

Cosmic strings and domain walls of the QCD quark condensate with and without a hidden axion

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The chiral-quark condensate of QCD, which spontaneously breaks the anomalous axial symmetry, gives rise to axionic type global string-wall systems. If a Peccei-Quinn-type axion exists in the theory, the axionic strings are in general accompanied by winding of the QCD quark condensate. Depending on the axion model the winding can proceed either in the η' or in the pion direction. This determines the structure of fermionic zero modes and the anomaly inflow which has important astrophysical consequences. We point out that η' and pion string-wall systems exist in pure QCD, independently of the hidden axion. Strikingly, even if a hidden axion exists, the early cosmology can be entirely dominated by string-wall systems formed by the QCD quark condensate. We also discuss their role in the QCD phase transition and in heavy-ion physics.

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I. INTRODUCTION

The chiral symmetry of quarks is spontaneously broken by the chiral-quark condensate. At the same time, this chiral symmetry is explicitly broken by quark masses as well as by the chiral anomaly. Due to the explicit breaking, the would-be massless Nambu-Goldstone bosons, such as the pions and the η' meson, gain nonzero masses. Since no spontaneously broken discrete subgroup survives this explicit breaking, the vacuum of the chiral symmetry is topologically trivial without truly stable topological defects. However, from the point of view of various cosmological and phenomenological implications the unstable defects are no less interesting.

In this work, we point out that, despite the above limitations, there exist nontrivial winding configurations corresponding to 2π -domain walls bounded by strings. The windings take place in two major flavor directions corresponding to the pseudo-Nambu-Goldstone bosons η' or π^0 . However, winding in other directions such as the η meson is also possible.

If a hidden Peccei-Quinn QCD axion exists in nature, the axionic strings and walls are in general accompanied by

windings of the phases of the QCD quark condensate. This has important implications for the zero mode content of axionic strings and, correspondingly, for their superconductivity properties [1] as well as the dynamics of the anomaly inflow [2].

However, more importantly, the QCD 2π -string-wall systems can exist regardless of the hidden axion. As we shall explain in detail, a simple way to understand this is that the η' meson of QCD represents an axion, albeit of poor quality, which directly probes the physics of the QCD θ -vacuum. In particular, in the limit of a vanishing quark mass, the η' meson fully cancels the vacuum θ -angle [3]. The nonzero quark mass offsets this cancellation but does preserve the axionlike nature of the η' meson. Correspondingly, the η' meson mirrors the structures usually discussed only in the context of a hidden axion, such as domain walls bounded by cosmic strings.

Since the vacuum of QCD is 2π -periodic in the phases of the quark condensate, the phases change across the wall by multiples of 2π . We shall refer to such domain walls as 2π -walls.

We discuss a sufficient condition for the existence of 2π -walls formed by the QCD quark condensate. The condition is the existence of a wall configuration across which the order parameter is nonzero everywhere. This guarantees that the phases are well-defined at every point across the wall. This suffices for the existence of a valid 2π wall configuration.

Moreover, since all phases are 2π -periodic, the vacua on the two sides of the wall are physically identical. This guarantees the existence of strings that bound the wall.

Of course, the solution for the string-core cannot be derived analytically, since in that region QCD is strongly

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coupled. However, this is not an obstacle for knowing that the solution exists.

Thus, the key question is whether the quark condensate remains nonzero across the wall.

For walls with a winding of the phases in π^0 -direction, this is guaranteed by the smallness of the quark masses. The backreaction from the explicit breaking term on the modulus of the chiral-quark condensate is negligible.

As we shall explain, the situation with η' -domain walls is more subtle. The reason is that across such a wall the effective vacuum angle θ_{eff} changes by 2π . Thus, the interior of the wall probes phases in which QCD dynamics are rather poorly understood.

In order to bypass this difficulty, we use an explicit solution of an η' -type domain wall obtained in [4] as a guideline. The role of η' in this solution is played by the phase of the gaugino condensate in $SU(N)$ supersymmetric Yang-Mills theory. The existence of such domain walls was originally suggested in [5], however the explicit solution was obtained in [4]. The advantage of the latter solution is that it is exact in $1/N$ and suits our purpose very well. This solution explicitly shows that the absolute value of the fermion condensate remains nonzero (and constant up to $1/N$) across the domain wall. It therefore gives supporting evidence for the existence of η' domain walls in ordinary QCD.

A planar infinite wall is classically stable. Although via a quantum tunneling process a hole can be punched through it, the probability is highly suppressed. Of course, in reality, infinite walls never form. Instead, they materialize during phase transitions in the form of closed bubbles or as walls bounded by cosmic strings. Such structures collapse and radiate energy in various modes, including electromagnetic radiation as well as gravitational waves.

String-wall systems formed by the QCD condensate can have particularly important cosmological implications in scenarios with an early epoch QCD phase transition [6].

From the low-energy perspective of the theory of mesons, the QCD condensate string-wall systems represent solitonic states. We discuss their potential manifestations in heavy-ion experiments.

II. STRONG- CP PUZZLE

In the standard discussion (which ignores gravity) the strong- CP puzzle is formulated as the following naturalness problem. QCD has a continuum of vacua conventionally labeled by the CP -violating vacuum angle θ [7,8].

These vacua belong to different superselection sectors. The choice of the vacuum is accounted for by the following term in the Lagrangian:

$$\frac{g^2}{32\pi^2} \bar{\theta} G_{\mu\nu} \tilde{G}^{\mu\nu}, \quad (1)$$

where $G_{\mu\nu}$ is the gluon field strength and $\tilde{G}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta}$ is its dual. In the rest of this paper, we will drop the constants in front, including the strong coupling constant g .

In a theory with massive quarks, the physically measurable parameter is the quantity $\bar{\theta} \equiv \theta + \arg(\det M_q)$, where $\arg(\det M_q)$ is the phase of the determinant of the quark-mass matrix.

In the quantum theory, the $\bar{\theta}$ -term induces an electric dipole moment of the neutron (nEDM) [9,10], denoted as d_n . The comparison of the resulting theoretical value with the current experimental limit, $|d_n| < 2.9 \times 10^{-26} \text{ e cm}$ [11–13], gives the bound

$$|\bar{\theta}| \lesssim 10^{-9}. \quad (2)$$

Notice that there exists an additional contribution to the nEDM coming from the breaking of CP -symmetry by the weak interaction [14–16]. However, this correction is too small to affect the current bound (2).

Thus, observations indicate that we live in a sector with a minuscule or zero $\bar{\theta}$. This is puzzling.

III. PECCEI-QUINN SOLUTION

Let us consider the Peccei-Quinn solution [17] which is based on removing $\bar{\theta}$ by an anomalous chiral symmetry $U(1)_{PQ}$. For simplicity, let us consider a single quark flavor, Ψ , that transforms under the symmetry as

$$\Psi \rightarrow e^{i\frac{1}{2}\alpha\gamma_5} \Psi. \quad (3)$$

Due to the chiral anomaly, the current exhibits an anomalous divergence,

$$\partial^\mu (\bar{\Psi} \gamma_\mu \gamma_5 \Psi) \propto G \tilde{G}. \quad (4)$$

Correspondingly, the Lagrangian shifts as

$$\delta \mathcal{L} \propto \alpha G \tilde{G}. \quad (5)$$

Thus, the $\bar{\theta}$ -term can be removed by the chiral $U(1)_{PQ}$ transformation,

$$\bar{\theta} \rightarrow \bar{\theta} - \alpha. \quad (6)$$

This indicates that $\bar{\theta}$ is unphysical.

The fact that, in the presence of an anomalous chiral symmetry, the $\bar{\theta}$ -term is unphysical can also be understood in the language of instanton zero modes. According to the index theorem, each spin-1/2 quark deposits a chiral zero mode in the instanton background. These modes kill the instanton effect.

However, in nature there exist no massless quarks. Therefore, the $U(1)_{PQ}$ -symmetry must be spontaneously broken by the vacuum expectation value (VEV) of a

complex scalar field $\Phi(x) = \rho(x)e^{i\theta_\phi(x)}$, which under the $U(1)_{PQ}$ transforms as

$$\Phi \rightarrow e^{i\alpha}\Phi. \quad (7)$$

The modulus of the scalar gets a VEV, $\langle \rho \rangle = f_\phi$, from the Mexican-hat (Goldstone) potential

$$V(\Phi) = \lambda^2 (\Phi^* \Phi - f_\phi^2)^2, \quad (8)$$

where λ is a coupling constant. This gives a mass, $M_\Psi = g_\Psi f_\phi$, to the fermion through a Yukawa coupling in the fermion Lagrangian,

$$\mathcal{L}_\Psi = i\bar{\Psi}_{L,R}\gamma^\mu D_\mu \Psi_{L,R} - g_\Psi \Phi^* \bar{\Psi}_L \Psi_R + \text{H.c.}, \quad (9)$$

where we have written the Dirac spinors $\Psi = \Psi_L + \Psi_R$ in terms of left-right components, g_Ψ is the Yukawa coupling constant and D_μ is the covariant derivative with respect to the gauge symmetry.

The Nambu-Goldstone phase, $\theta_\phi(x)$, is the axion [18], which in terms of a canonically normalized field $\phi(x)$ can be written as $\theta_\phi(x) \equiv \frac{\phi(x)}{\sqrt{2}f_\phi}$.

Taking the anomaly into account, the effective theory of the axion is¹

$$\mathcal{L}_{\text{axion}} = f_\phi^2 (\partial_\mu \theta_\phi(x))^2 - (\theta_\phi(x) - \bar{\theta}) G\tilde{G}. \quad (10)$$

Thus, the effective vacuum angle, $\theta_{\text{eff}}(x) := \theta_\phi(x) - \bar{\theta}$, becomes dynamical. The axion relaxes dynamically to the global minimum which, according to the Vafa-Witten theorem [19], is at $\theta_{\text{eff}} = 0$.

This can be seen by an explicit computation of the energy dependence on $\theta_{\text{eff}}(x)$. Indeed, nonperturbative effects (instantons) generate a potential for the axion. In the dilute instanton gas approximation,

$$V(\theta_\phi) = -\Lambda^4 \cos(\theta_\phi(x) - \bar{\theta}), \quad (11)$$

which has a minimum at $\theta_{\text{eff}} = 0$.

