

Thermal damping of mass-modulating scalars

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The cosmological evolution of a scalar field is shaped by Hubble damping. Any nongravitational couplings of the scalar with the primordial thermal bath generically contribute additional damping. Although often neglected, such thermal damping could be the dominant dissipative effect on the scalar. We derive approximate yet highly general thermal damping rates of scalar fields that modulate the masses of thermally populated particles. We extend previous results to cover cases of particular phenomenological interest where the scalar background oscillates sinusoidally but not necessarily slowly compared to the thermalization rates of the primordial bath. As applications, we estimate the thermal damping of scalars coupled linearly to neutrinos, quadratically to gluons, and linearly to Weakly Interacting Massive Particles, and demonstrate the importance of this effect in certain parameter spaces of these models. We also estimate the thermal damping rates in models of QCD axions.

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I. INTRODUCTION

A dynamical scalar field coupled to a thermal bath, given enough time, will thermalize. This tendency manifests initially through a thermal damping mechanism, which drains the scalar field's kinetic energy to heat up the thermal bath. While the details of the mechanism that leads to the scalar being damped are model dependent, thermal damping as a phenomenon is highly general. Thermal damping is known to play important roles in various contexts, including but not limited to warm inflation [1–5], reheating after inflation [6–8], (early) dark energy models [9–12], and cosmology of axions [13,14], supersymmetric flat directions [15,16], and neutrino-coupled scalars [17].

Although there are known exceptions [2,5,18], generically, the overall effect of a thermal environment on a scalar background at a given *instance* is to make it roll faster through a thermal mass-correction rather than to slow it down with thermal damping [19]. This is because compared to thermal mass, thermal damping is a higher-order

effect in coupling: the scalar-bath coupling is used once to bring the bath out of equilibrium and once more to get a backreaction. However, thermal damping is a dissipative effect. Even if it affects the instantaneous motion of a scalar only slightly, it can still be important in the long term. In many cases, thermal damping competes only with Planck-suppressed Hubble friction in order to be relevant. For this reason, thermal damping can have far-reaching consequences despite being a subdominant effect in the aforementioned sense.

Thermal damping of a scalar field has previously been considered mostly in high-temperature contexts. In this work, we instead explore thermal damping at later, colder cosmological epochs, where its effects are arguably more important compared to Hubble friction. At temperatures above all the relevant mass scales, the thermal damping rate Υ scales as $\Upsilon \propto T$ since T is the only scale, whereas during radiation domination, $H \propto T^2$. Thus, it is clear that, at least in this regime, thermal damping is more important at lower temperatures. This rough scaling breaks down at sufficiently low temperatures for various model-dependent reasons. Regardless, in order to properly assess the possible relevance of thermal damping, one needs to find the point in the cosmic history at which the ratio Υ/H is maximized. As we will see, this typically occurs close to the point where the scalar-coupled thermal bath fields decouple from the rest of the thermal bath. The need to quantify thermal damping in these relatively late epochs motivates us to

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generalize the thermal damping rate formula to regimes beyond what is typically considered.

Thermal damping is notoriously difficult to estimate due to its nonequilibrium nature. The standard derivation of the thermal damping rate of a scalar field is based on a resummation prescription known as the two-particle irreducible (2PI) formalism [20–32], from which one obtains an equation of motion for the Schwinger-Keldysh expectation value of the scalar field [33]. Thermal damping enters this equation through contributions to the scalar field’s 2PI-resummed effective action that are nonlocal due to the finite widths of the resummed propagators of the thermal bath fields. Although thermal damping as a subject has a long history, the scope in which it has been considered remains limited. The complexity of the standard calculation of thermal damping may have made it difficult to notice its relevance outside previously considered setups.

However, in most cases of phenomenological interest, the thermal bath fields are well approximated as a gas of (quasi-) particles with certain distribution functions. The nonequilibrium evolution of such systems can be described by a set of Boltzmann equations, which are not only conceptually simple and intuitive but also sufficient for deriving the damping rates [16,17,19,34–36]. In this picture, the dynamical scalar field distorts the thermal bath’s distribution functions away from equilibrium. The thermal bath then backreacts to the scalar field in a time-delayed manner, due to its finite response time, resulting in an effect proportional to the scalar’s velocity, a friction. In fact, as long as the thermal bath fields have perturbative couplings and adiabatically varying dispersion relation, the 2PI and Boltzmann approaches are equivalent. In these limits, the Kadanoff-Baym equations, i.e., the 2PI evolution equations for two-point functions, of the thermal bath fields, reduce to Boltzmann equations for a gas of quasiparticles [37,38].

Previous works on thermal damping focused on cases where the thermal bath is relativistic and thermalizes fast compared to the scalar dynamics. In contrast, we expect thermal damping to be most significant when the scalar-coupled thermal bath particles are on the verge of decoupling, upon which they are often semi- or nonrelativistic and thermalize relatively slowly. In this paper, we adopt the Boltzmann approach and generalize the thermal damping rates to include these previously unexplored cases. In the fast thermalization regime, the thermal bath quickly adapts to the scalar field’s motion, resulting in distortions of the thermal bath’s distribution functions, and hence the thermal damping rates Υ , being inversely proportional to the bath’s thermalization rate, $\Upsilon \propto \Gamma_{\text{th}}^{-1}$. In the opposite, slow-thermalization regime [39], the thermal bath does not have time to fully react to the changing scalar field. Instead, it responds proportionally to its thermalization rate, resulting in $\Upsilon \propto \Gamma_{\text{th}}$. We capture the smooth transition from fast thermalization to slow thermalization, and find that thermal damping is often most important in the intermediary

regimes. In addition, we consider the thermal damping due to multiple interacting species, and reveal new types of thermal damping where the scalar drives the thermal bath and receives backreaction through different couplings.

For concreteness, we focus on the case where the dynamical field is a scalar that modulates the masses of particles in the primordial thermal bath. We demonstrate through examples that thermal damping could, in fact, play important roles in commonly considered models at later epochs and lower temperatures, say, $T \sim 10$ MeV, than commonly considered, e.g., during preheating after inflation. Specifically, we consider three different scenarios: (i) a scalar with Yukawa coupling to active neutrinos, (ii) models of hadron-mass modulation at quadratic order: QCD axion and quadratically coupled scalar, and (iii) a scalar with Yukawa coupling to Weakly Interacting Massive Particle (WIMP) dark matter (DM). As pointed out in our companion paper [17], thermal damping can modify the expected relic abundance of a scalar field, ameliorate late-time limits on the scalar background, and, in certain models, motivate experimental targets.

This paper is organized as follows. In Sec. II, we provide an intuitive understanding of the thermal damping effect and illustrate its effect with a simple toy model. In Sec. III, we derive general thermal damping rates of scalar fields that modulate the masses of relativistic/nonrelativistic, scalar/fermion thermal bath fields that thermalize fast/slow and have interspecies couplings. In Sec. IV, we estimate the rates of thermal damping in several concrete models and point out instances where it has to be taken into account. Finally, we discuss future directions and conclude in Sec. V.

II. THERMAL DAMPING INTUITION

Consider an oscillating homogeneous scalar field $\phi(t)$ with a bare mass m_ϕ and a scalar or a (spin-1/2) fermion χ with a ϕ -dependent effective mass $M_\chi(\phi)$, as described by the following Lagrangian:

$$-\mathcal{L} \supset \frac{1}{2} m_\phi^2 \phi^2 + \begin{cases} M_\chi(\phi) \bar{\chi} \chi & (\text{fermion } \chi) \\ \frac{1}{2} M_\chi^2(\phi) \chi^2 & (\text{scalar } \chi) \end{cases}. \quad (1)$$

We assume the χ field is coupled to a large thermal bath whose temperature T is approximately constant. Hence, schematically, the setup is as follows:

$$\text{oscillating } \phi(t) \leftrightarrow \chi \leftrightarrow \text{thermal bath}.$$

When the scalar background $\phi(t)$ evolves in time, it changes the χ particles’ effective mass $M_\chi[\phi(t)]$ without directly affecting their momenta. This brings the χ particles’ *thermal equilibrium* distribution away from its distribution at that instance. In the presence of χ number-changing interactions, the χ particles will relax toward the new equilibrium state corresponding to the new effective mass that brought it out of the previous equilibrium state. As the scalar background continues to

evolve, the χ field remains out of equilibrium as it chases its elusive time-dependent equilibrium state. Each time χ rethermalizes and releases entropy, some energy gets dissipated irreversibly. Since χ 's departure from equilibrium comes at the expense of the scalar field's kinetic energy, the net effect is a damping of the scalar field with the corresponding heating of the thermal bath.

In this section, we show in a minimal toy model how the aforementioned entropically favored energy flow from an oscillating scalar field to the thermal bath manifests explicitly, from the perspectives of both the ϕ field and the χ field. The ϕ field experiences a nonconservative potential gradient whose slope is slightly steeper during an ascent ($d|\phi|/dt > 0$) than a descent ($d|\phi|/dt < 0$) of its potential. In certain limits, this results in an effect mathematically equivalent to a friction term in the equation of motion of ϕ . The χ particles are net-produced from the thermal bath relatively light during ϕ 's descent and net-depleted relatively heavy during ϕ 's ascent. During the net depletion of χ , the extra effective mass energy χ acquired from ϕ is dumped into the thermal bath.

A. Simple estimate

In this illustrative discussion, we assume for simplicity that the χ particles are nonrelativistic $M_\chi \gtrsim T$ but have appreciable abundance. In the absence of Hubble expansion, the coherent ϕ field evolves according to

$$\ddot{\phi} + m_\phi^2 \phi + M'_\chi(\phi) n_\chi = 0, \quad (2)$$

where n_χ is the number density of χ particles and a prime $\prime$ denotes $\partial/\partial\phi$. As the ϕ field evolves, it modulates χ 's thermal equilibrium number density as

$$n_\chi^{\text{eq}}(\phi) = \left(\frac{M_\chi(\phi) T}{2\pi} \right)^{3/2} e^{-M_\chi(\phi)/T}, \quad (3)$$

making it deviate from χ 's actual number density, $n_\chi \neq n_\chi^{\text{eq}}$. The out-of-equilibrium χ will then try to readjust to re-establish thermal equilibrium. However, it takes a finite time, set by the rates of χ number-changing processes, for n_χ to react. By the time the n_χ reaches the n_χ^{eq} of the previous instance, the n_χ^{eq} has moved to a different value. Thus, at any given time, there is a departure of the number density of χ from its equilibrium value, $n_\chi - n_\chi^{\text{eq}} \neq 0$. The equilibrium-seeking dynamics of n_χ can be approximately captured by the following momentum-integrated Boltzmann equation in the relaxation time approximation:

$$\dot{n}_\chi = \Gamma_{\text{th}} [n_\chi^{\text{eq}}(\phi) - n_\chi]. \quad (4)$$

The thermal damping effect can be seen most directly in the regime where the scalar field evolves slowly compared to χ 's thermalization rate, $|\dot{\phi}/\phi| \ll \Gamma_{\text{th}}$. In this case, the number density of χ stays close to its equilibrium value such that $\dot{n}_\chi \approx \dot{n}_\chi^{\text{eq}}$, and so the Boltzmann Eq. (4) gives

$$n_\chi \approx n_\chi^{\text{eq}} - \frac{\dot{n}_\chi^{\text{eq}}}{\Gamma_{\text{th}}}. \quad (5)$$

When substituted into the equation of motion Eq. (4), the equilibrium part, n_χ^{eq} , leads to a nondissipative contribution to ϕ 's effective potential gradient, $M'_\chi n_\chi^{\text{eq}}$, while the nonequilibrium part, $n_\chi - n_\chi^{\text{eq}} \approx -\dot{n}_\chi^{\text{eq}}/\Gamma_{\text{th}} \propto \dot{\phi}$, is what results in a thermal damping term $-(M'_\chi n_\chi^{\text{eq}}/\Gamma_{\text{th}})\dot{\phi}$,

$$\ddot{\phi} + m_\phi^2 \phi + M'_\chi n_\chi^{\text{eq}} - \frac{M'_\chi n_\chi^{\text{eq}}}{\Gamma_{\text{th}}} \dot{\phi} = 0. \quad (6)$$

