overrightarrow{\hat{p}_2} \end{array} \right| \\ \overleftarrow{\hat{p}_2} \end{array} + \left(\frac{ka^2}{4\pi} \right) \begin{array}{c} \text{---} \hat{A}_1 \text{---} \\ \left| \begin{array}{c} \text{---} \hat{A}_2 \text{---} \\ \text{---} \hat{A}_1 \text{---} \end{array} \right| \\ \text{---} \hat{A}_2 \text{---} \end{array} \end{array} \\ = & \hat{p}_{x;1} + \hat{p}_{x;2} - \hat{p}_{x-\hat{1};1} - \hat{p}_{x-\hat{2};2} + \frac{ka^2}{4\pi} \left(\square \hat{A}_{x;1,2} \right), \end{aligned} \quad (25)$$

for any lattice site x .

Equivalently,

$$e^{i\lambda\hat{G}}|\psi\rangle = |\psi\rangle, \quad \forall \lambda \in \mathbb{R}, \quad (26)$$

which is the Gauss' law or the local gauge transformation. (2)

$$e^{2\pi i \hat{L}_1}|\psi\rangle = e^{i\theta_1}|\psi\rangle, \quad (27)$$

where θ_1 is a constant global phase angle, and

$$\hat{L}_1 = \dots \leftarrow \begin{array}{c} \xleftarrow{-\frac{ka}{4\pi}\hat{A}_1} \\ \uparrow \frac{1}{a}\hat{p}_2 \end{array} \leftarrow \dots \quad (28)$$

$$= \sum_{x_1=0}^{N_1-1} \left(\frac{1}{a}\hat{p}_{(x_1,x_2);2} - \frac{ka}{4\pi}\hat{A}_{(x_1,x_2+1);1} \right),$$

for any $x_2 \in \{0, 1, \dots, N_2 - 1\}$. This loop operator generates one of the large gauge transformations. Here, by “large gauge transformation” we mean a gauge transformation that is not contractible to the identity, i.e., one that has a nontrivial winding number around a noncontractible cycle of the spatial torus, in contrast to the “small” (locally contractible) gauge transformations generated by the Gauss’ law. Note that such a loop can be locally deformed by adding or subtracting the Gauss’ law (constraint 1) (by which we mean redefining the path of $\hat{L}_1$ via products with the Gauss’ law constraint operator at intermediate sites, which leaves the action on physical states unchanged). Therefore, more generally we can write $\hat{L}_1$ for any topologically nontrivial horizontal loop that wraps through the boundary, i.e., any curve with the winding number (1,0).

(3)

$$e^{2\pi i \hat{L}_2} |\psi\rangle = e^{i\theta_2} |\psi\rangle, \quad (29)$$

where θ_2 is another constant global phase angle, and

$$\hat{L}_2 = \begin{array}{c} \vdots \\ \xrightarrow{\frac{1}{a}\hat{p}_1} \\ \xrightarrow{\frac{ka}{4\pi}\hat{A}_2} \\ \vdots \end{array} \quad (30)$$

$$= \sum_{x_2=0}^{N_2-1} \left(\frac{1}{a}\hat{p}_{(x_1,x_2);1} + \frac{ka}{4\pi}\hat{A}_{(x_1+1,x_2);2} \right),$$

for any $x_1 \in \{0, 1, \dots, N_1 - 1\}$. This loop operator generates the other large gauge transformations (i.e., those with nontrivial winding number around the second noncontractible cycle of the spatial torus). Note that such a loop can also be locally deformed by adding or subtracting the Gauss’ law (constraint 1) in the same sense as for $\hat{L}_1$ described above. Therefore, more generally, we can write $\hat{L}_2$ for any topologically nontrivial vertical loop that wraps through the boundary, i.e., any curve with the winding number (0,1).

The second and the third constraints compactify the theory. In fact, the two constraints $e^{2\pi i \hat{L}_i} |\psi\rangle = e^{i\theta_i} |\psi\rangle$ allow the spectrum of $\hat{L}_i = m_i + \frac{\theta_i}{2\pi} \in \mathbb{Z} + \frac{\theta_i}{2\pi}$, $i = 1, 2$. These two integers m_1 and m_2 are quantum numbers labeling the eigenfunctions of $\hat{L}_1$ and $\hat{L}_2$, respectively. For example, we can label the eigenfunctions of $\hat{L}_1$ by $|m_1, \theta_1\rangle$, which satisfies $\hat{L}_1 |m_1, \theta_1\rangle = (\frac{\theta_1}{2\pi} + m_1) |m_1, \theta_1\rangle$. Then, the function $|\psi\rangle$ that satisfies the constraint $e^{2\pi i \hat{L}_1} |\psi\rangle = e^{i\theta_1} |\psi\rangle$ can be most generally written as $|\psi\rangle = \sum_{m_1 \in \mathbb{Z}} c_{m_1} |m_1, \theta_1\rangle$, where c_{m_1} are arbitrary coefficients. However, in order to satisfy the other constraint $e^{2\pi i \hat{L}_2} |\psi\rangle = e^{i\theta_2} |\psi\rangle$, the coefficients are not arbitrary anymore but need to satisfy $c_{m_1} = c_{m_1+k}$. This fact is due to the commutation relation between $\hat{L}_1$ and $\hat{L}_2$. Therefore, the most general wave function satisfying both constraints can be written as $|\psi\rangle = \sum_{m_1 \in \{0, \dots, k-1\}} c_{m_1} \sum_{n \in \mathbb{Z}} |m_1 + nk, \theta_1\rangle$, i.e., the arbitrary linear combination of k basis wave functions $\sum_{n \in \mathbb{Z}} |m_1 + nk, \theta_1\rangle$, $m_1 = 0, \dots, k-1$. Each basis wave function $\sum_{n \in \mathbb{Z}} |m_1 + nk, \theta_1\rangle$, i.e., eigenfunction of $e^{2\pi i \hat{L}_1}$, is an equal weight summation of eigenfunctions of $\hat{L}_1$. A large gauge transformation $|\psi\rangle \rightarrow e^{2\pi i \hat{L}_j} |\psi\rangle$ changes $m_i \rightarrow m_i + k\epsilon_{ij}$ and leaves the basis wave functions invariant [see the commutator between $\hat{L}_i$ and $\hat{L}_j$ in Eq. (33)]. Therefore, we have a compact theory which is invariant under the large gauge transformation. This compactification is similar in spirit to the Villain approximation, where multiple non-periodic functions are summed with equal weights to become a periodic function.

After having formulated these three constraints, we need to ensure that the lattice Hamiltonian operator in Eq. (23) is well defined on this constrained Hilbert space. This is true because $[\hat{H}, \hat{G}] = 0$, and

$$[\hat{H}, \hat{L}_i] = 0, \quad i = 1, 2, \quad (31)$$

which can be checked straightforwardly. Moreover, we need to check the mutual compatibility between the three constraints, as discussed in the next subsection.

B. Quantization of Chern-Simons level

To check the compatibility between the three constraints introduced in the previous subsection, we need to ensure that the operators $e^{i\lambda \hat{G}}$, $e^{2\pi i \hat{L}_1}$, and $e^{2\pi i \hat{L}_2}$ commute with each other. The first operator commutes with the latter two for any λ because

$$[\hat{G}, \hat{L}_i] = 0, \quad i = 1, 2. \quad (32)$$

The latter two, instead, do not commute with each other for arbitrary values of k , since

$$[\hat{L}_1, \hat{L}_2] = -\frac{k}{2\pi} i, \quad (33)$$

and therefore,

$$e^{2\pi i \hat{L}_1} e^{2\pi i \hat{L}_2} = e^{2\pi i k} e^{2\pi i \hat{L}_2} e^{2\pi i \hat{L}_1}. \quad (34)$$

The compatibility of the constraints 2 and 3, which is necessary to obtain a nontrivial Hilbert space that contains more than the zero vector, seems to enforce the quantization of the Chern-Simons coupling constant k in integers, since $e^{2\pi i k} = 1$ implies $k \in \mathbb{Z}$.

The quantization of the Abelian Chern-Simons coupling constant k garnered significant interest in the late 1980s and early 1990s. The goal of this section is to clarify that it is incorrect to assert that k must be an integer. In fact, Chern-Simons theory is well defined for any rational $k = p/q$ where p and q are coprime integers, as demonstrated by Polychronakos [54,55]. This same result was obtained on the lattice by Eliezer and Semenoff [56], who showed that the theory remains consistent when the Chern-Simons level is taken to be a rational fraction and computed the ground-state degeneracy. The consistency of the Chern-Simons theory with rational k was rigorously derived in the context of a Chern-Simons theory coupled to fermions on a torus in the continuum by Iengo and Lechner [57]. Their analysis provided an exact demonstration of how rational values of k naturally emerge in such systems and confirmed that the resulting physical properties align with those of an equivalent theory with an effective coupling $\bar{k} = pq$. As we show, also in the Maxwell-Chern-Simons case on the lattice, a theory with a rational $k = p/q$ is well defined and equivalent to a theory with an effective coupling constant $\bar{k} = pq$. We use the term “effective coupling” because, after enlarging the Hilbert space to construct an irreducible representation of the rational- k algebra, the modified loop operators h_1, h_2 defined below in Eq. (36) obey the algebraic relation $h_1 h_2 = h_2 h_1 e^{2\pi i \bar{k}}$ in Eq. (37), which has the same form as the integer- k algebra in Eq. (34) with k replaced by $\bar{k}$. In this sense, $\bar{k} = pq$ plays the role of an integer-valued Chern-Simons level controlling the noncommutative loop algebra of the rational-level theory.

For rational k , to obtain gauge-invariant states, we have to enlarge the Hilbert space in order to construct an irreducible representation of the noncommutative algebra in Eq. (34). To this end, let us define the loop operators h_1 and h_2 as

$$h_1 = e^{2\pi i q \hat{L}_1}, \quad (35)$$

$$h_2 = e^{2\pi i q \hat{L}_2}, \quad (36)$$

which correspond to changing the compactification radius, $2\pi \rightarrow 2\pi/q$. Here, h_1 and h_2 are Casimir operators of the algebra, which satisfy the relation

$$h_1 h_2 = h_2 h_1 e^{2\pi i \bar{k}}, \quad \bar{k} = pq \in \mathbb{Z}, \quad (37)$$

making constraints 2 and 3 compatible with a rational Chern-Simons coupling constant.

Note that in our starting lattice action in Eq. (22), we require $k \in 2\mathbb{Z}$ in order to make the action periodic in the gauge field variables and thus gauge invariant. Therefore, to be rigorous our Hamiltonian formulation only works for the even- k Maxwell-Chern-Simons theory. Efforts of resolving the pure Chern-Simons lattice theory for any k have been made in Ref. [3]. The extension to odd- k Maxwell-Chern-Simons lattice theory is left for future work.

Nevertheless, an interesting observation can be made if we regardlessly put an odd k value into our Hamiltonian formulation. The odd- k Maxwell-Chern-Simons theory is a fermionic theory [47]. Surprisingly, we can probe the fermionic nature of the odd- k theory in our formulation. Note that $e^{2\pi i \hat{L}_1} e^{2\pi i \hat{L}_2} = e^{\pi i k} e^{2\pi i (\hat{L}_1 + \hat{L}_2)}$. In an odd- k theory, the three operators, $e^{2\pi i \hat{L}_1}$, $e^{2\pi i \hat{L}_2}$, and $e^{2\pi i (\hat{L}_1 + \hat{L}_2)}$, cannot be all imposed to be the identity consistently. More specifically, we have four choices of their values: $(+1, +1, -1)$, $(+1, -1, +1)$, $(-1, +1, +1)$, and $(-1, -1, -1)$. This reflects the fact that there is no standard choice of boundary conditions (spin structures) in a fermionic theory. For a fermionic theory on a torus, there are four spin structures. The result of the theory could depend on the choice of spin structure for an odd k level [47].

C. Degeneracy of states

One of the most interesting aspects of the Chern-Simons theory is its topological nature: pure Chern-Simons theories are metric independent and therefore independent of the geometry of the manifold on which they are defined; i.e., they depend only on the topology of the manifold. The ground state is degenerate on a manifold with nontrivial topology. On a manifold of genus g , the degeneracy is k^g for integer k , and $(pq)^g$ for rational $k = p/q$ [58]. As we show, the ground-state degeneracy is present also in the Maxwell-Chern-Simons theory. In fact, this degeneracy of the states can be directly seen from our construction of the constrained Hilbert space.

For integer k , let us consider the following two loop operators:

$$g_1^{1/k} \quad \text{and} \quad g_2^{1/k}, \quad (38)$$

where $g_1 = e^{2\pi i \hat{L}_1}$ and $g_2 = e^{2\pi i \hat{L}_2}$. These two operators are well defined on the constrained Hilbert space. They commute with the Gauss' law by Eq. (32). They also commute with large gauge transformations, for example,

$$\begin{aligned} g_1^{1/k} g_2 &= e^{2\pi i k/k} g_2 g_1^{1/k} \\ &= g_2 g_1^{1/k}; \end{aligned} \quad (39)$$

i.e., if $|\psi\rangle$ is a physical state in the constrained Hilbert space, then $g_1^{1/k}|\psi\rangle$ and $g_2^{1/k}|\psi\rangle$ are also physical states in the constrained Hilbert space. Furthermore, the new physical states are different from $|\psi\rangle$ because the two operators do not commute with each other,

$$g_1^{1/k} g_2^{1/k} = e^{2\pi i/k} g_2^{1/k} g_1^{1/k}, \quad (40)$$

unless $k = 1$. Because $e^{i\theta_1}|\psi\rangle = g_1|\psi\rangle$, where θ_1 is a global phase angle defined in Eq. (27), by applying different powers of the $g_1^{1/k}$ operator, we can obtain k different physical states,

$$\{|\psi\rangle, g_1^{1/k}|\psi\rangle, \dots, g_1^{(k-1)/k}|\psi\rangle\}. \quad (41)$$

These states span a k -dimensional subspace of the Hilbert space and can be distinguished by the $g_2^{1/k}$ operator resulting in a k -fold degeneracy of the ground state for each state in the spectrum.

In a modern language, $g_1^{1/k}$ is the generator of a discrete 1-form $\mathbb{Z}_k$ symmetry, and $g_2^{1/k}$ is the generator of another discrete 1-form $\mathbb{Z}_k$ symmetry. The noncommuting nature of these two symmetries results in the k -fold degeneracy.

For rational $k = p/q$, let us define the operators,

$$h_1^{1/\bar{k}} \quad \text{and} \quad h_2^{1/\bar{k}}, \quad (42)$$

with h_1 and h_2 defined in Eq. (36) and $\bar{k} = pq$. These two operators obey the same algebraic relations as $g_1^{1/k}$ and $g_2^{1/k}$ with $\bar{k}$ replacing k . As for integer k , by applying different powers of the $h_1^{1/\bar{k}}$ operator we can obtain $\bar{k} = pq$ different physical states,

$$\{|\psi\rangle, h_1^{1/\bar{k}}|\psi\rangle, \dots, h_1^{(\bar{k}-1)/\bar{k}}|\psi\rangle\}, \quad (43)$$

which span a $\bar{k}$ -dimensional subspace of the Hilbert space and which can be distinguished by the $h_2^{1/\bar{k}}$ operator. We have, thus, a $\bar{k} = pq$ -fold degeneracy of the ground state for each state in the spectrum. This confirms the degeneracy of states obtained for the Maxwell-Chern-Simons theory in the continuum in Refs. [54,55] and, on the lattice, for the ground-state of pure Chern-Simons theories in [56].

Note that the two operators defined in Eq. (38) commute with the Hamiltonian by Eq. (31). Therefore, if $|\psi\rangle$ is an energy eigenstate of the Hamiltonian, all k distinguishable physical states in Eq. (41) have the same energy. In conclusion, there is a k -fold degeneracy for each state in

the spectrum. The same holds for the two operators defined in Eq. (42), leading to a $\bar{k} = pq$ -fold degeneracy for each state in the spectrum.

In what follows, for simplicity, we concentrate on the case of integer k . The generalization to rational k is straightforward.