IV. η' MESON AS POOR-QUALITY AXION

Interestingly, QCD already contains an axion, albeit of poor quality. This axion is the η' meson. Consider QCD with N_f flavors of massless (or light) quarks ψ^i , where $i = 1, 2, \dots, N_f$ is the flavor index. Each flavor represents a Dirac fermion, $\psi^i = \psi_L^i + \psi_R^i$, with both left-handed and right-handed chiralities transforming as fundamental representations of the color group.

The term “light” refers to the quarks with masses below the QCD scale Λ . In the real world, $N_f = 3$, since there are three light quarks, u , d , s . However, let us keep the discussion more general and assume for now that these N_f light-quark flavors are massless.

In the case of massless quarks, QCD exhibits a global $U(N_f)_L \times U(N_f)_R$ symmetry. In reality, this symmetry is only approximate because it is explicitly broken by quark masses. One particularly interesting subgroup of this global symmetry group is the axial $U(1)_A$ symmetry, under which the quarks transform as

$$\psi_i \rightarrow e^{-i\frac{1}{2}\alpha\gamma_5}\psi_i. \quad (12)$$

The $U(1)_A$ symmetry is spontaneously broken by the quark condensate $\langle \bar{\psi}_i \psi_j \rangle = \Lambda^3 \delta_{ij}$. However, the corresponding massless (or light) Nambu-Goldstone boson, residing in $\bar{\psi}_i \gamma_5 \psi_j$, was nowhere to be found. This is the famous axial $U(1)_A$ problem.

This problem was solved by 't Hooft [20], who understood that the would-be Goldstone boson was getting mass from the $U(1)_A$ anomaly through instantons. The resulting potential for the phase degree of freedom, $\theta_\eta(x)$, in the dilute instanton gas approximation is

$$V(\eta') = -\Lambda^4 \cos(N_f \theta_\eta - \theta). \quad (13)$$

The phase of the condensate, $\theta_\eta(x) = \frac{\eta'(x)}{\sqrt{2}f_\eta}$ is related to the canonically normalized η' field through f_η , the decay constant of η' . The potential (13) gives a mass to η' , $m_\eta \sim \frac{\Lambda^2}{f_\eta}$. In the real world, with 3 colors and 3 light-quark flavors, $f_\eta \sim \Lambda$.

Interestingly, 't Hooft did not point out that simultaneously with generating a mass for η' , the potential gives $\theta_{\text{eff}} = 0$ and, therefore, solves the strong-CP problem.

In other words, in QCD with massless quarks, η' is a full-fledged axion [3,21,22]. Unfortunately (or fortunately?), in the real world quarks are massive [23], which explicitly breaks $U(1)_A$. Therefore, η' cannot enforce $\theta_{\text{eff}} = 0$. This can be seen from the effective theory obtained after inclusion of the quark masses. For simplicity, we consider a single light quark with the following mass term:

$$m_\psi \bar{\psi}_L \psi_R + \text{H.c.} \quad (14)$$

After the inclusion of the mass term, the effective potential of the phase degree of freedom becomes

$$V(\eta') = -\Lambda^4 \cos(\theta_\eta - \bar{\theta}) - \Lambda_m^4 \cos(\theta_\eta), \quad (15)$$

where $\Lambda_m^4 \simeq m_\psi \Lambda^3$. We do not take any other neutral mesons into account, as we are only working with one light flavor. Here, we included the phase of the mass,

¹Notice that, since in this simplified discussion there are no other quarks, $\bar{\theta} = \theta$.

θ_m , into $\bar{\theta} = \theta + \theta_m$. The minimum of the potential is now at

$$\theta_\eta = \arctan \left(\tan(\bar{\theta}) \left(1 + \frac{\Lambda_m^4}{\Lambda^4 \cos(\bar{\theta})} \right)^{-1} \right). \quad (16)$$

Assuming that $\bar{\theta}$ is small² such that we can use $\tan \bar{\theta} \approx \bar{\theta}$, the effective vacuum angle $\theta_{\text{eff}} = \theta_\eta - \bar{\theta}$ is

$$\theta_{\text{eff}} \simeq -\frac{|m_\psi|}{\Lambda} \bar{\theta}. \quad (17)$$

All observable CP -violating effects, such as nEDM, are controlled by θ_{eff} . Thus, in the case of massless quarks, η' turns $\theta_{\text{eff}} = 0$ and solves the strong-CP puzzle. However, since in the real world quarks are massive with $m_\psi/\Lambda \sim 10^{-2}$, η' cannot provide an explanation for the observed smallness of θ_{eff} . In other words, for $m_\psi \neq 0$, η' is a poor-quality axion.

Nevertheless, the lessons we learn from above are extraordinarily important:

- (1) The mass of η' represents an experimental proof of the existence of θ -vacua;
- (2) η' illustrates the reality of axion dynamics: η' would be an exact quality axion, if at least one quark would be massless [3];
- (3) Since quarks are massive, there must exist a good-quality axion.

V. COUPLED SYSTEM

In order to understand the mixing between the η' meson and the PQ axion field, we consider a simplified model that includes a single light quark, ψ , and a single heavy quark, Ψ . The latter gets its mass from the PQ field Φ . In the low-energy EFT this model gives rise to an axion and the η' meson. The simplest way to understand their coupling is through the 't Hooft determinant.

A. Massless-quark case

We first consider the case where the explicit mass term of the light quark, m_ψ , vanishes. The tree-level UV Lagrangian has the form,

$$\begin{aligned} \mathcal{L}_{\text{UV}} = & \partial_\mu \Phi^* \partial^\mu \Phi - \lambda^2 (\Phi^* \Phi - f_\Phi^2)^2 \\ & + i \bar{\Psi}_{L,R} \gamma^\mu D_\mu \Psi_{L,R} - g_\Psi \Phi^* \bar{\Psi}_L \Psi_R + \text{H.c.} \\ & + i \bar{\psi}_{L,R} \gamma^\mu D_\mu \psi_{L,R} - m_\psi \bar{\psi}_L \psi_R + \text{H.c.} \\ & - G_{\mu\nu} G^{\mu\nu} + \bar{\theta} G_{\mu\nu} \tilde{G}^{\mu\nu}, \end{aligned} \quad (18)$$

²This assumption is made for illustrative purposes, there is no physical reason why $\bar{\theta}$ would have to be small. The result that η' turns $\theta_{\text{eff}} = 0$ for massless quarks holds for any value of $\bar{\theta}$.

(irrelevant numerical factors are dropped for simplicity). We start by analyzing the theory for $m_\psi = 0$ and switch on this explicit mass term later.

For $m_\psi = 0$, the theory contains two interesting global $U(1)$ symmetries; the axial symmetry in (12) and the PQ symmetry in (3) and (7). We can combine them to get an anomalous axial symmetry, which we shall denote by $U(1)_A$,

$$\Phi \rightarrow e^{i\alpha} \Phi, \quad \Psi \rightarrow e^{i\frac{1}{2}\alpha\gamma_5} \Psi, \quad \psi \rightarrow e^{i\frac{1}{2}\alpha\gamma_5} \psi, \quad (19)$$

and the anomaly-free $U(1)_V$ symmetry,

$$\Phi \rightarrow e^{i\alpha} \Phi, \quad \Psi \rightarrow e^{i\frac{1}{2}\alpha\gamma_5} \Psi, \quad \psi \rightarrow e^{-i\frac{1}{2}\alpha\gamma_5} \psi. \quad (20)$$

Both symmetries are spontaneously broken by two sources; the VEV of the field $\langle |\Phi| \rangle = f_\Phi$ and the QCD condensate of the light quark, $\langle \bar{\psi}\psi \rangle = \Lambda^3$.

As previously, the associated phase degrees of freedom we shall denote by θ_ϕ and θ_η , respectively. They can be expressed through corresponding canonically normalized Nambu-Goldstone bosons and their decay constants as

$$\theta_\phi = \frac{\phi}{\sqrt{2}f_\phi}, \quad \theta_\eta = \frac{\eta'}{\sqrt{2}f_\eta}. \quad (21)$$

However, the anomalous symmetry $U(1)_A$ is explicitly broken by instantons. After taking into account the $U(1)_A$ -violating instanton effect the effective Lagrangian takes the form,

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & f_\phi^2 (\partial_\mu \theta_\phi \partial^\mu \theta_\phi) + f_\eta^2 (\partial_\mu \theta_\eta \partial^\mu \theta_\eta) \\ & + \Lambda^4 \cos(\theta_\phi + \theta_\eta - \bar{\theta}). \end{aligned} \quad (22)$$

The cosine potential comes from the dilute instanton gas approximation, which suffices for illustrating our point. The exact form of the potential is unimportant. The only essential information is that the potential is a periodic function with minimum at $\bar{\theta}_{\text{eff}} = 0$.

The phase degree of freedom that cancels $\bar{\theta}$ is $\theta_\phi + \theta_\eta$ which represents the Goldstone phase of the spontaneously broken axial $U(1)_A$ symmetry. In terms of the canonically normalized field, which we shall denote by a_η , it can be written as

$$\theta_\phi + \theta_\eta = \frac{a_\eta}{\sqrt{2}\tilde{f}}, \quad (23)$$

where

$$\tilde{f} \equiv \frac{f_\phi f_\eta}{f} \quad \text{and} \quad f = \sqrt{f_\phi^2 + f_\eta^2}. \quad (24)$$

The degree of freedom a_η is a combination of the η' and ϕ ,

$$a_\eta = \eta' \cos(\xi) + \phi \sin(\xi), \quad (25)$$

with mixing angle ξ given by $\sin(\xi) = f_\eta/f$. Since $f_\phi \gg f_\eta$, it is clear that the mixing angle is tiny, $\xi \simeq \frac{f_\eta}{f_\phi} \ll 1$. Correspondingly, the decay constant is $\tilde{f} \simeq f_\eta$ and the true axion,

$$a_\eta \simeq \eta' + \xi\phi, \quad (26)$$

is mostly made out of η' with a negligible $\mathcal{O}(\xi)$ admixture from ϕ . The mass of this particle is generated from the instanton term and is

$$m_\eta \simeq \frac{\Lambda^2}{f_\eta}. \quad (27)$$

The orthogonal degree of freedom,

$$a_\phi \equiv \phi \cos(\xi) - \eta' \sin(\xi) \simeq \phi - \xi\eta', \quad (28)$$

which is predominantly made out of ϕ , remains massless. This field corresponds to a Goldstone boson of the spontaneously broken $U(1)_V$ symmetry. Since the $U(1)_V$ is anomaly free and thus not explicitly broken by instantons, its Goldstone boson remains massless.

