

Attracting without being attracted: Dark matter as an aether wind

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(Received 9 May 2025; accepted 20 January 2026; published 10 February 2026)

We explore the possibility that part of what we call dark matter may be the mark of a preferred frame, revealing a breakdown of diffeomorphism invariance. In the nonrelativistic limit, this appears as a deviant matter source capable of attracting normal matter, but not feeling the attraction from other forms of matter or from itself. While this implies a violation of momentum conservation, no logical inconsistencies arise in this deviant “Newtonian” limit. In contrast, due to Bianchi identities, the relativistic theory must undergo core change, and we discuss a modification of Einstein’s gravity capable of coupling a nonconserved source to gravity. It results from fixing some of the spatial components of the metric, thereby constraining the possible diffeomorphisms and clipping some of the equations. Bianchi identities can always be used to refill the equations, but the effective Stückelberg stresses are so outlandish that this defines symmetry breakdown and violations of local energy-momentum conservation. We work out spherically symmetric solutions with static halos and flat rotation curves, with and without a central black hole. The model has the drawback that it can evade experimental constraints simply by setting to zero the local density of deviant matter (which is a nondynamic input). Its presence, in contrast, would leave inimitable signatures. We briefly discuss the Hamiltonian formulation of these models, where such dark matter appears as a central charge in the Poisson bracket of the Hamiltonian and the momentum.

DOI: [10.1103/rr4j-m53n](https://doi.org/10.1103/rr4j-m53n)

I. INTRODUCTION

For all its undeniable observational successes, the dark matter paradigm cannot fend off the impression that what we call dark matter might be a mere phenomenological proxy for a more fundamental theoretical construction. The model is also not without observational shortcomings. A case in point is galactic-scale dark matter. Dark matter explains observed gravitational pull by means of non-observed dark gravitational sources, but, for consistency, the dark sources should stay where they are needed under their own gravitational pull and that of visible forms of matter. This does not always happen without contortions, for example in the case of galactic halos (e.g., fuzzy dark matter [1] and triaxial bars [2], among others).

None of this caused undue consternation at the inception of the dark matter paradigm; indeed the *ad hoc* phenomenological nature of dark matter was initially cherished. At its most extreme (and possibly facetiously), the idea of

painted-on dark matter was informally floated: a *subdominant* dustlike component capable of attracting visible matter, but staying put where it was required (hence the moniker “painted on”), oblivious to its own gravity or that of other sources. Regardless of the phenomenological merits of this view, it will provide the broad inspiration for this paper. Could such subdominant painted-on dark matter be encrusted in, and so be the tell-tale signature of a preferred frame in the Universe? There is nothing more fundamental than deciding whether such a frame exists: this is a throwback to the old aether quest. Combining such a foundational issue with dark matter could therefore be a step forward in understanding its true nature.

Implementing such an idea in the context of general relativity, however, is far from trivial. Within the Newtonian perspective/limit there is *a priori* nothing inconsistent with an object attracting but not being attracted. It contradicts Newton’s third law and so leads to violations of momentum conservation, but no logical inconsistency arises. Indeed, at the Newtonian level, painted-on dark matter can be introduced by directly positing a dustlike fluid with active gravitational mass ($m_a \neq 0$) but no passive gravitational mass ($m_p = 0$), or alternatively to the latter, infinite inertial mass ($m_i = \infty$). Such a fluid would produce gravity according to the usual Newton law of gravity, but feel no force of gravity if $m_p = 0$, or fail to be budged by any force if $m_i = \infty$.

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The situation is dramatically different in general relativity, because concepts such as inertial mass and (active and passive) gravitational mass disappear, and are traded for geodesic motion for point particles. Crucially, such geodesic motion is not a postulate, but is implied by stress-energy tensor conservation, itself automatically enforced by the Bianchi identities for the Einstein tensor $G_{\mu\nu}$ and the gravitational field equations. Inserting painted-on dark matter (or any other matter source violating local energy-momentum conservation) into the Einstein equations leads to a logical contradiction.

There are several attitudes one can take towards this. One is to accept that an entirely different theory of gravity needs to be built to accommodate a nonconserved source. Another is to ask whether general relativity can be minimally modified to accommodate such an alien concept. In this paper, we take the latter route, skirting the perils of trivial solutions. If we do not radically depart from general relativity, the theory wants to force on us a solution effectively amounting to a standard conserved source, e.g., an equivalent of fuzzy dark matter. This threat will be present throughout this paper, but will be avoided by making such equivalent completions, reinstating conservation, sufficiently ludicrous. This is closely related to the Stückelberg method for reinstating symmetry and will be discussed later in this paper.

A bit of background. Our approach follows from a series of papers where the Hamiltonian constraint is violated in some regimes (high energy, early Universe, etc.) only to be restored in others. Invariably such symmetry restoration cannot avoid an indelible memory of the broken phase in the form of something that looks like, but is not dark matter. This happens in Hořava-Lifshitz theory [3,4], in theories where there is evolution (or time dependence) in the laws of physics [5,6], or as an outcome of quantization [7–10]. A passing connection with mimetic dark matter exists [11]. Finally, the fact that this effect can result from the interaction between a global, presiding Universe and local physics led to this component being called “Machian” dark matter in Ref. [12].

We acknowledge all of this past work, but stress that the novelty here is that we are proposing a permanent breakdown of diffeomorphism invariance wherever such violations of the Hamiltonian constraint took place. An analogy (in reverse) with topological defects will be drawn in the concluding remarks of this paper. We remind the reader that such symmetry breaking is not entirely new and has appeared in several (other) guises in past literature (e.g., Refs. [13–15]).

Throughout this paper we will use units such that $G = 1$ (rather than $8\pi G = 1$).

II. NONCONSERVED SOURCES AND WHAT THEY IMPOSE ON GRAVITY

Given our intention to abandon the redundancy of local energy-momentum conservation for “normal” matter due to

Bianchi identities, we start by imposing it as a separate postulate:

$$\nabla_\mu T^\mu{}_\Delta = 0. \quad (1)$$

We then add to normal matter a component of nonconserved matter (additively, so that the total is $T_{\mu\nu} = T^\mu{}_\Delta + T^\mu{}_M$). In this paper, we choose a component with energy-momentum tensor of the form:

$$T^\mu{}_\Delta = \frac{\Delta(x)}{\sqrt{h}} n^\mu n^\nu, \quad (2)$$

where x is a three-dimensional spatial coordinate on the preferred foliation Σ_t , and n^μ , $h = \det[h_{ij}]$ and other Arnowitt–Deser–Misner 3 + 1 variables to be used in this paper (e.g., $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$) follow the conventions and notation in Refs. [33,35]. Here, the normal to the preferred frame n^μ does not need to be geodesic (so the acceleration of the foliation, $a_\mu = n^\nu \nabla_\nu n^\mu$ can be nonzero). Importantly, $\Delta(x)$ is nondynamical: it is painted on the preferred frame. The reason for this choice is that it connects with historical/high-energy violations of the Hamiltonian constraint [4–7,9,11,12], with $\Delta(x)$ representing the value of the leftover Hamiltonian, once the driving mechanism switches off. The connection with the Hamiltonian framework will be explored in greater detail in Sec. IX.¹