If, for example, the effective mass of χ is

$$M_\chi^2(\phi) = m_\chi^2 + \lambda \phi^2, \quad (7)$$

with $m_\chi^2 \gg \lambda \phi^2, T^2$, then the thermal damping rate is

$$-\frac{M'_\chi n_\chi^{\text{eq}}}{\Gamma_{\text{th}}} \approx \frac{\lambda^2 \phi^2 n_\chi^{\text{eq}}}{m_\chi^2 \Gamma_{\text{th}} T}. \quad (8)$$

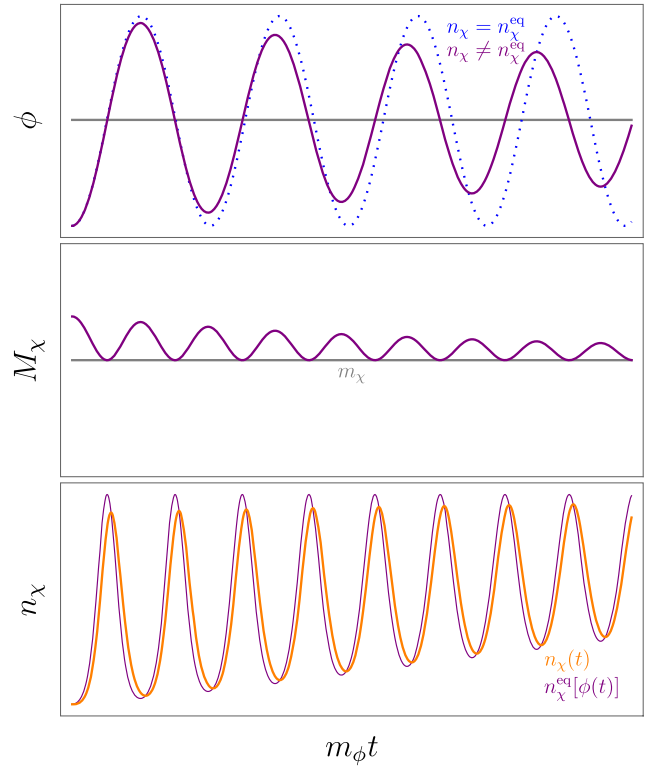


FIG. 1. Illustration of thermal damping of χ -mass modulating scalar field ϕ . As the scalar field ϕ oscillates (top), it modulates χ 's effective mass M_χ (middle, purple), which, in turn, modulates χ 's equilibrium number density n_χ^{eq} (bottom, purple). The χ field remains out of equilibrium, as its actual number density n_χ (bottom, orange) continually seeks the moving n_χ^{eq} (bottom, purple) and thereby backreacts dissipatively on ϕ (top, purple). The scalar ϕ evolution without thermal damping (top, dotted blue) is also shown.

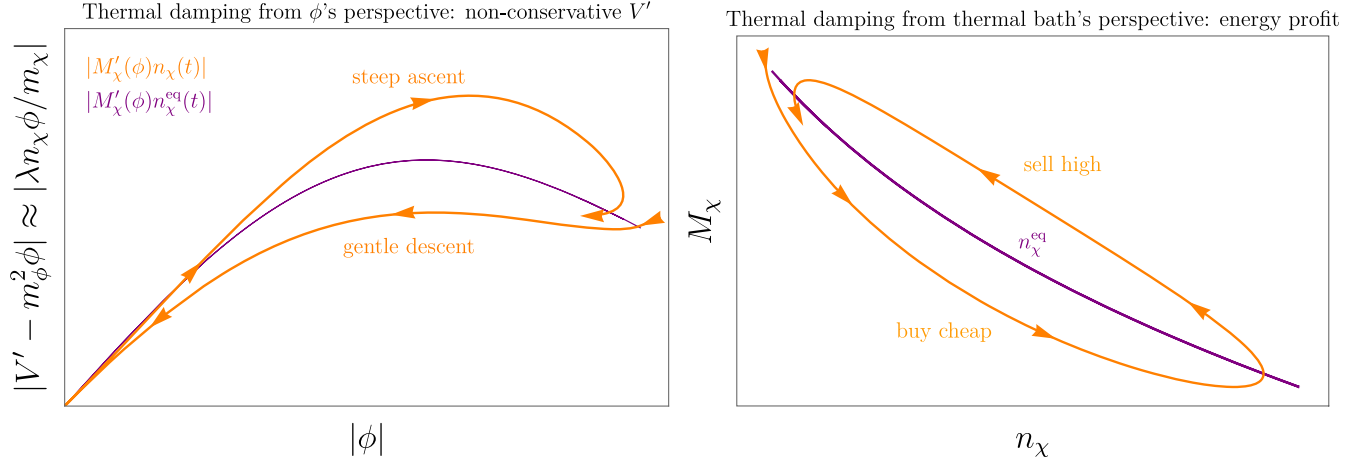


FIG. 2. Origin of thermal damping in the toy model of Sec. II. Here, the model parameters are chosen to be the same as in Fig. 1. Left: χ 's contribution to ϕ 's effective potential gradient $|V' - m_\phi^2 \phi| \approx |\lambda n_\chi \phi / m_\chi|$ plotted against the absolute value of the ϕ field. It shows that this potential gradient contribution is nonconservative: gentler during descent ($d|\phi|/dt < 0$), steeper during ascent ($d|\phi|/dt > 0$). The area enclosed by the orange curve is the magnitude of the nonconservative work on the ϕ field in one half-oscillation, $\approx \int_\Phi d\phi V'$. Right: the χ particles are produced ($\dot{n}_\chi > 0$) by the thermal bath when they are relatively light and decay/annihilate ($\dot{n}_\chi < 0$) to the thermal bath when they are relatively heavy. In other words, the thermal bath “buys” χ particles when they are energetically cheap and “sells” them later when their energy costs are relatively high. The area enclosed by the orange curve, $\oint dn_\chi M_\chi$, is the net energy gain (“profit”) of the thermal bath, which is equal to the dissipative energy loss of ϕ by energy conservation.

See Fig. 1 for a qualitative illustration of the thermal damping effect in this example setup.

B. Qualitative understanding

The dissipated kinetic energy of the coherent scalar field ϕ will be channeled into thermal bath degrees of freedom, as expected from the equipartition theorem. The origin of this entropically favored energy flow from the ϕ field to the thermal bath can be seen from the perspective of either sector.

From the perspective of the ϕ field, thermal damping arises from the *asymmetric backreaction* of the χ field [16]. The scalar field ϕ receives backreaction from the χ particles via its coupling to χ 's mass $M_\chi(\phi)$, which contributes an effective potential gradient in ϕ 's equation of motion of $V' \approx (m_\phi^2 + \lambda n_\chi / m_\chi) \phi$. The damping arises due to the nonconservative behavior of the potential gradient V' , which on average has a gentler slope during ϕ 's decent toward the origin ($\phi = 0$) and steeper during its ascent away from the origin, $|\bar{V}'|_{\text{ascent}} > |\bar{V}'|_{\text{descent}}$. This is illustrated on the left panel of Fig. 2. Alternatively, the damping can also be seen from ϕ 's perspective as due to the delayed backreaction from χ [19]. Due to the finite thermal equilibration rate Γ_{th} of χ , this backreaction occurs with a time delay $\tau_{\text{th}} \sim \Gamma_{\text{th}}^{-1}$. At a given time t , the number density of χ is responding to an earlier configuration of ϕ , $n_\chi(t) \approx n_\chi^{\text{eq}}[\phi(t - \tau_{\text{th}})] \approx n_\chi^{\text{eq}}(\phi) - \dot{\phi} \tau_{\text{th}} n_\chi^{\text{eq}}$. This results in a damping term $\propto \dot{\phi}$ in the equation of motion of ϕ .

The net heating of the thermal bath can be seen as due to what we refer to as the *buy cheap, sell high* effect, which

is similar to what occurs in the instant preheating scenario [16,40]. The equilibrium density of χ particles n_χ^{eq} being higher when its effective mass $M_\chi(\phi)$ is lower implies that the χ particles tend to be produced from the thermal bath when $M_\chi(\phi)$ is relatively light (energetically cheap) and decay/annihilate when $M_\chi(\phi)$ is relatively heavy (energetically costly). In between their creation and destruction, the χ particles have their effective mass $M_\chi(\phi)$ increased on average, thereby extracting some energy from the ϕ field. This extracted energy eventually ends up in the thermal bath when the χ particles decay/annihilate. In short, the χ particles are produced when they are relatively cheap energetically and deposit their energies to the thermal bath when their M_χ is relatively high, resulting in a net energy gain for the thermal bath. This is illustrated on the right panel of Fig. 2.

III. THERMAL DAMPING RATES

In this section, we estimate the general thermal damping rates of mass-modulating scalars ϕ with Lagrangian terms of the form shown in Eq. (1), in regimes where the thermal damping effect can be mathematically captured by a $\propto \dot{\phi}$ frictionlike term in the equation of motion of the scalar ϕ .

A. Effective equation of motion

If χ behaves as a gas of particles with distribution function f_p , the equation of motion of ϕ reads

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi + \int \frac{d^3p}{(2\pi)^3} \frac{M'_\chi M_\chi}{\sqrt{M_\chi^2 + p^2}} f_p = 0, \quad (9)$$

where $M'_\chi = \partial M_\chi / \partial \phi$ and the integral corresponds to $\langle \bar{\chi}\chi \rangle$ in the fermion χ case and $M_\chi \langle \chi^2 \rangle$ in the scalar χ case. The χ field is well approximated as a gas of quasiparticles if its effective mass M_χ varies only adiabatically, $|\dot{M}_\chi|/M_\chi^2 \ll 1$, and its interactions are sufficiently weak that their interaction mean free time is long compared to relevant interaction timescales (e.g., scattering or annihilation timescales). These conditions are typically satisfied by, e.g., the standard model (SM) particles in the primordial plasma.

The distribution function f_p of the χ particles can be separated into its equilibrium part f_p^{eq} and nonequilibrium part δf_p ,

$$f_p = f_p^{\text{eq}} + \delta f_p. \quad (10)$$

In the absence of chemical potentials,

$$f_p^{\text{eq}}(\phi) = \frac{g_{\text{dof}}}{e^{\sqrt{M_\chi^2(\phi) + p^2}/T} \pm 1}, \quad (11)$$

where g_{dof} is the number of degrees of freedom of the χ field and the upper (lower) sign refers to fermion (boson). For simplicity, we limit our analysis to cases where $\delta f_p \ll f_p^{\text{eq}}$. This corresponds to requiring $|M_\chi(\phi) - M_\chi(\phi=0)| \ll M_\chi(\phi=0)$ in the nonrelativistic limit and $|M_\chi| \ll T$ in the relativistic limit. The equation of motion of ϕ , Eq. (9), can then be written in the following effective form:

$$\ddot{\phi} + (3H + \Upsilon)\dot{\phi} + V'_{\text{eff}} = 0, \quad (12)$$

where we have collected the equilibrium part in an effective potential gradient term,

$$\begin{aligned} V'_{\text{eff}} &\equiv m_\phi^2\phi + M'_\chi \int \frac{d^3p}{(2\pi)^3} \frac{M_\chi}{\sqrt{M_\chi^2 + p^2}} f_p^{\text{eq}} \\ &= m_\phi^2\phi + g_{\text{dof}} M'_\chi \begin{cases} \xi n_\chi^{\text{eq}}, & M_\chi \gg T \\ \frac{c_{\text{b/f}}}{12} M_\chi T^2, & M_\chi \ll T \end{cases}, \end{aligned} \quad (13)$$

where $\xi = M_\chi/|M_\chi|$, $c_{\text{b}} = 1$, and $c_{\text{f}} = 1/2$, and the non-equilibrium part in a term of thermal damping form,

$$\Upsilon\dot{\phi} \equiv M'_\chi \int \frac{d^3p}{(2\pi)^3} \frac{M_\chi}{\sqrt{M_\chi^2 + p^2}} \delta f_p. \quad (14)$$

Note that this merely defines $\Upsilon\dot{\phi}$. The above integral does not always reduce to a friction term proportional to $\dot{\phi}$.