Thus, when the theory contains a massless quark, the axion is the η' meson [3], whereas the phase of the PQ field, ϕ , remains massless and plays no role in canceling $\bar{\theta}$.

B. Massive-quark case

In order to account for the effect of a nonzero mass of the light quark, we now deform the theory by putting $m_\psi \neq 0$. As a result, the theory possesses no spontaneously broken global symmetry spared of explicit breaking. Correspondingly, both Goldstone bosons become massive.

To analyze the vacuum structure, we study the effective Lagrangian of the phases which now takes the form,

$$\mathcal{L}_{\text{eff}} = f_\phi^2 (\partial_\mu \theta_\phi \partial^\mu \theta_\phi) + f_\eta^2 (\partial_\mu \theta_\eta \partial^\mu \theta_\eta) + \Lambda^4 \cos(\theta_\phi + \theta_\eta - \bar{\theta}) + \Lambda_m^4 \cos(\theta_\eta). \quad (29)$$

First of all, we notice that all CP -violating phases are fully canceled in the vacuum, which is achieved for

$$\theta_\eta = 0, \quad \theta_\phi = \bar{\theta}. \quad (30)$$

Due to the last term in (29), the previous mass eigenstates are slightly rearranged. The correction to the eigenstates (26) and (28) are $\sim \mathcal{O}(m_\psi/\Lambda)$. So they are still given mostly by η' and ϕ , respectively.

Additionally, the masses of the eigenstates a_η and a_ϕ change. The mass of a_η gets a correction,

$$\delta m_\eta^2 \simeq \frac{\Lambda_m^4}{f_\eta^2} \sim m_\psi \Lambda. \quad (31)$$

More importantly, the previously zero-mass eigenstate a_ϕ now gains the mass,

$$m_\phi \simeq \frac{\Lambda_m^2}{f_\phi}. \quad (32)$$

This is expected, since both $U(1)_V$ and $U(1)_A$ are now explicitly broken; $U(1)_A$ by the anomaly and $U(1)_V$ by the mass term.

The true axion, i.e., the mode that makes $\theta_{\text{eff}} = 0$, is now $\theta_\phi \sim a_\phi + \xi a_\eta$. Thus, the true axion now resides predominantly in the a_ϕ boson with a small admixture from a_η . This boson is what traditionally is called the axion, however this terminology hides the essence of the story.

Without ϕ , the η' would reduce the value of $\bar{\theta}_{\text{eff}}$ down to a fraction $\sim m_\psi/\Lambda$, as shown in Eq. (13). Thus, one can say that the η' meson is already doing part of the job. However, this is not nearly sufficient for reducing the $\bar{\theta}_{\text{eff}}$ to the phenomenologically acceptable value. Therefore, for solving the strong- CP puzzle, having ϕ is crucial.

VI. STRINGS AND WALLS

We shall now discuss the topological defects that arise in our model that takes into account the mixing between η' and the axion. Since by shifting the fields we can eliminate the CP -violating phase, without loss of generality we set $\bar{\theta} = 0$. In our analysis we repeat the same steps as before. Namely, we first find the topologically nontrivial solutions for $m_\psi = 0$ and then analyze how these structures are affected by $m_\psi \neq 0$.

A. Massless-quark case

We start with the case $m_\psi = 0$. The effective Lagrangian of the phase degrees of freedom is given in Eq. (22). In the absence of the anomaly, both spontaneously broken symmetries $U(1)_A$ and $U(1)_V$ would be exact, with the topology of the vacuum manifold being the two-torus $T_2 = S_1 \times S_1$. Such a vacuum would support two independent integer winding numbers, n_ϕ and n_η , in the phases of the two condensates, $\langle \Phi \rangle$ and $\langle \bar{\psi}\psi \rangle$. Correspondingly, there would exist two types of cosmic strings with uncorrelated windings in θ_ϕ and θ_η , respectively.

In polar coordinates, r, φ, z , around straight infinite strings along the z -axis the respective phases wind as

$$\theta_\eta = n_\eta \varphi, \quad \theta_\phi = n_\phi \varphi. \quad (33)$$

The lowest energy configurations are given by winding numbers $|n_\phi| = |n_\eta| = 1$.

However, the QCD anomaly changes the picture and correlates the two windings. The main term responsible for the correlation is the instanton-induced potential. The structure can be understood by analyzing the effective low-energy theory of the phase degrees of freedom.

The equations of motion are

$$\square\theta_\phi + \frac{\Lambda^4}{2f_\phi^2} \sin(\theta_\phi + \theta_\eta) = 0, \quad (34)$$

$$\square\theta_\eta + \frac{\Lambda^4}{2f_\eta^2} \sin(\theta_\phi + \theta_\eta) = 0. \quad (35)$$

There exist two classes of relevant solutions with correlated phases.

1. Axion- η' string of global $U(1)_V$

First, we focus on the $U(1)_V$ symmetry. As explained above, this is the $U(1)$ symmetry which is not explicitly but only spontaneously broken. Thus, there exist global strings of $U(1)_V$. Far away from the straight string the two phases wind oppositely,

$$\theta_\phi = -\theta_\eta = n_V \varphi, \quad (36)$$

where φ is the polar angle in the plane perpendicular to the string and n_V is the winding number. The lowest energy string is the one with unit winding number, $n_V = \pm 1$. Due to the instanton-induced potential, there exists no string around which only one phase winds.

This can be seen by evaluating the energy functional around a circle of radius R in the perpendicular plane,

$$\sigma = \int_0^{2\pi R} dl f_\phi^2 (\partial_l \theta_\phi)^2 + f_\eta^2 (\partial_l \theta_\eta)^2 - \Lambda^4 \cos(\theta_\phi + \theta_\eta), \quad (37)$$

where $l = R\varphi$. The uniform winding of only one phase would necessarily create a nonuniformity (domain wall) due to the last term.

In other words, a string solution without domain walls must not be explicitly broken by the cosine potential. Thus, we need a field configuration with $\theta_\phi + \theta_\eta = 0$, which requires that θ_ϕ and θ_η wind oppositely.

2. η' string-wall system of $U(1)_A$

Let's now turn to the anomalous $U(1)_A$ symmetry. This symmetry is spontaneously broken and, due to the anomaly, it is also explicitly broken. Thus, we have $U(1)_A$ strings attached to domain walls.

The domain wall appears because of the explicit breaking, i.e., the cosine potential. Energetically, the phase prefers to stay in the minimum as much as possible and

concentrate the entire winding in a small region, which is the domain wall.

Naively, around a $U(1)_A$ string both phases would wind equally, $\theta_\phi = \theta_\eta = n_A \varphi$. However, this is not favorable energetically. Instead, to leading order in the f_η/f_ϕ expansion, the lowest energy string winds only in the θ_η -phase. This winding corresponds to a transformation that combines $U(1)_A$ and $U(1)_V$.

The wall of η' , with the wall placed at $l = 0$ in the compact coordinate, is described by³

$$\theta_\eta = 4 \arctan(e^{m_\eta l}), \quad \theta_\phi = 0, \quad (38)$$

where the mass m_η was defined in Eq. (27).

The tension (energy per unit area) of the η' wall is given by

$$\sigma_\eta \sim \Lambda^2 f_\eta. \quad (39)$$

If instead of winding in θ_η we wind in θ_ϕ , the corresponding domain wall would have a much higher tension,

$$\sigma_\phi \sim \Lambda^2 f_\phi \gg \sigma_\eta. \quad (40)$$

Therefore, the energy cost of these domain walls is much larger.

To conclude, in the lowest energy topological defects of both types (strings as well as walls bounded by strings) the phase of the light-quark condensate, θ_η , winds. Correspondingly, in the core of the strings the quark condensate must vanish.

B. Massive light quark

We now deform the theory by putting $m_\psi \neq 0$. The effective Lagrangian is given by Eq. (29). The last term comes from the quark mass that breaks the $U(1)_V$ symmetry explicitly. In the following, we consider the effect of this explicit breaking on the structure of the topological defects.

1. η' string-wall system

For $m_\psi \ll \Lambda$, the effect of a light-quark mass term on the $U(1)_A$ string-wall system is insignificant, and it is still dominated by the winding of the θ_η phase. In the first approximation, the solution is still given by Eq. (38) with the mass m_η slightly shifted by the explicit breaking term as given by Eq. (31).

³This is the solution to the equation of motion in Eq. (34) above, which is the sine-Gordon equation. The solution is for an infinite radius and represents a good approximation for $R \rightarrow \infty$.

2. η' -axion string-wall system

The effect of $m_\psi \neq 0$ on the would-be $U(1)_V$ -string is more dramatic, since a domain wall gets attached to it. However, the phase θ_η continues to fully wind around the string.

This can be understood in simple terms by analyzing the energy balance among various terms in the energy functional around a large circle of radius $R \gg m_\phi^{-1}$. After the inclusion of the mass term it becomes

$$\sigma = \int_0^{2\pi R} dl f_\phi^2 (\partial_l \theta_\phi)^2 + f_\eta^2 (\partial_l \theta_\eta)^2 + \Lambda^4 (1 - \cos(\theta_\phi + \theta_\eta)) + \Lambda_m^4 (1 - \cos(\theta_\eta)), \quad (41)$$

where for convenience we normalized the energy of the vacuum to zero. Since the topological structure is determined by the asymptotic values of the fields far away from the string core, for large R , the variational problem is effectively one dimensional.

Let us examine what is traditionally called an axionic string. This string is usually defined as the one with nonzero winding number n_ϕ of the Peccei-Quinn field phase θ_ϕ . The uniform winding of only this phase, $\theta_\phi = l/R$, would cost energy $\sigma_R = 2\pi(f_\phi^2/R + R\Lambda^4)$, which diverges linearly with R .

The system can lower this divergent energy by two tricks:

- (1) The opposite winding in $\theta_\eta = -\theta_\phi$, which gets rid of the energy of the Λ^4 -term. Since $\Lambda \gg \Lambda_m$, the first cosine in Eq. (41) is much more costly in energy;
- (2) Limiting the nontrivial winding of the phases to fixed arches, while keeping the rest of the circle in the vacuum. This turns the axionic string into a string-wall system.