We see that

$$n_\mu \nabla_\nu T^\mu{}_\Delta = \frac{1}{\sqrt{-g}} \partial_i (N^i \Delta) \quad (3)$$

$$h_{i\mu} \nabla_\nu T^\mu{}_\Delta = \frac{1}{\sqrt{-g}} \Delta \partial_i N. \quad (4)$$

This may be proved directly, or else using the Hamiltonian framework, as was done in Refs. [5,6] [see also Eqs. (62) and (63) later in this paper]. The question is what can the gravitational field equations be, given these matter sources? We may consider a lhs with $H_{\mu\nu} \neq G_{\mu\nu}$ but closely related to $G_{\mu\nu}$ (for example, preserving some of its components but not others), so that the full set of independent equations reads

$$H^{\mu\nu} = 8\pi(T^\mu{}_\Delta + T^\mu{}_M), \quad (5)$$

$$\nabla_\mu T^\mu{}_\Delta = 0. \quad (6)$$

This requires that $H_{\mu\nu}$ violate Bianchi identities accordingly:

¹We note that unlike Hořava–Lifshitz (HL) theory [3], this theory never suffers the threat of Ostrogradski instabilities, since $\Delta(x)$ is nondynamical. In this sense, our model is technically closer to the Cuscuton model [36] than to HL or dynamical Einstein-aether theory [37].

$$n_\mu \nabla_\nu H^{\mu\nu} = \frac{8\pi}{\sqrt{-g}} \partial_i (N^i \Delta), \quad (7)$$

$$h_{i\mu} \nabla_\nu H^{\mu\nu} = \frac{8\pi}{\sqrt{-g}} \Delta \partial_i N. \quad (8)$$

There is an infinite number of such $H_{\mu\nu}$ and, as already flagged in the Introduction, we must avoid the trivial ones: those that simply result in $G_{\mu\nu}$ plus terms that effectively endow $T_\Delta^{\mu\nu}$ with standard local dynamical variables, rendering it a conserved matter source. In such a case, we would be back to square one: fuzzy dark matter or similar, in the context of galactic halos. As interesting as these approaches are, they are not the purpose of this paper.

III. CLIPPING

A possible solution is to decrease the number of Einstein equations by clipping some of them, reducing in tandem the number of metric variables, to keep the system well defined. This can break the contradiction between a non-conserved source and the Bianchi identities. In general, the new field equations will be

$$P_{\mu\nu}{}^{\alpha\beta} [G_{\alpha\beta} - 8\pi(T_{\alpha\beta}^\Delta + T_{\alpha\beta}^M)] = 0, \quad (9)$$

$$\nabla_\mu T_M^{\mu\nu} = 0, \quad (10)$$

where $P_{\mu\nu}{}^{\alpha\beta}$ projects out, or clips some components. In general the projector takes the form

$$P_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta \pm N_{\mu\nu} N^{\alpha\beta}, \quad (11)$$

(the sign depending on the nature of the clipping), with $N^{\mu\nu}$ encoding the components to be clipped.

The process of clipping could arise from constraining the metric *before* variations of the Einstein-Hilbert action are taken. Let the conditions (indexed by i) upon the metric take the general form

$$F_i(g_{\mu\nu}) = 0, \quad (12)$$

which need not be covariant and could involve differential operators. These conditions can be seen as the result of removing some diffeomorphisms. Specifically, for an infinitesimal diffeomorphism defined by $x^\mu \rightarrow x^\mu + \epsilon \xi^\mu$ (with $\epsilon \ll 1$), since the metric transforms according to the Lie derivative with respect to generator ξ^μ , condition (12) translates into a condition on the allowed generators, as follows:

$$F_i(\xi_{\mu;\nu}) = 0. \quad (13)$$

This is similar but fundamentally different from gauge fixing, in that it states that the corresponding gauge

symmetry never existed due to Δ . Furthermore, we insist on preserving full symmetry for normal matter, so that Eq. (1) remains valid.

Such clipping is the most conservative option (in the sense of the option closest to the spirit of general relativity). But we could also clip an Einstein equation arising from varying the action with respect to a different metric component than the one we have frozen. This is much less conservative. It would require a revision or even a definite abandonment of the action principle formulation. We will present one such example in Appendix.

Such clippings are not new. For example, unimodular gravity [16–24] can be built from

$$P_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta}, \quad (14)$$

(one does not vary with respect to the trace of the metric), whereas projectable Hořava-Lifshitz theory [3,4] and similar theories of evolution [5] employ

$$P_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - n_\mu n_\nu n^\alpha n^\beta, \quad (15)$$

(one does not vary with respect to a local lapse function defined on the preferred frame Σ_t). For a nonprojectable version of the theory [6], the momentum constraint is also removed, as follows:

$$P_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - n_{(\mu} h_{\nu)}^\gamma n^\alpha h_\gamma^\beta. \quad (16)$$

In these known examples, we find that, having *clipped* the Einstein equations, we can *refill* them using the Bianchi identities and Eq. (1), with the procedure adding to the equations an integration constant $S_{\mu\nu}$, as follows:

$$G_{\mu\nu} = 8\pi(T_{\mu\nu} + S_{\mu\nu}), \quad (17)$$

(where $T_{\mu\nu}$ is the total stress-energy tensor, containing normal matter and Δ). The new $S_{\mu\nu}$ satisfies

$$P_{\mu\nu}{}^{\alpha\beta} S_{\alpha\beta} = 0, \quad (18)$$

$$\nabla_\mu S^{\mu\nu} = -\nabla_\mu T^{\mu\nu}. \quad (19)$$

This integration constant is the cosmological constant Λ in the first case, a dustlike fluid in the second, and the unusual fluid described in Ref. [6] in the third case. In all these cases, Eq. (19) vanishes: as already stressed in the Introduction, this is one aspect in which painted-on matter is different from previous literature. We can interpret $S_{\mu\nu}$ as the Stückelberg fields restoring full diffeomorphisms invariance (for this interpretation regarding unimodular gravity, see Ref. [25]), and this will be critically appraised in our case later in this paper.

