

Generation structures and Yukawa couplings in magnetized $T^{2g}/\mathbb{Z}_N$ models

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We study fermion zero-mode wave functions with various chiralities in the magnetized T^{2g} , ($g = 2, 3$), torus. First, we consider the wave functions satisfying the Dirac equation and the boundary conditions on the magnetized torus. Second, we introduce the $SO(3)$ (or parity) transformations and derive the wave functions under the modular transformation. Additionally, we calculate the Yukawa couplings taking chirality into account. Lastly, we briefly review how to construct the $T^4/\mathbb{Z}_N$, ($N = 2, 3, 4, 6$), and $T^6/\mathbb{Z}_{12}$ twisted orbifold. Also, we explicitly analyze the number of the wave functions in $\mathbb{Z}_N$ sectors.

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I. INTRODUCTION

Superstring theory is a promising candidate for a unified theory of particle physics. When we start with superstring theory, we need to compactify extra six-dimensional (6D) space to realize four-dimensional (4D) low energy effective field theory. Specifically, we should consider 4D chiral theory derived from superstring theory because the standard model is chiral theory. We expect that string compactification can solve mysteries in particle physics, e.g., origins of three-generation structure and the mass hierarchy of quarks and leptons.

Many compactifications such as Calabi-Yau manifolds have been proposed. In particular, toroidal compactification T^6 is one of the simplest examples of 6D compactifications. However, 4D $\mathcal{N} = 4$ supersymmetry (SUSY) remains in the simple T^6 compactification. It has the $SU(4)_R$ symmetry and leads to 4D nonchiral theory. Orbifolding $T^6/\mathbb{Z}_N$ is a way to realize 4D chiral theory, where 4D SUSY can be broken to $\mathcal{N} = 1$ or $\mathcal{N} = 0$ and the $SU(4)_R$ symmetry is also broken [1,2], although orbifolding can also lead to 4D $\mathcal{N} = 2$ SUSY and it is not a chiral theory.

Introduction of background magnetic fluxes is another way to realize 4D chiral theory [3–8]. The number of zero-modes is determined by an amount of magnetic fluxes. Their profiles in the compact space can be written explicitly and they are nontrivial [9,10]. Yukawa couplings and higher dimensional couplings in 4D low energy effective field theory can be computed by the overlap integral of wave functions [9–12]. Furthermore, the modular transformation behavior of zero modes was also studied [13–22].

The modular symmetry appears in 4D low energy effective field theory and controls allowed couplings. In these analyses, it is essential that we can write explicitly the wave functions.

One can combine the above orbifold geometry and the introduction of background magnetic fluxes. That is, toroidal orbifold compactification with magnetic fluxes is also interesting. $T^2/\mathbb{Z}_N$ compactifications with magnetic fluxes [23,24] were studied intensively. Their extensions to $(T^2 \times T^2 \times T^2)/\mathbb{Z}_N$ and $(T^2 \times T^2 \times T^2)/(\mathbb{Z}_M \times \mathbb{Z}_N)$ are straightforward, when T^6 is factorized as $T^6 = T^2 \times T^2 \times T^2$. Three-generation models were classified [25,26] and realization of quark and lepton masses and their mixing angles was studied [27–32].

Such a study was also extended to nonfactorizable T^4 and T^6 and their orbifolds with generic magnetic flux background [9,10,33–35]. These compactifications lead to more rich structure and are important. For example, these compactifications include more degrees of freedom of moduli. As mentioned above, it is essentially necessary to find explicitly the wave functions in order to write down the 4D low energy effective field theory, although only the number of zero modes could be computed by some index theorem [36–38]. Detailed studies of the zero-mode wave functions have been done for the zero modes, whose $SO(6)$ chiralities on T^6 are $(+, +, +)$ and $(-, -, -)$ including the $SO(4)$ chiralities $(+, +)$ and $(-, -)$ on T^4 , although the wave functions with general chirality were studied in Refs. [9,10]. Note that the invariant combinations of $SO(6)$ scalar, spinor, and vector representations appear in allowed couplings of 4D low energy effective field theory. For example, in 4D $\mathcal{N} = 1$ SUSY theory, the invariant combinations of $SU(3)$ triplets appear in 4D Yukawa couplings, where $SU(3) \subset SU(4)_R$. Therefore, we have to write explicitly the wave functions of zero modes with various chiralities in order to obtain 4D low energy

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effective field theory. That is the purpose of this paper. When T^6 is factorized as $T^6 = T_1^2 \times T_2^2 \times T_3^2$ and magnetic flux backgrounds are nonvanishing on only each T_i^2 , the wave functions with generic chiralities are obtained as products of the wave functions $\psi_{\pm}^{(i)}$ on T_i^2 as $\psi_{\pm}^{(1)} \psi_{\pm}^{(2)} \psi_{\pm}^{(3)}$. That is straightforward, because $\psi_{\pm}^{(i)}$ are known already. We study the wave functions of zero-modes with various chiralities on generic T^6 with magnetic flux backgrounds, which cannot be realized as $T^6 = T_1^2 \times T_2^2 \times T_3^2$ with nonvanishing magnetic flux backgrounds on only each T_i^2 .

This paper is organized as follows. We first introduce background magnetic fluxes and boundary conditions on T^6 in Sec. II. In Sec. III, we obtain wave functions with various chiralities that satisfy the Dirac equation and the boundary conditions. We also obtain the wave functions through $SO(3)$ rotations in T^6 . In Sec. IV, we show how to introduce the modular transformation behavior of zero-mode wave functions. This is crucial to analyze the number of zero modes in $T^{2g}/\mathbb{Z}_N$, ($g = 2, 3$). In Sec. V, we analyze the Yukawa couplings in magnetized T^6 taking both the chirality and the $SO(6)$ symmetry into account. In Sec. VI, we find the numbers of zero modes in magnetized $T^{2g}/\mathbb{Z}_N$

to verify whether three-generation models can be realized or not. Section VII is devoted to the Conclusion. In Appendixes A and B, we summarize the details of spinor representation under parity transformation and diagonal twists in magnetized $T^4/\mathbb{Z}_N$.