The winding length can be estimated by balancing the gradient energy and the potential energy. To leading order, we approximate the windings with a linear ansatz,

$$\theta_\eta = \frac{l}{L_\eta} \quad \text{for } 0 \leq l \leq 2\pi L_\eta \quad (42)$$

and

$$\theta_\phi = -\frac{l}{L_\phi} \quad \text{for } 0 \leq l \leq 2\pi L_\phi, \quad (43)$$

while outside of the respective arches the phases stay constant and in the vacuum. The L_η and L_ϕ are the lengths where the phases wind. They are also the domain wall thickness. We expect $L_\phi \geq L_\eta$ because $f_\phi > f_\eta$. The functions are shown in Fig. 1.

By explicit minimization of the functional (41) (see the Appendix), we estimate that the energetically most

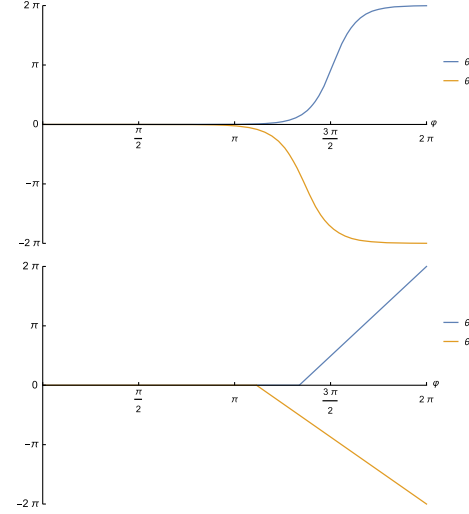


FIG. 1. Illustration of the wall profiles for the linear ansatz (42), (43) and for their smoothed out versions. This behavior is confirmed by numerical analysis [24].

favorable regime is $L_\phi = L_\eta \equiv L$ with the optimal winding length,

$$L = \frac{f}{\Lambda_m^2}, \quad (44)$$

where we used $f \equiv \sqrt{f_\phi^2 + f_\eta^2}$. The length of the winding arch (44) sets the thickness of the domain wall,

$$L_* = \frac{f}{\Lambda_m^2} \sim m_\phi^{-1}, \quad (45)$$

which matches the standard thickness of the axionic domain wall. The novelty is the simultaneous winding of θ_η .

We can refine the above energetic estimate by imposing the constraint $\theta_\phi = -\theta_\eta$ on the energy functional (41) and solving the equation of motion. For $R \gg m_\phi^{-1}$ this gives

$$\theta_\phi \simeq -\theta_\eta \simeq 4 \arctan(e^{m_\phi l}). \quad (46)$$

Of course, in reality, the above wall configuration has to be understood as the leading-order estimate in an m_ψ/Λ expansion.

We conclude that, in a theory in which the PQ field communicates to QCD via a heavy quark, what is traditionally called the axion string-wall system involves winding of the phase of the light-quark condensate, involving the η' meson. This has some important implications.

C. Implications for anomaly inflow

It is well known that fermions, which receive their mass from Yukawa couplings to the string field (such as the

Peccei-Quinn field Φ) with topologically nontrivial phase, deposit zero modes on the string [25].

From the point of view of the $(1 + 1)$ -dimensional theory on the string world-sheet these are massless fermions. Moreover, each Dirac flavor coupled to Φ with winding number n deposits n zero modes of definite chirality, determined by the sign of the winding number.

Witten showed [1] that when the fermions are electrically charged, their zero modes can carry a superconducting current along the string. That is, the string behaves as sort of a superconducting wire. This has important astrophysical implications [26].

When the global symmetry forming a string is anomalous, the charges of the localized fermions are not balanced. In such a case, despite the fact that the parent $(3 + 1)$ -dimensional theory has no gauge anomaly, the $(1 + 1)$ -dimensional theory of zero modes appears to be anomalous. This question was addressed by Callan and Harvey, who showed that the anomaly is canceled by the inflow of charge from the bulk onto the string [2].

It was discussed by Lazarides and Shafi that such a phenomenon must be exhibited by axionic strings, since the global symmetry forming it is (assumed to be) the anomalous Peccei-Quinn symmetry [27]. They suggested that the anomaly inflow must take place along the domain walls attached to the string (for more discussions, see [28,29]).

In general, the presumed anomaly on the axionic string can have important astrophysical implications due to the correlation between the charge and chirality of the current carriers [30–32].

Our analysis sheds a novel light on the question of anomaly inflow on the axionic string. Namely, in the considered prototype model with one heavy and one light quark, around the string of the lowest energy, the phase of the quark condensate, θ_η , winds oppositely to the phase of the Peccei-Quinn field Φ . Due to this, the set of deposited zero modes is anomaly free.

To see this, let us consider the UV-Lagrangian from Eq. (9). The phase of the PQ field θ_ϕ winds around the axionic string. Since the heavy quark Ψ has a Yukawa coupling with the PQ field, it deposits a zero mode. This zero mode has a definite chirality and it carries both electric charge as well as color, thereby creating an anomalous $(1 + 1)$ -dimensional current.

However, we have seen that, in the lowest energy state, the phase of the light-quark condensate, θ_η , winds in the opposite way. This has two effects. First, the quark condensate vanishes in the core of the string. Secondly, the light quark, ψ , deposits a zero mode in the core of the string.

The appearance of the light-fermion zero mode on the string can be seen in several ways. The shortcut is the index theorem.

It can also be seen directly, by noticing that via the 't Hooft determinant the instantons generate a direct Yukawa

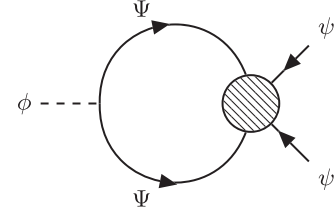


FIG. 2. A diagram describing how the effective Yukawa coupling (47) is generated by the 't Hooft vertex.

coupling between the Peccei-Quinn field, Φ , and the light fermions,

$$\Phi \bar{\psi}_L \psi_R + \text{H.c.} \quad (47)$$

The diagram generating this effective interaction is shown in Fig. 2. This coupling breaks $U(1)_A$ but it is invariant under the anomaly-free $U(1)_V$ symmetry. Hence, the Peccei-Quinn string with winding number $n = +1$ is seen by the light fermions as the antistring with winding number $n = -1$. Correspondingly, the localized zero mode from ψ has the opposite chirality from the one deposited by the heavy fermion, Ψ .

Since the zero modes of the ψ and Ψ have opposite chirality, their gauge anomalies cancel out. As a result, the fermion content of $(1 + 1)$ -dimensional theory on the string is anomaly free.

This is also apparent from the fact that around the minimal string the VEVs wind by the $U(1)_V$ transformation, which is an anomaly-free symmetry.

The explicit mass term m_ψ of the light quark does not affect the zero mode structure as long as $m_\psi \ll \Lambda$. This follows from the general consideration of the effect of an explicit bulk mass of the fermion on the zero-mode structure on the string [33].

In general, if the mass gap generated by the explicit breaking of the chiral symmetry is smaller than the gap generated from the spontaneous breaking, the zero mode fermion still exists. The effect of explicit breaking boils down to a small deformation of the wave-function profile of the zero mode.

Of course, for $m_\psi \neq 0$, simultaneously, domain walls get attached to the string, as already discussed. The thickness of the wall is much larger than the size of the string core.

As explained in the previous section, there exists a minimal η' string which winds only in η' . This winding involves the anomalous $U(1)_A$ symmetry such that the η' string has an anomaly.

The general message of this section is that the winding of the phase of the quark condensate around the string changes its zero-mode structure and, correspondingly, the anomaly content. In particular, even for a string formed by the PQ field, the energetics can force the phases to wind by an anomaly-free transformation, implying that the zero

mode content of the axion string is anomaly free. In general, the anomaly content of the string depends both on the quark-flavor structure of the model and on the specifics of their winding numbers.

D. More heavy-quark flavors

Let us now discuss how the existence of multiple heavy-quark flavors Ψ_i changes the story. The index i runs over the number of flavors N_f . At the same time, we keep a single light-quark flavor.

The anomaly-free symmetry, $U(1)_V$, acts on the fields as

$$\Phi \rightarrow e^{i\alpha}\Phi, \quad \Psi_i \rightarrow e^{i\frac{1}{2}\alpha\gamma_5}\Psi_i, \quad \psi \rightarrow e^{-i\frac{N_f}{2}\alpha\gamma_5}\psi. \quad (48)$$

At the same time, under the anomalous $U(1)_A$ symmetry the fields transform as

$$\Phi \rightarrow e^{i\alpha}\Phi, \quad \Psi_i \rightarrow e^{i\frac{1}{2}\alpha\gamma_5}\Psi_i, \quad \psi \rightarrow e^{i\frac{1}{2}\alpha\gamma_5}\psi. \quad (49)$$

The $U(1)_A$ possesses an anomaly-free Z_{2N_f} -subgroup, for $\alpha = \frac{2\pi k}{N_f}$ with $k = 1, 2, \dots, 2N_f$.

This is also visible from the effective Lagrangian of the phases, which takes the form

$$\begin{aligned} \mathcal{L} = & f_\phi^2 (\partial_\mu \theta_\phi \partial^\mu \theta_\phi) + f_\eta^2 (\partial_\mu \theta_\eta \partial^\mu \theta_\eta) \\ & + \Lambda^4 (\cos(N_f \theta_\phi + \theta_\eta - \bar{\theta}) - 1) \\ & + \Lambda_m^4 (\cos(\theta_\eta) - 1). \end{aligned} \quad (50)$$

For $\Lambda_m = 0$, the Lagrangian is invariant under $U(1)_V$ as well as under Z_{N_f} (the phases of the condensates do not transform under Z_2 that flips the signs of fermions). Again, without loss of generality we can set $\bar{\theta} = 0$.

Let us now consider topological defects. Setting first $\Lambda_m = 0$, we have two types of defects. The first type are $U(1)_V$ strings. Since the winding numbers must be integer, the string with minimal winding numbers has

$$n_\eta = \pm N_f \quad \text{and} \quad n_\phi = \mp 1. \quad (51)$$

Correspondingly, the phases are

$$\theta_\eta = \pm N_f \varphi, \quad \theta_\phi = \mp \varphi. \quad (52)$$

After switching on the Λ_m^4 term, this string becomes a junction of N_f domain walls. Across each wall, the changes of the phases are given by $\Delta\theta_\eta = 2\pi$ and $\Delta\theta_\phi = \frac{2\pi}{N_f}$ respectively, whereas in between the walls the phases stay constant.

The second type of defect is the η' -string wall system which behaves just as in the case of one heavy flavor. As before, it is energetically more favorable to only have winding in the η' direction instead of both.