IV. CLIPPERS FOR PAINTED-ON MATTER

Our case is different in that we start from a pregiven nondynamical $\Delta(x)$ on Σ_t (and so, Δ cannot be an integration constant due to refilling), implying that we cannot clip the Hamiltonian constraint (as in Refs. [4,5]), which is the only place where Δ appears. Hence,

$$P_{\mu\nu}{}^{\alpha\beta} T_{\alpha\beta}^{\Delta} = T_{\alpha\beta}^{\Delta}, \quad (20)$$

ruling out a term of the form (15). Instead, clipping is introduced in our case to remove the algebraic inconsistency arising from the nonconservation of $T_{\alpha\beta}^{\Delta}$ and the Bianchi identities for the unclipped Einstein equations.

Furthermore, since Δ is a nondynamical given, it does not need to be the most general field, that is, we can impose any conditions on Δ that we wish, each case corresponding to a different theory.² For example, we could impose that it must be a constant or piecewise constant,³ with regions of constant $\Delta \neq 0$ (in some spatial coordinates) acting as a defect in diffeomorphism invariance. This would give standard cold dark matter (CDM) if implemented on a cosmological scale ([4,5]; the cosmological frame is geodesic). On a particular nongeodesic frame, it would be something akin to painted-on galactic halos, as explored in Sec. VII, particularly Sec. VII A. This is a possibility and an interesting example, but many other options could be explored. At the most extreme, we could even require that the regions with constant $\Delta \neq 0$ be spherical.

Bearing these two important points in mind, we then note that not all clippers enforce consistency. For example, building the clipper from tensor products of $D_i \Delta$ would not work, since a covariantly constant Δ can lead to violations. Reciprocally, there is typically more than one set of clippers that can enforce consistency (for a given Δ). Each choice leads to a different theory.

For the sake of concreteness, we will explore one possible clipping mechanism (and defer to Sec. IX and further work, the Hamiltonian recipes for clipping). Face value, there are 2 degrees of freedom in the violations of local energy conservation (3) and (4), so one might think that two Einstein equations need to be clipped.⁴ As we will see, Eq. (3) does not appear in the Newtonian limit or in the leading post-Newtonian terms. In addition, it can be dealt with by fixing one of the shift functions N^i such that $\partial_i(N^i \Delta) = 0$. This restricts diffeomorphisms by

$$\partial_i[\Delta(x)N^2(\xi^{0;i} + N^i \xi^{0;0})] = 0, \quad (21)$$

²At its most basic, we could impose it to be C_0^2 , but simply having compact support is enough in many cases.

³Note that this condition explicitly breaks spatial diffeomorphism invariance, since Δ is a density, and a constant density is not covariantly constant. The statement is pegged to preferred three-dimensional coordinate frames.

⁴This may be overcounting, as the Hamiltonian perspective shows, but that is not the point here.

and this procedure, by refilling leads to the fluid in Ref. [6]. This is not where the novelty in this paper lies.

Let us focus instead on Eq. (4). Given that we are breaking four-dimensional diffeomorphism invariance, we can clip with purely spatial structures on Σ_t [that is, clip spatial Einstein equations, or equations that result from Hamilton's equations, rather than the constraints, as we have done for Eq. (3)]. An obvious structure on Σ_t is formed by the iso-lapse surfaces, that is, the two-dimensional surfaces Σ_N of constant lapse function N . [Note that if N is constant on Σ_t , there are no violations (4), the frame is geodesic, and this is just [4,5].] The normal to such two-dimensional spatial surfaces Σ_N is determined by the spatial part of the acceleration of Σ_t , $a_\mu = n^\alpha \nabla_\alpha n_\mu$, with $a_i = \partial_i \log N$. The clipper should therefore employ a normalized version of this acceleration:

$$u_\mu \equiv \frac{a_\mu}{\sqrt{a_\mu a^\mu}}. \quad (22)$$

We can also do a 2 + 1 Arnowitt–Deser–Misner decomposition based on surfaces Σ_N with lapse M , defining the normal $u_\mu = (-M, 0)$.

We can now define a variety of clippers based on u_μ . Each prescription leads to a different theory. In analogy with the unimodular cosmological constant, we could, for example, employ a transverse clipper:

$$P_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - \frac{1}{2} {}^{(2)}h_{\mu\nu} {}^{(2)}h^{\alpha\beta}, \quad (23)$$

with ${}^{(2)}h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu - u_\mu u_\nu$ the induced metric on the surfaces of constant N . Or we could use a longitudinal clipper

$$P_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - u_\mu u_\nu u^\alpha u^\beta, \quad (24)$$

or a combination of the two, or even more complex structures if need be.⁵ In this paper, we are not wedded to any of these theories but would rather explore the lot of them.

Returning to the second point made at the start of this section, we stress that there is no *a priori* reason why such clippers would enforce consistency for the most general Δ , but as pointed out, Δ does not need to be general. We can always impose a condition upon Δ such that each of these clippers is consistent.

⁵We have checked consistency in all applications presented in this paper, but given they mostly concern parameterized post-newtonian (PPN) and spherically symmetric situations, we could add ellipses to these clippers, to point out that extra terms might appear beyond these regimes.

V. CRITICISM OF REFILLING FOR PAINTED-ON MATTER

The case in hand is also different (compared to other clippings found in the literature) in that refilling now produces not $T_{\mu\nu}^\Delta$, but rather the relevant momentum and stresses to render it conserved. The integration constant $S_{\mu\nu}$ in Eq. (17) (the analog of Λ in unimodular gravity) is not Δ itself, but the spatial pressures and anisotropic stresses (and, to second order, the momentum as in Ref. [6]) required to render it conserved. This is an expression of the *superficial* power of Bianchi identities. If we start by accepting some components of the Einstein tensor and clip others (fewer than four), nothing can stop us, modulo technicalities, from refilling the latter by integrating the Bianchi identities, resulting in an integration constant to be inserted in the refilled Einstein equations. The integration constant can be thought of as the Stückelberg field restoring full diffeomorphism invariance, and in our case, stress-energy-momentum conservation.

But this can be misleading and even meaningless. Indeed, by making diffeomorphism invariance (and local energy-momentum conservation) *always* possible, the Bianchi-Stückelberg procedure has become content-free by itself. The situation is similar to the Einstein-Friedrichs debate [26]. Einstein used general covariance as the guiding principle leading to general relativity; however, any theory can be made generally covariant, including Galilean theory, as shown in Ref. [26]. The difference is that some theories (such as Galilean theories) require the introduction of forceful and artificial structures (effectively the Stückelberg fields adapted to Galilean theories). The absence of such structures is what sets apart general relativity and other truly diffeomorphism invariant theories.