D. Wilson loop operator

In the earlier discussion of the degeneracy of states, we saw that the two loop operators in Eq. (38) span the finite-dimensional Hilbert space. As usual, we can construct a Wilson loop operator,

$$\hat{W} = \exp \left(iQa \sum_{\text{loop}} \hat{A} \right), \quad (44)$$

where Q is the charge of the Wilson loop. The Wilson loop commutes with the gauge transformation operators $\hat{G}$, $e^{2\pi i \hat{L}_1}$, and $e^{2\pi i \hat{L}_2}$ if the charge Q is an integer. In particular, the loop can be chosen to be noncontractible. We define two noncontractible Wilson loops,

$$\begin{aligned} \hat{W}_1 &= \exp \left(iQa \sum_{x_1=0}^{N_1-1} \hat{A}_{(x_1,0);1} \right), \\ \hat{W}_2 &= \exp \left(iQa \sum_{x_2=0}^{N_2-1} \hat{A}_{(0,x_2);2} \right). \end{aligned} \quad (45)$$

However, the Wilson loop operator $\hat{W}$ does not commute with the Hamiltonian. When inserting $\hat{W}$, we shift the ground state to another topological sector. Let us evaluate the commutator between a noncontractible Wilson loop operator and the operators $e^{2\pi i \hat{L}_1/k}$, $e^{2\pi i \hat{L}_2/k}$ defined in Eq. (38),

$$\begin{aligned} \hat{W}_1 e^{2\pi i \hat{L}_1/k} &= e^{2\pi i \hat{L}_1/k} \hat{W}_1, \\ \hat{W}_1 e^{2\pi i \hat{L}_2/k} &= e^{-2\pi i Q/k} e^{2\pi i \hat{L}_2/k} \hat{W}_1, \end{aligned} \quad (46)$$

$$\begin{aligned} \hat{W}_2 e^{2\pi i \hat{L}_1/k} &= e^{-2\pi i Q/k} e^{2\pi i \hat{L}_1/k} \hat{W}_2, \\ \hat{W}_2 e^{2\pi i \hat{L}_2/k} &= e^{2\pi i \hat{L}_2/k} \hat{W}_2. \end{aligned} \quad (47)$$

We can see from these commutation relations that on the degenerate ground-state subspace, a horizontal noncontractible Wilson loop $\hat{W}_1$ with charge Q is equivalent to $e^{-2\pi i Q \hat{L}_1/k}$, which has exactly the same commutation relations as shown in Eq. (46). A vertical noncontractible Wilson loop $\hat{W}_2$ is equivalent to $e^{2\pi i Q \hat{L}_2/k}$, which has exactly the same commutation relations as shown in Eq. (47). In other words, the projections of $\hat{W}_1$, $\hat{W}_2$ on the degenerate ground-state subspace is proportional to $\hat{W}_1^{\text{TOP}} = e^{-2\pi i Q \hat{L}_1/k}$, $\hat{W}_2^{\text{TOP}} = e^{2\pi i Q \hat{L}_2/k}$, respectively, up to a

multiplicative renormalization factor.² Here we use “TOP” to highlight the topological nature: The ground-state sub-Hilbert space is also the subspace of the topological order, and the projected Wilson loops are also 1-form symmetry charge operators. We note that the Wilson loop here is different from the framed Wilson loop introduced in Ref. [2], but we still capture the framing topological feature of the Wilson loop. This is because its projection on the degenerate ground-state subspace, $\hat{W}_i^{\text{TOP}} = e^{\pm 2\pi i Q \hat{L}_i / k}$, has dangling links indicating the framing as shown in Eqs. (28) and (30). Furthermore, this ground-state projection can be used to show the perimeter scaling law of the Wilson loop operator $\hat{W}$, as explained in the following.

As in the continuum Maxwell-Chern-Simons theory, the Wilson loop operator always exhibits a perimeter law. For example, in Fig. 1, we consider a contractible Wilson loop consisting of two noncontractible Wilson loops across the spatial boundary, $\hat{W}_1$ and $\hat{W}_1^{-1}$, with a temporal distance of $|\tau_1 - \tau_0|$ between each other. This is not the standard rectangular Wilson loop in Euclidean spacetime: the two spatial pieces are noncontractible loops at different time slices, joined by Hamiltonian time evolution rather than by explicit temporal links. The physical content of the perimeter law is the same in both cases, however: the expectation value scales as the loop’s perimeter rather than its enclosed area, signaling the absence of confinement. The area of the Wilson loop is $|\tau_1 - \tau_0| \cdot L$, which is linearly dependent on $|\tau_1 - \tau_0|$. Here, L is the system size in the spatial direction. The perimeter of the Wilson loop is $2L$, which is independent of $|\tau_1 - \tau_0|$ since the spatial parts have a fixed length L and the temporal connection is provided by the gapped time evolution rather than by additional links. We can distinguish the scaling law of the Wilson loop by looking at how its expectation value $\langle \hat{W}_1^{-1}(\tau_1) \hat{W}_1(\tau_0) \rangle$ scales with $|\tau_1 - \tau_0|$. If $\hat{W}_1(\tau_0)$ is applied to one of the ground states which is an eigenstate of $e^{2\pi i \hat{L}_2 / k}$, it shifts the ground state to another topological sector (i.e., another eigenstate of $e^{2\pi i \hat{L}_2 / k}$ but with a different eigenvalue) by Eq. (46) and also excites the system to some higher energy state. For example, we start with one ground state and eigenstate of $e^{2\pi i \hat{L}_2 / k}$, $|\text{GS}_\alpha\rangle$. Suppose we get $\hat{W}_1(\tau_0)|\text{GS}_\alpha\rangle = |\text{GS}_\beta\rangle + \sum_i |E_i\rangle$, where $|\text{GS}_\beta\rangle$ is another ground state and eigenstate of $e^{2\pi i \hat{L}_2 / k}$, and $|E_i\rangle$ is the excited state with energy E_i (all states unnormalized). At time τ_1 , the resulted state evolves into $|\text{GS}_\beta\rangle + \sum_i e^{-E_i(\tau_1 - \tau_0)} |E_i\rangle$. Contract the time-evolved state with $\langle \text{GS}_\alpha | \hat{W}_1^{-1}(\tau_1)$, we get the expectation value $\langle \hat{W}_1^{-1}(\tau_1) \hat{W}_1(\tau_0) \rangle = \langle \text{GS}_\beta | \text{GS}_\beta \rangle + \sum_i e^{-E_i(\tau_1 - \tau_0)} \langle E_i | E_i \rangle$.

²The multiplicative renormalization factor is a term e^{-mL} in the lowest order, where m is the effective mass and L is the length of the loop. In the following analyses, this factor stays as a constant in each scenario, so it has no effect in the analyses.

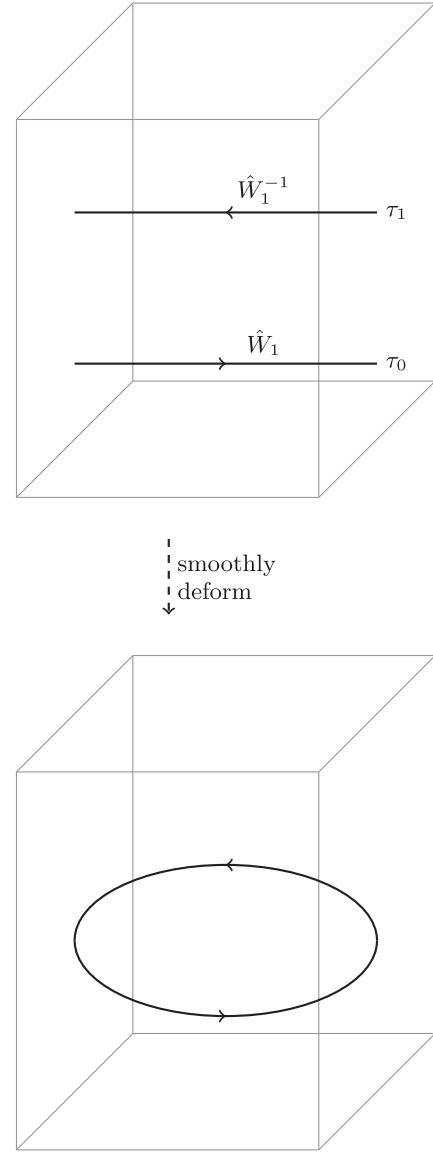


FIG. 1. An example of a contractible Wilson loop displaying the perimeter law. Note that with periodic boundary conditions, the two straight lines in the left panel represent two noncontractible Wilson loops $\hat{W}_1$ and $\hat{W}_1^{-1}$. Thus, they form one contractible loop. This implies that the upper panel can be smoothly deformed into the lower panel, which helps to visualize the topology. The upward direction is the positive time direction. Evaluating $\langle \hat{W}_1^{-1}(\tau_1) \hat{W}_1(\tau_0) \rangle$ gives a constant value independent of $|\tau_1 - \tau_0|$ when $\tau_1 \gg \tau_0$, which implies the perimeter law.

Because the system is gapped, the excited state components decay upon time evolution. When $|\tau_1 - \tau_0| \rightarrow \infty$, the Wilson loop expectation value $\langle \hat{W}_1^{-1}(\tau_1) \hat{W}_1(\tau_0) \rangle$ converges to a constant value $\langle \text{GS}_\beta | \text{GS}_\beta \rangle$ which is independent of $|\tau_1 - \tau_0|$. Therefore, the Wilson loop expectation value agrees with the perimeter law.

In addition to the perimeter scaling law, the Wilson loop operators also display nontrivial topological features. For example, we can consider two linked Wilson loops, as

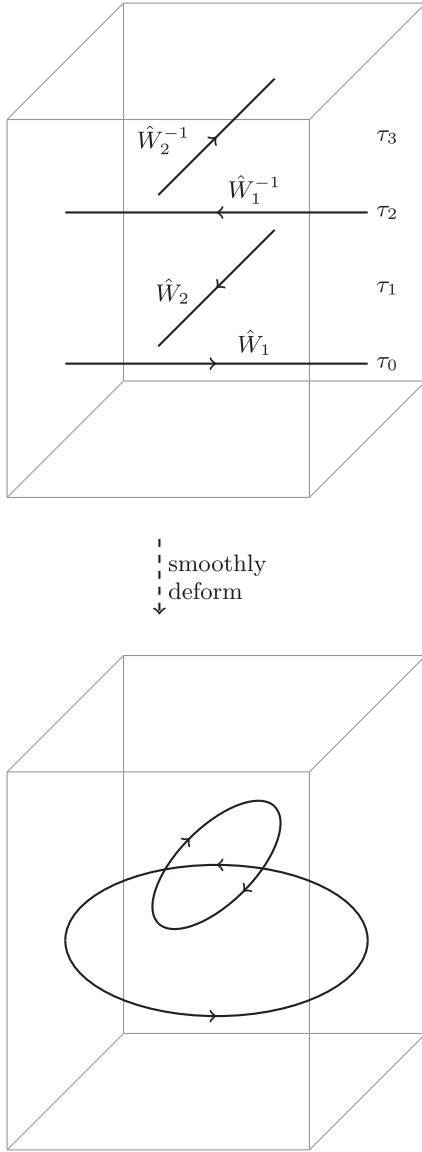


FIG. 2. Illustration of linked Wilson loops. The upward direction is the positive time direction. The pair of noncontractible Wilson loops $\hat{W}_1$ and $\hat{W}_1^{-1}$ together form a contractible loop. The pair of noncontractible Wilson loops $\hat{W}_2$ and $\hat{W}_2^{-1}$ together form another contractible loop. These two contractible loops are linked with each other and have a nonzero linking number between them. The upper panel can be smoothly deformed into the lower panel, which helps to visualize the topology. Evaluating $\hat{W}_2^{-1}(\tau_3)\hat{W}_1^{-1}(\tau_2)\hat{W}_2(\tau_1)\hat{W}_1(\tau_0)$ on the ground-state Hilbert space will give a nontrivial topological phase factor of $e^{2\pi i Q_1 Q_2/k}$, where k is the Chern-Simons level, Q_1 is the charge of $\hat{W}_1$, and Q_2 is the charge of $\hat{W}_2$. This nontrivial topological phase is due to the nonzero linking number between the two loops.

shown in Fig. 2. The pair of noncontractible loops $\hat{W}_1$ and $\hat{W}_1^{-1}$ form one contractible Wilson loop, and the other pair of noncontractible loops $\hat{W}_2$ and $\hat{W}_2^{-1}$ form a second contractible Wilson loop. The two contractible Wilson loops are linked with each other. To unlink the two loops,

we can exchange $\hat{W}_2$ and $\hat{W}_1^{-1}$, which are perpendicular to each other. If $\hat{W}_1$ has charge Q_1 and $\hat{W}_2$ has charge Q_2 , this exchange generates a commutator with a topological phase factor $e^{2\pi i Q_1 Q_2/k}$, as shown in the following:

$$\begin{aligned} & \hat{W}_2^{-1}(\tau_3)\hat{W}_1^{-1}(\tau_2)\hat{W}_2(\tau_1)\hat{W}_1(\tau_0) \\ & \sim e^{2\pi i Q_1 Q_2/k} \hat{W}_2^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1^{-1}(\tau_1)\hat{W}_1(\tau_0) \\ & \sim e^{2\pi i Q_1 Q_2/k}, \end{aligned} \quad (48)$$

where the times $\tau_0 < \tau_1 < \tau_2 < \tau_3$ are well separated, k is the Chern-Simons level, and in the last step $\hat{W}_2^{-1}(\tau_3)$ almost cancels with $\hat{W}_2(\tau_2)$, $\hat{W}_1^{-1}(\tau_1)$ almost cancels with $\hat{W}_1(\tau_0)$, by a similar argument as above when we discussed the perimeter law. Therefore, this example demonstrates that two linked Wilson loops with linking number 1 have a nontrivial ground-state expectation value with a topological phase factor of $e^{2\pi i Q_1 Q_2/k}$.

E. Anyon statistics

In Sec. III D, we have seen that our formulation supports Wilson loops, which display topological features. Here, we extend the discussion to open Wilson lines, which means that in Eq. (44) we integrate the gauge field on an open line instead of a loop, i.e.,

$$\hat{W}_{\text{open}} = \exp \left(i Q a \sum_{\text{line}} \hat{A} \right). \quad (49)$$

In order to have $\hat{W}_{\text{open}}$ commute with the large gauge transformations, $e^{2\pi i \hat{L}_1}$ and $e^{2\pi i \hat{L}_2}$, we need Q to be an integer, which is called the charge of the open Wilson line. Although a single open Wilson line is not locally gauge invariant [i.e., the Gauss' law in Eq. (25) does not commute with the open Wilson line at its two ends], the combination of an open Wilson line $\hat{W}_{\text{open}}$ with its inverse $\hat{W}_{\text{open}}^{-1}$ is gauge invariant, because the gauge noninvariant parts at their ends cancel with each other. For example, the combined operator $\hat{W}_{\text{open},1}\hat{W}_{\text{open},2}\hat{W}_{\text{open},1}^{-1}\hat{W}_{\text{open},2}^{-1}$ is gauge invariant: After a gauge transformation, the four open Wilson line operators generate c -number phases, and those phases generated by the operators and by their inverse operators cancel, respectively.

We are interested in open Wilson lines because they can be viewed as the world lines of anyons, which are excitations with fractional statistics [59,60]. Although our formulation does not involve any matter fields, we can still probe the mutual statistics and self-statistics of the anyons through the open Wilson lines. We briefly demonstrate the ideas below. The operators $\hat{W}$ in this subsection all refer to open Wilson lines, unless otherwise stated.

1. Mutual statistics

First, we look at the mutual statistics between two types of anyons, one with charge Q_1 and the other with charge

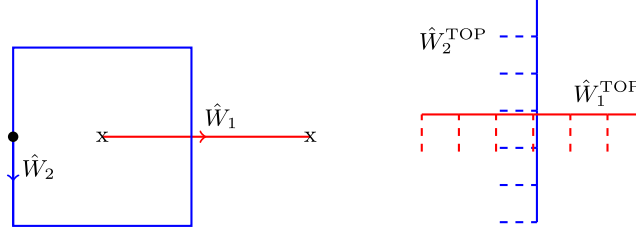


FIG. 3. Illustration of mutual statistics between anyons. In the left panel, $\hat{W}_1$ is an open Wilson line with charge Q_1 , and $\hat{W}_2$ is a Wilson loop with charge Q_2 . We illustrate the process of a charge- Q_2 anyon circling around a charge- Q_1 anyon as follows [62]. Suppose that initially there is a charge- Q_2 anyon (illustrated by the black dot). Then, we first create a charge- Q_1 anyon-antianyon pair located at two ends (“x”) with the open Wilson line $\hat{W}_1$ (red line). Second, we move the charge- Q_2 anyon around the charge- Q_1 anyon with $\hat{W}_2$ (blue line). Third, we annihilate the charge- Q_1 anyon-antianyon pair with $\hat{W}_1^{-1}$. Finally, we move back the charge- Q_2 anyon with $\hat{W}_2^{-1}$ to cancel the dynamical phase created by its movement. In the right panel, we zoom into the left panel to look at the intersection of the red and blue lines. $\hat{W}_1^{\text{TOP}}$ and $\hat{W}_2^{\text{TOP}}$ are the projections of the operators $\hat{W}_1$ and $\hat{W}_2$ on the degenerate ground-state Hilbert space, respectively. From the microscopic details of $\hat{W}_1^{\text{TOP}}$ and $\hat{W}_2^{\text{TOP}}$ on the lattice [i.e., the solid links are proportional to the gauge field $\hat{A}$, the dashed links are proportional to the canonical conjugate momentum $\hat{p}$, similar to $\hat{L}_i$ in Eqs. (28) and (30)], we can see that $\hat{W}_1^{\text{TOP}}$ and $\hat{W}_2^{\text{TOP}}$ do not commute with each other due to the fact that they share common links.