VII. AXION MODELS

The above-discussed simplified model with one heavy and one light quark captures the essence of the story for fully realistic axion models.

For the heavy-quark hidden axion model (also known as the “Kim-Shifman-Vainshtein-Zakharov” (KSVZ) model) [34], in which the PQ field couples to QCD via an exotic heavy quark, the connection is straightforward. In our model, the Ψ is the exotic heavy quark. The role of the light quark, ψ , is played by the up-quark, u , which is the lightest quark in the Standard Model.

The same is true for scenarios of “Dine-Fischler-Srednicki-Zhitnitsky” (DFSZ) type [35]. In these models no exotic quarks are introduced. Instead, the PQ field communicates with the QCD sector through the ordinary quarks via Higgs fields. For illustrative purposes, it suffices to consider a single generation realization of this scenario, in which we consider two flavors of light quarks, u and d .

To make the chiral $U(1)_{PQ}$ symmetry compatible with nonzero quark masses, one needs to introduce two designated Higgs doublets that give masses to the u and d quarks, respectively. We shall focus on the components with nonzero VEVs which are electrically neutral and we denote them by $H_u(x) = \rho_u(x)e^{i\chi_u(x)}$ and $H_d(x) = \rho_d(x)e^{i\chi_d(x)}$.⁴ The quark masses are generated from the following Yukawa couplings:

$$g_u H_u \bar{u}_L u_R + g_d H_d \bar{d}_L d_R + \text{H.c.}, \quad (53)$$

where g_u and g_d are the Yukawa coupling constants. H_u and H_d carry opposite gauge hypercharges. The above Yukawa couplings are invariant under the chiral PQ symmetry,

$$\begin{aligned} H_u &\rightarrow e^{i\alpha} H_u, & H_d &\rightarrow e^{i\alpha} H_d, \\ u &\rightarrow e^{-i\frac{1}{2}\alpha\gamma_5} u, & d &\rightarrow e^{-i\frac{1}{2}\alpha\gamma_5} d. \end{aligned} \quad (54)$$

The nonzero VEVs $\langle \rho_u \rangle = v_u$, $\langle \rho_d \rangle = v_d$ spontaneously break the $U(1)_{PQ}$ -symmetry and generate the quark masses, $m_u = g_u v_u$, $m_d = g_d v_d$. One combination of the phase degrees of freedom becomes a longitudinal component of the electroweak Z-boson, whereas the other would be an axion. However, it is well-known that since the VEVs v_u , v_d are bound from above by the weak-interaction scale, such an axion is phenomenologically inconsistent, as its interaction with the Standard Model would be too strong. In order to “hide” it, it is necessary to mix χ_u , χ_d with the phase of a PQ field, θ_ϕ , with much higher scale. This is achieved through the following phase-dependent coupling in the Lagrangian,

$$\mu \Phi H_u H_d + \text{H.c.} \quad (55)$$

⁴The electrically-charged components of the doublets are not relevant for the quark masses and play no role in the current discussion.

Under the PQ symmetry, Φ transforms as

$$\Phi \rightarrow e^{-i2\alpha}\Phi. \quad (56)$$

Below the QCD scale quarks condense. Since the quark masses are different, it is useful to keep track of their phase degrees of freedom separately; $\langle \bar{u}u \rangle = \Lambda_u^3 e^{i\theta_u(x)}$, $\langle \bar{d}d \rangle = \Lambda_d^3 e^{i\theta_d(x)}$, with scales $\Lambda_u \simeq \Lambda_d \sim \Lambda$.

After taking into account instanton effects, the effective Lagrangian for phase degrees of freedom becomes

$$\mathcal{L}_{\text{eff}} = f_\phi^2 (\partial_\mu \theta_\phi)^2 + \quad (57)$$

$$+ v_u^2 (\partial_\mu \chi_u - g Z_\mu)^2 + v_d^2 (\partial_\mu \chi_d + g Z_\mu)^2 + \quad (58)$$

$$+ f_u^2 (\partial_\mu \theta_u + g Z_\mu)^2 + f_d^2 (\partial_\mu \theta_d - g Z_\mu)^2 + \quad (59)$$

$$+ \mu f_\phi v_u v_d \cos(\theta_\phi + \chi_u + \chi_d) + \quad (60)$$

$$+ m_u \Lambda_u^3 \cos(\chi_u + \theta_u) + m_d \Lambda_d^3 \cos(\chi_d + \theta_d) + \quad (61)$$

$$+ m_d \Lambda_u^3 \cos(\theta_u - \chi_d - \bar{\theta}) + \quad (62)$$

$$+ m_u \Lambda_d^3 \cos(\theta_d - \chi_u - \bar{\theta}) + \quad (63)$$

$$+ m_u m_d \Lambda^2 \cos(\chi_u + \chi_d + \bar{\theta}) + \quad (64)$$

$$+ \Lambda^4 \cos(\theta_u + \theta_d - \bar{\theta}). \quad (65)$$

Let us make a few clarifying remarks. The first line represents the kinetic term of the phase of the PQ field. This field carries no gauge quantum number and does not couple directly to any of the gauge fields or fermions of the Standard Model.

The lines (58) and (59) describe the kinetic terms of the phases of the Higgses and the quark condensates, respectively. All these phases shift under the combination of the gauge hypercharge and weak isospin symmetry of the Standard Model, which acts on the Z-boson as

$$Z_\mu \rightarrow Z_\mu + \partial_\mu \beta, \quad (66)$$

where $\beta(x)$ is the gauge parameter. Under this symmetry the phases shift as

$$\chi_u \rightarrow \chi_u + \beta, \quad \chi_d \rightarrow \chi_d - \beta, \quad \theta_u \rightarrow \theta_u - \beta, \quad \theta_d \rightarrow \theta_d + \beta. \quad (67)$$

That is, the gauge symmetry (66) is Higgsed not only by the VEVs of the Higgs doublets, but also by the quark condensates. Correspondingly, the following combination of the phases:

$$v_u^2 \chi_u - v_d^2 \chi_d - f_u^2 \theta_u + f_d^2 \theta_d, \quad (68)$$

becomes the longitudinal component of the Z-boson.

Line (60) corresponds to the phase-dependent potential term emerging from the cross coupling (55) between PQ and Higgs fields. Line (61) describes the phase-dependent potential emerging from the Yukawa couplings between the Higgses and the respective quark condensates in Eq. (53).

The couplings (62), (63), and (64) describe the terms generated by combining various Yukawa couplings with the 't Hooft determinant. Finally, the term (65) is the quark-mass-independent instanton contribution.

As it is clear from Eq. (67), all the arguments of the phase-dependent potentials are gauge invariant, as they should be. It is clear that in the minimum of the potential all the CP-odd phases vanish and the strong-CP problem is solved. Therefore, without loss of generality, we can set $\bar{\theta} = 0$.

Let us now discuss the structure of cosmic strings and domain walls arising in this scenario. First, notice that while the axionic strings are expected to be global, they nevertheless carry a gauge magnetic flux [36]. This may come as a surprise, since the naive expectation is that around axion-like strings the phases wind by the global $U(1)_{PQ}$ transformation.

However, the requirement that each phase must be single-valued modulo 2π , implies that the topologically nontrivial background sources the gauge field. That is, around any string the winding number of each phase must be an integer. As shown in [36] this implies that the axionic string carries a magnetic flux of the Z-boson. This flux is given by

$$\text{flux} = \oint dx_\mu Z^\mu = \frac{1}{g_z} \frac{v_u^2 n_{\chi_u} - v_d^2 n_{\chi_d} - f_u^2 n_{\theta_u} + f_d^2 n_{\theta_d}}{v_u^2 + v_d^2 + f_u^2 + f_d^2}, \quad (69)$$

where the integral is taken around an asymptotic closed path around the string. This flux is noninteger in units of the inverse gauge coupling of the Z-boson, $1/g_z$. Thus, unlike the case of the Nielsen-Olesen string, the asymptotic pure-gauge form of the gauge field cannot entirely compensate the gradient energy of the winding phases. Correspondingly, the energy per unit length of the string is logarithmically divergent in the transverse direction. Thus, the axionic string shares properties of both local and global strings. Because of this, in [36] such strings were referred to as “semiglobal”.

Notice that due to the hierarchy between the weak and QCD scales, $v_u, v_d \gg f_u, f_d$, the contribution to the Z-flux from the phases of the quark condensate relative to the phases of the Higgs bosons are negligible ($\sim 10^{-4}$) and can be ignored. Therefore, the flux can be well-approximated by

$$\text{flux} \simeq \frac{1}{g_z} \left(n_{\chi_u} - \frac{v_d^2 (n_{\chi_u} + n_{\chi_d})}{v_u^2 + v_d^2} \right). \quad (70)$$

Notice that an accidental cancellation of the flux, due to an interplay between the VEVs and the winding numbers, is highly unlikely. Specifically, in the case of minimal winding, e.g., $n_{\chi_u} = \pm 1, n_{\chi_d} = 0$, the flux would be

$$\text{flux} \simeq \pm \frac{1}{g_z} \frac{v_u^2}{v_d^2 + v_u^2}, \quad (71)$$

and the cancellation of the flux is impossible even at the expense of fine tuning.

A. Massless up-quark case

Let us now discuss various winding configurations that give strings and walls. As in the previous sections, we start by setting the mass of the lightest quark, the up-quark, to zero. For $m_u = 0$, the effective potential is

$$\begin{aligned} V = & -\mu f_\phi v_u v_d \cos(\theta_\phi + \chi_u + \chi_d) \\ & - m_d \Lambda_d^3 \cos(\chi_d + \theta_d) - m_d \Lambda_u^3 \cos(\theta_u - \chi_d) \\ & - \Lambda^4 \cos(\theta_u + \theta_d). \end{aligned} \quad (72)$$

This potential is invariant under the following anomaly free global $U(1)_V$ symmetry,

$$\begin{aligned} \theta_\phi &\rightarrow \theta_\phi - 2\alpha, & \chi_u &\rightarrow \chi_u + \alpha, & \chi_d &\rightarrow \chi_d + \alpha, \\ \theta_u &\rightarrow \theta_u + \alpha, & \theta_d &\rightarrow \theta_d - \alpha. \end{aligned} \quad (73)$$

Notice that under this symmetry θ_u and θ_d shift in the opposite way. Since the symmetry is nonlinearly realized, there exists a corresponding massless Nambu-Goldstone boson, which resides predominantly in the combination $2\theta_\phi - \chi_u - \chi_d$,⁵ with a small admixture from the neutral pion, π^0 . At the same time, the axion that cancels $\bar{\theta}$ is the η' meson residing in $\theta_u + \theta_d$.