Likewise, in our context, we say that a theory has violations of diffeomorphism invariance and of local energy-momentum conservation if the $S_{\mu\nu}$ that return the theory to conservation are forceful/artificial, either by satisfying a nonlocal dynamics, or by breaking any reasonable energy condition, or by implying stresses depending on all other fluids and global environment [rather than a traditional equation of state, where the stresses would be local functions of $\Delta(x)$], or any such preposterous property. In our case, for a projector $P_{\mu\nu}^{\alpha\beta}$, the Stückelberg pressures $S^{\mu\nu}$ must satisfy

$$P_{\mu\nu}^{\alpha\beta} S_{\alpha\beta} = 0, \quad (25)$$

$$\nabla_\mu S^{\mu\nu} = -\nabla_\mu T_\Delta^{\mu\nu}, \quad (26)$$

with decompositions (7) and (8) for the rhs of Eq. (26). Hence, $S_{\mu\nu}$ depends nonlocally on all types of matter, not just locally on Δ (or else it can depend locally on Δ via a severely nonminimal coupling to gravity). This is not an equation of state as for a fluid, nor a local dynamics as for a

scalar field. In fact, the general form of the equations for the projectors discussed above is so cumbersome that we will not present them here (even though we will discuss their form in the spherically symmetric cases in Sec. VII and Appendix).

Note that using the tensor defined in Eq. (5), the field $S_{\mu\nu}$ reads (or is defined as)

$$H_{\mu\nu} = G_{\mu\nu} - 8\pi S_{\mu\nu}. \quad (27)$$

VI. NEWTONIAN AND POST-NEWTONIAN PERSPECTIVE

As already pointed out in the introduction, in the Newtonian theory, there is nothing self-contradictory about a source that attracts but is not attracted (with consequent momentum conservation violation). This can be seen in the Newtonian limit of a relativistic theory using the post-Newtonian expansion. Let the lapse function N and spatial metric h_{ij} admit expansions in $1/c$ (where c is the speed of light) of the form:

$$N = 1 + \frac{\phi}{c^2} + \mathcal{O}(c^{-4}), \quad (28)$$

$$h_{ij} = \gamma_{ij} \left(1 - \frac{2\psi}{c^2} \right) + \mathcal{O}(c^{-4}), \quad (29)$$

where ϕ and ψ are two post-Newtonian potentials, and γ_{ij} is the unperturbed flat metric in a generic coordinate system. Then the Einstein tensor $G^{\mu\nu}$ can be expanded accordingly as

$$G^{00} = \frac{2}{c^2} \Delta\psi + \mathcal{O}(c^{-4}),$$

$$G^{0i} = \frac{2}{c^3} [\Delta\zeta^i - D^i D_j \zeta^j - D^i \dot{\psi}] + \mathcal{O}(c^{-5}),$$

$$G^{ij} = -\frac{1}{c^2} [\Delta(\phi - \psi) \gamma^{ij} - D^i D^j (\phi - \psi)] + \mathcal{O}(c^{-4}).$$

As we can see by direct inspection, there is no contracted Bianchi identity to lowest order, $1/c^2$, (or rather the identity reads $0 = 0$). Hence, the Newtonian solution,

$$\nabla\psi = 4\pi\Delta, \quad (30)$$

$$\phi = \psi, \quad (31)$$

is fully unconstrained. The lowest-order nontrivial contracted Bianchi identities are of order $1/c^3$ (for $\dot{G}^{00}$) and to order $1/c^4$ (for $\dot{G}^{0i}$). They imply conditions upon the higher-order momentum and spatial stresses enforcing energy-momentum conservation. Their expression for the clippers proposed above is very complicated and makes

the point that they are artificial, but the issue can be understood qualitatively.

Consider, for example, momentum conservation for a blob of $\Delta(x)$ initially on its own. A normal matter source approaches, attracted to it, but the Δ blob does not budge, feeling no attraction to it. This can be seen as a violation of momentum conservation. It can also be reinterpreted as evidence of new stresses/pressure acting on the blob to counteract the gravitational forces of normal matter, with energy momentum conservation. However, the stresses depend on other forms of matter rather than on $\Delta(x)$, and worse of all, the dependence is nonlocal, since they must counter the Newtonian force generated by the whole (normal) matter source. The stresses required to make $T_{\Delta}^{\mu\nu}$ conserved can never arise from an equation of state.

VII. SPHERICALLY SYMMETRIC SOLUTIONS WITH TRANSVERSE CLIPPING

Let us now consider a static spherically symmetric spacetime with both Δ and normal matter, which takes the form of a fluid with equation of state

$$p_M = p_M(\rho_M). \quad (32)$$

We apply to the theory a transverse clipper (and leave the case of a radial clipper to Appendix). Let the conditions associated with the clipper force the two-dimensional metric transverse to the iso- N lines to be the Euclidean metric in spherical coordinates. Then, the metric is

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In addition, $\Delta = \Delta(r)$, implying

$$\rho_{\Delta} = \frac{\Delta(r)}{r^2 e^{\lambda}}, \quad (33)$$

where, we recall, Δ is a three-dimensional scalar density. We stress that we are *not* using gauge freedom to choose this metric (as in standard general relativity textbook treatments), but instead freezing the metric angular variables at the level of the action. Hence, there are no angular Einstein equations, as a particular case of Eq. (12) [and disallowing diffeomorphisms as in Eq. (13)]. The equations of motion (9) and (10) under spherical symmetry with a transverse clipper therefore read

$$e^{-2\lambda} \left[\frac{2}{r} \lambda' - \frac{1}{r^2} \right] + \frac{1}{r^2} = 8\pi(\rho_{\Delta} + \rho_M), \quad (34)$$

$$\left[\frac{2}{r} \Phi' + \frac{1}{r^2} \right] e^{-2\lambda} - \frac{1}{r^2} = 8\pi p_M, \quad (35)$$

$$-\Phi'(\rho_M + p_M) = p'_M, \quad (36)$$

(where, we recall, we have adopted $G = 1$). Just as with the Tolman-Oppenheimer-Volkov (TOV) system, by setting

$$e^{-2\lambda} = 1 - \frac{2m}{r}, \quad (37)$$

Eq. (34) gives

$$m' = 4\pi\rho r^2 = 4\pi \left(\rho_M r^2 + \Delta(r) \sqrt{1 - \frac{2m}{r}} \right), \quad (38)$$

with $\rho = \rho_M + \rho_{\Delta}$. Equation (35) then implies

$$\Phi' = \frac{m + 4\pi r^3 p_M}{r(r - 2m)}, \quad (39)$$

so that if $\Delta = 0$, we do recover the original three TOV ordinary differential equations, which together with the equation of state (32) can be solved for the four variables $m(r)$, $\rho_M(r)$, $p_M(r)$, and $\Phi(r)$ [with $\lambda(r)$ given separately by Eq. (37)]. Since Δ is *not* a variable, by adding it to the system we do not change its nature (i.e., the number of equations and variables).