However, we will identify two regimes where it does in the next section.

The effective potential V_{eff} , whose gradient is given in Eq. (13), is the usual finite-temperature effective potential for a scalar field $V_{\text{eff}}(\phi, T) = (T^4/2\pi^2)J_\pm[M_\chi(\phi)/T]$, with $J_\pm(y) = \int_0^\infty dx x^2 \ln |1 \mp e^{-\sqrt{x^2+y^2}}|$ and the upper (lower) sign referring to a bosonic (fermionic) χ field. This effective potential can be derived using the Matsubara formalism, as in [41,42]. However, such *equilibrium* finite-temperature formalisms would not produce the thermal damping term, Eq. (14), which is an inherently *nonequilibrium* effect. The nonequilibrium equation of motion for the expectation value of ϕ that includes the damping term can be derived formally through the 2PI formalism, which is based on the Schwinger-Keldysh formalism [20–32]. In this paper, we instead follow a more elementary route based on Boltzmann equations.

B. Solving Boltzmann's equation

In the relaxation time approximation, the Boltzmann equation in the absence of Hubble expansion reads $\dot{f}_p = -\Gamma_{\text{th}}(f_p - f_p^{\text{eq}})$, which can be rewritten using Eq. (10) as

$$\delta\dot{f}_p + \Gamma_{\text{th}}\delta f_p = -\dot{f}_p^{\text{eq}}, \quad (15)$$

where f_p^{eq} is given in Eq. (11) and

$$\dot{f}_p^{\text{eq}} = f_p^{\text{eq}}(1 \mp f_p^{\text{eq}}/g_{\text{dof}}) \frac{M_\chi \dot{M}_\chi}{T \sqrt{M_\chi^2 + p^2}}, \quad (16)$$

where the upper (lower) sign refers to fermion (boson). The thermalization rate Γ_{th} in Eq. (15) in general needs to be computed on a case-by-case basis. The general solution of Eq. (15) is

$$\delta f_p(t) = \delta f_p(t_i) e^{-\Gamma_{\text{th}}(t-t_i)} - \int_{t_i}^t d\tilde{t} \dot{f}_p^{\text{eq}}(\tilde{t}) e^{-\Gamma_{\text{th}}(t-\tilde{t})}, \quad (17)$$

where t_i is an initial time of reference and $\tilde{t}$ is a dummy time variable. The first term is a transient solution, which decays on a timescale of Γ_{th}^{-1} . This leaves us with the second, memory-integral term for $t - t_i \gg \Gamma_{\text{th}}^{-1}$.

Due to the nonlocal-in-time nature of the integral term in Eq. (17), the integral on the right-hand side of Eq. (14) does not in general lead to a local-in-time friction term that is proportional to the $\dot{\phi}$ at an instance. Even then, ϕ can still be subject to a thermal damping effect, although this time it can only be seen as a net energy loss of the ϕ field after integrating over each half-oscillation of ϕ [16]. The damping coefficient in this regime is best defined in terms of the fractional rate of change of the amplitude of ϕ on time-scales longer than the oscillation period of ϕ .

In what follows, we focus on two distinct, but non-exclusive, regimes where the integral in Eq. (17) does *mathematically* reduce to a result proportional to $\dot{f}_p^{\text{eq}}$ which, in turn, leads to a local friction term of the form $\propto \dot{\phi}$ on the right-hand side of Eq. (14).

1. Adiabatic regime

First, we consider the regime where the ϕ field varies adiabatically in the sense that its dynamics is slow compared to the thermal bath's thermalization rate,

$$\left| \frac{\dot{\phi}}{\phi} \right| \ll \Gamma_{\text{th}} \quad (\text{adiabatic}). \quad (18)$$

In this *adiabatic regime*, which has been the focus of previous works [20–29,31], the integral in Eq. (17) has support only for $t - \tilde{t} \lesssim \Gamma_{\text{th}}^{-1}$. Since in this regime, $f_p^{\text{eq}}(\tilde{t})$ is slowly changing on the timescale of Γ_{th}^{-1} , we can approximate $(\dot{f}_p^{\text{eq}})_{\text{at } \tilde{t}} \approx (\dot{f}_p^{\text{eq}})_{\text{at } t}$, yielding

$$\delta f_p \approx -\frac{\dot{f}_p^{\text{eq}}}{\Gamma_{\text{th}}}, \quad (19)$$

for $t - t_i \gg \Gamma_{\text{th}}^{-1}$.

2. Underdamped regime

Next, we point out a new regime, which we refer to as the *underdamped regime*, where the integral in Eq. (14) also reduces to a $\propto \dot{\phi}$ frictionlike term. This occurs when the ϕ background varies approximately sinusoidally,

$$\phi \approx \text{Re}[\Phi e^{iM_\phi t}], \quad (20)$$

where M_ϕ is the effective mass of the ϕ field (which accounts for possible thermal corrections). We define the underdamped regime by the condition that the amplitude Φ is slowly varying in the following sense:

$$\left| \frac{\dot{\Phi}}{\Phi} \right| \ll \min(M_\phi, \Gamma_{\text{th}}) \quad (\text{underdamped}). \quad (21)$$

In this underdamped regime, we rely on $f_p^{\text{eq}}(\phi)$ being nearly sinusoidal in evaluating the memory integral in Eq. (17) analytically. The requirement of sinusoidal $f_p^{\text{eq}}(\phi)$ restricts the possible forms of $M_\chi(\phi)$ in Eq. (1) for which our treatment applies. For relativistic ($M_\chi \gg T$) and non-relativistic ($M_\chi \ll T$) χ particles, the $\dot{f}_p^{\text{eq}}$ is sinusoidal if $\dot{M}_\chi$ and $M_\chi \dot{M}_\chi$ are sinusoidal, respectively. Using the ansatz $f_p^{\text{eq}} \propto e^{i\omega_f t}$, the memory integral of Eq. (17) evaluates, for $t - t_i \gg \Gamma_{\text{th}}^{-1}$, to

$$\delta f_p \approx -\text{Re} \left[\frac{\dot{f}_p^{\text{eq}}}{\Gamma_{\text{th}} + i\omega_f} \right]. \quad (22)$$

The angular frequency ω_f of the sinusoidal variation of f_p^{eq} is set by the functional form of M_χ . For instance, if $M_\chi \propto \phi^N$, then the angular frequency of $f_p^{\text{eq}} \propto e^{i\omega_f t}$ in relation to M_ϕ is $\omega_f = NM_\phi(2NM_\phi)$ for $M_\chi \gg T$ ($M_\chi \ll T$).

The underdamped approximation is valid in a wide range of cases of phenomenological interest because for scalars whose potentials are not protected by some symmetry [2,5,18], the in-medium effective potential correction term, $V'_{\text{eff}} - m_\phi^2 \phi$, is typically much stronger than the thermal damping term, $\Upsilon \dot{\phi}$ [19]. In these generic cases, the scalar would be underdamped by Υ when undergoing a periodic motion under its own effective mass M_ϕ . For such a scalar that is automatically underdamped, the underdamped approximation has a wider range of applicability than the adiabatic approximation. While the latter only covers the regime where $|\dot{\phi}/\phi| \sim M_\phi \ll \Gamma_{\text{th}}$, the former covers both this regime and the opposite regime where $M_\phi \gtrsim \Gamma_{\text{th}}$. More generally, the adiabatic results and the underdamped results are complementary. The adiabatic approximation covers part of the underdamped regime but also extends beyond it to overdamped cases, which are relevant for, e.g., warm inflation [1–5] and dark energy models [9–12]. Similarly, the underdamped approximation covers part of the adiabatic regime but also extends beyond it to non-adiabatic cases, which, as we will see, are relevant in a relatively low-temperature universe with correspondingly low Γ_{th} , where Υ/H is maximized in certain models.

We note that unlike in the adiabatic regime, in the $\Gamma_{\text{th}} \lesssim \omega_f$ part of the underdamped regime, the $\Upsilon \dot{\phi}$ damping term that will appear in the equation of motion of ϕ is actually not fundamentally proportional to $\dot{f}_p^{\text{eq}} \propto \dot{\phi}$. It is instead proportional to the time integral of a sinusoidal function involving $\dot{f}_p^{\text{eq}}$ with a nontrivial phase, as given in Eq. (17). The resulting damping term, Eq. (14), can be expressed as a $\propto \dot{\phi}$ frictionlike term owing to mathematical identities involving sinusoidal functions. Speaking in real numbers, this is essentially because a cosine/sine with a phase has a sine/cosine component and because a derivative and an integral both turn a cosine/sine into a sine/cosine.

C. Damping rates

In the adiabatic regime, as defined in Eq. (18), substituting Eq. (19) into Eq. (14) yields

$$\begin{aligned} \Upsilon^{(\text{adiabatic})} &= \frac{(M_\chi M'_\chi)^2}{2\pi^2 \Gamma_{\text{th}} T} \int \frac{p^2 dp}{M_\chi^2 + p^2} f_p^{\text{eq}} \left(1 \mp \frac{f_p^{\text{eq}}}{g_{\text{dof}}} \right) \\ &\sim \frac{(M'_\chi)^2 n_\chi^{\text{eq}}}{\Gamma_{\text{th}} T} \min \left(1, \frac{M_\chi^2}{T^2} \right). \end{aligned} \quad (23)$$

The damping coefficient Υ can also be computed analytically in the underdamped, but not necessarily adiabatic, regime, as defined in Eq. (21). Using Eq. (22) to evaluate Eq. (14), we find

$$\Upsilon^{(\text{underdamped})} = \Upsilon^{(\text{adiabatic})} \times \frac{\Gamma_{\text{th}}^2}{\Gamma_{\text{th}}^2 + \omega_f^2}, \quad (24)$$

where $\Upsilon^{(\text{adiabatic})}$ is the final *expression* for Υ , as given in Eq. (23), and ω_f is the angular frequency of the assumed sinusoidally varying $f_p^{\text{eq}} \propto e^{i\omega_f t}$. The latter is typically an integer multiple of the effective mass of the ϕ field, and so they are often of similar order $\omega_f \sim M_\phi$. Equations (23) and (24) are two of the main results of this work. In the next subsection, we will generalize them to cases with multiple thermal bath species that couple to ϕ at some level. In Sec. IV, we will apply these general damping rates to various specific cases.

As we can see, the underdamped thermal damping rate $\Upsilon^{(\text{underdamped})}$ in Eq. (24) has a Lorentzian form in the thermalization rate Γ_{th} . As Γ_{th} is increased from a tiny value, $\Gamma_{\text{th}} \ll \omega_f \sim M_\phi$, the damping initially gets stronger as the χ particles react to their ϕ -modulated effective mass $M_\chi(\phi)$ more, and consequently backreact more strongly to ϕ . In this regime, we have $\Upsilon \propto |\delta f| \propto \Gamma_{\text{th}}$, which is mildly analogous to how the dark matter abundance behaves in thermal freeze-in scenarios. The damping rate Υ is at its maximum when $\Gamma_{\text{th}} \sim \omega_f \sim M_\phi$. Going to the other extreme, $\Gamma_{\text{th}} \gg \omega_f \sim M_\phi$, the underdamped expression $\Upsilon^{(\text{underdamped})}$ reduces to the adiabatic one, $\Upsilon^{(\text{adiabatic})}$, as expected. Then, the rapid thermalization rate Γ_{th} weakens the nonequilibrium part of the distribution function δf because it keeps χ very close to thermal equilibrium. Hence, the damping rate in this regime scales as $\Upsilon \propto \Gamma_{\text{th}}^{-1}$.

Depending on the form of $M_\chi(\phi)$, the nature of the thermal damping can be quite different. In many cases, including those that we will discuss in the next section, we have an M_χ that is effectively of the following form:

$$M_\chi = m_\chi \left[1 + \frac{c_N}{N} \left(\frac{\phi}{\Lambda} \right)^N \right], \quad (25)$$

where m_χ is the bare mass of χ , c_N is a dimensionless constant, and Λ is a mass scale. For M_χ of this form, we obtain

$$\Upsilon^{(\text{adiabatic})} \sim \frac{c_N^2 m_\chi^2 \phi^{2(N-1)} n_\chi^{\text{eq}}}{\Lambda^{2N} \Gamma_{\text{th}} T} \min \left(1, \frac{M_\chi^2}{T^2} \right), \quad (26)$$

$$\Upsilon^{(\text{underdamped})} = \Upsilon^{(\text{adiabatic})} \frac{\Gamma_{\text{th}}^2}{(\Gamma_{\text{th}}^2 + N^2 M_\phi^2)}. \quad (27)$$

For $N = 1$ and $N = 2$, we reproduce the results of previous works, e.g., those of Refs. [16,17,19].