II. MAGNETIZED T^6 MODEL

Here we study $U(1)$ gauge theory on T^6 with background magnetic fluxes, which originated from the ten-dimensional (10D) supersymmetric Yang-Mills theory. We give a review of the background magnetic fluxes and the Dirac equation. We can extend this model to $U(N)$ gauge theory with background magnetic fluxes along the diagonal direction of $U(N)$.

A. Dirac operator

First, we review how to construct magnetized T^6 models [35]. The 6D torus $T^6 \simeq \mathbb{C}^3/\Lambda$ is constructed by the 6D lattice Λ . (See Refs. [39–44] for various 6D Lie lattices to construct T^6 and its orbifolds.) The 6D lattice Λ is spanned by six basis vectors e'_i , ($i = 1, \dots, 6$). We write them as

$$\begin{aligned} e'_1 &= 2\pi r_1 e_1 = 2\pi r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & e'_2 &= 2\pi r_2 e_2 = 2\pi r_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & e'_3 &= 2\pi r_3 e_3 = 2\pi r_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ e'_4 &= 2\pi r_i \Omega_{i1} e_1 = 2\pi \begin{bmatrix} r_1 \omega_{11} \\ r_2 \omega_{21} \\ r_3 \omega_{31} \end{bmatrix}, & e'_5 &= 2\pi r_i \Omega_{i2} e_2 = 2\pi \begin{bmatrix} r_1 \omega_{12} \\ r_2 \omega_{22} \\ r_3 \omega_{32} \end{bmatrix}, \\ e'_6 &= 2\pi r_i \Omega_{i3} e_3 = 2\pi \begin{bmatrix} r_1 \omega_{13} \\ r_2 \omega_{23} \\ r_3 \omega_{33} \end{bmatrix}, \end{aligned}$$

where $2\pi r_i$, ($i = 1, 2, 3$), are the scale factors and e_i denote the Euclidean standard basis. We define the complex structure moduli $\Omega = \Omega_{ij}$ on T^6 as

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix}. \quad (1)$$

We require $\Omega^T = \Omega$ to construct lattice of $T^{2g}/\mathbb{Z}_N$, ($g = 2, 3$). When we define complex coordinates $\tilde{Z}$ on $\mathbb{C}^3$ with real vectors $\vec{x}$, $\vec{y}$, we also define complex coordinates on T^6 written by $z^i = x^i + \Omega_{ij} y^j$,

$$Z^i \equiv 2\pi r_i (x^i + \Omega_{ij} y^j) = 2\pi r_i z^i, \quad (2)$$

where we impose the identifications $z^i \sim z^i + e_i \sim z^i + \Omega_{ij} e_j$ on T^6 , namely $x^i \sim x^i + e_i$ and $y^i \sim y^i + e_i$. Here, vector $\vec{v}$ may simply be denoted by v . In this paper, since we mainly consider $T^{2g}/\mathbb{Z}_N$ orbifolds with simply laced Lie Lattices, we restrict the scale factors $2\pi r_i = 1$.¹ We introduce the Siegel upper-half plane $\mathcal{H}^3$ defined as follows [46],

$$\mathcal{H}^3 = \{\Omega \in GL(3, \mathbb{C}) | \Omega^T = \Omega, \text{Im } \Omega > 0\}. \quad (3)$$

We will utilize Eq. (3) when we consider wave functions derived from the magnetic fluxes. In Sec. III, we will discuss how to obtain well-defined wave functions.

¹Magnetized T^6 model with the nontrivial scale factors $2\pi r_i$ is discussed in Appendix A of Ref. [45].

We introduce the following metric on $\mathbb{C}^3$:

$$ds^2 = \delta_{i,j} dZ^i d\bar{Z}^j \quad \text{for } i, j = 1, 2, 3. \quad (4)$$

We also define the Gamma matrices $\Gamma^{z^i}, \Gamma^{\bar{z}^i}$ to introduce the Dirac equation on T^6 ,

$$\begin{aligned} \Gamma^{z^1} &\equiv \frac{1}{2\pi r_1} \Gamma^{Z^1} = \frac{1}{2\pi r_1} (\sigma^z \otimes \sigma^3 \otimes \sigma^3), & \Gamma^{\bar{z}^1} &\equiv \frac{1}{2\pi r_1} \Gamma^{\bar{Z}^1} = \frac{1}{2\pi r_1} (\sigma^{\bar{z}} \otimes \sigma^3 \otimes \sigma^3), \\ \Gamma^{z^2} &\equiv \frac{1}{2\pi r_2} \Gamma^{Z^2} = \frac{1}{2\pi r_2} (\mathbf{1}_2 \otimes \sigma^z \otimes \sigma^3), & \Gamma^{\bar{z}^2} &\equiv \frac{1}{2\pi r_2} \Gamma^{\bar{Z}^2} = \frac{1}{2\pi r_2} (\mathbf{1}_2 \otimes \sigma^{\bar{z}} \otimes \sigma^3), \\ \Gamma^{z^3} &\equiv \frac{1}{2\pi r_3} \Gamma^{Z^3} = \frac{1}{2\pi r_3} (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^z), & \Gamma^{\bar{z}^3} &\equiv \frac{1}{2\pi r_3} \Gamma^{\bar{Z}^3} = \frac{1}{2\pi r_3} (\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^{\bar{z}}), \end{aligned} \quad (5)$$

where $\mathbf{1}_n$ represents $n \times n$ unit matrix and we define the Pauli matrices σ^i as follows