However, adding the (usually redundant) angular/transverse equation

$$e^{-2\lambda} \left[\Phi'' + \Phi'^2 - \Phi' \lambda' + \frac{\Phi'}{r} - \frac{\lambda'}{r} \right] = 8\pi p_M, \quad (40)$$

would render the system self-contradictory. This is the purpose of clipping.

A. The critical halo

It is curious that a constant⁶ $\Delta = \Delta_0$ (with a cutoff $r < r_{\max}$) leads to a halo with the correct profile to generate flat rotation curves. We call this profile the *critical halo*. In the absence of normal matter ($\rho_M = p_M = 0$), Eqs. (37) and (38) imply $m = m_0 r$ with

$$\Delta_0 = \frac{m_0}{4\pi\sqrt{1 - 2m_0}} \quad (41)$$

and $\lambda = \lambda_0$, with

$$e^{-2\lambda_0} = 1 - 2m_0. \quad (42)$$

⁶We again note that such Δ is constant as opposed to covariantly constant; recall that it is a tensor density. But since we are breaking covariance, this can hardly be used as an argument for what should be “naturally” constant in a uniform distribution of “matter.” Perhaps tensor densities survive the breakdown of diffeomorphism invariance better than traditional tensors.

We are assuming that $m_0 < 1/2$, so that the halo is everywhere outside its Schwarzschild radius. The energy density (33) for this profile is

$$\rho_\Delta = \frac{\Delta}{r^2 e^\lambda} = \frac{m_0}{4\pi r^2}. \quad (43)$$

Then, Eq. (39) leads to the Newtonian potential

$$\Phi = \frac{m_0}{1-2m_0} \ln \frac{r}{r_c}, \quad (44)$$

so that the metric is

$$ds^2 = -\left(\frac{r}{r_c}\right)^{\frac{2m_0}{1-2m_0}} dt^2 + \frac{dr^2}{1-2m_0} + r^2 d\Omega^2. \quad (45)$$

The solution outside the halo, $r > r_{\max}$, is the Schwarzschild solution for mass $M = m_0 r_{\max}$. Hence, the integration constant r_c should be adjusted to match Φ on both sides of $r = r_{\max}$. The above solution is exact, but one can take its Newtonian limit and prove that it leads to flat rotation curves, as clearly implied by Eq. (44).

In the above, the halo starts from $r = 0$, but it could start at any $0 < r < r_{\max}$, with similar results. If the halo does start at $r = 0$, then there is a naked singularity at $r = 0$ (the scalar ρ_Δ diverges). There are no horizons in this solution.

B. A critical halo around a black hole

Adding normal matter does not qualitatively change our results in the Newtonian limit, but beyond that we see that we lose superposition, given that we have to solve the differential equation (38) (which is no longer a first integral). Even the critical halo would stop being as simple as in Sec. VII A if normal matter were added on. As a pedantic exercise, we could consider a star of normal matter and a critical halo, and study the Oppenheimer-Volkov limit.

A more relevant problem is the extreme case where a black hole of normal matter forms, but the cutoff at $r < r_0$ for the halo is inserted to prevent the existence of a naked singularity prior to black hole formation. Throughout this paper, we have assumed a preferred spacelike surface Σ_t , so, if a black hole does form it cannot eat the Σ_t and all its leaves pile up at the horizon (a phenomenon already studied in Ref. [27]). A possible model follows from

$$m = M \quad \text{for } r < r_0, \quad (46)$$

$$m = M + m_0(r - r_0) \quad \text{for } r > r_0, \quad (47)$$

with $r_0 > 2M$ as the simplest choice.⁷

⁷Starting the halo from $r_0 = 0$ would simply shift M in what follows, but note that r becomes timelike inside the horizon.

For $r < r_0$, the metric is Schwarzschild with mass M . For $r_0 < r < r_{\max}$, integrating Eqs. (38) and (39) leads to

$$ds^2 = -\frac{[r - 2(M + m_0(r - r_0))]^{\frac{1}{1-2m_0}}}{r(r_0 - 2M)^{\frac{2m_0}{1-2m_0}}} dt^2 + \frac{dr^2}{1 - \frac{2}{r}(M + m_0(r - r_0))} + r^2 d\Omega^2. \quad (48)$$

The match point r_0 could be anywhere, but it is natural to choose it just outside the horizon $r_0 = 2M^+$, since it is where the Σ_t pile up.

This is an interesting black hole plus critical halo solution. One may object that there is an element of teleology in keeping the Σ_t outside the black hole's horizon, but there is already teleology in the definition of the concept of horizon.

C. The cost of refilling for these solutions

We now illustrate the points on refilling made in Sec. V, with reference to the solutions found in this section. Given Eq. (25), the only nonvanishing components of the Stückelberg fields $S_{\mu\nu}$ are those corresponding to a transverse pressure:

$$p_\perp^\Delta = \frac{S_{\theta\theta}}{r^2} = \frac{S_{\phi\phi}}{r^2 \sin^2 \theta}. \quad (49)$$

This can be found either by evaluating the rhs of Eq. (26) and integrating it, or, equivalently, by replacing

$$p_M \rightarrow p_M + p_\perp^\Delta \quad (50)$$

in the clipped equation (40), so that

$$p_\perp^\Delta = -p_M + \frac{e^{-2\lambda}}{8\pi} \left[\Phi'' + \Phi'^2 - \Phi' \lambda' + \frac{\Phi'}{r} - \frac{\lambda'}{r} \right], \quad (51)$$

with Φ and λ evaluated with the clipped equations, that is, as above. Either way we find

$$p_\perp^\Delta = \frac{r}{2} \rho_\Delta \Phi' = \frac{r}{2} \rho_\Delta \frac{4\pi \int dr^2 \rho + r^3 p_M}{r(r - 8\pi \int dr^2 \rho)}, \quad (52)$$

(where $\rho = \rho_M + \rho_\Delta$, we recall). As we see (and as announced in Sec. V) this is far from an equation of state [such as Eq. (32) for normal matter]. The Stückelberg pressure depends not only on Δ but also nonlocally on all other forms of matter. For a pure critical halo, we have the accidental relation

$$p_\perp^\Delta = \frac{m_0}{2(1-2m_0)} \rho_\Delta, \quad (53)$$

but this is not an equation of state (indeed it varies from halo to halo and the relation disappears altogether as soon as normal matter is added on).