Q_2 . To probe their mutual statistics, we want to move one anyon around the other and compare the phases between the initial and final states. We can move the charge- Q_1 anyon by applying a Wilson line $\hat{W}_1$ with charge Q_1 and the charge- Q_2 anyon by applying a Wilson line $\hat{W}_2$ with charge Q_2 . To make the movements, we apply the operator

$$\hat{W}_2^{-1}(\tau_4)\hat{W}_1^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1), \quad (50)$$

to a state with a charge- Q_2 anyon present, see Fig. 3 and Ref. [61]. Here, we consider the times $\tau_1 < \tau_2 < \tau_3 < \tau_4$, and the spacings between these times are large enough so that we only need to consider the ground-state properties of the system (i.e., the system is gapped and correlations from the excited states are exponentially suppressed for long times). The presence of the initial charge- Q_2 anyon can be created by some Wilson line on the ground state at $\tau \rightarrow -\infty$, and we can annihilate it by the inverse Wilson line at $\tau \rightarrow \infty$ to go back to the degenerate ground-state Hilbert space. In the following discussion, we neglect the creation and annihilation of this initial charge- Q_2 anyon and only briefly discuss it in footnote 3.

As shown in the left panel of Fig. 3, we probe the mutual statistics of the anyons by applying the operator $\hat{W}_2^{-1}(\tau_4)\hat{W}_1^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1)$ from Eq. (50). First, we create a pair of anyon and antianyon (illustrated by the black x’s) of charge Q_1 with the open Wilson line $\hat{W}_1(\tau_1)$ (illustrated by the red line). Second, we circle the existing charge- Q_2 anyon (illustrated by the black dot) around the charge- Q_1 anyon with $\hat{W}_2(\tau_2)$ (illustrated by the blue line). Third, we annihilate the charge- Q_1 anyon-antianyon pair with $\hat{W}_1^{-1}(\tau_3)$. Finally, we cancel the dynamical phase of the charge- Q_2 anyon with $\hat{W}_2^{-1}(\tau_4)$. The net effect of these operations results in a charge- Q_2 anyon circling around a charge- Q_1 anyon. Therefore, the ground-state expectation

value $\langle \hat{W}_2^{-1}(\tau_4)\hat{W}_1^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$ has a topological phase, showing the mutual statistics between charge- Q_1 and charge- Q_2 anyons.³

Next, we compute the ground-state expectation value $\langle \hat{W}_2^{-1}(\tau_4)\hat{W}_1^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$. For this, we consider a charge- Q open Wilson line,

$$\hat{W} = \exp \left(iQa \sum_{(x,i)} \hat{A}_{x,i} \right), \quad (51)$$

and a “dangling Wilson line,”

$$\begin{aligned} \hat{W}^{\text{TOP}} &= \exp \left[\frac{2\pi i Q}{k} \sum_{(x,i)} \left(\frac{ka}{4\pi} \hat{A}_{x,i} - \epsilon_{ij} \frac{1}{a} \hat{p}_{x-j,j} \right) \right] \\ &= \exp \left[iQ \sum_{(x,i)} \left(\frac{a}{2} \hat{A}_{x,i} - \epsilon_{ij} \frac{2\pi}{ka} \hat{p}_{x-j,j} \right) \right], \end{aligned} \quad (52)$$

which is a truncated line segment of the loop operator $e^{\pm 2\pi i Q \hat{L}_i / k}$, where $\hat{L}_i$ are the large gauge transformation operators in Eqs. (28) and (30). Here, $(x, i) \in \{(\text{site}, \text{direction})\}$ represent the sites in a line and the corresponding directions along the line. For visualization, see the correspondence between the left and right panels in Figs. 3 and 4, where the solid links contain the gauge field

³More precisely, we compare the phase between $\langle \hat{W}_0^{-1}(+\infty) \times \hat{W}_2^{-1}(\tau_4)\hat{W}_1^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1)\hat{W}_0(-\infty) \rangle$ and $\langle \hat{W}_0^{-1}(+\infty) \times \hat{W}_2^{-1}(\tau_4)\hat{W}_2(\tau_3)\hat{W}_1^{-1}(\tau_2)\hat{W}_1(\tau_1)\hat{W}_0(-\infty) \rangle$, where $\hat{W}_0(-\infty)$ prepares the initial charge- Q_2 anyon and $\hat{W}_0^{-1}(+\infty)$ annihilate it back to the ground state. Note that in the latter process each operator cancels with its inverse, and thus, the phase is zero. Therefore, this relative phase that shows the mutual statistics of the anyons is equivalent to the phase of $\langle \hat{W}_2^{-1}(\tau_4)\hat{W}_1^{-1}(\tau_3) \times \hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$.

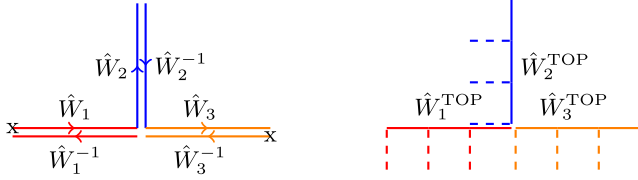


FIG. 4. Illustration of self-statistics of one type of anyon. In the left panel, $\hat{W}_1$, $\hat{W}_2$, and $\hat{W}_3$ are three open Wilson lines with charge Q . We illustrate the process of two charge- Q anyons exchanging their positions as follows [62]. Suppose initially there are two identical anyons located at the “x”s. We first move the left anyon to the middle by $\hat{W}_1$ (red line), second we move it upward by $\hat{W}_2$ (blue line), third we move the right anyon to the middle by $\hat{W}_3^{-1}$ (orange line), fourth we move it leftward by $\hat{W}_1^{-1}$ (orange line), fifth we move the original left anyon back to middle by $\hat{W}_2^{-1}$ (blue line), and finally we move it rightward by $\hat{W}_3$ (red line). Note that the displacements between parallel lines in the figure are for visual clarity. We have exchanged the position of the two anyons without them getting close to each other, and all dynamical phases created by anyon movements are canceled. In the right panel, we zoom in to look at the intersection of the red, blue, and orange lines. $\hat{W}_1^{\text{TOP}}$, $\hat{W}_2^{\text{TOP}}$, and $\hat{W}_3^{\text{TOP}}$ are the projections of the operators $\hat{W}_1$, $\hat{W}_2$, and $\hat{W}_3$ on the degenerate ground-state Hilbert space, respectively. From the microscopic details of $\hat{W}_1^{\text{TOP}}$, $\hat{W}_2^{\text{TOP}}$, and $\hat{W}_3^{\text{TOP}}$ on the lattice [solid links are proportional to the gauge field $\hat{A}$, dashed links are proportional to the canonical conjugate momentum $\hat{p}$, similar to $\hat{L}_i$ in Eqs. (28) and (30)], we can see that $\hat{W}_1^{\text{TOP}}$ and $\hat{W}_2^{\text{TOP}}$ do not commute with each other due to the fact that they share a common link.

$\hat{A}$ and the dashed links contain the canonical conjugate momentum $\hat{p}$. The relative signs between $\hat{A}$ and $\hat{p}$ in $\hat{W}^{\text{TOP}}$ follow the ones in $\hat{L}_i$ in Eqs. (28) and (30).

When the temporal spacings are large, we can compute $\langle \hat{W}_2^{-1}(\tau_4) \hat{W}_1^{-1}(\tau_3) \hat{W}_2(\tau_2) \hat{W}_1(\tau_1) \rangle$ by projecting everything onto the degenerate ground-state subspace. By an argument similar to the one we used in the previous discussion on the perimeter law in Sec. III D, $\hat{W}$ is equivalent to its projection $\hat{W}^{\text{TOP}}$ on the degenerate ground-state subspace when its temporal spacings between other operators are large enough, which ensures that the excited-state components exponentially decay out. In Sec. III D, this argument was presented for the closed Wilson loops $\hat{W}_1 \hat{W}_1^{-1}$. The same reasoning applies to the open Wilson lines appearing here, since the dominance of the topological projection at large temporal separation is dictated by the spectral gap and the locality of the Hamiltonian and is insensitive to whether the spatial line is closed or has end points—the end points contribute only to a local renormalization factor that is common to both sides of the relation below and cancels out of the topological phase. We therefore have

$$\begin{aligned} & \langle \hat{W}_2^{-1}(\tau_4) \hat{W}_1^{-1}(\tau_3) \hat{W}_2(\tau_2) \hat{W}_1(\tau_1) \rangle \\ & \sim \langle \hat{W}_2^{\text{TOP}-1} \hat{W}_1^{\text{TOP}-1} \hat{W}_2^{\text{TOP}} \hat{W}_1^{\text{TOP}} \rangle. \end{aligned} \quad (53)$$

We note that the operators in both sides of Eq. (53) are gauge invariant due to the gauge phase cancellation between the open Wilson lines and their corresponding inverse operators.

The right-hand side of Eq. (53) can be directly computed,

$$\begin{aligned} & \langle \hat{W}_2^{\text{TOP}-1} \hat{W}_1^{\text{TOP}-1} \hat{W}_2^{\text{TOP}} \hat{W}_1^{\text{TOP}} \rangle \\ & = e^{\frac{2\pi i}{k} Q_1 Q_2} \langle \hat{W}_2^{\text{TOP}-1} \hat{W}_2^{\text{TOP}} \hat{W}_1^{\text{TOP}-1} \hat{W}_1^{\text{TOP}} \rangle \\ & = e^{\frac{2\pi i}{k} Q_1 Q_2}, \end{aligned} \quad (54)$$

where we used the commutation relation,

$$\hat{W}_1^{\text{TOP}-1} \hat{W}_2^{\text{TOP}} = e^{\frac{2\pi i}{k} Q_1 Q_2} \hat{W}_2^{\text{TOP}} \hat{W}_1^{\text{TOP}-1}, \quad (55)$$

which can be seen by the overlapping red solid link ($\hat{A}_{x-\hat{i},1}$) and blue dashed link ($\hat{p}_{x-\hat{i},1}$) with the commutator $[\hat{A}_{x-\hat{i},1}, \hat{p}_{x-\hat{i},1}] = i$ and the blue solid link ($\hat{A}_{x-\hat{2},2}$) and the red dashed link ($\hat{p}_{x-\hat{2},2}$) with the commutator $[\hat{A}_{x-\hat{2},2}, \hat{p}_{x-\hat{2},2}] = i$. The variables on the red links, $\hat{A}_{x-\hat{i},1}$ and $\hat{p}_{x-\hat{2},2}$, are in $\hat{W}_1^{\text{TOP}-1}$, and the variables on the blue links, $\hat{A}_{x-\hat{2},2}$ and $\hat{p}_{x-\hat{i},1}$, are in $\hat{W}_2^{\text{TOP}}$. x is the intersection site of the red and blue solid lines, shown in the right panel in Fig. 3. In particular, the phase comes from

$$\begin{aligned} & \exp \left[-iQ_1 \left(\frac{a}{2} \hat{A}_{x-\hat{i},1} - \frac{2\pi}{ka} \hat{p}_{x-\hat{2},2} \right) \right] \\ & \times \exp \left[iQ_2 \left(\frac{a}{2} \hat{A}_{x-\hat{2},2} + \frac{2\pi}{ka} \hat{p}_{x-\hat{i},1} \right) \right] \\ & = e^{\frac{2\pi i}{k} Q_1 Q_2} \exp \left[iQ_2 \left(\frac{a}{2} \hat{A}_{x-\hat{2},2} + \frac{2\pi}{ka} \hat{p}_{x-\hat{i},1} \right) \right] \\ & \times \exp \left[-iQ_1 \left(\frac{a}{2} \hat{A}_{x-\hat{i},1} - \frac{2\pi}{ka} \hat{p}_{x-\hat{2},2} \right) \right], \end{aligned} \quad (56)$$

due to the commutator,

$$\begin{aligned} & \left[-iQ_1 \left(\frac{a}{2} \hat{A}_{x-\hat{i},1} - \frac{2\pi}{ka} \hat{p}_{x-\hat{2},2} \right) \right. \\ & \left. iQ_2 \left(\frac{a}{2} \hat{A}_{x-\hat{2},2} + \frac{2\pi}{ka} \hat{p}_{x-\hat{i},1} \right) \right] = \frac{2\pi i}{k} Q_1 Q_2, \end{aligned} \quad (57)$$

where the minus sign in the first term comes from the inverse (-1) in $\hat{W}_1^{\text{TOP}-1}$.

In summary, we have computed that $\langle \hat{W}_2^{-1}(\tau_4) \hat{W}_1^{-1}(\tau_3) \times \hat{W}_2(\tau_2) \hat{W}_1(\tau_1) \rangle$ has a nontrivial topological phase $e^{\frac{2\pi i}{k} Q_1 Q_2}$, which demonstrates the mutual statistics between charge- Q_1 and charge- Q_2 anyons.

2. Self-statistics

Next, we study the anyon self-statistics using open Wilson lines. To probe the self-statistics, we want to exchange the position of two anyons with the same charge Q and compare the phase between the initial and final

states. To make the movements, we apply the composite operator,

$$\hat{W}_3(\tau_6)\hat{W}_2^{-1}(\tau_5)\hat{W}_1^{-1}(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1), \quad (58)$$

to a state with two charge- Q anyons present at the end of the open Wilson lines $\hat{W}_1$ and $\hat{W}_3$, see Fig. 4 and Ref. [62]. As before, we consider times $\tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 < \tau_6$, and the spacing between these times are large enough so that we only need to consider the ground-state properties. Similar to the earlier discussion on the mutual statistics, the creation and annihilation of the anyons are performed at $\tau \rightarrow -\infty$ and $\tau \rightarrow \infty$, respectively, and they are not important to our discussion.

As shown in Fig. 4, we apply the operator $\hat{W}_3(\tau_6) \times \hat{W}_2^{-1}(\tau_5)\hat{W}_1^{-1}(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1)$ to operate the anyons as follows: Initially, we have two anyons locating at the two crosses, respectively, and we want to exchange their location without having them close to each other. First, we move the anyon on the left to the middle with the open Wilson line $\hat{W}_1(\tau_1)$. Second, we move it upward with $\hat{W}_2(\tau_2)$, in order to make space for the other anyon. Third, we move the other anyon on the right to the middle with $\hat{W}_3^{-1}(\tau_3)$. Forth, we move it to the left cross with $\hat{W}_1^{-1}(\tau_4)$. Fifth, we move the original left anyon downward back to the middle with $\hat{W}_2^{-1}(\tau_5)$. Finally, we move it to the right cross with $\hat{W}_3(\tau_6)$. With these six steps, we exchange the locations of the two anyons.

Therefore, the expectation value of the process $\langle \hat{W}_3(\tau_6)\hat{W}_2^{-1}(\tau_5)\hat{W}_1^{-1}(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$ compared to the expectation value of the trivial process $\langle \hat{W}_1^{-1}(\tau_6)\hat{W}_2^{-1}(\tau_5)\hat{W}_3(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$ has a topological phase showing the self-statistics between two charge- Q anyons. The former anyon-exchanged ground-state expectation value can be computed similarly with the projections of the open Wilson lines on the degenerate ground-state subspace. On the right panel in Fig. 4, we show the corresponding projections $\hat{W}_i^{\text{TOP}}$ at the junction point of the three open Wilson lines. The formula of the open Wilson lines and their projections are written in Eqs. (51) and (52), respectively.