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (6)$$

We also define $\sigma^z = \sigma^1 + i\sigma^2$, $\sigma^{\bar{z}} = \sigma^1 - i\sigma^2$ using the Pauli matrices. We find that the gamma matrices satisfy the Clifford algebra,

$$\begin{aligned} \{\Gamma^{Z^i}, \Gamma^{Z^j}\} &= \{\Gamma^{\bar{Z}^i}, \Gamma^{\bar{Z}^j}\} = 0, \\ \{\Gamma^{Z^i}, \Gamma^{\bar{Z}^j}\} &= 4\delta^{i,j}. \end{aligned} \quad (7)$$

In particular, we can determine the chirality of wave functions on magnetized T^6 by the following matrix Γ^7 :

$$\Gamma^7 = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 = \text{diag}[+, -, -, +, -, +, +, -]. \quad (8)$$

Then, the Dirac operator $\not{D}$ defined on T^6 is given by

$$i\not{D} \equiv i(\Gamma^{z^j} D_{z^j} + \Gamma^{\bar{z}^j} \bar{D}_{\bar{z}^j}), \quad (9)$$

where we define following covariant derivatives $D_{z^j}, \bar{D}_{\bar{z}^j}$. Here, we consider $U(1)$ gauge field theory, where the gauge fields $A_z, A_{\bar{z}}$ are coupled to fermions with $U(1)$ unit charge $q = +1$,

$$\begin{aligned} D_{z^j} &\equiv D_j = \partial_{z^j} - iA_{z^j}, \\ \bar{D}_{\bar{z}^j} &\equiv \bar{D}_j = \partial_{\bar{z}^j} - iA_{\bar{z}^j}. \end{aligned} \quad (10)$$

In order to show both the gauge fields written in Eq. (10) and boundary conditions of spinor wave functions on T^6 explicitly, we will introduce background magnetic fluxes in the next Subsection II. 2.

B. Magnetic fluxes and boundary condition on T^6

Here, $U(1)$ background magnetic fluxes F on T^6 are written as follows:

$$\begin{aligned} F &\equiv \frac{1}{2} (p_{xx})_{ij} dx^i \wedge dx^j + \frac{1}{2} (p_{yy})_{ij} dy^i \wedge dy^j + (p_{xy})_{ij} dx^i \wedge dy^j \\ &= \frac{1}{2} (F_{zz})_{ij} dz^i \wedge dz^j + \frac{1}{2} (F_{\bar{z}\bar{z}})_{ij} d\bar{z}^i \wedge d\bar{z}^j + (F_{z\bar{z}})_{ij} (idz^i \wedge d\bar{z}^j), \end{aligned} \quad (11)$$

where

$$\begin{aligned} (F_{zz})_{ij} &= [(\bar{\Omega} - \Omega)^{-1}]^T (\bar{\Omega}^T p_{xx} \bar{\Omega} + p_{yy} + p_{xy}^T \bar{\Omega} - \bar{\Omega}^T p_{xy}) (\bar{\Omega} - \Omega)^{-1})_{ij}, \\ (F_{\bar{z}\bar{z}})_{ij} &= [(\bar{\Omega} - \Omega)^{-1}]^T (\Omega^T p_{xx} \Omega + p_{yy} + p_{xy}^T \Omega - \Omega^T p_{xy}) (\bar{\Omega} - \Omega)^{-1})_{ij}, \\ (F_{z\bar{z}})_{ij} &= i[(\bar{\Omega} - \Omega)^{-1}]^T (\bar{\Omega}^T p_{xx} \Omega + p_{yy} + p_{xy}^T \Omega - \bar{\Omega}^T p_{xy}) (\bar{\Omega} - \Omega)^{-1})_{ij}. \end{aligned} \quad (12)$$

In what follows, we suppose $p_{xx} = p_{yy} = 0$ for simplicity. We then consider the Hermitian Yang-Mills equation to keep SUSY in 10D $\mathcal{N} = 1$ super Yang-Mills theory (see Refs. [9,47]). That is, since F has to be (1,1)-form, we then find

$$p_{xy}^T \Omega = \Omega^T p_{xy}. \quad (13)$$

Also, since we can define the integer fluxes N as $p_{xy} = 2\pi N^T$ through the Dirac quantization condition, we find

$$(N\Omega)^T = N\Omega. \quad (14)$$

This is called the F -term SUSY condition, which realizes F -flat vacua satisfying the Hermitian Yang-Mills equation. In particular, when both $N^T = N$ and $\Omega^T = \Omega$ hold under Eq. (14), we find $[N, \Omega] = 0$. That is, N and Ω can be diagonalized simultaneously. We will restrict ourselves to the symmetric matrices, namely $N^T = N$ and $\Omega^T = \Omega$.² We use the following background magnetic fluxes:

$$\begin{aligned} F &= \pi[N^T(\text{Im } \Omega)^{-1}]_{ij}(dz^i \wedge d\bar{z}^j) \\ &\equiv F_{i\bar{j}}(dz^i \wedge d\bar{z}^j). \end{aligned} \quad (15)$$

We find the corresponding gauge potential A defined as $F = dA$,

$$\begin{aligned} A(z, \bar{z}) &= \pi \text{Im}[(N(\bar{z} + \bar{\eta}))^T (\text{Im } \Omega)^{-1} dz] \\ &= -\frac{i\pi}{2}[(N(\bar{z} + \bar{\eta}))^T (\text{Im } \Omega)^{-1}]_i dz^i \\ &\quad + \frac{i\pi}{2}[(N(z + \eta))^T (\text{Im } \Omega)^{-1}]_i d\bar{z}^i \\ &\equiv A_{z^i} dz^i + A_{\bar{z}^i} d\bar{z}^i, \end{aligned} \quad (16)$$

where η denotes the Wilson line. For simplicity, we use the vanishing Wilson line, namely $\eta = \vec{0}$. Then the values of the gauge fields A_{z^i} , $A_{\bar{z}^i}$ lead to the following commutation relations with Eq. (10)

$$\begin{aligned} [D_i, \bar{D}_j] &= F_{i\bar{j}}, \\ [D_i, D_j] &= F_{ij} = 0, \\ [\bar{D}_i, \bar{D}_j] &= F_{\bar{i}\bar{j}} = 0. \end{aligned} \quad (17)$$