D. Multiple thermal dampers

In realistic cases, the thermal environment may involve multiple coupled species. The total thermal damping in the

presence of multiple thermal dampers χ_i is given by summing Eq. (14) over the species χ_i ,

$$\Upsilon \dot{\phi} \equiv \sum_i M'_{\chi_i} \int \frac{d^3 p}{(2\pi)^3} \frac{M_{\chi_i}}{\sqrt{M_{\chi_i}^2 + p^2}} \delta f_p^{(i)}. \quad (28)$$

The $\delta f_p^{(i)}$ can be found by solving a set of Boltzmann equations for all $\delta f_p^{(i)}$, which are in general coupled. If the thermal bath particles are close to equilibrium and the Hubble expansion is negligible, these coupled Boltzmann equations can be written in the following linearized form:

$$\vec{f}_p = -\mathbf{M}_\Gamma \delta \vec{f}_p, \quad (29)$$

where $\delta \vec{f}_p = (\delta f_p^{(1)}, \delta f_p^{(2)}, \dots, \delta f_p^{(i)}, \dots)^T$ with $\delta f_p^{(i)} \equiv f_p^{(i)} - f_p^{\text{eq},(i)}$, $\vec{f}_p$ is defined similarly to $\delta \vec{f}_p$ with $\delta f_p \rightarrow f_p$, and $\mathbf{M}_\Gamma$ is a matrix that quantifies the rates at which χ_i thermalize between themselves (relative thermalization) as well as with the rest of the thermal bath (absolute thermalization). The thermalization rate matrix $\mathbf{M}_\Gamma$ is diagonal if the different species χ_i do not interact directly. For SM χ_i , the matrix $\mathbf{M}_\Gamma$ typically has non-diagonal entries. In the *adiabatic* limit, we have

$$\delta \vec{f}_p \approx -\mathbf{M}_\Gamma^{-1} \vec{f}_p^{\text{eq}}. \quad (30)$$

In this regime, it is useful to define the quantity

$$\Gamma_{\text{th}}^{(i \rightarrow j)} = \frac{1}{(\mathbf{M}_\Gamma^{-1})_{ij}}, \quad (31)$$

which can be interpreted as the rate at which the distribution function of χ_i responds to modulations in the mass (and hence equilibrium distribution) of χ_j . The total thermal damping in this adiabatic regime can be written as

$$\Upsilon \sim \sum_i \sum_j \frac{M'_{\chi_i} M'_{\chi_j} n_{\chi_j}^{\text{eq}}}{\Gamma_{\text{th}}^{(i \rightarrow j)} T} \min \left(1, \frac{M_{\chi_i} M_{\chi_j}}{T^2} \right). \quad (32)$$

The double sum can be understood as follows. Given a pair of species, χ_i and χ_j , the species j has its equilibrium distribution function $f_p^{\text{eq},(j)}(t)$ modulated by ϕ . The other species, χ_i , responds to this $f_p^{\text{eq},(j)}(t)$ with the rate $\Gamma_{\text{th}}^{(i \rightarrow j)}$ and backreacts to ϕ . In the presence of multiple species with comparable $M_{\chi_i} \approx M_\chi$ and $M'_{\chi_i} \approx M'_\chi$, the largest contribution to thermal damping rate Υ comes from the weakest link, namely, the $i - j$ combination that yields the slowest $\Gamma_{\text{th}}^{(i \rightarrow j)}$. In the *underdamped* limit, we instead have

$$\delta \vec{f}_p \approx -\text{Re}[(\mathbf{M}_\Gamma + i\omega_f)^{-1} \vec{f}_p^{\text{eq}}], \quad (33)$$

where $\omega_f = \text{diag}(\omega_{f_1}, \dots, \omega_{f_j}, \dots)$, which yields

$$\Upsilon \sim \text{Re} \sum_i \sum_j \frac{M'_{\chi_i} M'_{\chi_j} n_{\chi_j}^{\text{eq}}}{T} [(\mathbf{M}_\Gamma + i\omega_f)^{-1}]_{ij} \times \min \left(1, \frac{M_{\chi_i} M_{\chi_j}}{T} \right). \quad (34)$$

IV. EXAMPLES

In this section, we consider the effect of thermal damping in several models: a scalar coupled to neutrinos (Sec. IV A), a scalar quadratically coupled to gluons (Sec. IV B 1), the QCD axion and lighter QCD axion (Sec. IV B 2), and the scalar mediator of long-range dark matter forces (Sec. IV C). These cases represent different types of χ fields: relativistic and nonrelativistic χ , fermionic and bosonic χ , single species χ and multiple coupled species χ_i , and different functional forms of ϕ -dependent effective mass $M_{\chi_i}(\phi)$. In all the scenarios we consider in this section, the underdamped results of the previous section are applicable.

A. Neutrino-mass modulators

Consider a scalar field ϕ of bare mass m_ϕ coupled diagonally to the (assumed Majorana) active neutrinos mass-eigenstates ν_i according to the following effective Lagrangian below the electroweak scale [17,43–58]:

$$-\mathcal{L} \supset \frac{1}{2} m_\phi^2 \phi^2 + \sum_i M_{\nu_i}(\phi) \nu_i \nu_i + \text{H.c.}, \quad (35)$$

where $i = 1, 2, 3$ and $M_{\nu_i} = m_{\nu_i} - g\phi$. We will assume for simplicity that the neutrinos have the same bare mass equal to $m_{\nu_i} = m_\nu = 0.1$ eV and thermalize with the weak interaction rate $\Gamma_{\text{th}} = G_F^2 T^5$, where G_F is the Fermi constant. From Eq. (13), we find that the effective mass of the ϕ field in this model is

$$M_\phi^2 = V''_{\text{eff}} = m_\phi^2 + \frac{1}{4} g^2 T^2. \quad (36)$$

Using Eqs. (26) and (27), we find that in the limit $M_\nu \ll T$ the thermal damping rate of the scalar field ϕ oscillating with amplitude Φ is

$$\Upsilon_\nu = \begin{cases} \frac{3g^2 m_\nu^2}{2\pi^2} \frac{\Gamma_{\text{th}}}{M_\phi^2 + \Gamma_{\text{th}}^2}, & \Phi \ll m_\nu/g \\ \frac{3g^4 \phi^2}{2\pi^2} \frac{\Gamma_{\text{th}}}{4M_\phi^2 + \Gamma_{\text{th}}^2}, & \Phi \gg m_\nu/g \end{cases}. \quad (37)$$

Here, we have restored the $\mathcal{O}(1)$ factors omitted in Eqs. (26) and (27), see [17] for details on this.

In Fig. 3, we plot a representative time evolution of the neutrino thermalization rate Γ_{th} , the scalar field's effective

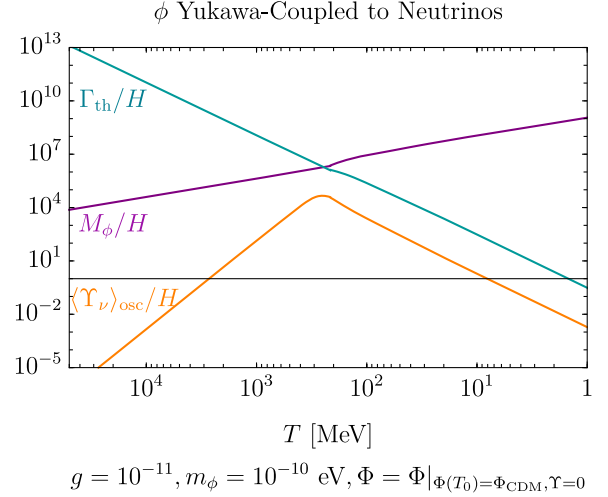


FIG. 3. The neutrino thermalization rate Γ_{th} , scalar effective mass M_ϕ , and oscillation-averaged thermal damping rate $\langle \Upsilon_\nu \rangle_{\text{osc}}$ in the neutrino-coupled scalar model of Eq. (35). Here, we set $g = 10^{-11}$, $m_\phi = 10^{-10}$ eV, and that ϕ makes up the total dark matter abundance today, $\Phi = \Phi_{\text{CDM}}$. In evaluating the thermal damping rate, we take this present-day value of Φ and evolve Φ backward in time accounting for the thermal mass of ϕ but assuming $\langle \Upsilon_\nu \rangle_{\text{osc}} = 0$.

mass M_ϕ , and the ϕ -oscillation-averaged thermal damping rate $\langle \Upsilon \rangle_{\text{osc}} \equiv [\int_0^{\pi M_\phi^{-1}} dt \Upsilon \dot{\phi}^2 / (\pi M_\phi^{-1})] / (M_\phi^2 \Phi^2 / 2)$. Here, we fix $g = 10^{-11}$, $m_\phi = 10^{-10}$ eV, and assume the scalar amplitude Φ today corresponds to the correct ϕ dark matter abundance. The displayed $\langle \Upsilon_\nu \rangle_{\text{osc}}$ is evaluated on the backward extrapolation of the scalar amplitude Φ assuming that thermal damping is absent, $\Upsilon_\nu = 0$. The plot shows that, given these assumptions, the $\langle \Upsilon_\nu \rangle_{\text{osc}}/H$ is significantly greater than unity in some temperature range, demonstrating the necessity to consider the effect of thermal damping. It also shows that the $\langle \Upsilon_\nu \rangle_{\text{osc}}/H$ is maximized at around $\Gamma_{\text{th}} \sim M_\phi$ and becomes smaller in either extremes of $\Gamma_{\text{th}} \ll M_\phi$ and $\Gamma_{\text{th}} \gg M_\phi$, in accordance to the expectations of Sec. III C. This behavior is not captured by the adiabatic approximation, thus highlighting the advantage of the underdamped approximation.

We discuss in detail the phenomenological implications of thermal damping in this model in our companion paper [17]. We summarize the key points here. Thermal damping allows the scalar field ϕ to have large amplitudes in the past without running afoul with late-time constraints, such as those from Cosmic Microwave Background (CMB) and Big Bang Nucleosynthesis (BBN). The time evolution of the scalar amplitude Φ in the presence of thermal damping by the neutrinos exhibits a dynamical attractor. As a result, a wide range of ϕ initial conditions converge to the same late-time evolution dictated only by the model parameters g and m_ϕ . There is a combination of g and m_ϕ , corresponding to a line in the g vs m_ϕ parameter space, that

leads to ϕ constituting the total dark matter today. This parameter-space line serves as a compelling target to aim for experimentally.

B. Hadron-mass modulators

In this section, we discuss the case of a scalar coupled to the hadronic sector of the SM, leading to a time-dependent hadronic mass due to the scalar background. Depending on the CP property of the scalar field, the leading mass modulation can happen at the linear order or at the quadratic order in the scalar field. Examples of models with linear modulation include a light dilaton [59,60], relaxion [61,62], or a generic light Higgs portal model [63], whereas the QCD axion [64,65], the lighter QCD axion [66–68], and a quadratically coupled scalar field [69–73] induce hadron-mass modulations at quadratic order in the field. As shown in Eq. (26) and Eq. (27), the thermal damping rate takes a different form for these types of interactions. For instance, in the adiabatic case, it is independent of the field amplitude for a linear coupling, whereas it depends linearly on the field amplitude for the case of a quadratic coupling.

As linearly coupled light scalars mediate long-range forces, they are subjected to strong constraints coming from fifth-force and/or equivalence principle violation searches [74–77],¹ independently of their cosmic abundances. In contrast, the most stringent limits on some quadratically coupled scalars are based on their cosmic abundances [72,78,79]. Thermal damping can relax these limits because it reduces the scalar amplitudes. Thus, in order to demonstrate the effect of thermal damping we will focus in what follows on the quadratically coupled case, where its effect is arguably more pronounced.