Regarding the black hole plus halo solution in Sec. VII B, we further note that there are no divergences at $r = 2M$ in the clipped theory, since the components of the Einstein tensor that would have diverged have been replaced by $H_{\theta\theta} = H_{\phi\phi} = 0$. If we refill, i.e., restore diffeomorphism invariance with Stückelberg fields $S_{\mu\nu}$, then these do diverge, with

$$p_{\perp}^{\Delta} \rightarrow \frac{m_0(1 + 2m_0)}{2(r - 2M)}, \quad (54)$$

as $r \rightarrow 2M$. Thus, in this reinterpretation there is a pressure singularity at the horizon, not dissimilar to the cosmological singularities studied by Barrow [28], but on a timelike or near-null surface, rather than a spacelike surface.

We close this section by stressing that all theories in this paper have the same Newtonian limit, but different predictions beyond the Newtonian limit, depending on the clipper used. To illustrate this point, in Appendix we briefly repeat most of the exercise in this section with a radial (instead of a transverse) clipper.

VIII. TOWARDS PHENOMENOLOGY

The models we have just presented have the drawback that they can evade any experimental constraints simply by setting the nondynamical Δ to zero wherever the test was carried out (since there is no pretension at universal gravitation). This valid criticism is counteracted by the fact that, reciprocally, if $\Delta \neq 0$, these models would imply inimitable signatures. In this section, we give examples of this in the Newtonian and PPN limits [30] (with a criticism of the PPN formalism in this context) and beyond.

A. Proof of concept for testability

It is not difficult to devise stylized situations with the unmistakable imprint of Δ matter, which can be in the realm of gedanken experiments or not. For these tests to be generic (and not restricted to a given relativistic theory and its specific clipping) they should probe the subverted Newtonian limit only.

As an example, consider a Newtonian binary system, where one observes the wobble of a luminous object due to a dark companion. If the dark companion is made of Δ matter with (active gravitational) mass m_2 and the visible object has mass m_1 , then for circular orbits with radius a_1 , we would have the Keplerian relation for the orbital frequency ω , as follows:

$$\omega^2 a_1^3 = m_2, \quad (55)$$

whatever the value of m_1 . In contrast to this subverted Newtonian limit, the usual Newtonian theory abiding by momentum conservation displays

$$\omega^2 a_1^3 = \frac{m_2^3}{(m_1 + m_2)^2}. \quad (56)$$

The point is that we can conceive situations where m_1 , a_1 , and ω are measurable, so we would know the difference between the two theories, particularly if $m_2 \sim m_1$.

In a future publication [32], we will examine the effect of complicating this setup as much as wanted (elliptic orbits, several objects, etc.). A relation with the methods of Ref. [31] and other tests of momentum conservation is evident. However, as we stress below, the usefulness of PPN parameters (ζ_2 in the case of Ref. [31]) can be questionable, as we will explain further below. Note that it is enough to remark that PSR 1913 + 16 is not made of Δ matter for the PPN parameters to revert to general relativity values in this theory.

B. Lensing (and γ_{PPN})

Beyond the Newtonian limit, different clippings have different predictions (so a radial clipping, as outlined in Appendix, leads to different results). In addition, an analysis along the lines of Ref. [34] (in the unrelated context of the khronon) may be more useful than standard PPN parameters. It is easy to see that the basic assumptions leading to the PPN parametrization (see, for example, Refs. [30,33]) break down for this theory (which is metric, nonetheless). The PPN parameters vary in space and depend on the configuration of Δ and of other forms of matter. For example, for the solutions in Sec. VII, we could read off:

$$\gamma_{PPN}(r) = -\frac{m(r)}{r\Phi(r)}, \quad (57)$$

with different functions associated with Eqs. (45) and (48), and all PPN parameters reverting their general relativity value where $\Delta = 0$. Any lensing experiment is far from local and would probe γ_{PPN} along a light path, so the usefulness of the concept is dubious.

It makes more sense to consider directly the lens equation following from integrating the geodesic equation for the various metrics obtained for these models. For example, for metrics (45) and (48), with A and B defined from $ds^2 = -A^2 dt^2 + B^2 dr^2 + r^2 d\Omega^2$, one would need to integrate

$$u'^2 = \frac{E^2}{L^2 A^2 B^2} - \frac{u^2}{B^2}, \quad (58)$$

(with $u = 1/r$, $l = d/d\phi$, and standard constants of motion E and L), for light rays that penetrate the halo. For those that do not, we get the standard general relativity formulas.

Once again, this simplistic setup can be complicated as much as needed to allow comparison with observations and this will be studied elsewhere [32] together with the effect on Einstein's rings.

C. Gravity waves and strong gravity tests

In the weak field limit (since gravity then satisfies superposition and the refilled theory is equivalent to general relativity with a very unusual nondynamical medium), the gravitational wave solutions remain the same. Beyond the weak field limit, there might be interesting effects for gravitational waves propagating through a Δ halo (possibly related to other work on gravity waves propagating through a medium, e.g., Ref. [38]).

Note that within the refilled picture, the Δ matter must react to the wave to remain unmoved by its tidal stresses, so a wave of Stückelberg stresses is expected and could backreact on the wave creating an interesting resonance. This could happen in particular near the horizon of a black hole with a Δ halo in attendance, as studied in Sec. VII B, where the effects are already nonlinear. More generally, in future work [32], we will examine the impact on strong gravity phenomenon, for example, regarding black hole formation.

IX. HAMILTONIAN PERSPECTIVE

There is a close connection between painted-on dark matter and the Hamiltonian framework. As pointed out in different forms in Refs. [4–10], the effective dark matter fluid can be seen as the legacy effect of a violation of the Hamiltonian constraint. This may be variously labeled as a violation of diffeomorphism invariance or a (local) Lorentz invariance violation (LIV; see also related literature in flat [39,40] and curved [41] spacetime). This violation can be historical (for example, due to past evolution [29] or any other past interaction with a foliation [5,6]), or else a high-energy effect [3,4]. A combination of the two is also possible. The question then is what to do with the nonvanishing Hamiltonian once the LIV effects cease, if they do cease.

If the underlying theory satisfies the Dirac hypersurface deformation algebra (HDA; [42–44]) after symmetry restoration, then it is possible to fully restore the symmetry, with the effects of the symmetry breaking phase limited to a putative effective matter component, the total satisfying the Hamiltonian constraint *and* Dirac's HDA. This matter component is a dust CDM fluid if the preferred foliation is geodesic (e.g., Refs. [4,5]), and something more complicated for nongeodesic foliations (e.g., Ref. [6]).

But what if the damage done to four-dimensional diffeomorphism invariance is permanent, as is the case envisaged towards the end of Ref. [12], where the residual Hamiltonian gets frozen in, as a time-independent Δ ?