Also, the $U(1)$ gauge background $A(z, \bar{z}) = A(z)$ satisfies the following boundary conditions:

$$\begin{aligned} A(z + e_k) &= A(z) + d\zeta_{e_k}(z), \\ A(z + \Omega e_k) &= A(z) + d\zeta_{\Omega e_k}(z), \end{aligned} \quad (18)$$

where $k = 1, 2, 3$. Thus we obtain

²The discussion holds even if we introduce asymmetric N and Ω under the (1,1)-form background magnetic fluxes and the F -term SUSY condition $(N\Omega)^T = N\Omega$.

$$\begin{aligned} \zeta_{e_k}(z) &= \pi[N^T(\text{Im } \Omega)^{-1} \text{Im } z]_k, \\ \zeta_{\Omega e_k}(z) &= \pi \text{Im}[(N\bar{\Omega})^T (\text{Im } \Omega)^{-1} z]_k \\ &= \pi[N(\text{Re } \Omega(\text{Im } \Omega)^{-1} \text{Im } z - \text{Re } z)]_k, \end{aligned} \quad (19)$$

and impose boundary conditions of spinor wave functions in magnetized T^6 under $(N\Omega)^T = N\Omega$

$$\begin{aligned} \psi(z + e_k) &= e^{i\zeta_{e_k}(z)} \psi(z) = e^{i\pi[N^T(\text{Im } \Omega)^{-1} \text{Im } z]_k} \psi(z), \\ \psi(z + \Omega e_k) &= e^{i\zeta_{\Omega e_k}(z)} \psi(z) = e^{i\pi[N(\text{Re } \Omega(\text{Im } \Omega)^{-1} \text{Im } z - \text{Re } z)]_k} \psi(z). \end{aligned} \quad (20)$$

III. WAVE FUNCTIONS WITH VARIOUS CHIRALITIES IN THE MAGNETIZED T^6 MODEL

Here, we study wave functions of zero modes with various chiralities on T^6 with background magnetic fluxes.

A. Direct way to introduce wave functions with various chiralities

We denote components of the spinor field by

$$\begin{aligned} \Psi &\equiv \begin{bmatrix} + \\ - \end{bmatrix} \otimes \begin{bmatrix} + \\ - \end{bmatrix} \otimes \begin{bmatrix} + \\ - \end{bmatrix} \\ &= [\psi_{+++}, \psi_{++-}, \psi_{+-+}, \psi_{+--}, \psi_{-++}, \psi_{-+-}, \psi_{--+}, \psi_{---}]^T \\ &\equiv [\psi'_{1+}, \psi_{3-}, \psi_{2-}, \psi_{1+}, \psi_{1-}, \psi_{2+}, \psi_{3+}, \psi'_{1-}]^T. \end{aligned} \quad (21)$$

We define ψ_{j+} and ψ_{j-} , ($j = 1, 2, 3$), as the positive and negative chirality components, respectively. Also, we define $\psi'_{j+} = \psi_{+++}$ and $\psi'_{j-} = \psi_{---}$. The Dirac equation, $i\mathcal{D}_6\Psi = 0$, is written explicitly by components as

$$\begin{cases} D_3\psi_{3-} + D_2\psi_{2-} + D_1\psi_{1-} = 0, \\ \bar{D}_3\psi'_{1+} - D_2\psi_{1+} - D_1\psi_{2+} = 0, \\ \bar{D}_2\psi'_{1+} + D_3\psi_{1+} - D_1\psi_{3+} = 0, \\ -\bar{D}_2\psi_{3-} + \bar{D}_3\psi_{2-} + D_1\psi'_{1-} = 0, \\ \bar{D}_1\psi'_{1+} + D_3\psi_{2+} + D_2\psi_{3+} = 0, \\ -\bar{D}_1\psi_{3-} + \bar{D}_3\psi_{1-} - D_2\psi'_{1-} = 0, \\ -\bar{D}_1\psi_{2-} + \bar{D}_2\psi_{1-} + D_3\psi'_{1-} = 0, \\ \bar{D}_1\psi_{1+} - \bar{D}_2\psi_{2+} + \bar{D}_3\psi_{3+} = 0. \end{cases} \quad (22)$$

First, we briefly review the solution that only $\psi_{+++} = \psi'_{1+}$ is nonvanishing and the others are vanishing. For such a solution, the Dirac equations can be written by

$$\bar{D}_1\psi'_{1+} = \bar{D}_2\psi'_{1+} = \bar{D}_3\psi'_{1+} = 0. \quad (23)$$

Its solution is written by [9,10,35]

$$\psi_{+++}^J = \mathcal{N} \cdot e^{i\pi[Nz]^T(\text{Im}\Omega)^{-1}\text{Im}z} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (Nz, N\Omega), \quad (24)$$

where $J \in \Lambda_{N^T}$ denotes lattice vectors in lattice Λ_{N^T} spanned by $N^T e_i$, ($i = 1, 2, 3$). Thus we find $|\det N|$ of independent degenerated zero modes. We have introduced the Riemann-Theta function with characters defined as

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) \equiv \sum_{m \in \mathbb{Z}^3} e^{i\pi(m+a)^T \Omega (m+a)} e^{2\pi i(m+a)^T (z+b)}, \quad a, b \in \mathbb{R}^3, \quad \Omega \in \mathcal{H}^3. \quad (25)$$