1. Scalar quadratically coupled to gluon

In this section, we consider the case of a scalar field ϕ of bare mass m_ϕ , coupled quadratically to the gluon field strength tensor, $G_{\mu\nu}^a$ (a being the color index),² as

$$\mathcal{L} \supset -\frac{\phi^2}{2M_{\text{Pl}}^2} \frac{d\beta(g_s)}{2g_s} G_{\mu\nu}^a G^{a\mu\nu}, \quad (38)$$

where $\beta(g_s) = -(11 - 2/3N_f)g_s^3/(16\pi^2)$ is the QCD beta function with N_f being the number of light quarks, g_s is the gauge coupling constant, d is the coupling strength, and $M_{\text{Pl}} = 2.4 \times 10^{18}$ GeV is the reduced Planck scale. Due to the above interaction, the QCD scale, Λ_{QCD} , depends on the scalar field ϕ as

¹The scalar linearly coupled to neutrinos model, Eq. (35), is an exception to this, as neutrinos cannot be contained by ordinary matter.

²One can extend the ϕ^2 coupling to other SM fields as well. Here, we consider one example to demonstrate the effect of the thermal damping.

$$\frac{\partial \ln \Lambda_{\text{QCD}}}{\partial(\phi^2)} = \frac{d}{2M_{\text{Pl}}^2}, \quad (39)$$

which in turn leads to ϕ -dependent hadron masses. Thus, a time-dependent scalar background leads to modulations in nucleon (m_N), and in neutral and charged pion masses as [69]

$$M_\pi \approx m_\pi \left[1 + \frac{d\phi^2}{4M_{\text{Pl}}^2} \right], \quad (40)$$

$$M_N \approx m_N \left[1 + 0.9 \frac{d\phi^2}{2M_{\text{Pl}}^2} \right], \quad (41)$$

where $m_N = 931$ MeV, and we denote m_π as generic pion mass with $m_{\pi^0} \approx 135$ MeV and $m_{\pi^\pm} \approx 140$ MeV. For simplicity, we limit our analysis to temperatures $T \lesssim 100$ MeV, where the QCD sector is confined and dominated by nonrelativistic pions and nucleons. The thermal damping rate of ϕ is given by Eq. (34) as

$$\Upsilon_\pi \sim \frac{d^2 m_\pi^2 \phi^2}{4M_{\text{Pl}}^4 T} \sum_i \sum_j [(\mathbf{M}_\Gamma + iM_\phi \mathbf{I})^{-1}]_{ij} n_j^{\text{eq}}, \quad (42)$$

where the summation indices, i and j , take two “values,” π^0 and $(\pi^- + \pi^+)$, $\mathbf{I}$ is a unit 2×2 matrix, and $\mathbf{M}_\Gamma$ is the thermalization rate matrix, as defined in Eq. (29). In this model, there can be “cross-damping,” where ϕ perturbs the pion number densities with, for instance, its π^0 coupling, and gets backreaction from $\pi^\pm$, or vice versa. The above expression for Υ_π involves the effective mass M_ϕ of ϕ and the thermalization rate matrix $\mathbf{M}_\Gamma$ [defined in Eq. (31)] for hadrons. Using Eq. (13), we find the effective mass of ϕ to be

$$\begin{aligned} M_\phi^2 &= V''_{\text{eff}} \\ &= m_\phi^2 + \delta m_{\pi^0}(0) n_{\pi^0}^{\text{eq}}(0) + 2\delta m_{\pi^\pm}(0) n_{\pi^\pm}^{\text{eq}}(0), \end{aligned} \quad (43)$$

where we have neglected the contributions to ϕ mass from protons and neutrons, as they are less abundant than the pions. As ϕ modulates the masses of neutral and charged pions, calculating the thermalization rate matrix $\mathbf{M}_\Gamma$ requires simultaneous solving of multiple Boltzmann equations, which we describe in detail in the Appendix. In the same Appendix, we also derive the thermalization rates of nucleons. We confirm that both the thermal-mass and thermal damping effects from nucleons are negligible in the cases discussed here. In the adiabatic limit, the thermal damping rate simplifies as

$$\Upsilon_\pi \sim \frac{d^2 m_\pi^2 \phi^2}{4M_{\text{Pl}}^4 T} \frac{n_\pi^{\text{eq}}}{\Gamma_{\text{th}}^{\text{eff}}}, \quad (44)$$

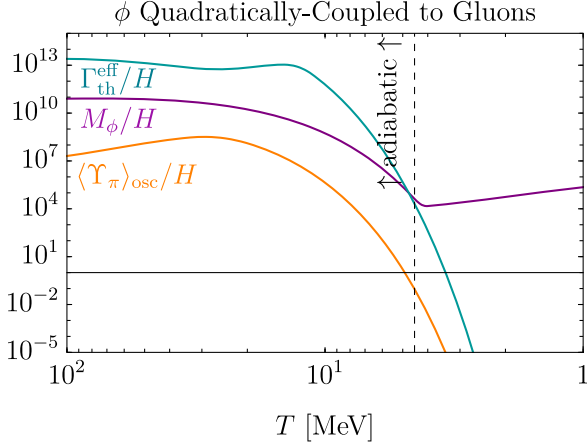
where $\Gamma_{\text{th}}^{\text{eff}}$ is the effective thermalization rate, whose analytical expression is given in the [Appendix](#).

We plot in [Fig. 4](#) the scalar field's effective thermalization rate $\Gamma_{\text{th}}^{\text{eff}}$, the effective mass M_ϕ , and the oscillation-averaged thermal damping rate $\langle \Upsilon_\pi \rangle_{\text{osc}}$, defined as $\langle \Upsilon_\pi \rangle_{\text{osc}} = [(\pi M_\phi^{-1})^{-1} \int_0^{\pi M_\phi^{-1}} dt \Upsilon_\pi \dot{\phi}^2] / [M_\phi^2 \Phi^2 / 2]$ by turquoise, purple, and orange lines, respectively. We have normalized these quantities by the Hubble scale H . We take $m_\phi = 10^{-10}$ eV and set the initial amplitude of the ϕ oscillation to be $\Phi = M_{\text{Pl}}/\sqrt{d}$ with $d = 10^{23}$. We see that at around $T \sim 10$ MeV, the thermal damping effect starts to become negligible compared to Hubble damping. On the other hand, M_ϕ^2 remains dominated by the $\propto n_\pi^{\text{eq}}$ thermal pion contributions down to $T \sim 6$ MeV, whereupon bare mass starts dominating. This behavior is seen by the change of slope of the purple line, representing M_ϕ/H , from an exponentially falling off behavior to a $m_\phi/H \propto T^{-2}$ scaling.

To understand the long-term evolution of ϕ in the presence of the thermal mass and damping, we write the field in terms of its slowly varying amplitude $\Phi(t)$ as

$$\phi(t) = \text{Re}[\Phi(t)e^{iM_\phi t}]. \quad (45)$$

Plugging this into the Equation of Motion (EOM) derived in [Eq. \(12\)](#), multiplied by $\dot{\phi}$ and integrated over an



$$d = 10^{23}, m_\phi = 10^{-10} \text{ eV}, \Phi = M_{\text{Pl}}/\sqrt{d} \approx 2.4 \times 10^7 \text{ GeV}$$

FIG. 4. The pion effective thermalization rate $\Gamma_{\text{th}}^{\text{eff}}$, scalar effective mass M_ϕ , and oscillation-averaged thermal damping rate $\langle \Upsilon_\pi \rangle$ as a function of temperature in the scalar quadratically coupled to gluons model, [Eq. \(38\)](#). Here, assume $d = 10^{23}$ and $m_\phi = 10^{-10}$ eV, and fix the scalar amplitude to $\Phi = M_{\text{Pl}}/\sqrt{d}$. The vertical dashed line indicates the rough boundary between the adiabatic and nonadiabatic regimes. The displayed $\Gamma_{\text{th}}^{\text{eff}}$ is meaningful only in the adiabatic regime, where the adiabatic thermal damping formula, [Eq. \(26\)](#), applies. In the nonadiabatic regime, the thermal damping rate is given by the more general [Eq. \(27\)](#), in which there is no simple notion of thermalization rate.

oscillation, we get the EOM for the amplitude of ϕ as

$$\ddot{\Phi} + \left[3H + \langle \Upsilon_\pi \rangle_{\text{osc}} + \frac{\dot{M}_\phi}{M_\phi} \right] \frac{\Phi}{2} = 0. \quad (46)$$

In estimating the above formula, we have also assumed $|\dot{\Phi}/\Phi| \ll M_\phi$. To solve the EOM, we start by noting that [Eq. \(46\)](#) can be recast as $\dot{n}_\phi = -(3H + \langle \Upsilon_\pi \rangle_{\text{osc}})n_\phi$, where $n_\phi = M_\phi \Phi(t)^2$ is the time-averaged ϕ number density. Thus, in the absence of the thermal friction, n_ϕ redshifts as $n_\phi \propto 1/a^3$. When $\langle \Upsilon_\pi \rangle_{\text{osc}} \gtrsim H$, on a timescale much less than a Hubble time, and over which M_ϕ is almost constant, we have $\dot{n}_\phi/n_\phi \approx 2\dot{\Phi}/\Phi \approx -\langle \Upsilon_\pi \rangle_{\text{osc}} \propto -\Phi^2$ in the adiabatic case. It leads to $\Phi \sim \Phi_i [\langle \Upsilon_\pi \rangle_{\text{osc}}|_{t_i} (t - t_i)]^{-1/2}$, for some initial time t_i and initial thermal damping $\langle \Upsilon_\pi \rangle_{\text{osc}}|_{t_i}$. Thus, in the presence of thermal friction, the field amplitude and the number density decay as a power law, not exponentially.

In general, if $\langle \Upsilon_\pi \rangle_{\text{osc}} \gtrsim H$ at some point during the evolution history, then thermal damping decreases the ϕ number density significantly. Since the $\langle \Upsilon_\pi \rangle_{\text{osc}}$ depends on the scalar's amplitude Φ , there is a critical amplitude $\Phi_Y(T)$ at which $\langle \Upsilon_\pi \rangle_{\text{osc}} \sim H(T)$. Upon reaching Φ_Y , the amplitude Φ may track the Φ_Y for a while. Once Φ becomes significantly smaller than Φ_Y , the friction becomes unimportant, and the number density of the field redshifts as $n_\phi \propto a^{-3}$. Suppose Φ goes below Φ_Y first at T_Y . The abundance of the field at $T \lesssim T_Y$ can then be found by simply redshifting the number density as $n_\phi(T) \sim n_\phi(T_Y)(T/T_Y)^3$.

In [Fig. 5](#), we show the evolution of the oscillation amplitude of the scalar field Φ (left) and its number density n_ϕ (right) as a function of temperature T . The solid and dashed lines depict the evolution with and without the thermal damping term in the EOM, respectively. The turquoise, purple, and the dark orange lines show the evolution of Φ with initial amplitudes of $\Phi_i = 10^6, 10^5, 3 \times 10^3$ GeV, respectively. The light red colored band shows the values of Φ for which $\Upsilon(\Phi_Y) \sim H$. For any Φ_i larger than Φ_Y at some point during the evolution, the field experiences thermal damping, and its final abundance reduces. This is shown by the thinner solid line. The temperature at which Φ_Y is minimized corresponds roughly to that at which Υ/H is maximized for a fixed Φ , as can be seen from [Fig. 4](#). Also, for our chosen data point ($d = 10^{23}$ and $m_\phi = 10^{-10}$ eV), the thermal damping leads to a universal late-time abundance independent of the initial amplitude Φ_i , as can be seen on both panels of [Fig. 5](#). Such a dynamical attractor is also found in the neutrino-coupled scalar model, [Eq. \(35\)](#), as detailed in our companion paper [\[17\]](#).