When the temporal spacings are large, we can compute the ground-state expectation value $\langle \hat{W}_3(\tau_6)\hat{W}_2^{-1}(\tau_5) \times \hat{W}_1^{-1}(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$ by projecting everything onto the degenerate ground-state subspace, similar to what we did above in the discussion on the mutual statistics in Eqs. (53) and (54). We have

$$\begin{aligned} & \langle \hat{W}_3(\tau_6)\hat{W}_2^{-1}(\tau_5)\hat{W}_1^{-1}(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle \\ & \sim \langle \hat{W}_3^{\text{TOP}}\hat{W}_2^{\text{TOP-1}}\hat{W}_1^{\text{TOP-1}}\hat{W}_3^{\text{TOP-1}}\hat{W}_2^{\text{TOP}}\hat{W}_1^{\text{TOP}} \rangle \\ & \sim e^{\frac{\pi i}{k}Q^2} \langle \hat{W}_1^{\text{TOP-1}}\hat{W}_2^{\text{TOP-1}}\hat{W}_3^{\text{TOP}}\hat{W}_3^{\text{TOP-1}}\hat{W}_2^{\text{TOP}}\hat{W}_1^{\text{TOP}} \rangle \\ & \sim e^{\frac{\pi i}{k}Q^2}, \end{aligned} \quad (59)$$

where we use the fact that $\hat{W}_3^{\text{TOP}}$ commutes with the other open Wilson lines' projections because of no overlapping links (see the right panel in Fig. 4), and we use the commutation relation,

$$\hat{W}_2^{\text{TOP-1}}\hat{W}_1^{\text{TOP-1}} = e^{\frac{\pi i}{k}Q^2}\hat{W}_1^{\text{TOP-1}}\hat{W}_2^{\text{TOP-1}}, \quad (60)$$

which can be seen by the overlapping red solid link ($\hat{A}_{x-\hat{i};1}$) and blue dashed link ($\hat{p}_{x-\hat{i};1}$) with the commutator $[\hat{A}_{x-\hat{i};1}, \hat{p}_{x-\hat{i};1}] = i$. The variable $\hat{A}_{x-\hat{i};1}$ is in $\hat{W}_1^{\text{TOP-1}}$, and the variable $\hat{p}_{x-\hat{i};1}$ is in $\hat{W}_2^{\text{TOP-1}}$. x is the junction site of the three solid lines, shown in the right panel in Fig. 4. In particular, the phase comes from

$$\begin{aligned} & \exp \left[-iQ \frac{2\pi}{ka} \hat{p}_{x-\hat{i};1} \right] \exp \left[-iQ \frac{a}{2} \hat{A}_{x-\hat{i};1} \right] \\ & = e^{\frac{\pi i}{k}Q^2} \exp \left[-iQ \frac{a}{2} \hat{A}_{x-\hat{i};1} \right] \exp \left[-iQ \frac{2\pi}{ka} \hat{p}_{x-\hat{i};1} \right], \end{aligned} \quad (61)$$

due to the commutator,

$$\left[-iQ \frac{2\pi}{ka} \hat{p}_{x-\hat{i};1}, -iQ \frac{a}{2} \hat{A}_{x-\hat{i};1} \right] = \frac{\pi i}{k} Q^2, \quad (62)$$

where the minus sign in both terms come from the inverse (-1) in $\hat{W}_i^{\text{TOP-1}}$.

In summary, we have computed that the expectation value $\langle \hat{W}_3(\tau_6)\hat{W}_2^{-1}(\tau_5)\hat{W}_1^{-1}(\tau_4)\hat{W}_3^{-1}(\tau_3)\hat{W}_2(\tau_2)\hat{W}_1(\tau_1) \rangle$ has a nontrivial topological phase $e^{\frac{\pi i}{k}Q^2}$, which demonstrates the self-statistics between two charge- Q anyons. Note that this self-statistics phase $e^{\pi i Q^2/k}$ differs from the naive setting $Q_1 = Q_2 = Q$ in the mutual-statistics result $e^{2\pi i Q_1 Q_2/k}$ by a factor of 2 in the exponent. This factor of 2 has the standard topological origin: a self-exchange of two identical anyons is a half braiding, while the mutual statistics involves a full braiding (one particle going completely around the other).

Note that for the minimally charged anyons with $Q = 1$, the self-statistics has the exchange phase $e^{\frac{\pi i}{k}}$. We know the bosonic statistics has the phase $(+1)$, and the fermionic statistics has the phase (-1) . Therefore, for $k \geq 2$, this phase $e^{\frac{\pi i}{k}}$ demonstrates a fractional statistics (i.e., neither bosonic nor fermionic statistics) of anyons.

We discussed earlier in Sec. III B that an odd- k Chern-Simons level implies the fermionic nature of the theory. Now from the self-statistics, we can directly see that the theory contains fermions when k is odd. Consider the anyons with charge $Q = k$. Their self-statistics has the exchange phase $e^{\frac{\pi i}{k}Q^2} = e^{i\pi k}$. When k is even, the charge- k anyons are bosons. When k is odd, the charge- k anyons are fermions.

From the calculation of the anyon statistics above, we demonstrate that our lattice Hamiltonian formulation

captures the nontrivial braiding of Wilson lines and therefore realizes the framing anomaly [2,3,63–65]. In our formulation, the projections of the Wilson lines on the degenerate ground-state subspace, as shown in Eq. (52), explicitly include a well-defined point-splitting regularization. The dangling dashed links shown in Figs. 3 and 4, on which the canonical conjugate momentum operators $\hat{p}$ are sitting, define the framing of the Wilson loops. This agrees with the observation in Refs. [2,3] that the physical operators are ribbons or framed Wilson loops. There, the authors construct the framed Wilson loops by two displaced Wilson loops [2,3]. Here, with the Maxwell term introduced, we can define independent canonical conjugate momentum operators $\hat{p}$, and we construct the framed Wilson loops by the “dangling Wilson loops” as shown in Figs. 3 and 4.

IV. ANALYTICAL SOLUTION

A. Constant mode

We know from the continuum model that, when the Maxwell-Chern-Simons theory is defined on a torus, in the Hodge decomposition of the gauge field A , there are harmonic parts which corresponds to global constant modes [54,66]. These modes are topological modes with flat connections corresponding to the nontrivial homology cycles of the torus. They define the topological sector of the theory and they are, thus, the relevant objects to study its topological properties. Quantizing the theory in the $A_0 = 0$ gauge, we can decompose A in the sum of nonflat local modes and global flat modes,

$$\bar{A}(\bar{x}, t) = \tilde{A}(\bar{x}, t) + \bar{a}(t) = A_{\text{nf}} + A_{\text{f}}. \quad (63)$$

The flat modes satisfy trivially the Gauss’ law constraint and have to be quantized independently. Since in the Hodge decomposition A_{nf} and A_{f} are orthogonal, the Hamiltonian can be written as

$$H = H_{\text{nf}} + H_{\text{f}}, \quad (64)$$

and the Hilbert space is the direct product of two Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_{\text{nf}} \otimes \mathcal{H}_{\text{f}}. \quad (65)$$

The wave function is, thus, given by

$$\Psi = \Psi_{\text{nf}} \otimes \Psi_{\text{f}}. \quad (66)$$

To study the constant modes solution on the lattice, we start from Hamiltonian Eq. (23). For constant modes, only the first term survives, since the magnetic field B acting on constant modes is zero. So no Villain approximation is required to solve the theory in the topological sector.

The Hamiltonian Eq. (23) for constants modes is

$$H = \frac{e^2}{2a^2} S \left[\left(p_1 - \frac{ka^2}{4\pi} a_2 \right)^2 + \left(p_2 + \frac{ka^2}{4\pi} a_1 \right)^2 \right], \quad (67)$$

with $S = N_1 N_2$ the (dimensionless) area of the flat lattice torus, $k = p/q$ and p_i are the constant modes’ canonical momenta in the zero-lattice-momentum sector. Note that since k is rational we have $(a_1, a_2) \simeq (a_1 + 2\pi q/a, a_2 + 2\pi q/a)$. Equation (67) is equivalent to the Hamiltonian of a charged particle of mass $m = (a^2/e^2 S)$ moving in a fictitious magnetic field: $B = ka^2/2\pi$ on a torus of area $A = (2\pi)^2 q^2/a^2$, where a_1 and a_2 are the coordinates and $p_i = -i \frac{\partial}{\partial a_i}$ [note that the area of the torus with coordinates a_1 and a_2 has dimensions of a $(length)^{-2}$, so all other dimensions are scaled accordingly],

$$H = \frac{1}{2m} S [(p_1 - \mathcal{A}_1)^2 + (p_2 + \mathcal{A}_2)^2], \quad (68)$$

with, in the symmetric gauge, $\mathcal{A}_1 = -Ba_2/2$, $\mathcal{A}_2 = Ba_1/2$. In the Landau gauge, $\mathcal{A}_1 = 0$, $\mathcal{A}_2 = Ba_1$, which is used in our following solution.

This is a well-known problem whose solution was found in Ref. [67]. We, thus, present only the results. A detailed derivation can be found in Ref. [68]. On the infinite plane, the spectrum consists of Landau levels determined by the eigenvalue of a shifted harmonic oscillator, with frequencies $\omega = ke^2/2\pi$, given by the Chern-Simons mass in our case [58]. The states are infinitely degenerate since the energy does not depend on one of the momenta (in the present case p_2 in the Landau gauge). On the torus, although the Hamiltonian which describes the system is the same, the energy spectrum has a finite degeneracy given by the number of fluxes N_ϕ piercing the surface,

$$\Phi = N_\phi \phi_0, \quad (69)$$

where Φ is the total flux, and $\phi_0 = 2\pi$ is the flux quantum. For consistency of the theory on the torus, N_ϕ must be an integer. In our case, we have $N_\phi = pq \in \mathbb{Z}$, giving exactly the same degeneracy $\bar{k}$ we found in Sec. III C for a genus 1 surface. This degeneracy can be computed looking at the algebra of magnetic translation operators (see Ref. [68]).

In what follows, we consider the case of integer k with $(a_1, a_2) \simeq (a_1 + 2\pi/a, a_2 + 2\pi/a)$. The vector potential is periodic up to a gauge transformation,

$$\mathcal{A}_i(a_1 + 2\pi/a, a_2) = \mathcal{A}_i(a_1, a_2) - \partial_i \phi_1(a_2), \quad (70)$$

$$\mathcal{A}_i(a_1, a_2 + 2\pi/a) = \mathcal{A}_i(a_1, a_2) - \partial_i \phi_2(a_1), \quad (71)$$

with

$$\phi_1(a_2) = \theta_1 - \frac{2\pi}{a} Ba_2, \quad (72)$$

$$\phi_2(a_1) = \theta_2, \quad (73)$$

where θ_1 and θ_2 are fixed phase angles defined in Sec. III C. The phase angles ϕ_i satisfy the cocycle consistency condition,

$$\phi_2\left(a_1 + 2\frac{\pi}{a}\right) + \phi_1(a_2) - \phi_1\left(a_2 + 2\frac{\pi}{a}\right) - \phi_2(a_1) = 2\pi k. \quad (74)$$

The periodicity in the torus with coordinates a_i is equivalent to a large gauge transformation. The wave function transforms, thus, as

$$\begin{aligned} \Psi(a_1 + 2\pi/a, a_2) &= e^{i\phi_1(a_2)} \Psi(a_1, a_2) \\ &= e^{i(\theta_1 - \frac{2\pi}{a} B a_2)} \Psi(a_1, a_2), \end{aligned} \quad (75)$$

$$\begin{aligned} \Psi(a_1, a_2 + 2\pi/a) &= e^{i\phi_2(a_1)} \Psi(a_1, a_2) \\ &= e^{i\theta_2} \Psi(a_1, a_2). \end{aligned} \quad (76)$$

These phase angles θ_i can be reabsorbed in a redefinition of the magnetic translation generators and distinguish different superselection sector of the theory [68]. In our discussion, we set $\theta_i = 0$.

Reintroducing the original gauge and positing the complex topological components [66,69],

$$b = \frac{1}{2\pi}(ia_1 + a_2); \quad \bar{b} = \frac{1}{2\pi}(-ia_1 + a_2), \quad (77)$$

we can rewrite Eq. (67) as

$$H = \frac{e^2 S}{2a^2} \left(-\frac{1}{\pi} \frac{\partial}{\partial b} + \frac{ka^2}{2} \bar{b} \right) \left(\frac{1}{\pi} \frac{\partial}{\partial \bar{b}} + \frac{ka^2}{2} b \right) + \text{const} \quad (78)$$

$$= \frac{e^2 S}{2a^2} B^+ B + \text{const}, \quad (79)$$

with

$$B \equiv \frac{1}{\pi} \frac{\partial}{\partial \bar{b}} + \frac{ka^2}{2} b, \quad [B, B^+] = \frac{ka^2}{\pi}. \quad (80)$$

Equation (79) is the Hamiltonian of an harmonic oscillators, and the ground state Ψ^0 is given by $B\Psi^0 = 0$. Imposing the periodicity conditions Eqs. (75) and (76), we see that there are k independent ground states Ψ_s^0 with $s = 0, \dots, k-1$. The ground-state wave function is

$$\Psi_s^0 = e^{\frac{\pi k a^2}{2} b^2} e^{-\frac{\pi k a^2}{2} b \bar{b}} \theta \left[\begin{matrix} s/k \\ 0 \end{matrix} \right] (kb|ik), \quad (81)$$

expressed in terms of the Jacobi θ function. In terms of the a_i fields, the ground-state wave function is

$$\Psi_s^0 = e^{\frac{ka^2}{8\pi}(ia_1+a_2)^2} e^{-\frac{ka^2}{8\pi}(a_1^2+a_2^2)} \sum_{m \in \mathbb{Z}} e^{\frac{\pi}{k}(mk+s)^2} e^{a(mk+s)(-a_1+ia_2)}. \quad (82)$$

These k independent ground states are consistent with our earlier discussion on the k -fold degeneracy of states in Sec. III C.

The excited states can be obtained applying B^+ to the ground-state wave function Eq. (82) and have energies of the order ω , the Chern-Simons mass. In the limit in which the Chern-Simons mass goes to infinity (strong coupling limit), we recover the topological theory.