The factor $\mathcal{N}$ denotes normalization constant,

$$\mathcal{N} = (\det N)^{\frac{1}{4}} \cdot [\text{Vol}(T^6)]^{-\frac{1}{2}} = (\det N)^{\frac{1}{4}} \cdot [\det(\text{Im}\Omega)]^{-\frac{1}{2}}, \quad (26)$$

if $\det \text{Im}\Omega > 0$. Here, we normalize the wave function as

$$\int_{T^6} d^3z d^3\bar{z} (\psi^J)^* \psi^K = [\det(2\text{Im}\Omega)]^{-1/2} \delta_{J,K}, \quad (27)$$

following Ref. [21]. Note that ψ_{+++} shown in Eq. (24) are well defined only if all of three eigenvalues of $F_{i\bar{j}} = \pi[N^T(\text{Im}\Omega)^{-1}]_{ij}$ are positive. For other $F_{i\bar{j}} = \pi[N^T(\text{Im}\Omega)^{-1}]_{ij}$, the mode ψ_{+++} has no zero mode, but other modes like $\psi_{\underline{+++}}$ have zero modes, where the underline denotes all of the possible permutations.

We can also obtain the solution for only the component ψ_{---} by flipping all chiralities in ψ_{+++} . We find that the sign of $\det N$ has to be flipped. Then ψ_{---} is obtained as

$$\psi_{---}^J = \mathcal{N} \cdot e^{i\pi[N\bar{z}]^T(\text{Im}\Omega)^{-1}\text{Im}z} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (N\bar{z}, N\bar{\Omega}), \quad (28)$$

under the following replacement:

$$z \rightarrow \bar{z}, \quad \Omega \rightarrow \bar{\Omega}, \quad N \rightarrow -N \equiv N. \quad (29)$$

Next, we study the solutions that other components are nonvanishing. Hereafter, we focus on wave functions of $\psi_{1+} = \psi_{+--}$, $\psi_{2+} = \psi_{-+-}$, and $\psi_{3+} = \psi_{--+}$. The others of $\psi_{1-} = \psi_{-++}$, $\psi_{2-} = \psi_{+-+}$, and $\psi_{3-} = \psi_{++-}$ are obtained by the replacement in Eq. (29). Now, let us study the solution that only ψ_{1+} is nonvanishing and the others are vanishing. The corresponding Dirac equations are written by

$$D_2\psi_{1+} = D_3\psi_{1+} = \bar{D}_1\psi_{1+} = 0. \quad (30)$$

Note that both D_i and $\bar{D}_i$ appear in this equation, while only $\bar{D}_i$ appear in Eq. (23). By using Eq. (17), we find $F_{12}\psi_{1+} = F_{13}\psi_{1+} = 0$. We cannot obtain a nontrivial

solution in the generic case including $F_{12}, F_{13} \neq 0$. When $F_{12}, F_{13} = 0$, the first plane z^1 and the other are decomposed to the factorizable torus, $T^2 \times T^4$. In order to understand the nonfactorizable magnetized T^6 , we study the solution that $\psi_{1+} = \psi_{+--}$, $\psi_{2+} = \psi_{-+-}$, and $\psi_{3+} = \psi_{--+}$ are excited simultaneously. These states with different chiralities are mixed with each other. Following Ref. [10], we make the ansatz for the solution that these components can be written by a single function ψ , i.e., $\psi_{i+} = \alpha_i \psi$, ($i = 1, 2, 3$), where α_i are coefficients. By use of this ansatz, the Dirac equations can be written as

$$\begin{cases} (\alpha_1 D_2 + \alpha_2 D_1)\psi = 0, \\ (\alpha_1 D_3 - \alpha_3 D_1)\psi = 0, \\ (\alpha_2 D_3 + \alpha_3 D_2)\psi = 0, \\ (\alpha_1 \bar{D}_1 - \alpha_2 \bar{D}_2 + \alpha_3 \bar{D}_3)\psi = 0. \end{cases} \quad (31)$$

Note that the first three equations are not independent of each other. This is because

$$\begin{aligned} \alpha_3(\alpha_1 D_2 + \alpha_2 D_1)\psi + \alpha_2(\alpha_1 D_3 - \alpha_3 D_1)\psi \\ = \alpha_1(\alpha_2 D_3 + \alpha_3 D_2)\psi = 0. \end{aligned} \quad (32)$$

Suppose $\alpha_1 \neq 0$. Then, we use the ratios $q_1 = \frac{\alpha_2}{\alpha_1}$, $q_2 = -\frac{\alpha_3}{\alpha_1}$ and the notation $\vec{q} = (q_1, q_2)^T$.³ Equation (31) leads to the following integrable conditions

$$\begin{cases} F_{2\bar{1}} - q_1^2 F_{1\bar{2}} + q_1(F_{1\bar{1}} - F_{2\bar{2}}) - q_2 F_{2\bar{3}} - q_1 q_2 F_{1\bar{3}} = 0, \\ F_{3\bar{1}} - q_1 F_{3\bar{2}} - q_2(F_{3\bar{3}} - F_{1\bar{1}}) - q_1 q_2 F_{1\bar{2}} - q_2^2 F_{1\bar{3}} = 0, \end{cases} \quad (33)$$

so as to obtain a nontrivial solution of ψ . Since F is the (1,1)-form, $F_{i\bar{j}} = \pi[N^T(\text{Im}\Omega)^{-1}]_{ij}$ is a real symmetric matrix. It has six degrees of freedom, but there are two constraints due to Eq. (33). As a result, we have four degrees of freedom in F . We parametrize them by a real number $\hat{N}'$ and 2×2 real symmetric matrix $\tilde{N}'$. By use of them, we can write $F_{i\bar{j}}$ with the coefficients q_i as