Since the symmetry (73) is exact, it can give rise to anomaly-free PQ strings, similar to the ones we discussed previously.

Consider a string with unit winding numbers, $n_\phi = \pm 1$. In order to keep the potential (72) vanishing while retaining the single-valued phases, only one of the Higgs phases can wind around the string. As explained, in order to reconcile the topological nontriviality with single valued phases, winding cannot be purely by a global $U(1)_V$ transformation but must be combined with a $U(1)_Z$ -gauge transformation (67). According to [36], this leads to a nonvanishing magnetic Z-flux in the core of the string, given by (69) [or approximately by (70)].

⁵This is the combination that is not getting a mass from the term (55). In terms of canonically normalized fields, this combination is predominantly made out of ϕ , due to the hierarchy $f_\phi \gg v_u, v_d$.

With only one of the Higgs phases winding, there exist two winding options; $n_{\chi_u} = \mp 1, n_{\chi_d} = 0$ or vice versa $n_{\chi_u} = 0, n_{\chi_d} = \mp 1$. In both cases there is a Z-boson magnetic flux trapped in the string with the value (70).

In the first case, the quark condensates do not wind $n_{\theta_u} = n_{\theta_d} = 0$. In the second case, the winding numbers are $n_\phi = -n_{\chi_d} = -n_{\theta_u} = n_{\theta_d}$, in order to keep the potential vanishing. This winding pattern of the quark phases around the string corresponds to a neutral pion, π^0 .

Analogously, we can consider strings with higher winding numbers. All configurations in which χ_d winds are accompanied by windings of the quark condensates with the winding numbers satisfying the relation $n_{\chi_d} = n_{\theta_u} = -n_{\theta_d}$.

It is instructive to compare this to our discussion of the $U(1)_V$ PQ string in the KSVZ-type model (18) of the previous section. There, in order to keep the potential vanishing, it was necessary for θ_η to wind around the string oppositely to θ_ϕ . In the present case of a DFSZ-type model, the situation is different, as the PQ field does not directly couple to the quarks. The winding of θ_ϕ is compensated by the Higgs phases. Therefore, as long as one of the quarks is massless, depending on which of the Higgs phases winds, the winding of the quark phases may or may not be necessary. However, when necessary, both phases wind simultaneously in the opposite directions, $\theta_u = -\theta_d$, thereby, realizing the π^0 winding pattern in the flavor space.⁶

B. Massive up-quark case

Now we wish to investigate what happens to the $U(1)_V$ strings, once we switch on $m_u \neq 0$. This mass induces an explicit breaking of the $U(1)_V$ symmetry via the following additional terms in the effective potential:

$$\begin{aligned} V_{m_u} \equiv & -m_u \Lambda_u^3 \cos(\chi_u + \theta_u) - m_u \Lambda_d^3 \cos(\theta_d - \chi_u) \\ & - m_u m_d \Lambda^2 \cos(\chi_u + \chi_d). \end{aligned} \quad (74)$$

The explicit breaking gives rise to domain walls, which get attached to the strings.

Let us take another look at the situation with $n_\phi = 1$. In the massless up-quark case we found that, in order to cancel the winding of the PQ phase, at least one of the Higgs phases must wind. For $n_{\chi_d} = 0$ (at least in the lowest energy configuration) there was no winding necessary in the quark phases, whereas the case $n_{\chi_d} \neq 0$ would induce winding in π^0 .

In the massive case, due to the additional terms (74) induced by m_u , the winding in χ_u can stimulate additional windings (or nontrivial angular dependences) in the quark

⁶In fact, since the quark phases contribute to the overall energy via the kinetic terms and the gauge flux, the case of χ_u winding is slightly favored compared to χ_d winding. However, the contribution from the quark phases is $\sim f_u^2 - f_d^2$, which is negligible. Similarly, the contribution from the flux is $\sim f_u^2 + f_d^2$.

phases. However, in a sector with given n_{χ_u} the ground state configuration is determined by a more detailed energy balance.

We observe that, just as in the case of KSVZ-type models with heavy exotic quarks, also in DFSZ-like theories the winding in the PQ axion is in general accompanied by winding in the phases of quark condensate.

VIII. INTERNAL STRUCTURE OF QCD WALLS

As we have seen, the topological structure of the QCD vacuum supports the existence of 2π domain walls. Across the wall, the phase of the quark condensate winds by 2π . The winding can assume different patterns of flavor dependence. Focusing on the quark condensates of the first generation, $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$, there exist windings in two basic flavor directions.

In the case $n_{\theta_u} = -n_{\theta_d} \neq 0$, we can say that the winding takes place in the direction of π^0 . In the case of $n_{\theta_u} + n_{\theta_d} \neq 0$, we can say that the winding involves the η' meson.

In both cases the domain wall can come in the form of a closed bubble of a certain radius, or as a membrane bounded by a string. A planar (infinite) wall can also exist. However, since the vacua on the both sides of the wall are identical, a hole can be punched through it. The edge of this hole represents a loop of a cosmic string. Thus, for each category (π^0 or η') the existence a 2π domain wall implies the existence of a corresponding string that bounds it.

Let us now discuss the internal structure of these walls. The important question that we would like to address is whether the order parameter in form of the quark condensate remains nonzero throughout the wall. This is a necessary condition for the phase to be well-defined. When we pass through the wall, the quark condensate can rotate in the flavor space nontrivially. As long as its absolute value stays nonzero, the winding of the phase is well-defined and the existence of the wall is guaranteed by topology.

Notice that the answer to this question is important for η' domain walls formed by the quark condensate in pure QCD, without an external axion. For axionic domain walls of the PQ symmetry, the question is less pressing, since the PQ phase θ_ϕ is well-defined everywhere and is guiding the phases of the light-quark condensates.

The reason why the question is nontrivial for the η' domain wall is that through such a wall θ_{eff} changes by 2π . Correspondingly, QCD probes phases for which the back-reaction to the order parameter may become significant. In particular, the corrections from beyond the dilute instanton gas approximation can become important. Notice that the analogous question does not arise for domain walls with π^0 winding.

Of course, an immediate technical obstacle for resolving the internal structure of the pure-QCD η' domain wall is that within the wall the effective theory of the order parameter breaks down. However, we are not interested

in an exact solution but only in an assurance that the condensate is nonzero across the wall.

To answer this question, a very useful guideline is provided by domain walls in $\mathcal{N} = 1$ supersymmetric QCD (SQCD) [5]. These walls are the closest cousins to the η' domain walls discussed above.

The closest relative to QCD is an $\mathcal{N} = 1$ supersymmetrized version of $SU(N)$ Yang-Mills (SYM). The role of the quarks in this theory is played by the fermionic partners of the gauge bosons, the gauginos, λ . Unlike the quarks in QCD, which transform in the fundamental representation, the gauginos transform in the adjoint representation of $SU(N)$. However, we expect that this difference is not essential for our discussion.

The theory possess an anomalous chiral symmetry, the $U(1)_R$ symmetry,

$$\lambda \rightarrow e^{i\alpha} \lambda. \quad (75)$$

Since the gauginos transform in the adjoint representation of the $SU(N)$ gauge group, the Z_N subgroup of $U(1)_R$ is anomaly free.

It is well known that similarly to ordinary QCD, $SU(N)$ SYM theory confines and breaks the anomalous $U(1)_R$ symmetry through the gaugino condensate $\langle \bar{\lambda}\lambda \rangle \neq 0$. The 't Hooft determinant generated by instantons represents a $2N$ -gaugino vertex.

As pointed out in [5], this breaking gives rise to domain walls since the condensate spontaneously breaks the anomaly-free Z_N symmetry,

$$\langle \bar{\lambda}\lambda \rangle \rightarrow e^{i\frac{2\pi}{N}} \langle \bar{\lambda}\lambda \rangle. \quad (76)$$

Moreover, due to BPS properties of the wall, the wall tension can be computed exactly in terms of the gaugino condensate.

It was conjectured in [37] that for large N the wall acts as a sort of a D -brane for large- N QCD string theory. This was further supported by the analysis of [4,38] and of many subsequent papers.

Now, while across the elementary wall the phase of the condensate changes by $2\pi/N$, the effective phase of the 't Hooft determinant changes by 2π . Correspondingly, across the wall θ_{eff} changes by 2π similarly to the η' domain walls in ordinary QCD. Due to this, the question of validity of the phase degree of freedom within the two cases should be similar.

For our purposes, the solution derived in [4] is most informative. In this paper, the functional dependence of the gaugino condensate across the domain wall was evaluated explicitly. The solution is exact in large N . In this solution the gaugino condensate is nonzero across the wall and the phase degree of freedom (the analog of η') is well-defined everywhere. Extrapolating this result to our case of ordinary QCD with quarks in the fundamental representation,

we get the assurance that η' remains well-defined across the wall.

We thus conclude that pure QCD can support at least two types of string-wall systems, determined by the winding of π^0 and η' , respectively.

IX. COSMOLOGY

As we have seen, topologically nontrivial η' or π^0 winding configurations of the QCD quark condensate will in general accompany axionic strings. Moreover, η' or π^0 string-wall defects exist in pure QCD without involving a hidden axion. This allows for an entirely different view of θ -vacuum cosmology that, to our knowledge, has not been considered previously.

Strings can only have a cosmological relevance if the phase transition in which they are formed takes place after (or towards the end of) inflation. In most inflationary scenarios the reheating temperature after inflation is larger than the present day QCD scale. This separation of present-day scales usually leads to the assumption that QCD effects play no role in the early cosmology of the θ vacuum. In particular, the standard picture is based on two assumptions:

- (1) Early string cosmology is dominated by PQ strings of the hidden axion;
- (2) At the moment of string-formation the axion potential vanishes. Therefore, strings are free of domain walls.

We argue that the above picture is only valid in a restricted parameter space within a limited class of scenarios, whereas more generic scenarios are drastically different. In particular, the defects that dominate the early cosmology of the θ vacuum can be the string-wall systems produced by the QCD condensate, corresponding to η' or pion windings. Due to this, in our discussion we will commonly refer to early defects as θ defects, rather than specifically as PQ strings.