A. LIV and let LIV in the Hamiltonian perspective

If this is the case, then it is clear that the algebra of constraints must be modified in such regions. Recall [42–44] that the Dirac HDA results from the extended Hamiltonian splitting as $\mathcal{H}_E = N\mathcal{H} + N^i\mathcal{H}_i$, with algebra as follows:

$$\{H_i(f^i), H_j(g^j)\} = H_i((\mathcal{L}_f g)^i), \quad (59)$$

$$\{H_i(f^i), H(f)\} = H(f^i \partial_i f), \quad (60)$$

$$\{H(f), H(g)\} = H_i(h^{ij}(f \partial_j g - g \partial_j f)), \quad (61)$$

for smeared Hamiltonian $H(f) = \int d^3y f(y) \mathcal{H}(y)$ and momentum $H_i(f^i) = \int d^3y f^i(y) \mathcal{H}_i(y)$ constraints (where $\mathcal{L}$ is the Lie derivative). The HDA is equivalent to four-dimensional diffeomorphism invariance. Restoration of diffeomorphism invariance in the presence of a nonvanishing Hamiltonian (and possibly momentum [5,6]) then amounts to evolving the Hamiltonian and momentum according to this algebra, and then redefining the total Hamiltonian (and possibly momentum) absorbing the nonvanishing values into a fluid so that the total vanishes [5,6]. This is equivalent to the conservation of the Stückelberg fields $S^{\mu\nu}$ defined in Eq. (17). Indeed, as proved in Ref. [6], we have

$$n_\mu \nabla_\nu S^{\mu\nu} = 0 \Leftrightarrow \dot{\mathcal{H}} = \{\mathcal{H}, \mathbf{H}\}_{HDA} = \partial_i(N^i \mathcal{H}) + \partial_i(\mathcal{H}^i N) + \mathcal{H}^i \partial_i N, \quad (62)$$

$$h_{\alpha\mu} \nabla_\nu S^{\mu\nu} = 0 \Leftrightarrow \dot{\mathcal{H}}_i = \{\mathcal{H}_i, \mathbf{H}\}_{HDA} = \mathcal{H} \partial_i N + \partial_j(N^j \mathcal{H}_i) + \mathcal{H}_j \partial_i N^j, \quad (63)$$

with the total Hamiltonian defining the time evolution given by

$$\mathbf{H} = \int_{\Sigma_t} d^3x \mathcal{H}_E(x). \quad (64)$$

From inspecting these equations, it is therefore clear that for a clipped theory with a nonconserved stress energy tensor associate with a dustlike Δ matter frozen in on a nongeodesic frame, the HDA must be broken. The last term in Eq. (62) precludes the time independence of Δ .

In fact, such $\Delta(x)$ can be seen as a central charge in the Dirac HDA, as we now show.

B. Painted-on matter as a central charge in the Dirac HDA

Let us consider the effect of a nondynamical frozen-in (time-independent) legacy Hamiltonian Δ which by construction is only a function of x on a preferred foliation Σ_t , as follows:

$$\Delta = \Delta(x). \quad (65)$$

In a theory in which we recover the normal matter and gravity Lagrangian, we also recover the HDA for the sum of their Hamiltonians and momenta, represented by GM in the expression for the extended Hamiltonian:

$$\begin{aligned} \mathcal{H}_E &= N\mathcal{H} + N^i\mathcal{H}_i \\ &= \mathcal{H}_{GME} + N\Delta = N(\mathcal{H}_{GM} + \Delta) + N^i\mathcal{H}_i^{GM}. \end{aligned} \quad (66)$$

Hence the total Hamiltonian (including Δ) and the momentum cannot satisfy the Dirac HDA. With generic smearing functions, we must have instead

$$\{H_i(f^i), H_j(g^j)\} = H_i((\mathcal{L}_f g)^j), \quad (67)$$

$$\{H_i(f^i), H(f)\} = (H - \Delta)(f^i \partial_i f), \quad (68)$$

$$\{H(f), H(g)\} = H_i(h^{ij}(f \partial_j g - g \partial_j f)), \quad (69)$$

implying on-shell relations (once one varies with respect to N and N^i), as follows:

$$\{H_i(f^i), H_j(g^j)\} \approx 0, \quad (70)$$

$$\{H_i(f^i), H(f)\} \approx - \int d^3x \Delta(x) f^i \partial_i f, \quad (71)$$

$$\{H(f), H(g)\} \approx 0. \quad (72)$$

Alternatively, the unsmeared algebra can be written as [42–44]

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_i(y) \delta_{,j}(x, y) + \mathcal{H}_j(x) \delta_{,i}(x, y), \quad (73)$$

$$\{\mathcal{H}_i(x), \mathcal{H}(y)\} = (\mathcal{H}(x) - \Delta(x)) \delta_{,i}(x, y), \quad (74)$$

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = h^{ij}(\mathcal{H}_i(x) \delta_{,j}(x, y) + \mathcal{H}_j(y) \delta_{,i}(x, y)), \quad (75)$$

where $\delta_{,i}(x, y) = \frac{\partial}{\partial x^i} \delta(x, y)$, and δ is a bidensity of weight zero (i.e., a scalar) in the first argument and of weight one in the second [45]. Smeared or not, the relevant Poisson bracket is the one involving the Hamiltonian and the momentum, with on-shell anomalous result as follows:

$$\begin{aligned} \{H_i(f^i), H(N)\} &\approx - \int d^3x \Delta(x) f^i \partial_i N \\ \{\mathcal{H}_i(x), \mathcal{H}(y)\} &\approx -\Delta(x) \frac{\partial}{\partial x^i} \delta(x, y). \end{aligned} \quad (76)$$

In the face of this result, going through the Dirac procedure described in Ref. [42] offers two major clarifications. These result from forks in the procedure, which arise not because the procedure is ambiguous, but because the way we laid out the theory has several options which have to be clarified, leading to such forks. Some of these possibilities do not serve the purpose of this paper.

First, as already stressed at the start of this paper, Δ must be nondynamical, which in the canonical framework means it must not be seen as a phase space variable (with a conjugate momentum Π_Δ , which does not appear anywhere). Otherwise, Dirac's procedure implies $\Delta = 0$ as a secondary constraint, and we do not want that. Therefore, in the canonical formalism, $\Delta(x)$ should be seen as a nondynamical central charge in the Dirac HDA. We can keep it general or not, since it is an external, given function. With this proviso, Dirac's HDA therefore contains second-class constraints.