³If $\alpha_1 = 0$, we use the ratio $q = \frac{\alpha_3}{\alpha_2}$. Then, we can carry out a similar discussion.

$$\begin{aligned}
F_{i\bar{j}} &\equiv \pi[\hat{N}^T(\text{Im } \Omega)^{-1}]_{ij} + \pi[\tilde{N}^T(\text{Im } \Omega)^{-1}]_{ij}, \\
\pi[\hat{N}^T(\text{Im } \Omega)^{-1}]_{ij} &= \hat{N}' \begin{bmatrix} 1 & -\vec{q}^T \\ -\vec{q} & \vec{q}(\vec{q})^T \end{bmatrix} = \hat{N}' \begin{bmatrix} 1 & -q_1 & -q_2 \\ -q_1 & q_1^2 & q_1 q_2 \\ -q_2 & q_1 q_2 & q_2^2 \end{bmatrix}, \\
\pi[\tilde{N}^T(\text{Im } \Omega)^{-1}]_{ij} &= \begin{bmatrix} \vec{q}^T \tilde{N}' \vec{q} & \vec{q}^T \tilde{N}' \\ \tilde{N}' \vec{q} & \tilde{N}' \end{bmatrix} \\
&= \begin{bmatrix} \tilde{N}'_{11} q_1^2 + 2\tilde{N}'_{12} q_1 q_2 + \tilde{N}'_{22} q_2^2 & \tilde{N}'_{11} q_1 + \tilde{N}'_{12} q_2 & \tilde{N}'_{12} q_1 + \tilde{N}'_{22} q_2 \\ \tilde{N}'_{11} q_1 + \tilde{N}'_{12} q_2 & \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{12} q_1 + \tilde{N}'_{22} q_2 & \tilde{N}'_{12} & \tilde{N}'_{22} \end{bmatrix}, \tag{34}
\end{aligned}$$

such that they satisfy the condition (33), where $N \equiv \hat{N} + \tilde{N}$.

One can rotate $F_{i\bar{j}}$ as

$$F_{i\bar{j}}^{\text{part}} \equiv P_{12}^{-1} F_{i\bar{j}} P_{12} = \begin{bmatrix} F_{1\bar{1}}^{\text{part}} & 0 & 0 \\ 0 & F_{2\bar{2}}^{\text{part}} & F_{2\bar{3}}^{\text{part}} \\ 0 & F_{2\bar{3}}^{\text{part}} & F_{3\bar{3}}^{\text{part}} \end{bmatrix}, \tag{35}$$

by the following rotation:

$$P_{12} = \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{36}$$

where

$$q_1 = -\frac{\tan \theta_1}{\cos \theta_2}, \quad q_2 = \tan \theta_2. \tag{37}$$

The explicit entries in $F_{i\bar{j}}^{\text{part}}$ are obtained as

$$\begin{aligned}
F_{1\bar{1}}^{\text{part}} &= [F_{1\bar{1}} + F_{2\bar{2}} q_1^2 + F_{3\bar{3}} q_2^2 - 2(F_{1\bar{2}} q_1 - F_{2\bar{3}} q_1 q_2 + F_{1\bar{3}} q_2)] \cos^2 \theta_1 \cos^2 \theta_2, \\
F_{2\bar{2}}^{\text{part}} &= [2(F_{1\bar{2}} - F_{2\bar{3}} q_2) q_1 + (-2F_{1\bar{3}} q_2 + F_{1\bar{1}} + F_{3\bar{3}} q_2^2) q_1^2 \cos^2 \theta_2] \cos^2 \theta_2 + F_{2\bar{2}}] \cos^2 \theta_1, \\
F_{3\bar{3}}^{\text{part}} &= [F_{1\bar{1}} q_2^2 + F_{3\bar{3}} + 2F_{1\bar{3}} q_2] \cos^2 \theta_2, \\
F_{2\bar{3}}^{\text{part}} &= [2F_{1\bar{3}} + (F_{1\bar{1}} - F_{3\bar{3}}) q_2] q_1 \cos^2 \theta_2 + (F_{1\bar{2}} q_2 + F_{2\bar{3}} - F_{1\bar{3}} q_1) \cos \theta_1 \cos \theta_2. \tag{38}
\end{aligned}$$

Therefore, the parameters q_1 and q_2 as well as α_i correspond to parameters of $SO(6)$ rotation, in particular the holomorphic rotation among z^i ($\bar{z}^i$). In this rotated basis, the solution ψ corresponds to ψ_{+--} if $F_{1\bar{1}}^{\text{part}} > 0$. If $F_{1\bar{1}}^{\text{part}} < 0$, the solution ψ corresponds to the combination of ψ_{-+-} and ψ_{--+} . Furthermore, we can diagonalize the bottom right 2×2 matrix of $F_{i\bar{j}}^{\text{part}}$ by the rotation P_3 of z^2 and z^3 in the above basis, i.e.,

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_3 & -\sin \theta_3 \\ 0 & \sin \theta_3 & \cos \theta_3 \end{bmatrix}. \tag{39}$$