In order to explain the shortcomings of the standard scenario, let us note that the above assumptions are based on the large hierarchy of the present-day scales,

$$\frac{f_\phi}{\Lambda}|_{\text{today}} \gg 1. \quad (77)$$

However, the reasoning that maps this hierarchy onto the sequence of cosmological phase transitions contains a logical gap, implicitly assuming that the present day hierarchy of scales (77) can be extrapolated to the post-inflationary epoch. This assumption is not supported by any well-established understanding of the early cosmology and has already been challenged in [6]. On the contrary, it is much more generic to assume that the hierarchy between different scales in the early Universe differs significantly from the hierarchy today. This is mainly due to the fact that

physics in the early universe, particularly during or towards the end of inflation can strongly depend on the dynamics of fields that have settled in their minima early on in the evolution of the Universe. In order to understand this, let us focus on an immediate postinflationary scenario of defect formation.

First, notice that a postinflationary formation of θ defects does not necessarily correspond directly with the reheating temperature, as the transition could have taken place nonthermally, through direct interactions with the inflaton field, which we denote by Σ .

In general, within an effective field theory, the potential of the PQ field is expected to depend on couplings to the inflaton (or other fields) of the type

$$\mathcal{U}\left(\frac{\Sigma}{M}\right)\Phi^*\Phi, \quad (78)$$

where $\mathcal{U}$ is a generic function of Σ and M is a scale. Such couplings can change the VEV of the PQ field in a Σ -dependent way, thereby influencing the time of the phase transition as well as the tension of the PQ strings.

However, the most important thing for the present discussion is the value of the QCD scale during the spontaneous breaking of the axial $U(1)_A$ symmetry. As shown in [6], the QCD gauge coupling could have become strong during or towards the end of inflation, resulting in a potential for θ_{eff} . As pointed out there, the gauge coupling can strongly depend on the inflaton or other fields (which themselves can strongly depend on the inflaton). This can cause the effective coupling strengths to vary significantly throughout the (post)inflationary epoch, either due to large excursions of the expectation value of the inflaton (and its subsequent decay) or through fields whose VEVs depends on the inflaton [see e.g., (78)]. Even if such a field dependence is not taken into account by tree-level renormalizable operators (which has no *a priori* justification), it is expected to be generated both by loops as well as by high-dimensional operators (for early discussions see [39]).

This situation is generic in inflationary scenarios emerging from string theory, such as e.g., brane inflation [40,41]. This is, because in string theory, gauge couplings are dependent on fields, such as the dilaton. These fields, as argued in [6], become easily displaced during the inflationary epoch, due to their interactions with the inflaton. One important consequence of this displacement is a substantial change of the gauge couplings as compared to their present day values.

However, it is not necessary to make any assumptions about the UV structure of the theory. Due to the EFT framework, it is natural to expect the appearance of operators like

$$\mathcal{W}\left(\frac{\Sigma}{M}\right)\text{tr}G_{\mu\nu}G^{\mu\nu}, \quad (79)$$

where $\mathcal{W}$ is a generic function of Σ and M is a scale. These operators effectively set the value of the gauge coupling through the relation,

$$\frac{1}{g^2} = \mathcal{W}\left(\frac{\Sigma}{M}\right). \quad (80)$$

Due to this, it is reasonable to expect that the gauge coupling depends on the inflaton, either directly or through other fields.

Furthermore, it is natural to assume that its value during inflation differed significantly from its current value. Hence, it is not justified to assume that the value of the QCD coupling today and during the inflationary epoch were the same.

Of course, whether at the moment of symmetry breaking and string formation the gauge coupling was shifted towards stronger or weaker values, depends on the specific scenario. However, since both the variety as well as the parameter space of inflationary models is large, the likelihood of actualization of the strong early epoch QCD scenario of [6] appears rather high.

Thus, it is possible that the early epoch hierarchy between f_ϕ and Λ is drastically different from its present-day value (77). In fact, the hierarchy could be as extreme as

$$\left.\frac{f_\phi}{\Lambda}\right|_{\text{early}} \ll 1. \quad (81)$$

In this case, not only the potential of θ_{eff} is generated in the early epoch, but QCD dynamics fully dominates the cosmology of θ -defects.⁷

In particular, there are the following important implications. First, the axial $U(1)_A$ (or $U(1)_{PQ}$) cosmic strings, whether formed by the PQ or the QCD condensate, are immediately accompanied by the domain walls. More importantly, the string dynamics could have been dominated not by the PQ hidden axion but entirely by the phase of the QCD condensate. In fact, the hierarchy (81) could have been so extreme that, at the moment of quark condensation, the PQ field could have been in the symmetric vacuum, $\Phi = 0$. This would imply that the main driving force of the string-wall dynamics is not the PQ axion, but rather the quark composites, such as η' or π^0 .

⁷This has immediate implication of lifting out the previous cosmological bound on the axion scale, $f_\phi > 10^{12}$ GeV [42–44], since in the scenario of [6], the axion's coherent oscillations start way before the ordinary QCD temperatures. Correspondingly, even for a maximal initial amplitude, the axion energy gets efficiently red-shifted in the early epoch. For more recent implementations of this scenario, see [45] and references therein.

A. Pion strings

While the η' strings would always be accompanied by domain walls, the story with pion strings is more subtle and deserves special attention. Their precise nature depends on the VEV of the Higgs field at the time of quark condensation. Since in this discussion we are not interested in a hidden axion, we restrict it to a pure SM with a single neutral Higgs denoted by $H = \rho(x)e^{i\chi(x)}$.⁸ In order to clarify various regimes, let us consider the Yukawa couplings, restricting for simplicity, to a single generation of quarks,

$$g_u H \bar{u}_L u_R + g_d H^* \bar{d}_L d_R + \text{H.c.} \quad (82)$$

The only exact chiral symmetry respected by these couplings is the local symmetry gauged by the Z-boson (66). Under it, the phases of the Higgs and the quark condensates transform as

$$\chi \rightarrow \chi + \beta, \quad \theta_u \rightarrow \theta_u - \beta, \quad \theta_d \rightarrow \theta_d + \beta. \quad (83)$$

In the ordinary (present-day) SM vacuum, the VEV of the Higgs, v , is much larger than the quark condensate. Correspondingly, the combination of the phases that becomes the longitudinal component of the Z-boson,

$$v^2 \chi - f_u^2 \theta_u + f_d^2 \theta_d, \quad (84)$$

consists predominantly of the Higgs phase, with a tiny admixture from the phases of the quark condensate. The orthogonal degree of freedom corresponds to π^0 , which gets its mass from the above Yukawa couplings.

Now, in the early Universe the story can be very different. In particular, if in the epoch of quark condensation, the Higgs boson had a smaller VEV, the Goldstone boson eaten up by the Z-boson would be π^0 , with a small admixture from χ .⁹ The cosmic strings formed in this phase transition would be local strings carrying the magnetic flux of the Z-boson,

$$\text{flux} = \frac{1}{g_z} \frac{v^2 n_\chi - f_u^2 n_{\theta_u} + f_d^2 n_{\theta_d}}{v^2 + f_u^2 + f_d^2}, \quad (85)$$

Since all the phases wind by the gauge transformation of the Z-boson, the flux is integer. For example, a minimal string with winding numbers $n_{\theta_u} = -n_{\theta_d} = -n_\chi = \pm 1$ carries a unit magnetic flux $= \mp 1/g_z$.

⁸The other component of the Higgs doublet plays no role for our considerations, see also Sec. VI.

⁹This admixture would be nonzero since even if the mass square of the Higgs, m_H^2 , is large and positive, the quark condensate would nevertheless source a small VEV of the Higgs $v \sim \Lambda^3/m_H^2$.

In light of the integer Z -flux, such pion strings share similarities with semilocal electroweak strings [46]. However, the crucial difference is that the pion strings are formed by the winding of the phases of the QCD quark condensate.

If, at the moment of quark condensation, we have $v \gg f_u, f_d$, the story is opposite. In this case, the low-energy theory of π^0 has an approximate global symmetry, which is not touching the Higgs phase χ . Under this symmetry global strings can form. For example, around such a string the winding numbers can be $n_{\theta_u} = -n_{\theta_d} = \pm 1$ and $n_\chi = 0$. The magnetic flux of the Z -boson carried by such strings will be negligible. These strings will be the boundaries of 2π domain walls, formed due to explicit breaking of symmetry by the quark masses.

A priori, both scenarios are equally plausible. This is due to the fact that, while the QCD gauge coupling can become strong through interactions of the type described by Eq. (79), the inflaton can also dynamically alter the VEV of the Higgs field by coupling to it via (78). Therefore, it is entirely model-dependent which of the two scenarios will play out.

The cosmology of such string-wall systems formed by π^0 or η' in an epoch of early strong QCD [6], would be very different from the standard one. In particular, they would have a much higher wall tension, as the value of the condensate would necessarily be much larger. This leads to a much faster collapse, resulting in intense radiation, including in gravitational waves.

To summarize, the new key point we would like to bring with this discussion is that, instead of the axion, η' and pion string walls can be the main driving force of the early cosmology of the θ vacuum.

Notice, that since the early QCD scale can be much larger than the explicit breaking of $U(1)_A$ due to Yukawa couplings, other, heavier mesons, such as η could be equally (or more) important for the winding. In fact, in scenarios with $\Lambda/v|_{\text{early}} \gg 1$ the number of flavors could even go beyond $N_f = 3$, as now even the heavy quarks (those with $m_Q \gtrsim 1$ GeV) might condense during the phase transition. Thus, there might be a much higher number of mesons involved than in the minimal picture.¹⁰

B. Implications for QCD phase transition

The string-wall systems produced by the QCD chiral condensate can have implications for the ordinary, thermal QCD phase transition. We assume that by this epoch the inflaton and other heavy VEVs are settled to their vacua, and the formation of the QCD condensate takes place according to the standard picture. Even in this case, a

topologically nontrivial winding of the quark condensate phases can affect the story.

As there is no significant hierarchy of scales, the strings and walls form essentially simultaneously below the QCD temperatures. That is, upon the formation of the quark condensate, its phase already possesses a well-defined minimum. Therefore, below the critical temperature, the absolute value of the order parameter starts growing and at the same time its phase finds the minimum.

However, the phases θ_u, θ_d are ill-defined before the start of the transition. After the onset of the transition they are defined modulo 2π and each phase can relax towards the minimum from a different initial value, which they obtain at the start of the transition. This allows for the formation of domain walls, across which some of the phases change by 2π .