B. Energy spectrum

In this section, we present the solution of all lattice momentum modes to the Hamiltonian in Eq. (23). Since the Hamiltonian is quadratic, we can use the Fourier transform to decompose it into independent momentum sectors. We rewrite the Hamiltonian as

$$\begin{aligned} \hat{H} = & \sum_{\substack{x=(x_1, x_2) \\ x_1 \in \{0,1,\dots,N_1-1\} \\ x_2 \in \{0,1,\dots,N_2-1\}}} \left[\frac{e^2}{2a^2} \sum_{\mu, \nu \in \{1,2\}} \left(\hat{p}_{x;\mu} - \epsilon_{\mu\nu} \frac{ka^2}{4\pi} \hat{A}_{x-\hat{\nu};\nu} \right)^2 \right. \\ & \left. + \frac{1}{2e^2} (\square \hat{A}_{x;1,2})^2 \right]. \end{aligned} \quad (83)$$

We apply the following Fourier transform:

$$\hat{A}_{q;\mu} = \frac{1}{\sqrt{N_1 N_2}} \sum_{\substack{x=(x_1, x_2) \\ x_1 \in \{0,1,\dots,N_1-1\} \\ x_2 \in \{0,1,\dots,N_2-1\}}} e^{i\vec{q} \cdot \vec{x}} \hat{A}_{x;\mu}, \quad (84)$$

$$\hat{p}_{q;\mu} = \frac{1}{\sqrt{N_1 N_2}} \sum_{\substack{x=(x_1, x_2) \\ x_1 \in \{0,1,\dots,N_1-1\} \\ x_2 \in \{0,1,\dots,N_2-1\}}} e^{i\vec{q} \cdot \vec{x}} \hat{p}_{x;\mu}, \quad (85)$$

with inverse Fourier transform,

$$\hat{A}_{x;\mu} = \frac{1}{\sqrt{N_1 N_2}} \sum_{\substack{q=(q_1, q_2) \\ q_1 \in \{0, \frac{2\pi}{N_1}, \dots, \frac{2\pi}{N_1}(N_1-1)\} \\ q_2 \in \{0, \frac{2\pi}{N_2}, \dots, \frac{2\pi}{N_2}(N_2-1)\}}} e^{-i\vec{q} \cdot \vec{x}} \hat{A}_{q;\mu}, \quad (86)$$

$$\hat{p}_{x;\mu} = \frac{1}{\sqrt{N_1 N_2}} \sum_{\substack{q=(q_1, q_2) \\ q_1 \in \{0, \frac{2\pi}{N_1}, \dots, \frac{2\pi}{N_1}(N_1-1)\} \\ q_2 \in \{0, \frac{2\pi}{N_2}, \dots, \frac{2\pi}{N_2}(N_2-1)\}}} e^{-i\vec{q} \cdot \vec{x}} \hat{p}_{q;\mu}. \quad (87)$$

The commutation relation between the operators is

$$[\hat{A}_{x;\mu}, \hat{p}_{y;\nu}] = i\delta_{x,y} \delta_{\mu,\nu}, \quad (88)$$

$$[\hat{A}_{q;\mu}, \hat{p}_{s;\nu}] = i\delta_{q+s, 2\pi\mathbb{Z}} \delta_{\mu,\nu}. \quad (89)$$

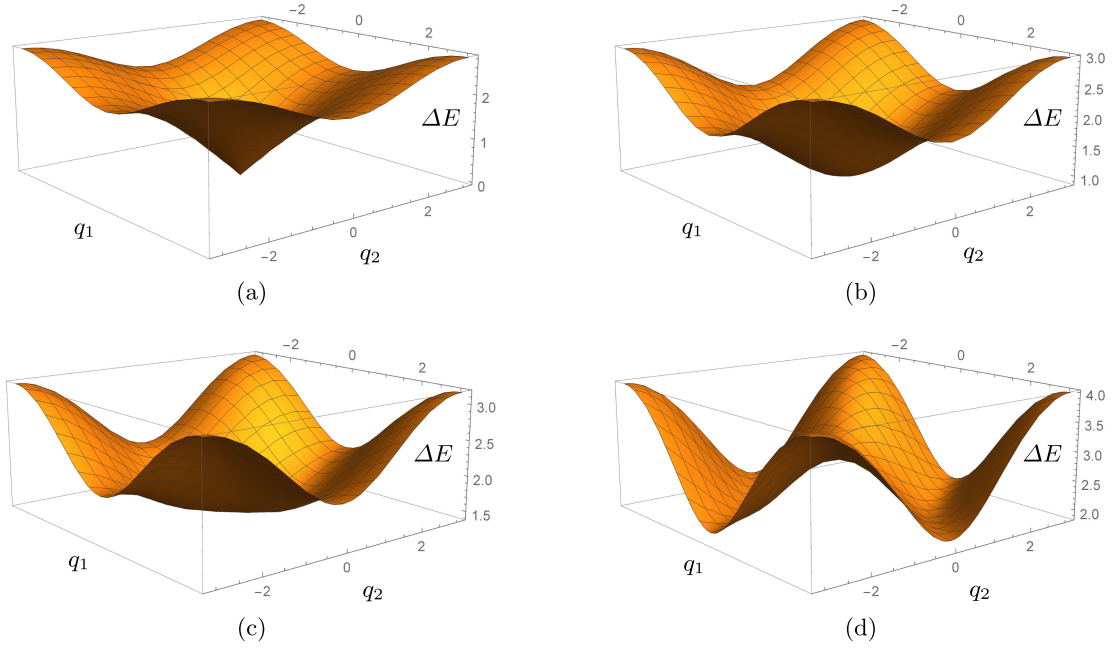


FIG. 5. 3D plot of the band structure of the analytical solution to the Maxwell-Chern-Simons Hamiltonian. The vertical axis is the energy of the first excited state ΔE above the ground state. Two horizontal axes are the two components of the lattice momentum $q = (q_1, q_2)$. Each subfigure corresponds to a different coupling strength $\lambda = (ke^2 a/4\pi)^2$. (a) $\lambda = 0$, this corresponds to the pure Maxwell theory, in which we can see a linear-dispersion gapless photon. (b) $\lambda = 0.2$, the Chern-Simons term gaps out the theory and gives the photon a mass. (c) $\lambda = 0.5$, this is the special λ value beyond which the band bottom is no longer at $q = (0, 0)$. (d) $\lambda = 2.0$, for large λ value the band looks like the one for a pure Chern-Simons theory, in which the band bottom are two lines specified by $q_1 + q_2 = \pi + 2\pi\mathbb{Z}$. These two lines correspond to the “staggered momentum modes” mentioned in Ref. [3].

The Fourier transformed Hamiltonian in Eq. (83) looks like

$$\begin{aligned} \hat{H} = & \sum_{\substack{q=(q_1, q_2) \\ q_1 \in \{0, \frac{2\pi}{N_1}, \dots, \frac{2\pi}{N_1}(N_1-1)\} \\ q_2 \in \{0, \frac{2\pi}{N_2}, \dots, \frac{2\pi}{N_2}(N_2-1)\}}} \frac{e^2}{2a^2} \left[\left(\hat{p}_{q;1} - \frac{ka^2}{4\pi} e^{iq_2} \hat{A}_{q;2} \right) \left(\hat{p}_{-q;1} - \frac{ka^2}{4\pi} e^{-iq_2} \hat{A}_{-q;2} \right) \right. \\ & + \left(\hat{p}_{q;2} + \frac{ka^2}{4\pi} e^{iq_1} \hat{A}_{q;1} \right) \left(\hat{p}_{-q;2} + \frac{ka^2}{4\pi} e^{-iq_1} \hat{A}_{-q;1} \right) \Big] \\ & + \frac{1}{2e^2} [(1 - e^{-iq_2}) \hat{A}_{q;1} - (1 - e^{-iq_1}) \hat{A}_{q;2}] [(1 - e^{iq_2}) \hat{A}_{-q;1} - (1 - e^{iq_1}) \hat{A}_{-q;2}]. \end{aligned} \quad (90)$$

After some change of variables and applying the Gauss' law Eq. (25) (see more details in the Appendix F), the Hamiltonian in the lattice momentum $q = (q_1, q_2)$ sector corresponds to a simple harmonic oscillator with angular frequency,

$$\omega^2 = \frac{1}{a^2} [2(1 - \cos q_1) + 2(1 - \cos q_2)] + \left(\frac{ke^2}{4\pi} \right)^2 [2 + 2 \cos(q_1 + q_2)]. \quad (91)$$

The energy of the corresponding first excited state above the ground state is

$$\Delta E = \omega = \sqrt{\frac{1}{a^2} [2(1 - \cos q_1) + 2(1 - \cos q_2)] + \left(\frac{ke^2}{4\pi} \right)^2 [2 + 2 \cos(q_1 + q_2)]}, \quad (92)$$

which is plotted in Fig. 5. The different subplots in Fig. 5 correspond to different values of a combined parameter $(ke^2 a/4\pi)^2$. Each state in the band is k -fold degenerate.

There are two interesting limits of this energy spectrum. First, we note that the dimensionless lattice momentum $q = a\tilde{q}$, where $\tilde{q}$ is the dimensionful momentum. Then, by taking the continuum limit $a \rightarrow 0$, we get

$$\omega^2 \rightarrow |\tilde{q}|^2 + \left(\frac{ke^2}{2\pi}\right)^2, \quad (93)$$

which describes a massive particle with mass equals to $ke^2/2\pi$; i.e., the photon becomes massive, and the theory opens a gap. The $|\tilde{q}|^2$ part comes from the first term $\frac{2}{a^2}(1 - \cos a\tilde{q}_i)$ in Eq. (91). This energy dispersion agrees with the one in the continuous Maxwell-Chern-Simons theory [5].

Another limit is when $k \rightarrow \infty$. The Chern-Simons term dominates. The band structure looks similar to the one of the pure continuous Chern-Simons theory, with the band bottom located at two lines specified by $q_1 + q_2 = \pi + 2\pi\mathbb{Z}$. These two lines correspond to the “staggered momentum modes” mentioned in Ref. [3].

One interesting observation is that the band bottom changes location, although this does not mean a phase transition because the band is showing the first excited state but not the ground state.

There is also a distinction between the $q = (0, 0)$ zero mode and the other modes. The Gauss’ law (constraint 1) acts trivially on the zero mode and therefore leaves a free degree of freedom. Constraints 2 and 3 require both the momentum and the coordinate of this free degree of freedom to be quantized, which is equivalent to the tight-binding model of a single particle on a ring with finitely many sites. As shown in the Appendix F, there are exactly k sites on the ring. Therefore, the Hilbert space for the zero mode is k dimensional and results in the k -fold degeneracy of every state. This is consistent with our earlier discussion on constant modes in Sec. IV A.

V. CONCLUSION

In this paper, we derive the Hamiltonian formulation of Maxwell-Chern-Simons theory on a lattice in $2 + 1$ D. Compared to the well-known $2 + 1$ D compact lattice QED Hamiltonian, the modification in our Hamiltonian formulation is as follows. First, the instantons in the theory are suppressed as a hard constraint. Similar to the Witten effect in $3 + 1$ D [46], an instanton (i.e., a magnetic monopole in $2 + 1$ D) must have an electric charge under the Chern-Simons action; thus, our pure gauge theory cannot allow instantons. We further show that the instanton suppression can turn the $2 + 1$ D compact lattice QED Hamiltonian into an unusual form (see Appendix E). This instanton-suppressed compact lattice QED Hamiltonian has quadratic terms in the gauge field $\hat{A}$, and it is indeed a compact theory by the Villain approximation with two 1-form constraints. The two 1-form constraints ensure that the theory is invariant under large gauge transformations in the two spatial directions. Furthermore, this

instanton-suppressed quadratic Hamiltonian is analytically solvable, showing a gapless theory with massless photons. Unlike the instanton-allowed compact lattice QED, which is gapped in $2 + 1$ D, the instanton-suppressed version corresponds to the Maxwell theory in the continuum, which is gapless.

The second modification is to introduce the Chern-Simons term into this instanton-suppressed Hamiltonian. The effect of this step is that a gauge field operator $\hat{A}$ on a shifted site and in the perpendicular direction is attached to the conjugate momentum operator $\hat{p}$, wherever $\hat{p}$ appears. Therefore, we see the modification in the quadratic $\hat{p}$ terms in the Hamiltonian in Eq. (23), the Gauss’ law constraint in Eq. (28), and the two 1-form constraints in Eqs. (28) and (30).

We demonstrate the topological features of our Hamiltonian formulation. The compatibility between the constraints requires the quantization of the Chern-Simons level k . The existence of two 1-form operators generates the k -fold degeneracy of the spectrum. The Wilson loop expectation values in the theory always show a perimeter law. Linked Wilson loops have nontrivial topological phases evaluated on the degenerate Hilbert subspace. Anyons as the excitations at the ends of open Wilson lines display fractional mutual statistics and self-statistics. These topological features agree with the ones in a continuum Maxwell-Chern-Simons theory, providing strong evidence that our lattice formulation belongs to the same universality class as the continuum theory. We note, however, that a fully rigorous proof of universality is beyond the scope of this work.

The continuum limit of our lattice Hamiltonian formulation can also be directly seen from the analytical solution of the theory. Since the Hamiltonian in Eq. (23) is quadratic, we perform the Fourier transform and solve its band structure in the Brillouin zone. We can see the correct band gap $ke^2/2\pi$ and the massive photon dispersion that agree with the continuum Maxwell-Chern-Simons theory.

In our work, we add the Maxwell term to the Chern-Simons theory, which allows us to construct the independent canonical conjugate momentum operator $\hat{p}$. Our formulation accepts the gauge field configurations as the natural basis for the Hilbert space. This is in contrast to the recent work that presents a canonical quantization of lattice Chern-Simons theory [3], which does not have Maxwell action and the canonical quantization unavoidably has nontrivial commutation between operators on different spatial locations. This “fuzzy” space makes it difficult to construct a good orthonormal basis for the Hilbert space and apply numerical methods. Therefore, our lattice Hamiltonian formulation opens the door to numerical methods to study this topological theory. It also paves the way for designing quantum algorithms for simulating Maxwell-Chern-Simons theory based on our lattice Hamiltonian formulation. Numerical simulations with

tensor-network ansatz or machine-learning approaches are likewise applicable to our formulation. An important future direction is to extend the simulations to our model coupled to matter fields.

For future study, our Hamiltonian formulation provides a foundation for quantum simulation of Maxwell–Chern–Simons theory using both classical and quantum computational approaches. On the classical side, tensor-network methods and machine-learning-based variational algorithms could be explored to investigate the topological and dynamical properties of the theory. On the quantum side, the local and gauge-invariant structure of our lattice Hamiltonian makes it a candidate for digital quantum simulation schemes such as trotterized time evolution [70], variational quantum eigensolvers [71,72], and quantum phase estimation [73], with potential applications to studying topological mass generation, anyonic excitations, and nonperturbative gauge dynamics on both near-term and fault-tolerant quantum hardware. Furthermore, the compact gauge degrees of freedom can be efficiently encoded using qudit or continuous-variable representations, allowing for systematic evaluation of quantum resource requirements such as qubit counts, circuit depths, and error thresholds needed to maintain gauge invariance and topological order. By combining these algorithmic developments with resource analyses, future work could establish Maxwell–Chern–Simons theory as an important example for quantum simulations of topological lattice gauge theory, helping to bridge high-energy and condensed-matter physics with the frontier of quantum computation. We emphasize, however, that the practical realization of these directions will require substantial further algorithmic development and resource analysis, and the extent to which Maxwell–Chern–Simons theory can serve as a useful target for quantum simulation will only become clear once such studies are carried out.

Note added. In Ref. [74], the authors study the same Lagrangian as ours and derived a Hamiltonian formulation that does not include the final gauge fixing step (this step is outlined in Appendix E). By doing so, they extend the discussions of the Maxwell–Chern–Simons theory to sectors beyond the zero total magnetic flux sector. This zero total magnetic flux sector is what we focus on in this work, because we study the pure Maxwell–Chern–Simons theory without matter. We construct a local and uniform Hamiltonian formulation in this sector in perspective of Hamiltonian simulation. In Ref. [74], efforts have been made to construct a local and uniform Hamiltonian formulation for spacetime manifolds with arbitrary topology and arbitrary flux sectors. The focus of Ref. [74] is to derive the chiral properties of the theory, in contrast to our emphasis on future quantum computing implementations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: REVIEW OF INSTANTONS IN COMPACT MAXWELL THEORY

In this section, we briefly review the effect of instantons on a $2 + 1$ D compact $U(1)$ gauge theory (i.e., Maxwell theory) [76,77]. The theory contains instantons if it has a finite cutoff scale, which is our lattice spacing here. On the lattice, “discontinuities” should be understood in the following precise sense: in the compact Villain formulation, the integer degrees of freedom n on the plaquettes allow configurations in which the magnetic flux through a closed surface differs from the naive finite-difference flux $\Delta_1 A_2 - \Delta_2 A_1$ by integer multiples of 2π . These configurations carry nontrivial topological charge and are the lattice analogues of the smooth-spacetime instantons (i.e., monopoles in $2 + 1$ D), which become genuine discontinuities of the gauge field in the continuum limit. With this understanding, instantons are discontinuities in the gauge field configuration where the total flux through a plane suddenly changes by 2π . In $2 + 1$ D, an instanton configuration has a finite action. Therefore, when computing the contribution to the

partition function $e^{-S(A)}$, instanton configurations have nonzero weight. Because we need to take into account the instanton configurations, the theory is no longer Gaussian quadratic around the $A = 0$ minimum, i.e., no longer a free theory. Moreover, because an instanton changes the flux permanently after its appearance, its long-range feature has significant influence on the correlation functions. In particular, it makes the Wilson loops scale as the area law and confines charged particles [76].

The effect of instantons can be seen clearly using the duality between the U(1) gauge theory and the XY model [78]. In the duality, the flux in the U(1) gauge theory maps to the charge in the XY model. Therefore, the instanton operator, as creation/annihilation operator of the flux, maps to $e^{\pm i\theta}$ in the XY model. Summing over instanton configurations in the partition function maps to the following Lagrangian in the XY model:

$$\mathcal{L} = \frac{\chi}{2}(\partial_\mu \theta)^2 - K \cos \theta, \quad (\text{A1})$$

where χ and K are coefficients related to the coupling strength and instanton fugacity, respectively.

Because the $\cos \theta$ term is always relevant in $2 + 1\text{D}$ [79], the instanton operator gaps out the theory. The $\cos \theta$ term also breaks the $U(1)_{XY}$ symmetry, leaving the theory with a unique ground state and massive excitations. The linear potential between charges in the U(1) gauge theory maps to the linear potential between vortices in the XY model, which is due to the $\cos \theta$ energy cost at the $\theta = \pi$ or $-\pi$ branch-cut line between the vortices [78].