Thus, the flux background is written by the diagonal matrix $(F_{1\bar{1}}^{\text{diag}}, F_{2\bar{2}}^{\text{diag}}, F_{3\bar{3}}^{\text{diag}})$, where

$$\begin{aligned}
F_{1\bar{1}}^{\text{diag}} &= F_{1\bar{1}}^{\text{part}}, \\
F_{2\bar{2}}^{\text{diag}} &= [\sin \theta_3 \tan 2\theta_3 + \cos \theta_3] \cos \theta_3 F_{2\bar{2}}^{\text{part}} + [-\cos \theta_3 \tan 2\theta_3 + \sin \theta_3] \sin \theta_3 F_{3\bar{3}}^{\text{part}}, \\
F_{3\bar{3}}^{\text{diag}} &= [-\cos \theta_3 \tan 2\theta_3 + \sin \theta_3] \sin \theta_3 F_{2\bar{2}}^{\text{part}} + [\sin \theta_3 \tan 2\theta_3 + \cos \theta_3] \cos \theta_3 F_{3\bar{3}}^{\text{part}},
\end{aligned} \tag{40}$$

and

$$\tan 2\theta_3 = \frac{2F_{2\bar{3}}^{\text{part}}}{F_{2\bar{2}}^{\text{part}} - F_{3\bar{3}}^{\text{part}}}, \tag{41}$$

if $F_{2\bar{2}}^{\text{part}} \neq F_{3\bar{3}}^{\text{part}}$. If $F_{2\bar{2}}^{\text{part}} = F_{3\bar{3}}^{\text{part}}$, we find

$$F_{2\bar{3}}^{\text{part}} (\cos^2 \theta_3 - \sin^2 \theta_3) = 0. \tag{42}$$

Since we consider the most general case where $F_{2\bar{3}}^{\text{part}} \neq 0$, we find $\tan^2 \theta_3 = 1$. The diagonal basis corresponds to

$$w^i = (P^{-1})^{ij} z^j, \tag{43}$$

where $P = P_{12}P_3$. In this basis, the solution ψ corresponds to a single component of $\psi_{+++}, \psi_{+--}, \psi_{--+}$. Note that P_3 depends on values of the flux. Furthermore, θ_1 and θ_2 depend on q_1 and q_2 .

By use of the background magnetic fluxes, we can write the solutions of the Dirac equation, which are consistent with the boundary conditions Eq. (20) by⁴

$$\psi_{M,N}^J = \mathcal{N} \cdot f(z, \bar{z}) \cdot \hat{\Theta}(z, \bar{z}), \tag{44}$$

where

$$\begin{aligned}
f(z, \bar{z}) &\equiv \exp [i\pi((\hat{N}z)^T (\text{Im } \Omega)^{-1} \text{Im } z - (\tilde{N}\bar{z})^T (\text{Im } \Omega)^{-1} \text{Im } \bar{z})] \\
&= \exp [i\pi(\hat{N}z + \tilde{N}\bar{z})^T (\text{Im } \Omega)^{-1} \text{Im } z] \\
&= \exp [i\pi(N\text{Re}z + iM \text{Im } z)^T (\text{Im } \Omega)^{-1} \text{Im } z],
\end{aligned} \tag{45}$$

$$\begin{aligned}
\hat{\Theta}(z, \bar{z}) &\equiv \sum_{m \in \mathbb{Z}^3} e^{\pi i(m+J^T N^{-1})^T (\hat{N}\Omega + \tilde{N}\bar{\Omega})(m+J^T N^{-1})} e^{2\pi i(m+J^T N^{-1})^T (\hat{N}z + \tilde{N}\bar{z})} \\
&= \sum_{m \in \mathbb{Z}^3} e^{\pi i(m+J^T N^{-1})^T (N\text{Re}\Omega + iM \text{Im } \Omega)(m+J^T N^{-1})} e^{2\pi i(m+J^T N^{-1})^T (N\text{Re}z + iM \text{Im } z)} \\
&= \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (N\text{Re}z + iM \text{Im } z, N\text{Re}\Omega + iM \text{Im } \Omega).
\end{aligned} \tag{46}$$

We have defined

$$M \equiv \hat{N} - \tilde{N}. \tag{47}$$

Similar to ψ_{+++} , the normalization constant $\mathcal{N}$ in Eq. (44) is written by

$$\mathcal{N} = |\det N|^{\frac{1}{4}} \cdot [\text{Vol}(T^6)]^{-\frac{1}{2}} = |\det N|^{\frac{1}{4}} \cdot [|\det (\text{Im } \Omega)|]^{-\frac{1}{2}}. \tag{48}$$

In addition, we can replace $|\det N|$ by $|\det M|$ in the above equation of $\mathcal{N}$, because of $|\det N| = |\det M|$.

In this paper, we impose the F -term SUSY condition $(N\Omega)^T = N\Omega$ in Eq. (14) and the Riemann condition. From Eq. (3), when we consider the Riemann condition $(N\text{Re}\Omega + iM \text{Im } \Omega)^T = N\text{Re}\Omega + iM \text{Im } \Omega$ and $M \text{Im } \Omega > 0$, namely $N\text{Re}\Omega + iM \text{Im } \Omega \in \mathcal{H}^3$, the spinor wave functions shown in Eq. (44) converge and satisfy both Dirac equation and the boundary conditions.

(See Appendixes B and C of Ref. [45].) Thus, we can obtain the wave functions with chiralities $(+--)$.

In addition, when $N = M$, the wave functions in Eq. (44) correspond to ψ_{+++} in Eq. (24). Also, wave functions' flipped chirality are derived from appropriate transformations such as $z \rightarrow \bar{z}$, $\Omega \rightarrow \bar{\Omega}$. We therefore find all spinor wave functions on T^6 with nonfactorizable background magnetic fluxes F in Eq. (34).