Understanding the characteristic size of the wall requires a more precise knowledge of the phase transition. Applying the intuition coming from the analysis of domain walls in weakly-coupled scalar theories [47–49], their size can reach the Hubble scale. Beyond this scale, it should be expected that these walls are bounded by strings, as the Hubble scale sets the upper limit on the correlation length.

Due to the gradual Hubble expansion, soon after their formation, those strings will enter the horizon and cause the walls to collapse. Their energy is released in hadrons as well as in gravitational waves.

X. IMPLICATIONS FOR HEAVY ION COLLIDERS?

From the point of view of the low-energy theory of mesons, the string-wall systems formed by the QCD condensate represent solitonic states. An important question is the possibility of observing them in high-energy collisions. In general, the formation of solitons or other nonperturbative states during the collision of a small number of high-energy quanta is exponentially suppressed [50–62].

This suppression can be understood by thinking of such processes in terms of effective $2 \rightarrow N$ S -matrix transitions. In a weakly interacting theory, a soliton of some characteristic size L and mass $M \gg 1/L$ can be thought of as a coherent state. The constituent quanta of the state have characteristic energy $E \sim 1/L$ and occupation number $N \sim (ML) \gg 1$. Hence, the formation of a soliton in a high-energy two-particle scattering process represents a transition process from two hard quanta into N soft quanta (of energy $\sim 1/L$). On general grounds, it can be proven [61] that for a well-defined transition such a matrix element is bounded from above by a factor e^{-N} .

Overcoming this suppression is possible, but only if the final state soliton has a maximal microstate degeneracy. For N -particle coherent bound states, the maximal possible degeneracy scales as $\sim e^N$ [61]. Correspondingly, such states have maximal microstate entropy $S \sim N$. There exists

¹⁰Using KSVZ-type models, this would potentially even allow for $N_f \geq 6$.

evidence [63] that such a compensation takes place for certain multi-gluon states called the “color glass condensate” [64]. However, for the string-wall solitons discussed in the present work, the degeneracy factor requires further investigation.

Given our present understanding, promising places to look for the experimental manifestations of these objects can be heavy ion collisions, such as at the LHC [65] and at the RIHC [66]. Of course, at best, these collisions could produce small defects, not much exceeding the QCD length. However, any chance of catching glimpses of string-wall resonances would be of extraordinary importance.

XI. COMMENT ON THE GAUGE AXION

In [3] an alternative formulation of the axion was introduced, not relying on any anomalous global symmetry. Instead, in this formulation the axion represents an intrinsic part of the QCD gauge redundancy. Hence the term “gauge axion”. Under the QCD gauge transformation of the gluons, $A_\mu \rightarrow U(x)A_\mu U^\dagger(x) + U\partial_\mu U^\dagger$ with $U(x) \equiv e^{-i\omega(x)^b T^b}$, the gauge axion shifts as

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \frac{1}{f}\Omega_{\mu\nu}, \quad (86)$$

where $\Omega_{\mu\nu} = \text{tr} A_{[\mu} \partial_{\nu]} \omega$ and f is the gauge axion scale.

The gauge axion $B_{\mu\nu}$ enters the Lagrangian via a unique gauge invariant combination,

$$C_{\mu\nu\alpha} - f\partial_{[\mu} B_{\nu\alpha]}, \quad (87)$$

where $C_{\mu\nu\alpha} \equiv \text{tr}(A_{[\mu} \partial_\nu A_{\alpha]} + \frac{2}{3} A_{[\mu} A_\nu A_{\alpha]})$ is the Chern-Simons 3-form of QCD, which under the QCD gauge transformation shifts as

$$C_{\mu\nu\alpha} \rightarrow C_{\mu\nu\alpha} + \partial_{[\mu} \Omega_{\nu\alpha]}. \quad (88)$$

The gauge redundancy of $B_{\mu\nu}$ makes $\bar{\theta}$ unphysical and solves the strong- CP problem to all orders in the operator expansion [3,67].

At low energies, the gauge axion can be dualized into a pseudoscalar phase with an effective Lagrangian given by (10). Correspondingly, the structure of domain walls discussed in the present work goes through the gauge-axion formulation.

However, as mentioned in a number of papers [3,22,67,68], the equivalence is broken by the explicit breaking of the PQ symmetry through arbitrary $U(1)_{PQ}$ -violating operators as well as by the UV completion of the theory. The explicit breaking terms generate a mass for the PQ axion beyond what it receives from QCD instantons. Such an axion cannot be dualized into $B_{\mu\nu}$.

Likewise, unlike the PQ axion, the gauge axion cannot be UV completed into a phase of complex scalar field. A detailed discussion of both points can be found in [3,22,67,68].

Due to this, the cosmic string structure of the gauge axion is different and requires a separate discussion. This will be given in the upcoming work [69].

Finally, we wish to comment that the structure of string-wall systems produced purely via the QCD condensate, without the involvement of a hidden axion, such as η' or π^0 string-wall systems, are not sensitive to the PQ versus gauge nature of the axion and remain valid regardless of its presence.

XII. SUMMARY AND OUTLOOK

In this paper we have explored the winding configurations of QCD quark condensates. In particular, the 2π domain walls bounded by cosmic strings, formed by windings of η' or π^0 mesons. If a hidden axion exists in the theory, such windings in general accompany the axionic string-wall system, affecting the physics of fermionic zero modes and hence anomaly inflow and superconductivity properties.

However, the QCD condensate string walls can exist independently of the hidden axion. This is particularly relevant in scenarios where in the early Universe, the standard hierarchy $f_\phi \gg \Lambda$ is either softened or flipped entirely. It is important to understand that in a generic cosmological scenario the cross-couplings between the inflaton and the Peccei-Quinn field (78), as well as the coupling between the inflaton and gluons (79) must be taken into account. Such couplings dramatically modify the hierarchy between the QCD and the PQ scales in the early epoch [6] (for more recent constructions, see [45]). In these generic scenarios it is certainly possible that the QCD defects play a dominant role. Of course, the precise parameter range must be determined on a model-by-model basis, taking into account other constraints. Regardless, it is clear that the effect of the cross couplings on the early-epoch value of the hierarchy (81) is rather dramatic, making the role of the QCD condensate string-walls in that epoch highly prominent. In particular, they can be a potential source of gravitational waves and electromagnetic radiation.

In the low-energy theory of mesons, such objects appear as solitonic states. We discussed prospects of their production in heavy ion collisions.

The string-wall structures described in the present paper are fully supported by numerical results, which shall be discussed in the future work [24].

Finally, we would like to comment on possible string-wall structures formed by fermion condensates in other gauge sectors of the Standard Model and gravity.

In this respect, we wish to point out that string-wall defects emerging in a theory of neutrino mass [70] have

been studied in [71]. These defects are formed due to the spontaneous breaking of a non-Abelian neutrino-flavor symmetry by the neutrino condensate [22].

Furthermore, we would like to comment that, according to the arguments given in [72], the gravitational Eguchi-Hanson instantons [73], imply the presence of the condensate in spin-3/2 [74] (rather than in spin-1/2) fermions. This condensate breaks the anomalous supersymmetric R -symmetry spontaneously. The defects emerging from this breaking are R -string-wall systems. Since R -charges are shared by the Standard Model particles, it is generic for R -strings to carry a Z -boson flux [75].

On a separate point, it was argued recently [76], that the electroweak sector of the Standard Model is accompanied by a new pseudo-scalar meson, η_w , that shifts under the anomalous $(B + L)$ -symmetry. This degree of freedom mixes with the phase of a $(B + L)$ -violating quark and lepton condensate formed by electroweak instantons. It would be natural to investigate the associated string-wall systems.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX

In the massive-quark case, the broken $U(1)_V$ gives rise to strings attached to domain walls. This can be seen from the energy of the system given in Eq. (41).

The usual axionic string solution $\theta_\phi = l/R$ has energy density $\sigma_R = 2\pi(f_\phi^2/R + R\Lambda^4)$. The energy of a string wall system also winding in η' has lower energy.

The ansatz in Eqs. (42) and (43) approximate a string wall system for η' and ϕ . We perform the integration in the

energy functional from Eq. (41) to find

$$\begin{aligned} \sigma(L_\phi, L_\eta) = & 2\pi f_\phi^2 \frac{1}{L_\phi} + 2\pi f_\eta^2 \frac{1}{L_\eta} + 2\pi \Lambda_m^4 L_\eta \\ & + 2\pi \Lambda^4 L_\phi \left(1 + \frac{L_\phi}{2\pi(L_\phi - L_\eta)} \sin \left(2\pi \frac{L_\eta}{L_\phi} \right) \right). \end{aligned} \quad (\text{A1})$$

To get an idea about the energy balance, we compare two regimes.

First let us take $L_\phi = L_\eta \equiv L$. In this regime the η' and ϕ wind exactly opposite. The minimization of energy with respect to L gives the optimal winding length,

$$L = \frac{f}{\Lambda_m^2}, \quad (\text{A2})$$

where we used $f \equiv \sqrt{f_\phi^2 + f_\eta^2}$. The energy of this solution is

$$\sigma_1 = 4\pi f \Lambda_m^2 \simeq 4\pi f \phi \Lambda_m^2. \quad (\text{A3})$$

The second regime is $L_\eta \ll L_\phi$, which approximately (i.e., in leading order in L_η/L_ϕ expansion) gives,

$$\begin{aligned} \sigma(L_\phi, L_\eta) \simeq & 2\pi f_\phi^2 \frac{1}{L_\phi} + 2\pi f_\eta^2 \frac{1}{L_\eta} + 2\pi \Lambda_m^4 L_\eta \\ & + 2\pi \Lambda^4 L_\phi \left(1 + \frac{L_\eta}{L_\phi} \right). \end{aligned} \quad (\text{A4})$$

Extremizing the energy gives

$$L_\phi = \frac{f_\phi}{\Lambda^2}, \quad L_\eta = \frac{f_\eta}{\sqrt{\Lambda_m^4 + \Lambda^4}}, \quad (\text{A5})$$

with corresponding energy

$$\sigma_2 = 4\pi f_\phi \Lambda^2 + 4\pi f_\eta \sqrt{\Lambda_m^4 + \Lambda^4}. \quad (\text{A6})$$

Taking into account that $\Lambda \gg \Lambda_m$, we have

$$\frac{\sigma_2}{\sigma_1} \simeq \frac{\Lambda^2}{\Lambda_m^2} \gg 1, \quad (\text{A7})$$

from which we conclude that the regime $L_\phi = L_\eta \equiv L$ is energetically more favorable.

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