When the Chern-Simons action is present, an instanton will carry charge, which is similar to the Witten effect in $3 + 1\text{D}$ [46]. Moreover, when the Chern-Simons level k is an odd integer, the instanton operator will be fermionic. We do not consider the matter field in this work, and therefore, we suppress the instantons.

When instantons are suppressed, it is equivalent to set $K = 0$ in Eq. (A1). The $U(1)_{XY}$ symmetry is spontaneously broken, and we have massless Goldstone modes, which are the massless photons in the U(1) gauge theory. However, we can no longer view the gauge field $A \in [0, 2\pi/a)$ on each link for the instanton-suppressed compact U(1) gauge theory. The equivalence between $A = 0$ and $A = 2\pi/a$ makes the flux only well defined by mod 2π , which naturally allows the existence of instantons. We have to use the Villain approximation to formulate an instanton-suppressed theory.

APPENDIX B: VILLAIN APPROXIMATION

A theory with compact variables means that the partition function Z is periodic in these compact variables θ_i . For such a theory, it is natural to include in the action terms like $\cos(\sum_i c_i \theta_i)$, where c_i are the coefficients that match the periodicity of the compact variable θ_i . However, these

cosine terms are nonlinear and are difficult to handle analytically. The Villain approximation simplifies the analysis by replacing the cosine terms with a periodic Gaussian potential [39],

$$e^{\beta \cos(\theta)} \approx \sum_{n=-\infty}^{\infty} e^{-\frac{\tilde{\beta}}{2}(\theta + 2\pi n)^2}, \quad (\text{B1})$$

where $\tilde{\beta}$ is a function of β , and—without loss of generality—we write θ as 2π periodic here.

A clarification on the term “approximation” is in order. Strictly speaking, the right-hand side is not a Taylor or perturbative truncation of the left-hand side; rather, it is an alternative compact action that preserves the exact 2π periodicity of the original cosine and shares its continuum limit and universality class. In some specific cases, such as the quantum rotor, the Villain action is in fact a “perfect” lattice action: it is the renormalization-group fixed-point action that reproduces the exact continuum spectrum at any lattice spacing, with no lattice artifacts [80]. We adopt the term “Villain approximation” here to follow the standard nomenclature in the lattice gauge theory literature, but the reader may equivalently read it as “Villain formulation.” In practice, the implementation of this formulation introduces, on each plaquette where a cosine term originally appeared, an additional integer degree of freedom n that is summed over in the partition function; this turns the action into a quadratic form in the gauge field plus integer shifts, which is what enables the analytical treatment in the following Appendixes.

APPENDIX C: SINGLE QUANTUM ROTOR

We go step by step to see how to construct a Hamiltonian formulation for a compact lattice field theory. We start with the action of a single quantum rotor, i.e., a $0 + 1\text{-D}$ compact quantum field. A careful construction of its Hamiltonian formulation shows the spirit of the later steps.

We use Euclidean time formulation, discretize the time, and set up periodic boundary conditions. The action of the rotor is

$$S(\theta) = -\frac{\beta}{d\tau} \sum_{i=0}^{T-1} \cos(\theta_{i+1} - \theta_i), \quad (\text{C1})$$

where β is a coefficient, $d\tau$ is the time spacing, $\theta_i \in [0, 2\pi)$ is the rotor angle at the i th time slice, $\theta_T = \theta_0$, T is the total number of time slices.

The partition function of the rotor is

$$Z = \sum_{\theta_0, \theta_1, \dots, \theta_{T-1} \in [0, 2\pi)} e^{-S(\theta)}, \quad (\text{C2})$$

where we have used the summation symbol to represent the integral over continuous variables θ 's.

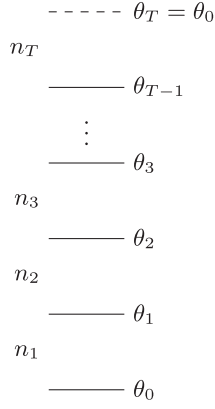


FIG. 6. Illustration of the variables in a single quantum rotor model, i.e., a 0 + 1-D compact quantum field. The vertical direction is time. There are T number of discrete time slices, and we have periodic boundary conditions.

We can apply the Villain approximation to the cosine terms in the action. By doing this, we introduce T number of integer degrees of freedom, $n_1, n_2, \dots, n_T$, each between a pair of θ 's, as illustrated in Fig. 6. The approximated action with these extra degrees of freedom looks like

$$S(n, \theta) = \frac{\tilde{\beta}}{2d\tau} \sum_{i=0}^{T-1} (\theta_{i+1} - \theta_i + 2\pi n_{i+1})^2, \quad (C3)$$

where $\tilde{\beta}$ is a function of β . When $d\tau \rightarrow 0$, $\beta/d\tau \rightarrow \infty$, the asymptotic behavior of the Villain approximation says $\tilde{\beta} \rightarrow \beta$ [81].

The approximated partition function with the extra degrees of freedom looks like

$$Z = \sum_{\substack{\theta_0, \theta_1, \dots, \theta_{T-1} \in [0, 2\pi) \\ n_1, n_2, \dots, n_T \in \mathbb{Z}}} e^{-S(n, \theta)}. \quad (C4)$$

The approximated action $S(n, \theta)$ holds discrete transformations that keeps it invariant. For every $i \in 0, 1, \dots, T-1$, the following transformation

$$\begin{aligned} \theta_i &\rightarrow \theta_i + 2\pi, \\ n_i &\rightarrow n_i - 1, \\ n_{i+1} &\rightarrow n_{i+1} + 1, \end{aligned}$$

keeps the approximated action, and thus the approximated partition function, invariant.

Writing down the transformation invariance of the approximated action explicitly, and taking $i = 1$, we have

$$S(n_1 - 1, n_2 + 1, \theta_1 + 2\pi, *) = S(n_1, n_2, \theta_1, *), \quad (C5)$$

where $*$ represents all the other variables in $S(n, \theta)$, which are fixed.

We can apply this transformation multiple times, until we make $n_1 = 0$. That is to say,

$$S(0, n_2 + n_1, \theta_1 + 2\pi n_1, *) = S(n_1, n_2, \theta_1, *). \quad (C6)$$

Note that $\theta_1 + 2\pi n_1$ takes value in $[2\pi n_1, 2\pi(n_1 + 1))$. When we sum over $n_1 \in \mathbb{Z}$, it is equivalent to sum over $\theta_1 \in (-\infty, +\infty)$. In other words, by absorbing n_1 , θ_1 is lifted from $[0, 2\pi)$ to $(-\infty, +\infty)$. More rigorously, we define new variables,

$$\tilde{\theta}_1 = \theta_1 + 2\pi n_1, \quad \tilde{\theta}_1 \in (-\infty, +\infty), \quad (C7)$$

$$\tilde{n}_2 = n_2 + n_1, \quad \tilde{n}_2 \in \mathbb{Z}, \quad (C8)$$

with the inverse,

$$n_1 = \left\lfloor \frac{\tilde{\theta}_1}{2\pi} \right\rfloor, \quad (C9)$$

$$\theta_1 = \tilde{\theta}_1 - 2\pi \left\lfloor \frac{\tilde{\theta}_1}{2\pi} \right\rfloor, \quad \theta_1 \in [0, 2\pi), \quad (C10)$$

$$n_2 = \tilde{n}_2 - \left\lfloor \frac{\tilde{\theta}_1}{2\pi} \right\rfloor, \quad (C11)$$

where $\lfloor \cdot \rfloor$ is the largest integer less than or equal to a given number.

The approximate partition function now looks like

$$\begin{aligned} Z &= \sum_{\substack{\theta_0, \theta_1, \dots, \theta_{T-1} \in [0, 2\pi) \\ n_1, n_2, \dots, n_T \in \mathbb{Z}}} e^{-S(n_1, n_2, \theta_1, *)} \\ &= \sum_{\substack{\theta_0, \theta_1, \dots, \theta_{T-1} \in [0, 2\pi) \\ n_1, n_2, \dots, n_T \in \mathbb{Z}}} e^{-S(0, n_2 + n_1, \theta_1 + 2\pi n_1, *)} \\ &= \sum_{\substack{\theta_0, \theta_2, \dots, \theta_{T-1} \in [0, 2\pi) \\ \theta_1 \in (-\infty, +\infty), \tilde{n}_2 \in \mathbb{Z} \\ n_3, n_4, \dots, n_T \in \mathbb{Z}}} e^{-S(0, \tilde{n}_2, \tilde{\theta}_1, *)} \\ &= \sum_{\substack{\theta_0, \theta_2, \dots, \theta_{T-1} \in [0, 2\pi) \\ \theta_1 \in (-\infty, +\infty), \tilde{n}_2 \in \mathbb{Z} \\ n_3, n_4, \dots, n_T \in \mathbb{Z}}} e^{-\tilde{S}(\tilde{n}_2, \tilde{\theta}_1, *)}, \end{aligned} \quad (C12)$$

where $\tilde{S}(\tilde{n}_2, \tilde{\theta}_1, *) = S(0, \tilde{n}_2, \tilde{\theta}_1, *)$; i.e., n_1 is dropped from the dependencies.

We can repeat this process to absorb $\tilde{n}_2$ by

$$\tilde{\theta}_2 = \theta_2 + 2\pi \tilde{n}_2, \quad \tilde{\theta}_2 \in (-\infty, +\infty), \quad (C13)$$

$$\begin{aligned} \tilde{n}_3 &= n_3 + \tilde{n}_2, \quad \tilde{n}_3 \in \mathbb{Z}, \\ &\dots \end{aligned} \quad (C14)$$

until

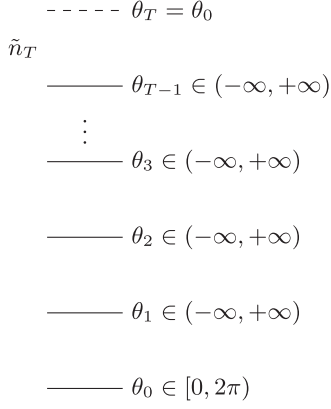


FIG. 7. Illustration of the variables in a single quantum rotor model, i.e., a 0 + 1-D compact quantum field. The vertical direction is time. There are T number of discrete time slices, and we have periodic boundary conditions. After absorbing most of the extra integer degrees of freedom introduced by the Villain approximation, most of the compact variables are lifted to $(-\infty, +\infty)$. However, the variable in the bottom layer, θ_0 , is not lifted. Also, the integer in the last layer, $\tilde{n}_T$, cannot be absorbed.

$$\tilde{\theta}_{T-1} = \theta_{T-1} + 2\pi\tilde{n}_{T-1}, \quad \tilde{\theta}_{T-1} \in (-\infty, +\infty), \quad (\text{C15})$$

$$\tilde{n}_T = n_T + \tilde{n}_{T-1} = \sum_{i=1}^T n_i, \quad \tilde{n}_T \in \mathbb{Z}, \quad (\text{C16})$$

which transforms the approximate partition function into

$$Z = \sum_{\substack{\theta_0 \in [0, 2\pi) \\ \tilde{\theta}_1, \tilde{\theta}_2, \dots, \tilde{\theta}_{T-1} \in (-\infty, +\infty), \\ \tilde{n}_T \in \mathbb{Z}}} \times \exp \left\{ -\frac{\tilde{\beta}}{2d\tau} [(\tilde{\theta}_1 - \theta_0)^2 + (\tilde{\theta}_2 - \tilde{\theta}_1)^2 + \dots + (\tilde{\theta}_{T-1} - \tilde{\theta}_{T-2})^2 + (\theta_0 - \tilde{\theta}_{T-1} + 2\pi\tilde{n}_T)^2] \right\}. \quad (\text{C17})$$

Note that θ_0 is not lifted, and the extra integer degree of freedom on the last layer, $\tilde{n}_T$, cannot be absorbed. It cannot be absorbed because $\tilde{n}_T = \sum_{i=1}^T n_i$ has a physical meaning of the total flux in the Polyakov loop and thus is gauge invariant.

Now with this quadratic formulation of the partition function, we can apply the standard method to get the Hamiltonian formulation. More explicitly, we set up the Hilbert space, write down the transfer matrix, construct the Hamiltonian operator, and recognize the nonabsorbable degree of freedom as constraint on the Hilbert space.

When setting up the Hilbert space, we choose the set of basis to be the configurations of a middle time layer. In our example here, the set of basis is $\{|\tilde{\theta}\rangle : \tilde{\theta} \in (-\infty, +\infty)\}$. Note that even if we start from a compact model, after the Villain approximation and absorption of degrees of

freedom, we end up with a seemingly noncompact set of basis. We do not need to worry at this moment, as later the constraint on the Hilbert space will bring us back to a compact model.

We also note that the first time layer (i.e., the bottom layer in Fig. 7) has fewer configurations. Because of the periodicity of the variable θ_0 , we can duplicate the partition function to extend its configuration space to match the one of the middle layers. In a nonrigorous way, we can write

$$\sum_{\theta_0 \in [0, 2\pi)} = \frac{1}{|\mathbb{Z}|} \sum_{\tilde{\theta}_0 \in (-\infty, +\infty)}. \quad (\text{C18})$$

Here, $|\mathbb{Z}|$ denotes the (formally infinite) cardinality of the integers; this notation is a shorthand for the corresponding well-defined limit, in which a finite cutoff is taken on the integer sums and removed at the end. The factor $1/|\mathbb{Z}|$ keeps the duplicated partition function equal to the original, and it cancels out below when summing over $\tilde{n}_T$ produces an identical $|\mathbb{Z}|$ factor (the borrowed degree of freedom is returned). All physical quantities derived from this construction are finite and well defined.

Given the Hilbert space, we want to write down the transfer matrix $\hat{T}$. For two neighboring middle time layers, we have the matrix elements of the transfer matrix as

$$\langle \tilde{\theta}_{i+1} | \hat{T} | \tilde{\theta}_i \rangle = e^{-\frac{\tilde{\beta}}{2d\tau} (\tilde{\theta}_{i+1} - \tilde{\theta}_i)^2}, \quad (\text{C19})$$

where $\tilde{\theta}_{i+1}, \tilde{\theta}_i \in (-\infty, +\infty)$ are configurations on the two neighboring middle time layers.

For the last time layer (i.e., the top layer in Fig. 7), we have a different transfer matrix $\hat{T}_{\text{top}}$ with matrix elements as

$$\langle \tilde{\theta}_0 | \hat{T}_{\text{top}} | \tilde{\theta}_{T-1} \rangle = e^{-\frac{\tilde{\beta}}{2d\tau} (\tilde{\theta}_0 - \tilde{\theta}_{T-1} + 2\pi\tilde{n}_T)^2}. \quad (\text{C20})$$

In terms of the transfer matrices, the partition function looks like

$$Z = \frac{1}{|\mathbb{Z}|} \sum_{\tilde{n}_T \in \mathbb{Z}} \text{Tr}(\hat{T}_{\text{top}} \hat{T}^{T-1}), \quad (\text{C21})$$

where the $|\mathbb{Z}|$ factor was borrowed earlier to match the configuration spaces.