The presence of the $\dot{M}_\phi/M_\phi < 0$ term in [Eq. \(46\)](#) leads to an antifriction effect that causes the oscillation amplitude

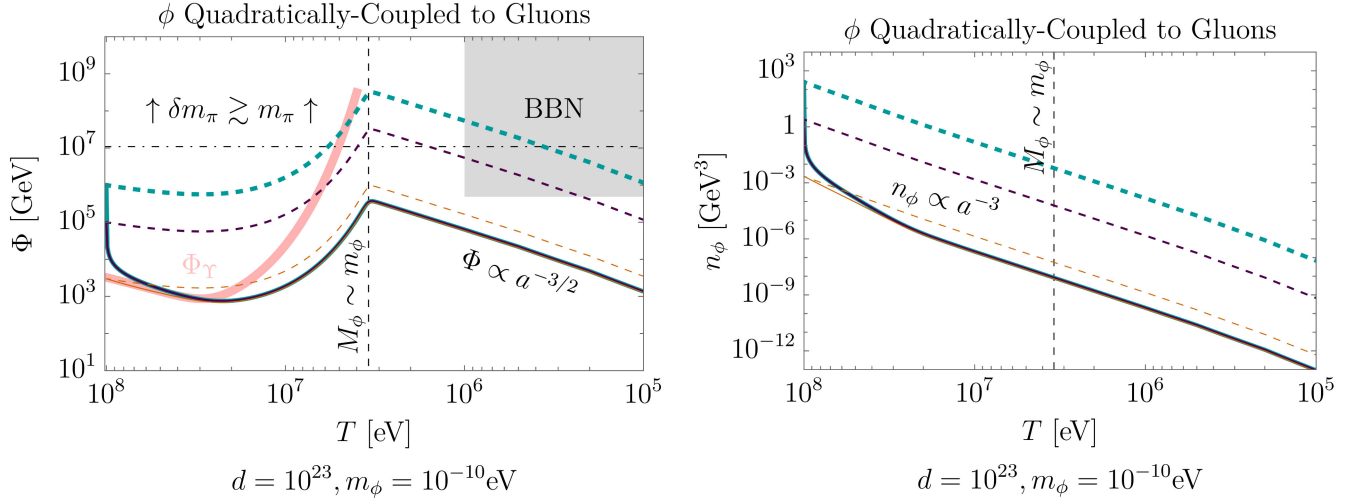


FIG. 5. Evolution of the scalar field amplitude Φ (left) and number density $n_\phi = M_\phi^2 \Phi$ (right) in the scalar quadratically coupled to gluons model, Eq. (38). Here, we assume the same model parameters as in Fig. 4 and pick initial conditions such that $\Phi = 10^6$ GeV (turquoise), 10^5 GeV (purple), and 3×10^3 GeV (orange) at $T = 100$ MeV. The evolutions of Φ and n_ϕ are shown as solid lines, which in these figures merge almost completely to a single line. Their evolution in the absence of thermal damping is shown in dashed lines for comparison. On the left plot, the thick pink line is the scalar amplitude Φ_Υ at which $\Upsilon \sim H$, and the approximate BBN limit ($\delta\Lambda_{\text{QCD}}/\Lambda_{\text{QCD}} \lesssim 2 \times 10^{-3}$ [80]) is shown as a gray shaded region.

of ϕ to increase when M_ϕ decreases. This effect dominates over Hubble, $|\dot{M}_\phi/M_\phi|/H \sim m_\pi/T \gg 1$, when ϕ is dominated by its thermal mass. As a result, in cases with $\Phi_i = 10^6$ GeV and 10^5 GeV, in the *absence* of thermal damping, the ϕ amplitude increases to the point where the ϕ -induced change in pion mass becomes larger than the bare pion mass ($\delta m_\pi > m_\pi$) during the course of evolution (shown by the turquoise and purple dashed lines). Thus, for those initial conditions the theory loses its predictive power. However, thermal damping can overcome both antifriction and Hubble, and reduce the oscillation amplitude enough to cure the theory from such pathological behavior. Similarly, the overclosure bound, BBN bound on the variation of the QCD scale [80,81], and the bounds from the MICROSCOPE experiment [72] are also relaxed due to the thermal damping. For the chosen parameters in Figs. 4 and 5, the BBN bound from Ref. [80] shown in Fig. 5 is far more stringent than the limits from MICROSCOPE and overclosure.

Note that the benchmark point considered here may lie in a tuned region of the parameter space. Such tuning can, however, be mitigated in concrete UV completions (see, e.g., Ref. [82]).

2. The QCD axion and the lighter QCD axion

In this section, we discuss another example of hadron-mass modulator, namely, the models of QCD and light QCD axions, which induce hadron-mass variation at the quadratic order as well [65,72]. To see how that works, note that at the leading order the effective axion (a) potential can be written as [64,83]

$$V(a, \pi^0) = -M_{\pi^0}^2 f_\pi^2, \quad (47)$$

where $f_\pi = 93$ MeV, and f_a is the axion decay constant, and

$$M_{\pi^0} = m_{\pi^0} \sqrt{1 - \frac{4m_u m_d \sin^2(\theta/2)}{(m_u + m_d)^2}}, \quad (48)$$

where m_u (m_d) is the up (down) quark mass. Note that we also employ the pseudoaxionic shift symmetry to set the effective strong- CP angle to zero, and define $\theta \equiv a/f_a$. Expanding around the axion vacuum expectation value, we obtain the change of the neutral pion mass due to the axion as

$$\delta m_{\pi^0} \equiv M_{\pi^0} - m_{\pi^0} \simeq -7 \text{ MeV} \times \theta^2, \quad (49)$$

where we use small-angle approximation. Compared to the case discussed in Sec. IV B 1, the axion only modulates the mass of neutral pions at the leading order.³ Starting from Eq. (49), and using Eq. (13), we find the effective mass of the axion as

³The nucleon masses depend on the pion masses at the chiral loop order, and thus an axion-dependent pion mass also leads to axion-dependent nucleon masses as well. However, as $\partial \ln m_N / \partial \ln m_\pi^2 \simeq 0.06$ and the nucleon number density is small compared to that of the pions, we neglect the nucleon contribution here.

$$M_a^2 = V''_{\text{eff}} = m_a^2 - \frac{14 \text{ MeV}}{f_a^2} n_{\pi^0}^{\text{eq}}, \quad (50)$$

where $m_a^2 \simeq m_{\pi^0}^2 f_{\pi}^2 / f_a^2$ for the QCD axion and arbitrary for the lighter QCD axion. Using Eq. (26), we find the thermal damping coefficient in the underdamped limit ($|\dot{a}/a| \ll \min[\Gamma_{\text{th}}, M_a]$) as

$$\Upsilon_{\pi} = \frac{n_{\pi^0}^{\text{eq}} \Gamma_{\text{th}}}{(\Gamma_{\text{th}}^2 + M_a^2) T} \left(\frac{14 \text{ MeV}}{f_a} \right)^2 \theta^2, \quad (51)$$

for some effective thermalization rate Γ_{th} . In Appendix, we discuss in detail about how to calculate Γ_{th} .

Let us first discuss the QCD axion. We find that for the vanilla misalignment scenario the ranges of initial conditions and f_a values for which the thermal damping plays any important role lead to the Universe being overclosed by the QCD axion at late times. Although, we note that this conclusion depends on the detailed thermal history. For example, models that involve an entropy dump at some point before matter-radiation equality would weaken the overclosure constraint [84,85]. In such an alternative cosmology, a simple backward extrapolation of the overclosure bound does not work, and thermal damping could be important at some point in the past. Furthermore, there are other scenarios that involve the axion decay constant [84,86,87] or the QCD scale [88,89] changing as a function of time or the presence of large pion chemical potential stemming from lepton asymmetry [90–92]. In general, whenever the hadronic sector behaves differently from the vanilla case, the thermal damping effects needs to be re-evaluated, and they might turn out to be important then. There is also a chance that, even in the vanilla case, some form of thermal damping might be important in determining the axion-string contribution to the axion abundance. Simulations of axion strings around the QCD phase transition traditionally assume that the axion field evolves according to its *thermal equilibrium* effective potential, as given by the topological susceptibility. Since thermal damping is a *nonequilibrium* effect, its effect is not captured by such simulations.

Another possibility is to consider models of the lighter QCD axions [66,67,93]. For those models, the mass of the axion is independent of f_a and the finite temperature correction to the axion mass can make the axion's effective potential be minimized at $\theta = \pi$ instead of $\theta = 0$. This happens at some high temperature if $m_a f_a \lesssim (14 \text{ MeV } n_{\pi}^{\text{eq}}(0))^{1/2}$. If the axion potential is minimized at $\theta = \pi$ initially, then the initial axion misalignment could effectively be $\theta(T_i) \approx \pi$ [94,95]. In those cases, the thermal damping could be more important. In principle, solving the evolution of the lighter QCD axion, as considered in, e.g., [94], but with the inclusion of thermal damping, could change some of the conclusions found in the literature. However, a large θ leads to various

complications, such as the anharmonicity of the axion potential [96] and modified hadron interactions [97–99]. The difference between the lighter QCD axion and quadratically coupled scalar cases is that in the axion case the damping is important when θ is oscillating around π , which tends to have little effect on the final abundance of the axion, while in the quadratically coupled scalar case the damping reduces the amplitude of ϕ and weakens various bounds, as discussed previously.

C. WIMP-mass modulators

Next, we briefly consider a scalar ϕ coupled to a fermionic WIMP dark matter χ , with the following Lagrangian:

$$-\mathcal{L} \supset \frac{1}{2} m_{\phi}^2 \phi^2 + (m_{\chi} - g_{\chi} \phi) \bar{\chi} \chi. \quad (52)$$

This could serve as a model for long-range dark matter forces [100]. While χ is close to a thermal equilibrium, the rate of number-changing process that restores its equilibrium is roughly given by $\Gamma_{\text{th}} \sim n_{\chi}^{\text{eq}} \langle \sigma v \rangle_{\text{ann}}$, where $\langle \sigma v \rangle_{\text{ann}}$ is χ 's total annihilation cross section to SM particles, whose limits can be found in, e.g., Ref. [101]. We will assume the WIMP-miracle annihilation cross section $\langle \sigma v \rangle_{\text{ann}}$ that leads to the correct DM abundance

$$\langle \sigma v \rangle_{\text{ann}}^{\text{DM}} \sim 3 \times 10^{-26} \text{ cm}^3/\text{s}. \quad (53)$$

The thermal damping rate of ϕ under the underdamped approximation can then be estimated as follows:

$$\Upsilon_{\chi} \sim \frac{(M'_{\chi})^2 n_{\chi}^{\text{eq}}}{\Gamma_{\text{th}} T} \frac{\Gamma_{\text{th}}^2}{M_{\phi}^2 + \Gamma_{\text{th}}^2} \min \left[1, \frac{M_{\chi}^2}{T^2} \right]. \quad (54)$$

Note that, for $|\phi| \ll m_{\chi}/g_{\chi}$, the Υ_{χ} is independent of the amplitude of the ϕ field. If this rate is greater than Hubble, then it would lead to an *exponential* damping of the ϕ amplitude. For instance, if we take $m_{\chi} = 100 \text{ GeV}$, then we have $\Upsilon_{\chi} \gtrsim H$ at $T = m_{\chi}$ (well before the WIMPs freeze out) if $\alpha_{\chi\chi} \equiv (g_{\chi}^2/4\pi)/Gm_{\chi}^2 \gtrsim 10^{12}$ and $m_{\phi} \ll \text{MeV}$. Further, as long as $\alpha_{\chi\chi} \gg 10^{14}$, the ϕ field begins oscillating before $T = m_{\chi}$ under its thermal mass, $M_{\phi} \sim g_{\chi} T$. Such dark forces with coupling strength relative to gravity $\alpha_{\chi\chi} \gg 10^{14}$ and $\lambda_{\phi} \equiv m_{\phi}^{-1} \gtrsim (\text{MeV})^{-1} \sim 100 \text{ fm}$ are still viable [100]. Note that, generically, Υ_{χ}/H is far from being maximized at $T = m_{\chi}$, so the regime where the Υ_{χ} is important extends well beyond the regime we have identified. The interactions that allow χ to annihilate to SM particles with the cross section given in Eq. (53) would generically generate interactions between the scalar ϕ and SM particles, which leads to fifth forces between ordinary matter at some level. Such fifth forces are strongly constrained for sufficiently large λ_{ϕ} [74]. However, those

interactions are model dependent. In some models, such fifth forces might be very suppressed.