C. Clipping as a Dirac bracket solution to the second-class constraints

The second clarification provided by Dirac's recipes [42] is that we should not eliminate the second-class constraints solely by restricting the Lagrange multipliers N and N^i . Second-class constraints may be solved by imposing conditions upon the Lagrange multipliers; in our case, these could be obtained from the secondary constraints:

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathbf{H}_E\} \approx -\partial_i(N^i \Delta) = 0, \quad (77)$$

$$\dot{\mathcal{H}}_i = \{\mathcal{H}_i, \mathbf{H}_E\} \approx -\Delta \partial_i N = 0, \quad (78)$$

implying that wherever $\Delta \neq 0$, one must have:

$$\partial_i N = 0, \quad (79)$$

$$\partial_i(N^i \Delta) = 0. \quad (80)$$

But this is not what we want, since it would set to zero the source terms in Eqs. (7) and (8), which are the point of this paper. Indeed, imposing Eqs. (79) and (80) implies a geodesic Σ_t and a conserved dust fluid. This is old news: it is the IR limit of projectable HL theory [4] (with violations in the UV as described in Ref. [46]), or the dust fluid appearing in the evolution scenarios in Ref. [5] and also in Refs. [8–10].

We want the possibility of a nonconserved dust fluid in a nongeodesic Σ_t . The Dirac bracket [42] offers a way in, and is equivalent to a clipping procedure as described in Secs. III and IV. This can be seen appealing to the

Maskawa-Nakajima theorem [47] which states that it is always possible to find a canonical transformation such that the second-class constraints take the form $q_r = p_r = 0$ for some of the new canonical pairs of variables (with the nonclosing algebra reduced to $\{q_r, p_s\} = \delta_{rs}$). Then, the Dirac bracket is equivalent to the Poisson bracket evaluated with respect to all variables except these, which are to be ignored. This is equivalent to clipping out these variables.

The number of pairs of second-class constraints equals the number of canonical pairs to be clipped, and so the number of spatial (h_{ij}) metric components to be clipped. In contrast, if we eliminate the Lagrange multipliers, the number of second-class constraints equals the number of multipliers eliminated. A hybrid approach is also possible [as advocated in Sec. IV; see comments around Eq. (21)]. The number of second-class constraints could also depend on any conditions imposed on Δ .

Face value, the implication is that it should be possible to clip a single spatial metric variable in our problem, and this would be the solution provided by the Dirac bracket, but we leave this exercise to future work. Notice also that the procedure in Sec. III is more general, and allows for clippings that do not emerge from the Dirac bracket, as well as for overclipping.

X. CONCLUDING REMARKS

In this paper, we embraced the unorthodox notion of painted-on dark matter: a matterlike component which is not matter at all but behaves like esoteric dust encrusted in a preferred foliation, which may be nongeodesic so that such dust violates conservation laws, and is a signature of an aether wind. Its theoretical origins may lie in violations of the Hamiltonian constraint, but in contrast with Refs. [4–10], by freezing the leftover Hamiltonian, we are entertaining the possibility that diffeomorphism invariance is never recovered in such regions, leading to violations of local energy-momentum conservation. A reverse analogy with topological defects can be drawn here. Topological defects arise when symmetry breaking takes place at low temperatures, the defect representing a region where the field remained stuck in the high temperature symmetric phase. We have the opposite setup: at low energies (or at late times), we restore the very fundamental diffeomorphism symmetry, which is broken at high energies (or in the early Universe). We then envisage the possibility that defects linger on, where (for nontopological reasons) the broken phase survives, and that this might be part of what we call dark matter.

In closing, we accept that further conceptual work is warranted on the problems associated with coupling gravity to a nonconserved source in a relativistic theory. In this paper, we suggested a solution: *clipping*, which can take the form of a Dirac bracket in the Hamiltonian formulation. But the Bianchi identities are so powerful that they always allow for *refilling* of the clipped equations with suitable

(Stückelberg) stresses. Nonconservation can then masquerade as conservation supplemented by appropriate stresses, but, as we showed, by making the equations for these stresses suitably ludicrous, we arrive at a possible definition of broken diffeomorphism invariance. Is this radical enough? Is there a more fundamental construction for a theory with nonconserved sources?

ACKNOWLEDGMENTS

We thank Dick Bond, Bianca Dittrich, and Kostas Skordis for help with this paper. J. M. is grateful to the participants of the Caribbean Futures Of Science Symposium (where this work was first presented) for equal measures of encouragement and acerbic criticism. R. I. was supported by a Bell-Burnell Fellowship and J. M. partly supported by STFC Consolidated Grant No. ST/T000791/1.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: A BRIEF OUTLINE OF RADIAL CLIPPING

As an example of how different the theory could be with an alternative clipping prescription, in this appendix, we consider a radial clipping, while fixing the transverse metric, so that the spherically symmetric equations are reduced to

$$e^{-2\lambda} \left[\frac{2}{r} \lambda' - \frac{1}{r^2} \right] + \frac{1}{r^2} = 8\pi(\rho_\Delta + \rho_M), \quad (\text{A1})$$

$$e^{-2\lambda} \left[\Phi'' + \Phi'^2 - \Phi' \lambda' + \frac{\Phi'}{r} - \frac{\lambda'}{r} \right] = 8\pi p_M, \quad (\text{A2})$$

$$-\Phi'(\rho_M + p_M) = p'_M, \quad (\text{A3})$$

as opposed to the system (34), (35), and (36). The integration (37) and (38) remains the same, but the equation for Φ is different. For a critical halo, we have the same expression for λ but Φ loses sight of Δ , as follows:

$$\Phi'' + \Phi'^2 + \frac{\Phi'}{r} = 0, \quad (\text{A4})$$

with solution:

$$\Phi = \Phi_0 + \ln[\ln r/r_0], \quad (\text{A5})$$

where Φ_0 is to be determined by the boundary conditions. Instead of Eq. (45), the critical halo metric is

$$ds^2 = -e^{2\Phi_0} \ln^2 \frac{r}{r_0} dt^2 + \frac{dr^2}{1 - 2m_0} + r^2 d\Omega^2. \quad (\text{A6})$$

The Newtonian limit of this solution is identical to that obtained with a transverse clipping (in particular it creates flat rotation curves), but the post-Newtonian and strong gravity predictions outlined in Sec. VIII are different.

The refilling Stückelberg stresses described in Sec. VII C are also different. We can define an anisotropic radial stress completion giving us back Eq. (35) with an rhs modified by

$$p_M \rightarrow p_M + p_{\Delta r} \quad (\text{A7})$$

such that

$$\frac{1}{r^2} (r^2 p_{\Delta r})' = -\Phi'(\rho_{\Delta} + p_{\Delta r}). \quad (\text{A8})$$

This refilling is to be contrasted with Eqs. (50) and (51) for a transverse clipping. Then, just as before, we could find the Φ field integrating the usual radial equation with this radial stress added on. Again, this is hiding the fact that $p_{\Delta r}$ is not related to Δ via any equation of state or local dynamics minimally coupled to gravity.

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