Now, we construct the Hamiltonian operator. For a general function $f(\hat{p})$ of the momentum operator $\hat{p}$, its matrix elements can be calculated by inserting the complete momentum basis $\mathbb{I} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} |p\rangle \langle p|$. Therefore, for any $\tilde{\theta}', \tilde{\theta} \in (-\infty, +\infty)$,

$$\langle \tilde{\theta}' | f(\hat{p}) | \tilde{\theta} \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(\tilde{\theta}' - \tilde{\theta})} f(p), \quad (\text{C22})$$

from which we can do the Fourier transform to get back the function $f(\cdot)$,

$$f(p) = \int_{-\infty}^{\infty} d\tilde{\theta} e^{-ip\tilde{\theta}} \langle \tilde{\theta} | f(\hat{p}) | 0 \rangle. \quad (\text{C23})$$

Perform a Fourier transform on the transfer matrix elements,

$$\begin{aligned} \int_{-\infty}^{\infty} d\tilde{\theta} e^{-ip\tilde{\theta}} \langle \tilde{\theta} | \hat{T} | 0 \rangle &= \int_{-\infty}^{\infty} d\tilde{\theta} e^{-ip\tilde{\theta}} e^{-\frac{\beta}{2d\tau} \tilde{\theta}^2} \\ &\propto e^{-\frac{d\tau}{2\beta} p^2}, \end{aligned} \quad (\text{C24})$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} d\tilde{\theta} e^{-ip\tilde{\theta}} \langle \tilde{\theta} | \hat{T}_{\text{top}} | 0 \rangle &= \int_{-\infty}^{\infty} d\tilde{\theta} e^{-ip\tilde{\theta}} e^{-\frac{\beta}{2d\tau} (\tilde{\theta} + 2\pi\tilde{n}_T)^2} \\ &\propto e^{ip2\pi\tilde{n}_T} e^{-\frac{d\tau}{2\beta} p^2}. \end{aligned} \quad (\text{C25})$$

The partition function now becomes

$$\begin{aligned} Z &= \frac{1}{|\mathbb{Z}|} \sum_{\tilde{n}_T \in \mathbb{Z}} \text{Tr} \left(e^{i\hat{p}2\pi\tilde{n}_T} e^{-\frac{d\tau}{2\beta} \hat{p}^2 T} \right) \\ &= \text{Tr}(\hat{C} e^{-\hat{H}\tau}), \end{aligned} \quad (\text{C26})$$

where $\hat{H} = \frac{1}{2\beta} \hat{p}^2$ is the Hamiltonian operator, $\tau = d\tau T$ is the total Euclidean time, and

$$\begin{aligned} \hat{C} &= \frac{1}{|\mathbb{Z}|} \sum_{\tilde{n}_T \in \mathbb{Z}} e^{i\hat{p}2\pi\tilde{n}_T} \\ &= \frac{1}{|\mathbb{Z}|} \sum_{\tilde{n}_T \in \mathbb{Z}} (e^{2\pi i \hat{p}})^{\tilde{n}_T} \\ &= \begin{cases} \mathbb{I} & \text{if } e^{2\pi i \hat{p}} = \mathbb{I}, \\ 0 & \text{if } e^{2\pi i \hat{p}} \neq \mathbb{I}. \end{cases} \end{aligned} \quad (\text{C27})$$

Note that the $|\mathbb{Z}|$ factor is canceled. This sum over non-absorbable degree of freedom $\tilde{n}_T$ sets up a constraint on the physical Hilbert space. In order to have a nonzero partition function, the allowed physical states $|\psi\rangle$ in the Hilbert space need to satisfy

$$e^{2\pi i \hat{p}} |\psi\rangle = |\psi\rangle, \quad (\text{C28})$$

which tells us that the momentum $\hat{p}$ is quantized, which is expected for a single quantum rotor. This constraint also tells us that the physical wave functions are periodic $\psi(\theta + 2\pi) = \psi(\theta)$. We are back to a compact variable θ .

In conclusion, for a single quantum rotor, this road map of constructing lattice Hamiltonian finally leads to the following result:

- (i) Hilbert space basis: $\{|\tilde{\theta}\rangle : \tilde{\theta} \in (-\infty, +\infty)\}$,
- (ii) Hamiltonian operator: $\hat{H} = \frac{1}{2\beta} \hat{p}^2$,
- (iii) Constraint on physical states: $e^{2\pi i \hat{p}} |\psi\rangle = |\psi\rangle$.

APPENDIX D: GENERAL ROAD MAP FROM LATTICE ACTION TO LATTICE HAMILTONIAN

From the example of deriving the Hamiltonian for a single quantum rotor, we can summarize the procedure to go from a time-discrete action to the Hamiltonian, involving the Villain approximation. The following steps are a conceptual summary of the procedure.

- (i) Start from a compact lattice action.
- (ii) Apply Villain approximation: add an integer degree of freedom for every cosine term in the action. Turn the cosine terms into quadratic.
- (iii) Gauge fix the variables to get rid of all gauge degrees of freedom. Absorb some integer degrees of freedom introduced earlier, to lift some compact variables into $(-\infty, +\infty)$.
- (iv) Set up the Hilbert space with the set of basis to be the configurations in a middle time layer.
- (v) Usually the first time layer (bottom layer) has fewer configurations. Duplicate the partition function to match the configuration space with the one in a middle layer. The degrees of freedom borrowed here are returned later.
- (vi) Usually the last time layer (top layer) has extra degrees of freedom. Summing over these degrees of freedom becomes constraints on the Hilbert space. Check that the number of extra degrees of freedom in the top layer equals to the borrowed degrees of freedom by the bottom layer.
- (vii) Write down the transfer matrix elements for a pair of neighboring middle layers and for the top layer.
- (viii) Fourier transform the transfer matrix elements for middle layers to see the Hamiltonian operator.
- (ix) Fourier transform the transfer matrix elements for the top layer to see the constraints on the physical Hilbert space.

Following this road map, one can derive the Hamiltonian of the three cases mentioned in the main text:

- (1) Pure Maxwell lattice theory with instantons $\rightarrow$ Eq. (20)
- (2) Pure Maxwell lattice theory without instantons $\rightarrow$ Eq. (21)
- (3) Maxwell-Chern-Simons lattice theory without instantons $\rightarrow$ Eq. (23)

and their corresponding Hilbert space constraints. Detailed summaries of the derivations are presented in the Appendix E, with full step-by-step calculations available in Sec. 5.7 of Ref. [82].

APPENDIX E: DERIVATION OF THE LATTICE HAMILTONIANS

In this Appendix, we summarize the key ideas and technical steps involved in deriving the three lattice Hamiltonians listed in the Appendix D. While the Hamiltonians for the pure Maxwell theory (both with

and without instantons) are known results, the transfer matrix derivation and the gauge-fixing procedure presented here are original to this work. These technical methods are essential for the derivation of our main result: the Hamiltonian of compact Maxwell-Chern-Simons theory in Sec. E 3. Full step-by-step calculations with all intermediate expressions and gauge-fixing figures can be found in Sec. 5.7 of Ref. [82].

1. Pure Maxwell with instantons

We start from the compact lattice Maxwell action on a 3D cubic lattice with periodic boundary conditions. The original action contains cosine terms $\cos(\text{curl of } A \text{ around each plaquette})$ for both timelike and spacelike plaquettes. After applying the Villain approximation (Appendix B), each cosine term $e^{\beta \cos \theta}$ is replaced by a periodic Gaussian $\sum_{n \in \mathbb{Z}} e^{-\tilde{\beta}(\theta + 2\pi n)^2/2}$, introducing one integer degree of freedom n per plaquette. The resulting action is quadratic in the combination (gauge field curl $+ 2\pi n$) on each plaquette.

We then gauge fix: first, local gauge transformations set certain gauge field components to zero along a maximal tree of the lattice; then, discrete gauge transformations absorb some of the integer degrees of freedom on nearby plaquettes, lifting the remaining compact gauge field variables into $(-\infty, +\infty)$. After gauge fixing, the remaining integer degrees of freedom on the plaquettes encode the instanton numbers in each cubic unit cell, as well as the total magnetic flux through each plane orientation of the 3D torus.

The transfer matrix is constructed between neighboring time slices. After gauge fixing, the timelike gauge fields and their associated integer degrees of freedom are eliminated, so the timelike plaquette terms in the action reduce to $(A' - A)^2$, where A' and A are the spatial gauge field configurations on two adjacent time slices. In the transfer matrix $\hat{T} = e^{-\hat{H}d\tau}$, Fourier transforming $(A' - A)$ into the conjugate momentum $\hat{p}$ yields the electric field energy $\frac{e^2}{2a^2} \hat{p}^2$. Crucially, summing over the instanton integers on the spacelike plaquettes in the partition function reverses the Villain approximation and wraps the quadratic magnetic terms back into the familiar cosine potential.

The Hilbert space constraints arise from the top layer of the lattice, which has extra unfixed degrees of freedom: the temporal gauge fields A_0 and the timelike integer plaquettes m . These are summed over in the partition function. Integrating over the $\mathbb{R}$ -ranged A_0 produces delta functions that enforce the lattice divergence of $\hat{p}$ to vanish at every site, which is the Gauss' law. Summing over the integers $m \in \mathbb{Z}$ enforces $e^{i2\pi \hat{p}/a} = 1$, i.e., $\hat{p} \in a\mathbb{Z}$ on the links under the extra timelike plaquettes. Since $\hat{p}$ is the conjugate momentum to $\hat{A}$, quantizing $\hat{p} \in a\mathbb{Z}$ means the gauge field $\hat{A}$ is periodic with period $2\pi/a$; i.e., the gauge field remains compact. Combined with Gauss' law, this compactness

condition propagates to all links. The resulting Hamiltonian is Eq. (20).

2. Pure Maxwell without instantons

The derivation proceeds identically up to the gauge-fixing step. The key difference is that the zero-instanton constraint eliminates the per-cube integer degrees of freedom. Only three global integer degrees of freedom remain, corresponding to the total magnetic flux through each plane orientation (one timelike, two spacelike).

Without the instanton sum, the spacelike plaquette terms cannot be wrapped back into cosine terms and remain quadratic, yielding the Hamiltonian in Eq. (21). The constraints on the Hilbert space now include Gauss' law and two additional 1-form (loop) operators, which arise from summing over the two spatial-plane integer degrees of freedom. The total flux quantum number n labels distinct sectors; despite the quadratic form, the theory remains compact due to these 1-form constraints.

3. Maxwell-Chern-Simons without instantons

Building on the transfer matrix method and gauge-fixing procedure developed for the pure Maxwell cases above, we now derive the main original result of this work: the lattice Hamiltonian for compact Maxwell-Chern-Simons theory.

The starting point is the combined action $S = S_{\text{Maxwell}} + S_{\text{CS}}$ in Eq. (22). The Chern-Simons action includes angular variables $\varphi \in [0, 2\pi)$ on each lattice site, which serve as Lagrange multipliers: summing over φ in the partition function enforces the total flux through every unit cell to be zero, thereby suppressing all instantons [see Eq. (11) and the discussion below it]. This instanton suppression is equivalent to eliminating the per-cube integer degrees of freedom, exactly as in the instanton-suppressed pure Maxwell case. Consequently, the gauge-fixing procedure and the resulting degree-of-freedom structure are identical: after gauge fixing, the physical degrees of freedom are $\mathbb{R}$ -ranged gauge fields on all spatial links, together with the same three global integer degrees of freedom (total flux through each plane orientation).

Compared to the pure Maxwell case, the transfer matrix element between two neighboring middle layers acquires an additional purely imaginary term from the Chern-Simons action. This term couples the spatial gauge field A on one time slice to the time difference $(A' - A)$ on the neighboring link in the perpendicular direction, taking the form $\frac{ika^2}{4\pi} \sum_x [-A_{x-1;1}(A'_{x;2} - A_{x;2}) + A_{x-2;2}(A'_{x;1} - A_{x;1})]$. When Fourier transforming $(A' - A)$ into the conjugate momentum $\hat{p}$, this imaginary cross term shifts the momentum as $\hat{p}_{x;i} \rightarrow \hat{p}_{x;i} - \epsilon_{ij} \frac{ka^2}{4\pi} \hat{A}_{x-\hat{j};j}$, which is the hallmark of the Chern-Simons modification.

The top-layer constraints are derived by the same procedure as in the pure Maxwell case—integrating over A_0 and summing over the timelike integers m_1, m_2 —but the

Chern-Simons action contributes additional terms that couple A_0 and m to the spatial gauge field operators $\hat{A}$. As a result, all three constraints are modified. First, integrating over A_0 now enforces a modified Gauss' law: the lattice divergence of $\hat{p}$ no longer vanishes by itself but is set equal to $-\frac{ka^2}{4\pi}\square\hat{A}$, the Chern-Simons correction [Eq. (25)]. Second, summing over m_1 and m_2 yields two modified 1-form operators $\hat{L}_1$ and $\hat{L}_2$: whereas in the pure Maxwell case these involve only sums of $\hat{p}$ along non-contractible dual loops, the Chern-Simons term attaches gauge field contributions $\hat{A}$ in the perpendicular direction to each $\hat{p}$ [Eqs. (28) and (30)].

A key result concerns the total magnetic flux n . To understand this, recall that during gauge fixing, each integer degree of freedom on a plaquette can be absorbed into a neighboring gauge field variable, lifting that variable from its original compact range to $(-\infty, +\infty)$. This absorption proceeds sequentially through the lattice. As we saw in the single quantum rotor model (Appendix C), at the end of such an absorption chain there is always one compact variable that cannot be lifted, because it encodes gauge-invariant global information of the Polyakov loop. The situation here is analogous: on the top layer, most temporal gauge fields A_0 absorb nearby integer degrees of freedom on timelike plaquettes and are lifted to $\mathbb{R}$; integrating these over $(-\infty, \infty)$ produces the Gauss' law delta functions discussed above. However, at the end of the absorption chain, the site $x = (0, 0)$ has no remaining integer degree of freedom available for absorption—all timelike integer plaquettes have already been absorbed by the other A_0 fields. Therefore, A_0 at $x = (0, 0)$ retains its original compact range $[0, 2\pi/d\tau]$, encoding the global

information of the temporal Polyakov loop. The modified Gauss' law at all other sites, together with the identity $\sum_x (\text{lattice divergence of } \hat{p} + \frac{ka^2}{4\pi}\square\hat{A}) = 0$, implies that the integrand at $x = (0, 0)$ reduces to $\exp(id\tau A_0 kn)$. For this integral over $[0, 2\pi/d\tau]$ to be nonvanishing, we need $kn = 0$; since $k \neq 0$, the total magnetic flux must be $n = 0$. Setting $n = 0$ yields exactly the Hamiltonian in Eq. (23).

For the complete derivations with all intermediate steps, including the detailed gauge-fixing figures and transfer matrix computations, see Sec. 5.7 of Ref. [82].

APPENDIX F: DETAILS OF ANALYTICAL SOLUTION

In this Appendix, we add some details about the analytical solution mentioned in Sec. IV B. In Eq. (90), we saw that in the momentum sector, $q = (q_1, q_2)$, the Fourier transformed Hamiltonian was

$$\begin{aligned} \hat{H}_q = & \frac{e^2}{2a^2} \left[\left(\hat{p}_{q;1} - \frac{ka^2}{4\pi} e^{iq_2} \hat{A}_{q;2} \right) \left(\hat{p}_{-q;1} - \frac{ka^2}{4\pi} e^{-iq_2} \hat{A}_{-q;2} \right) \right. \\ & + \left. \left(\hat{p}_{q;2} + \frac{ka^2}{4\pi} e^{iq_1} \hat{A}_{q;1} \right) \left(\hat{p}_{-q;2} + \frac{ka^2}{4\pi} e^{-iq_1} \hat{A}_{-q;1} \right) \right] \\ & + \frac{1}{2e^2} [(1 - e^{-iq_2}) \hat{A}_{q;1} - (1 - e^{-iq_1}) \hat{A}_{q;2}] \\ & \times [(1 - e^{iq_2}) \hat{A}_{-q;1} - (1 - e^{iq_1}) \hat{A}_{-q;2}]. \end{aligned} \quad (\text{F1})$$

We define the following change of variables (the operator notation $\hat{\cdot}$ is omitted for simplicity):

$$X = \frac{(1 - e^{-iq_2}) \hat{A}_{q;1} + (1 - e^{iq_2}) \hat{A}_{-q;1} - (1 - e^{-iq_1}) \hat{A}_{q;2} - (1 - e^{iq_1}) \hat{A}_{-q;2}}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F2})$$

$$\tilde{X} = \frac{(1 - e^{iq_1}) \hat{A}_{q;1} + (1 - e^{-iq_1}) \hat{A}_{-q;1} + (1 - e^{iq_2}) \hat{A}_{q;2} + (1 - e^{-iq_2}) \hat{A}_{-q;2}}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F3})$$

$$Y = \frac{(1 - e^{-iq_2}) \hat{A}_{q;1} - (1 - e^{iq_2}) \hat{A}_{-q;1} - (1 - e^{-iq_1}) \hat{A}_{q;2} + (1 - e^{iq_1}) \hat{A}_{-q;2}}{2i\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F4})$$

$$\tilde{Y} = \frac{(1 - e^{iq_1}) \hat{A}_{q;1} - (1 - e^{-iq_1}) \hat{A}_{-q;1} + (1 - e^{iq_2}) \hat{A}_{q;2} - (1 - e^{-iq_2}) \hat{A}_{-q;2}}{2i\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F5})$$

$$P_X = \frac{(1 - e^{-iq_2}) \hat{p}_{q;1} + (1 - e^{iq_2}) \hat{p}_{-q;1} - (1 - e^{-iq_1}) \hat{p}_{q;2} - (1 - e^{iq_1}) \hat{p}_{-q;2}}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F6})$$

$$\tilde{P}_X = \frac{(1 - e^{iq_1}) \hat{p}_{q;1} + (1 - e^{-iq_1}) \hat{p}_{-q;1} + (1 - e^{iq_2}) \hat{p}_{q;2} + (1 - e^{-iq_2}) \hat{p}_{-q;2}}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F7})$$

$$P_Y = \frac{(1 - e^{-iq_2})\hat{p}_{q;1} - (1 - e^{iq_2})\hat{p}_{-q;1} - (1 - e^{-iq_1})\hat{p}_{q;2} + (1 - e^{iq_1})\hat{p}_{-q;2}}{2i\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F8})$$

$$\tilde{P}_Y = \frac{(1 - e^{iq_1})\hat{p}_{q;1} - (1 - e^{-iq_1})\hat{p}_{-q;1} + (1 - e^{iq_2})\hat{p}_{q;2} - (1 - e^{-iq_2})\hat{p}_{-q;2}}{2i\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}. \quad (\text{F9})$$

Given the commutation relation in Eq. (89), one can check that $[X, P_X] = [\tilde{X}, \tilde{P}_X] = [Y, P_Y] = [\tilde{Y}, \tilde{P}_Y] = i$, and all other commutators are zero.