V. DISCUSSION AND CONCLUSION

When a scalar field evolves in the early Universe, it drives otherwise-thermalized particles with significant coupling to it out of thermal equilibrium. These particles then backreact to the scalar in two qualitatively different ways. Even in complete quasithermal equilibrium, the particles' free energy contributes to the scalar's effective potential. This effect is nondissipative. In addition, due the nonequilibrium nature of the driven particles they also backreact dissipatively on the scalar, leading to a thermal damping of the scalar's kinetic energy. In this work, we clarify the origin of such thermal damping effect in a simple toy model, estimate its rates for scalars that modulate the masses of thermally populated particles, and demonstrate its importance in several models.

In some cases, the manifestation of thermal damping in the equation of motion of the scalar field ϕ simplifies to a frictionlike term of the form $\Upsilon\dot{\phi}$. Previous studies that found such an $\Upsilon\dot{\phi}$ term were limited to the adiabatic regime where the thermal bath particles thermalize essentially instantaneously on the timescale of ϕ 's dynamics. We found a new regime where the thermal damping also reduces to a $\Upsilon\dot{\phi}$ term, namely, when the ϕ field evolves sinusoidally in an underdamped manner, but not necessarily slowly compared to the thermal bath's equilibration rates. The adiabatic and underdamped regimes overlap but extend beyond each other. We obtained in Eqs. (23) and (24) the general thermal damping rates in both regimes, covering cases with scalar and fermionic, relativistic and nonrelativistic thermal bath particles. We also considered a thermal bath with multiple intercoupled species, in which case the scalar can drive the thermal bath and receives backreaction from it through different couplings, resulting in a new type of thermal damping, as given in Eqs. (32) and (34).

To illustrate the implications of thermal damping, we considered, as separate examples, three types of thermal dampers, namely, neutrinos ν , hadrons (pions and nucleons), and WIMP dark matter χ , with couplings to a scalar ϕ of the form $\phi\nu\nu$, $\phi^2 GG$ (with gluon field strength G), and $\phi\bar{\chi}\chi$, respectively. We derived the thermal damping rates of ϕ in each of these models and pointed out instances consistent with existing limits where $\Upsilon/H \gtrsim 1$. We also derived the thermal damping rate of (lighter) QCD axion by pions and nucleons. It would be interesting to consider other types of mass-modulating scalars ϕ , such as those with direct couplings to the electron e , downquark d , and upquark u of the forms $m_e(\phi)\bar{e}e$, $m_d(\phi)\bar{d}d$, and $m_u(\phi)\bar{u}u$, respectively, where the latter two couplings may not preserve the isospin symmetry. Other concrete variants include a scalar ϕ that modulates the electromagnetic

coupling e through $F_{\mu\nu}F^{\mu\nu}/e^2(\phi)$ and a scalar ϕ that couples democratically to all SM particles. In the latter case, the thermal damping by different species may become dominant at different times, as they decouple in turn. One could also consider the thermal damping of a vector field in place of a scalar field ϕ .

We have focused on the thermal damping due to the backreaction of a thermal bath with slightly distorted distribution functions. There are in principle other types of damping that may appear concurrently in a given model. These include damping due to thermal bath particles scattering off of the coherent scalar, damping mediated by virtual particles (which occurs even when the scalar-coupled particles are at zero temperature), and damping due to nonperturbative effects [2,7,8,20,21]. In estimating the thermalization rate of the thermal bath, we assumed the relaxation time approximation, which is equivalent to assuming that the nonequilibrium distribution function f_p has the same shape as the equilibrium distribution f_p^{eq} . While we expect such simplified estimates to give the correct order of magnitude for the thermal damping rates, it would be interesting to consider more realistic, momentum-dependent relaxation rates in a future work.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: THERMALIZATION RATES

In this Appendix, we calculate the thermalization rates relevant for estimating the thermal damping rates considered in the main text. The results are summarized in Fig. 6.

1. Pions

Here, we derive the thermalization rates of pions at temperatures $\mathcal{O}(1\text{--}10)$ MeV. In this temperature range, the

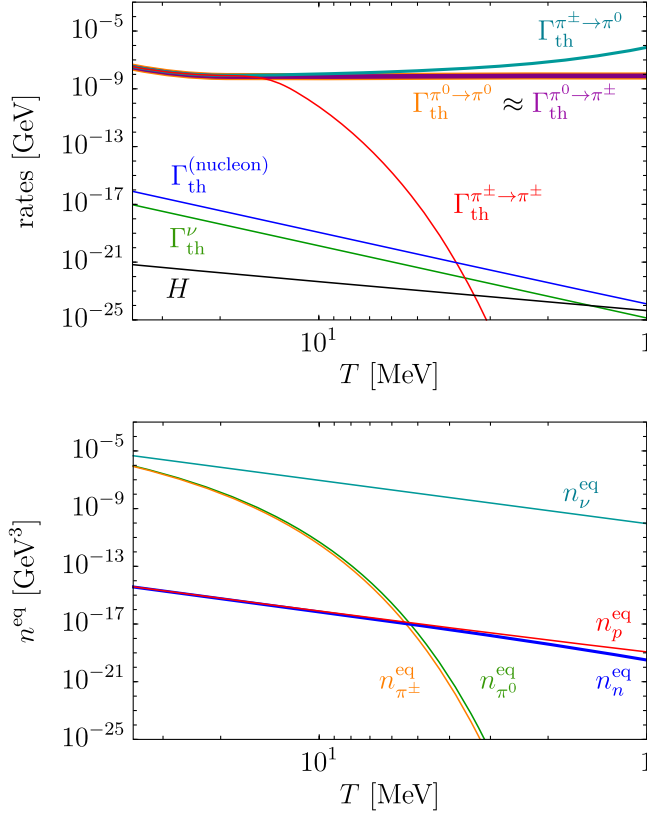


FIG. 6. Top: thermalization rates in different models. The thermalization rate in the neutrino-coupled scalar model is given by the typical electroweak interaction rate $\Gamma_{\text{th}}^{\nu} \simeq G_F^2 T^5$. For scalar quadratically coupled to gluon and (lighter) QCD axion, the four adiabatic thermalization rates of pions, $\Gamma_{\text{th}}^{\pi^0 \rightarrow \pi^0}$, $\Gamma_{\text{th}}^{\pi^0 \rightarrow \pi^{\pm}}$, $\Gamma_{\text{th}}^{\pi^{\pm} \rightarrow \pi^0}$, and $\Gamma_{\text{th}}^{\pi^{\pm} \rightarrow \pi^{\pm}}$, are obtained in Eqs. (A17), (A18), (A19), and (A20). In the latter models, the thermalization rates of protons and neutrons are approximately equal and given by $\Gamma_{\text{th}}^{(\text{nucleon})}$, as found in Eq. (A37). Also shown for comparison is the Hubble rate H accounting for the SM species only. Bottom: the thermal equilibrium number densities n^{eq} of the SM thermal dampers considered in this work: active neutrinos ν , pions $\pi^{0,\pm}$, protons p , and neutrons n .

most important reactions determining the pion densities $n_{\pi^{0,\pm}}$ are neutral pion decays to photon pair, charged pion pair annihilation to photons and neutral pions, as well as the inverse processes [102],

$$\pi^0 \leftrightarrow \gamma\gamma, \quad \pi^+\pi^- \leftrightarrow \gamma\gamma, \pi^0\pi^0. \quad (\text{A1})$$

Accounting for these processes, and neglecting Hubble expansion, the momentum-integrated Boltzmann equations read

$$\begin{aligned} \dot{n}_{\pi^0} = & -\Gamma_0 n_{\pi^0} + \langle \sigma v \rangle_{\gamma\gamma \rightarrow \pi^0} n_{\gamma}^2 - \langle \sigma v \rangle_{\pi^0\pi^0 \rightarrow \pi^+\pi^-} n_{\pi^0}^2 \\ & + \langle \sigma v \rangle_{\pi^+\pi^- \rightarrow \pi^0\pi^0} n_{\pi^+} n_{\pi^-}, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \dot{n}_{\pi^{\pm}} = & -\langle \sigma v \rangle_{\pi^+\pi^- \rightarrow \gamma\gamma} n_{\pi^+} n_{\pi^-} + \langle \sigma v \rangle_{\gamma\gamma \rightarrow \pi^+\pi^-} n_{\gamma}^2 \\ & - \langle \sigma v \rangle_{\pi^+\pi^- \rightarrow \pi^0\pi^0} n_{\pi^+} n_{\pi^-} + \langle \sigma v \rangle_{\pi^0\pi^0 \rightarrow \pi^+\pi^-} n_{\pi^0}^2, \end{aligned} \quad (\text{A3})$$

where angled brackets $\langle \dots \rangle$ denote thermal averaging, σ denotes cross section, and

$$\Gamma_0 = \langle \gamma_{\pi^0}^{-1} \tau_{\pi^0}^{-1} \rangle \quad (\text{A4})$$

is the thermal-averaged neutral pion's decay rate time-dilated by the Lorentz factor γ_{π^0} . We assume to first order that these quantities are given by their SM values and that the dominant radiation made of light SM particles remains in thermal equilibrium. In the cases we consider, the scalar field modulates hadron masses but not lepton masses, so the equilibrium number densities of leptons are not directly modulated by ϕ . Any indirect distortion of the radiation bath caused by out-of-equilibrium pions is suppressed relative to $\delta n_{\pi}/n_{\pi}$ by the small leptonic branching ratios and by the Boltzmann-suppressed pion energy fraction $\rho_{\pi}/\rho_{\text{rad}} \ll 1$ in the temperature range of interest. Although we do not expect the results to change significantly, it would be interesting to perform a more complete study of thermalization including the Boltzmann equations for species other than pions in the future.

The chemical equilibrium number densities of non-relativistic pions are

$$n_{\pi^0}^{\text{eq}} = \left(\frac{M_{\pi^0} T}{2\pi} \right)^{3/2} e^{-\frac{M_{\pi^0} - \mu_{\pi^0}}{T}}, \quad (\text{A5})$$

$$n_{\pi^{\pm}}^{\text{eq}} = \left(\frac{M_{\pi^{\pm}} T}{2\pi} \right)^{3/2} e^{-\frac{M_{\pi^{\pm}} - \mu_{\pi^{\pm}}}{T}}, \quad (\text{A6})$$

where we have capitalized the pion masses $M_{\pi^{0,\pm}}$ to indicate that these are their *effective* masses in the presence of a coherent scalar field. Due to the high abundances of photons and charged leptons, π^0 and $\pi^{\pm}$ stay close to chemical equilibrium down to temperatures $T \approx 6$ MeV, despite being severely Boltzmann suppressed. We will thus neglect the pion chemical potentials, $\mu_{\pi^{0,\pm}} \approx 0$, which implies $n_{\pi^+} \approx n_{\pi^-}$. Detailed balance of the aforementioned processes implies

$$\frac{\langle \sigma v \rangle_{\gamma\gamma \rightarrow \pi^0}}{\Gamma_0} = \frac{n_{\pi^0}^{\text{eq}}}{(n_{\gamma}^{\text{eq}})^2}, \quad (\text{A7})$$

$$\frac{\langle \sigma v \rangle_{\pi^+\pi^- \rightarrow \gamma\gamma}}{\langle \sigma v \rangle_{\gamma\gamma \rightarrow \pi^+\pi^-}} = \left(\frac{n_{\gamma}^{\text{eq}}}{n_{\pi^{\pm}}^{\text{eq}}} \right)^2, \quad (\text{A8})$$

$$\frac{\langle \sigma v \rangle_{\pi^+\pi^- \rightarrow \pi^0\pi^0}}{\langle \sigma v \rangle_{\pi^0\pi^0 \rightarrow \pi^+\pi^-}} = \left(\frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^{\pm}}^{\text{eq}}} \right)^2. \quad (\text{A9})$$

After linearizing, $\delta n_{\pi^{0,\pm}} \ll n_{\pi^{0,\pm}}^{\text{eq}}$, and assuming $\delta \dot{n}_{\pi^0} \ll \dot{n}_{\pi^0}^{\text{eq}}$, the Boltzmann equations for pion number densities, Eqs. (A2) and (A3), reduce to

$$\begin{pmatrix} \dot{n}_{\pi^0} \\ \dot{n}_{(\pi^++\pi^-)} \end{pmatrix} \approx -\mathbf{M}_\Gamma \begin{pmatrix} \delta n_{\pi^0} \\ \delta n_{(\pi^++\pi^-)} \end{pmatrix}, \quad (\text{A10})$$

$$\mathbf{M}_\Gamma \equiv \begin{pmatrix} \Gamma_0 + \Gamma_{0\leftrightarrow\pm} & -\Gamma_{0\leftrightarrow\pm} \frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^\pm}^{\text{eq}}} \\ -\Gamma_{0\leftrightarrow\pm} & \Gamma_\pm + \Gamma_{0\leftrightarrow\pm} \frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^\pm}^{\text{eq}}} \end{pmatrix}, \quad (\text{A11})$$

where we have defined $n_{(\pi^++\pi^-)} = n_{\pi^+} + n_{\pi^-} = 2n_{\pi^\pm}$ and

$$\Gamma_0 \equiv \langle \gamma_{\pi^0}^{-1} \tau_{\pi^0}^{-1} \rangle, \quad (\text{A12})$$

$$\Gamma_\pm \equiv 2n_{\pi^\pm}^{\text{eq}} \langle \sigma v \rangle_{\pi^+\pi^- \rightarrow \gamma\gamma}, \quad (\text{A13})$$

$$\Gamma_{0\leftrightarrow\pm} \equiv 2n_{\pi^0}^{\text{eq}} \langle \sigma v \rangle_{\pi^0\pi^0 \rightarrow \pi^+\pi^-}, \quad (\text{A14})$$

$$\delta n_{\pi^0} \equiv n_{\pi^0} - n_{\pi^0}^{\text{eq}}, \quad (\text{A15})$$

$$\delta n_{\pi^\pm} \equiv n_{\pi^\pm} - n_{\pi^\pm}^{\text{eq}}. \quad (\text{A16})$$

In our numerical computations, we use the thermal-averaged cross sections, as given in Refs. [102–106]. The adiabatic regime thermalization rates, as defined in Eq. (31), are

$$\Gamma_{\text{th}}^{\pi^0 \rightarrow \pi^0} = \frac{\Gamma_{0\leftrightarrow\pm} \Gamma_\pm}{\Gamma_\pm + \Gamma_{0\leftrightarrow\pm} \frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^\pm}^{\text{eq}}}} + \Gamma_0, \quad (\text{A17})$$

$$\Gamma_{\text{th}}^{\pi^0 \rightarrow \pi^\pm} = \frac{\Gamma_\pm (\Gamma_0 + \Gamma_{0\leftrightarrow\pm})}{\Gamma_{0\leftrightarrow\pm} \frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^\pm}^{\text{eq}}}} + \Gamma_0, \quad (\text{A18})$$

$$\Gamma_{\text{th}}^{\pi^\pm \rightarrow \pi^0} = \frac{\Gamma_0 \Gamma_\pm}{\Gamma_{0\leftrightarrow\pm}} + \Gamma_0 \frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^\pm}^{\text{eq}}} + \Gamma_\pm, \quad (\text{A19})$$

$$\Gamma_{\text{th}}^{\pi^\pm \rightarrow \pi^\pm} = \frac{\Gamma_0 \Gamma_{0\leftrightarrow\pm} \frac{n_{\pi^0}^{\text{eq}}}{n_{\pi^\pm}^{\text{eq}}}}{\Gamma_0 + \Gamma_{0\leftrightarrow\pm}} + \Gamma_\pm. \quad (\text{A20})$$

We stress that these rates can be meaningfully interpreted as thermalization rates only if the adiabatic approximation holds, i.e., as long as the relevant diagonal entries of $\mathbf{M}_\Gamma$, as given in Eq. (A11), are significantly greater than the $|\dot{\phi}/\phi| \sim M_\phi$. Outside this regime, one needs to invoke the more-involved underdamping formula, Eq. (27), which dissolves the meaning of Eq. (31).

a. QCD and lighter QCD axion

The (lighter) QCD axion only modifies the mass of the neutral pion. Thus, the Boltzmann Eq. (A11) reduces in the adiabatic limit to

$$\begin{pmatrix} \delta n_{\pi^0} \\ \delta n_{(\pi^++\pi^-)} \end{pmatrix} \approx -\mathbf{M}_\Gamma^{-1} \begin{pmatrix} \dot{n}_{\pi^0}^{\text{eq}}(a) \\ 0 \end{pmatrix}. \quad (\text{A21})$$

Given the thermal damping formula in Eq. (32), there is only one relevant thermalization rate, namely, $\Gamma_{\text{th}} = \Gamma_{\pi^0 \rightarrow \pi^0}$. Assuming $n_{\pi^0}^{\text{eq}} \approx n_{\pi^\pm}^{\text{eq}}$, this rate simplifies to

$$\Gamma_{\text{th}} \approx \begin{cases} \Gamma_\pm, & T \gtrsim 30 \text{ MeV} \\ \Gamma_0, & T \lesssim 30 \text{ MeV} \end{cases}. \quad (\text{A22})$$

b. Scalar quadratically coupled to gluons

For the case of a scalar ϕ field quadratically coupled to gluons, the pion masses are modified democratically, and the Boltzmann Eq. (A11) can be written in the adiabatic limit as

$$\begin{pmatrix} \delta n_{\pi^0} \\ \delta n_{(\pi^++\pi^-)} \end{pmatrix} \approx -\mathbf{M}_\Gamma^{-1} \begin{pmatrix} \dot{n}_{\pi^0}^{\text{eq}}(\phi) \\ \dot{n}_{(\pi^++\pi^-)}^{\text{eq}}(\phi) \end{pmatrix}. \quad (\text{A23})$$

If we neglect the small differences between the pion masses, i.e., $m_{\pi^0} \approx m_{\pi^\pm}$ and $\dot{m}_{\pi^0}^{\text{eq}}(\phi) \approx \dot{m}_{\pi^\pm}^{\text{eq}}(\phi)$, the thermal damping formula, Eq. (32), simplifies considerably, and it is then useful to define the effective thermalization $\Gamma_{\text{th}}^{\text{eff}}$ as follows:

$$\frac{1}{\Gamma_{\text{th}}^{\text{eff}}} \equiv \frac{1}{\Gamma_{\text{th}}^{\pi^0 \rightarrow \pi^0}} + \frac{2}{\Gamma_{\text{th}}^{\pi^0 \rightarrow \pi^\pm}} + \frac{1}{\Gamma_{\text{th}}^{\pi^\pm \rightarrow \pi^0}} + \frac{2}{\Gamma_{\text{th}}^{\pi^\pm \rightarrow \pi^\pm}}. \quad (\text{A24})$$

Assuming $n_{\pi^0}^{\text{eq}} \approx n_{\pi^\pm}^{\text{eq}}$, this effective rate simplifies to

$$\Gamma_{\text{th}}^{\text{eff}} = \frac{\Gamma_0 \Gamma_{0\leftrightarrow\pm} + \Gamma_\pm (\Gamma_0 + \Gamma_{0\leftrightarrow\pm})}{2\Gamma_0 + 6\Gamma_{0\leftrightarrow\pm} + \Gamma_\pm}, \quad (\text{A25})$$

$$\approx \begin{cases} \frac{1}{6} \Gamma_\pm & T \gtrsim 30 \text{ MeV} \\ \frac{1}{2} \Gamma_{0\leftrightarrow\pm} & T \lesssim 30 \text{ MeV} \end{cases}. \quad (\text{A26})$$

2. Nucleons

Here, we derive the thermalization rates of nucleons relevant for estimating the thermal damping rate at temperatures $T = \mathcal{O}(1\text{--}10)$ MeV. In the absence of Hubble expansion, the Boltzmann equations for the number densities of neutrons and protons, n_n and n_p , are [107]

$$\dot{n}_n = -\Gamma_{n \rightarrow p} n_n + \Gamma_{p \rightarrow n} n_p, \quad (\text{A27})$$

$$\dot{n}_p = -\Gamma_{p \rightarrow n} n_p + \Gamma_{n \rightarrow p} n_n, \quad (\text{A28})$$

where the neutron-to-proton conversion rate $\Gamma_{n \rightarrow p}$ is given by

$$\Gamma_{n \rightarrow p} = \Gamma_{n\nu_e \rightarrow p e^-} + \Gamma_{ne^+ \rightarrow p \bar{\nu}_e} + \tau_n^{-1}, \quad (\text{A29})$$

$$\Gamma_{n\nu_e \rightarrow pe^-} = \frac{1 + 3g_A^2}{2\pi^3} G_F^2 q^5 J(1, \infty), \quad (\text{A30})$$

$$\Gamma_{ne^+ \rightarrow p\bar{\nu}_e} = \frac{1 + 3g_A^2}{2\pi^3} G_F^2 q^5 J\left(-\infty, -\frac{m_e}{q}\right), \quad (\text{A31})$$

$$J(a, b) = \int_a^b \frac{x^2(x-1)^2 dx}{(1 + e^{q(x-1)/T_v})(1 + e^{-qx/T})} \times \sqrt{1 - \frac{(m_e/q)^2}{x^2}}, \quad (\text{A32})$$

where τ_n is the neutron's decay lifetime, G_F is the Fermi constant, $g_A \approx 1.26$, $q = m_n - m_p \approx 1.293$ MeV, and x is a dummy integration variable. As long as the neutrons and protons are close to thermal equilibrium, the proton-to-neutron conversion rate $\Gamma_{p \rightarrow n}$ is related to $\Gamma_{n \rightarrow p}$ as

$$\frac{\Gamma_{p \rightarrow n}}{\Gamma_{n \rightarrow p}} \approx \frac{n_n^{\text{eq}}}{n_p^{\text{eq}}} = e^{-Q/T}, \quad (\text{A33})$$

where we have defined $Q = q + \delta q$ for notational simplicity, $\delta q = \delta(m_n - m_p)$ is the correction to the neutron-proton mass difference due to interaction with a scalar field background, and we have neglected the small electron chemical potential $\mu_e \approx 6n_p/T^2$. Using the above relations, we can rewrite the Boltzmann equations in the form of Eq. (29),

$$\dot{n}_n = -(e^{-Q/T} + 1)\Gamma_{n \rightarrow p}\delta n_n, \quad (\text{A34})$$

$$\dot{n}_p = -(e^{-Q/T} + 1)\Gamma_{n \rightarrow p}\delta n_p. \quad (\text{A35})$$

Thus, in this case the thermalization rate matrix $\mathbf{M}_\Gamma$ is diagonal and given by

$$\mathbf{M}_\Gamma = \Gamma_{\text{th}}^{(\text{nucleon})} \mathbf{I}, \quad (\text{A36})$$

where $\mathbf{I}$ is a 2×2 unit matrix and

$$\Gamma_{\text{th}}^{(\text{nucleon})} = (e^{-Q/T} + 1)\Gamma_{n \rightarrow p}. \quad (\text{A37})$$

Note that the thermal damping formulas in Eqs. (32) and (34) assume that the thermal bath species have no chemical potentials. In contrast, the number densities of neutrons and protons at $T \ll \text{GeV}$ are predominantly set by their chemical potentials associated with the baryon asymmetry of the Universe. To calculate the thermal damping in this case we should start with the more general damping formula in Eq. (28). We would then find that the thermal damping rate is $\Upsilon \propto (M'_n - M'_p)\dot{n}_n^{\text{eq}}/\Gamma_{\text{th}}^{(\text{nucleon})}$, where $M_{n/p}$ is the effective mass of neutrons/protons, and we have assumed $\dot{n}_n^{\text{eq}} + \dot{n}_p^{\text{eq}} = \dot{n}_B \sim Hn_B$ is negligible on the time-scales much less than H^{-1} . Thus, the leading-order contribution to Υ is noncanceling only if the scalar-nucleon coupling breaks isospin. This is the case for the (lighter) QCD axion but not so for the scalar quadratically coupled to gluons.

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