The Gauss' law Eq. (25) after the Fourier transform becomes

$$\left\{ (1 - e^{iq_1})\hat{p}_{q;1} + (1 - e^{iq_2})\hat{p}_{q;2} + \frac{ka^2}{4\pi} [(1 - e^{-iq_2})\hat{A}_{q;1} - (1 - e^{-iq_1})\hat{A}_{q;2}] \right\} |\psi\rangle = 0, \quad (\text{F10})$$

for any physical state $|\psi\rangle$ in the Hilbert space, and any $q \in \{0, \frac{2\pi}{N_1}, \dots, \frac{2\pi}{N_1}(N_1 - 1)\} \times \{0, \frac{2\pi}{N_2}, \dots, \frac{2\pi}{N_2}(N_2 - 1)\}$. Note that Eq. (F10) is trivially satisfied for the zero mode $q = (0, 0)$. The zero mode is restricted by the two 1-form constraints Eq. (28) and Eq. (30). We analyze the zero mode separately later. For nonzero modes, the two 1-form constraints do not introduce further restrictions.

Note that Eq. (F10) also works for $-q$, which says

$$\left\{ (1 - e^{-iq_1})\hat{p}_{-q;1} + (1 - e^{-iq_2})\hat{p}_{-q;2} + \frac{ka^2}{4\pi} [(1 - e^{iq_2})\hat{A}_{-q;1} - (1 - e^{iq_1})\hat{A}_{-q;2}] \right\} |\psi\rangle = 0. \quad (\text{F11})$$

Linearly combining the Gauss' law constraints for q and $-q$, [Eq. (F10) + Eq. (F11)] $\Rightarrow$

$$\left(\tilde{P}_X + \frac{ka^2}{4\pi} X \right) |\psi\rangle = 0, \quad (\text{F12})$$

and [Eq. (F10) - Eq. (F11)] $\Rightarrow$

$$\left(\tilde{P}_Y + \frac{ka^2}{4\pi} Y \right) |\psi\rangle = 0. \quad (\text{F13})$$

We then do another change of variables,

$$Q_X = P_X + \frac{ka^2}{4\pi} \tilde{X}, \quad (\text{F14})$$

$$\tilde{Q}_X = \tilde{P}_X + \frac{ka^2}{4\pi} X, \quad (\text{F15})$$

$$Q_Y = P_Y + \frac{ka^2}{4\pi} \tilde{Y}, \quad (\text{F16})$$

$$\tilde{Q}_Y = \tilde{P}_Y + \frac{ka^2}{4\pi} Y. \quad (\text{F17})$$

One can check that $[X, Q_X] = [\tilde{X}, \tilde{Q}_X] = [Y, Q_Y] = [\tilde{Y}, \tilde{Q}_Y] = i$, and all other commutators are zero. With the new variables, the Gauss' law constraints Eq. (F12) and Eq. (F13) is further simplified,

$$\tilde{Q}_X |\psi\rangle = 0, \quad (\text{F18})$$

$$\tilde{Q}_Y |\psi\rangle = 0. \quad (\text{F19})$$

The inverse change of variables is

$$\hat{A}_{q;1} = \frac{(1 - e^{iq_2})(X + iY) + (1 - e^{-iq_1})(\tilde{X} + i\tilde{Y})}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F20})$$

$$\hat{A}_{-q;1} = \frac{(1 - e^{-iq_2})(X - iY) + (1 - e^{iq_1})(\tilde{X} - i\tilde{Y})}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F21})$$

$$\hat{A}_{q;2} = \frac{-(1 - e^{iq_1})(X + iY) + (1 - e^{-iq_2})(\tilde{X} + i\tilde{Y})}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F22})$$

$$\hat{A}_{-q;2} = \frac{-(1 - e^{-iq_1})(X - iY) + (1 - e^{iq_2})(\tilde{X} - i\tilde{Y})}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F23})$$

$$\hat{p}_{q;1} = \frac{(1 - e^{iq_2})(P_X + iP_Y) + (1 - e^{-iq_1})(\tilde{P}_X + i\tilde{P}_Y)}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F24})$$

$$\hat{p}_{-q;1} = \frac{(1 - e^{-iq_2})(P_X - iP_Y) + (1 - e^{iq_1})(\tilde{P}_X - i\tilde{P}_Y)}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F25})$$

$$\hat{p}_{q;2} = \frac{-(1 - e^{iq_1})(P_X + iP_Y) + (1 - e^{-iq_2})(\tilde{P}_X + i\tilde{P}_Y)}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F26})$$

$$\hat{p}_{-q;2} = \frac{-(1 - e^{-iq_1})(P_X - iP_Y) + (1 - e^{iq_2})(\tilde{P}_X - i\tilde{P}_Y)}{2\sqrt{(1 - \cos q_1) + (1 - \cos q_2)}}, \quad (\text{F27})$$

$$P_X = Q_X - \frac{ka^2}{4\pi} \tilde{X}, \quad (\text{F28})$$

$$\tilde{P}_X = \tilde{Q}_X - \frac{ka^2}{4\pi} X, \quad (\text{F29})$$

$$P_Y = Q_Y - \frac{ka^2}{4\pi} \tilde{Y}, \quad (\text{F30})$$

$$\tilde{P}_Y = \tilde{Q}_Y - \frac{ka^2}{4\pi} Y. \quad (\text{F31})$$

Inserting everything into the Fourier transformed Hamiltonian in Eq. (F1), after some algebra, we get

$$\begin{aligned} \hat{H}_q = & \frac{a^2}{4e^2} \left\{ \frac{1}{a^2} [2(1 - \cos q_1) + 2(1 - \cos q_2)] \right. \\ & + \left(\frac{ke^2}{4\pi} \right)^2 [2 + 2 \cos(q_1 + q_2)] \} (X^2 + Y^2) \\ & + \frac{e^2}{4a^2} \left\{ Q_X^2 + Q_Y^2 + \tilde{Q}_X^2 + \tilde{Q}_Y^2 \right. \\ & - \frac{ka^2}{4\pi} [2(1 - \cos q_1) + 2(1 - \cos q_2)] (X\tilde{Q}_X + Y\tilde{Q}_Y) \\ & \left. + \frac{ka^2}{4\pi} 2 \sin(q_1 + q_2) (Y\tilde{Q}_X - X\tilde{Q}_Y) \right\}. \quad (\text{F32}) \end{aligned}$$

Applying the Gauss' law constraints Eq. (F18) and Eq. (F19), we can set $\tilde{Q}_X = \tilde{Q}_Y = 0$. We define an angular frequency, which also appeared in Eq. (91) in the main text,

$$\begin{aligned} \omega^2 = & \frac{1}{a^2} [2(1 - \cos q_1) + 2(1 - \cos q_2)] \\ & + \left(\frac{ke^2}{4\pi} \right)^2 [2 + 2 \cos(q_1 + q_2)]. \quad (\text{F33}) \end{aligned}$$

The Hamiltonian in sector q finally looks simple and familiar,

$$\hat{H}_q = \frac{e^2}{4a^2} (Q_X^2 + Q_Y^2) + \frac{a^2}{4e^2} \omega^2 (X^2 + Y^2). \quad (\text{F34})$$

Note that we actually have combined the q and $-q$ sectors together in the analysis above. On both sectors, the total Hamiltonian is

$$\hat{H}_q + \hat{H}_{-q} = \frac{e^2}{2a^2} (Q_X^2 + Q_Y^2) + \frac{a^2}{2e^2} \omega^2 (X^2 + Y^2). \quad (\text{F35})$$

Simply rescaling the momentum and coordinate, we see that the q and $-q$ sectors become two decoupled simple harmonic oscillators, both having the same angular frequency ω described in Eq. (F33).

In the special case when $q = (0, \pi)$, $(\pi, 0)$, or (π, π) , i.e., when q and $-q$ are identical, one can proceed a similar analysis with half of the degrees of freedom (note that all Y -related variables become zero). The result is that the sector is equivalent to one simple harmonic oscillator with an angular frequency given by Eq. (F33) as well (without the Y, Q_Y degrees of freedom). Since the angular frequency for these special momentum values is given by the same equation, the band structure is smooth and has no singularity on these special momentum values.

The zero mode $q = (0, 0)$, however, needs to be treated separately.

First, for $q = (0, 0)$, the Hamiltonian in Eq. (F1) does not have the magnetic potential term, and the electric kinetic terms are quadratic,

$$\begin{aligned} \hat{H}_{q=(0,0)} = & \frac{e^2}{2a^2} \left[\left(\hat{p}_{q=(0,0);1} - \frac{ka^2}{4\pi} \hat{A}_{q=(0,0);2} \right)^2 \right. \\ & \left. + \left(\hat{p}_{q=(0,0);2} + \frac{ka^2}{4\pi} \hat{A}_{q=(0,0);1} \right)^2 \right]. \quad (\text{F36}) \end{aligned}$$

Second, the Gauss' law constraint acts trivially on the zero mode. However, the two 1-form constraints Eq. (28) and Eq. (30) act nontrivially on the zero mode,

$$\exp \left[i2\pi \sqrt{\frac{N_1}{N_2}} \left(\frac{1}{a} \hat{p}_{q=(0,0);2} - \frac{ka}{4\pi} \hat{A}_{q=(0,0);1} \right) \right] |\psi\rangle = e^{i\theta_1} |\psi\rangle, \quad (\text{F37})$$

$$\exp \left[i2\pi \sqrt{\frac{N_2}{N_1}} \left(\frac{1}{a} \hat{p}_{q=(0,0);1} + \frac{ka}{4\pi} \hat{A}_{q=(0,0);2} \right) \right] |\psi\rangle = e^{i\theta_2} |\psi\rangle, \quad (\text{F38})$$

for any physical state $|\psi\rangle$ in the Hilbert space. Here N_1, N_2 are the number of lattice sites in the 1 and 2 directions, respectively. θ_1 and θ_2 are two constant global phase angles.

Third, the change of variables defined earlier does not work for the zero mode. We need to define a separate change of variables for the zero mode purpose (again, the operator notation $\hat{\cdot}$ is omitted for simplicity),

$$X_0 = \frac{\hat{A}_{q=(0,0);1} + \frac{4\pi}{ka^2} \hat{p}_{q=(0,0);2}}{\sqrt{2}}, \quad (\text{F39})$$

$$Y_0 = \frac{\hat{A}_{q=(0,0);1} - \frac{4\pi}{ka^2} \hat{p}_{q=(0,0);2}}{\sqrt{2}}, \quad (\text{F40})$$

$$P_{X_0} = \frac{\hat{p}_{q=(0,0);1} - \frac{ka^2}{4\pi} \hat{A}_{q=(0,0);2}}{\sqrt{2}}, \quad (\text{F41})$$

$$P_{Y_0} = \frac{\hat{p}_{q=(0,0);1} + \frac{ka^2}{4\pi} \hat{A}_{q=(0,0);2}}{\sqrt{2}}. \quad (\text{F42})$$

One can check that $[X_0, P_{X_0}] = [Y_0, P_{Y_0}] = i$, and all other commutators are zero.

Inserting the change of variables, the zero mode Hamiltonian in Eq. (F36) becomes

$$\begin{aligned}\hat{H}_{q=(0,0)} &= \frac{e^2}{2a^2} \cdot 2 \left[P_{X_0}^2 + \left(\frac{ka^2}{4\pi} \right)^2 X_0^2 \right] \\ &= \frac{1}{2} \left(\frac{2e^2}{a^2} \right) P_{X_0}^2 + \frac{1}{2} \left(\frac{a^2}{2e^2} \right) \left(\frac{ke^2}{2\pi} \right)^2 X_0^2, \quad (\text{F43})\end{aligned}$$

which can be recognized as a simple harmonic oscillator with effective mass $\frac{a^2}{2e^2}$ and the angular frequency

$$\omega_0 = \frac{ke^2}{2\pi}. \quad (\text{F44})$$

This is the same result as inserting $q_1 = q_2 = 0$ into the band dispersion Eq. (F33). Therefore, the band is smooth and has no singularity at $q = (0, 0)$ either.

What makes the zero mode special is that the constraints do not remove the degrees of freedom like in Eq. (F18) and Eq. (F19) for the nonzero modes. Instead, the constraints for the zero mode, Eq. (F37) and Eq. (F38), after change of variables, say

$$\exp \left(i2\pi \sqrt{\frac{N_1}{N_2}} \frac{ka}{4\pi} \sqrt{2} Y_0 \right) |\psi\rangle = e^{i\theta_1} |\psi\rangle, \quad (\text{F45})$$

$$\exp \left(i2\pi \sqrt{\frac{N_2}{N_1}} \frac{1}{a} \sqrt{2} P_{Y_0} \right) |\psi\rangle = e^{i\theta_2} |\psi\rangle, \quad (\text{F46})$$

for any physical state $|\psi\rangle$ in the Hilbert space.

To make things cleaner, define a constant $b = \sqrt{\frac{N_2}{N_1}} \frac{4\pi}{ka} \frac{1}{\sqrt{2}}$. We rewrite the constraints above and get

$$e^{i2\pi Y_0/b} |\psi\rangle = e^{i\theta_1} |\psi\rangle, \quad (\text{F47})$$

$$e^{ikbP_{Y_0}} |\psi\rangle = e^{i\theta_2} |\psi\rangle, \quad (\text{F48})$$

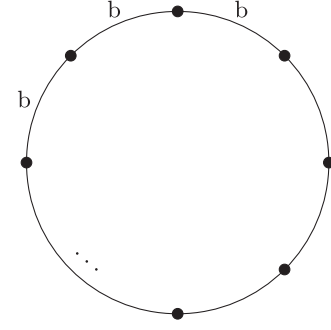


FIG. 8. Illustration of the zero mode degree of freedom which generates the degenerate Hilbert space. It is equivalent to the tight-binding model of a particle on a ring of k equally spaced sites, with distance b between neighboring sites and total perimeter kb .

which implies the spectrum of the operator $Y_0 \in b\mathbb{Z} + \frac{b\theta_1}{2\pi}$, and the spectrum of the operator $P_{Y_0} \in \frac{2\pi}{kb}\mathbb{Z} + \frac{\theta_2}{kb}$. The latter further implies that, in the Y_0 basis, the wave function has to be periodic with the period kb , i.e., $\psi(Y_0 + kb) = e^{i\theta_2} \psi(Y_0)$.

These constraints are very familiar to us. They are equivalent to the tight-binding model of a single quantum-mechanical particle on a discrete ring of k sites, with lattice spacing b between neighboring sites and total perimeter kb . See Fig. 8 for an illustration.

Because Y_0 does not appear in the Hamiltonian in Eq. (F43), it is a decoupled degree of freedom with zero Hamiltonian. Therefore, its Hilbert space is a degenerate subspace and is direct product to the other Hilbert spaces of the modes on the band structure.

Because the number of sites on the ring has to be an integer, we naturally get the quantization of k [otherwise the discreteness $Y_0 \in b\mathbb{Z} + \frac{b\theta_1}{2\pi}$ and the periodicity $\psi(Y_0 + kb) = e^{i\theta_2} \psi(Y_0)$ are incompatible] The number of independent states in the degenerate Hilbert subspace equals to the number of sites on the ring. Therefore, we also see the k -fold degeneracy of states from this analysis.

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