We can relate the above wave function solution with chiralities $(+--)$ to the wave function solution ψ_{+++} by six-dimensional rotation. Suppose that $F_{1\bar{1}}^{\text{diag}} > 0$, $F_{2\bar{2}}^{\text{diag}} < 0$, and $F_{3\bar{3}}^{\text{diag}} < 0$. Then, the solution corresponds to ψ_{+--} in the basis of w^i . We can rotate further $w'^i = \text{Re} w^i + i(U \text{Im } w)^i$, where

⁴The solution with these chiralities was studied for $\Omega = i\mathbf{1}_n$ in Ref. [10].

$$U \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (49)$$

In the basis of w'^i , each of the fluxes is positive, and the solution corresponds to ψ_{+++} . Thus, we can relate the solution ψ_{+++} for $F_{11}^{\text{diag}} > 0$, $F_{22}^{\text{diag}} < 0$, and $F_{33}^{\text{diag}} < 0$ to the solution ψ_{+++} for $F_{11}^{\text{diag}} > 0$, $F_{22}^{\text{diag}} > 0$, and $F_{33}^{\text{diag}} > 0$. In generic signs of the fluxes, we use the following U matrix:

$$U \equiv \begin{bmatrix} \text{sgn}(F_{11}^{\text{diag}}) & 0 & 0 \\ 0 & \text{sgn}(F_{22}^{\text{diag}}) & 0 \\ 0 & 0 & \text{sgn}(F_{33}^{\text{diag}}) \end{bmatrix}, \quad (50)$$

to relate them, where $\text{sgn}(x) = 1$ and -1 for $x > 0$ and $x < 0$, respectively. By use of the original coordinate z^i , we denote

$$z' \equiv \text{Re}z + iR \text{Im}z, \quad (51)$$

where $R \equiv PUP^{-1}$ with $R^2 = \mathbf{1}_3$. In addition, we define

$$\Omega' \equiv \text{Re}\Omega + iR \text{Im}\Omega, \quad (52)$$

where $z' = x + \Omega'y$. By use of them, we can relate the solution of $\psi'^J_{+++}(z', \Omega')$ with the chirality $(+++)$ in z' system to the solution $\psi^J_{M,N}(z, \Omega)$ with one of chiralities $(+--)$ as

$$\begin{aligned} \psi'^J_{+++}(z', \Omega') &= \mathcal{N} \cdot e^{\pi i [Nz']^T (\text{Im}\Omega')^{-1} \text{Im}z'} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (Nz', N\Omega') \\ &= \mathcal{N} \cdot e^{\pi i [N\text{Re}z + iM \text{Im}z]^T (\text{Im}\Omega)^{-1} \text{Im}z} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (N\text{Re}z + iM \text{Im}z, N\text{Re}\Omega + iM \text{Im}\Omega) \\ &= \psi^J_{M,N}(z, \Omega). \end{aligned} \quad (53)$$

Note that $R = N^{-1}M$ and this denotes the $SO(3)$ rotation for $\text{Im}z$ direction.

Here we give a comment about the metric in the z' system. Specifically, we have introduced the metric Eq. (4) in the z system. In the real basis, we can write the metric in the z system, g , as

$$g = dz^i d\bar{z}^{\bar{i}} = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \mathbf{1}_3 & \text{Re}\Omega \\ \text{Re}\Omega & \Omega\bar{\Omega} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \quad (54)$$

Meanwhile, a metric in the z' system, g' , can be described as

$$g' = dz'^i d\bar{z}'^{\bar{i}} = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \mathbf{1}_3 & \text{Re}\Omega' \\ \text{Re}\Omega' & \Omega'\bar{\Omega}' \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \quad (55)$$

Since we can see $\text{Re}\Omega' = \text{Re}\Omega$, and the $SO(3)$ rotation keeps the metric invariant, we can get $\Omega\bar{\Omega} = \Omega'\bar{\Omega}'$. In what follows, we suppose $[\text{Re}\Omega, \text{Im}\Omega] = 0$, or $[\Omega, \bar{\Omega}] = 0$. Then, we can find $[R, \Omega] = 0$ because of $[\Omega, \Omega'] = [\Omega, \bar{\Omega}] = 0$, which leads to $\Omega\bar{\Omega} = \Omega'\bar{\Omega}'$,

$$\Omega'\bar{\Omega}' = (\text{Re}\Omega)^2 + R^2(\text{Im}\Omega)^2 = \Omega\bar{\Omega}. \quad (56)$$

In the z' system, the chirality operator $\Gamma^{7'}$ is defined as

$$\Gamma^{7'} \equiv \sigma^3 \otimes (-\sigma^3) \otimes (-\sigma^3) = \text{diag}[+, -, -, +, -, +, +, -] \quad (57)$$

by using $U = \text{diag}(+, -, -)$. This $\Gamma^{7'}$ is the same as Γ^7 in the six-dimensional theory way. Similarly to Eq. (5), we can define the gamma matrices in z' system satisfying the Clifford algebra under the same metric $g = g'$ and the vielbein. Then, we will consider the wave functions with positive chirality $(+++)$ in the z' system, ψ'_{+++} , as solutions of the Dirac equation described in the z' system. Furthermore, by making use of the ψ'_{+++} , we will find the eigen equation of the Laplacian and the D -term SUSY condition in the z' system.

1. Laplacian and D -term SUSY condition

We define the Laplacian Δ in the z system as follows

$$\Delta \equiv -2 \sum_{j=1}^3 \{D_{z^j}, \bar{D}_{\bar{z}^j}\}. \quad (58)$$

If all the three eigenvalues of $F_{i\bar{j}} = \pi[N^T(\text{Im}\Omega)^{-1}]_{ij}$ are positive, the solution ψ_{+++} satisfies the Dirac equation. One can check that it also satisfies the eigen equation of the Laplacian,

$$\Delta\psi_{+++} = 2 \left[\sum_{j=1}^3 F_{z^j\bar{z}^j} \right] \psi_{+++} = 2 \cdot (\text{tr}F_{z^i\bar{z}^j})\psi_{+++}, \quad (59)$$

where the eigenvalue is non-negative, i.e., $\text{tr}F_{z^i\bar{z}^j} \geq 0$. This is consistent with the positive semidefiniteness of Δ , which

is generally true on a compact manifold. We can make use of Δ when we analyze mass spectrum of bosonic modes. The eigenvalue corresponds to the mass squared in scalar field on T^6 . The vector modes $(A_{z^i}, A_{\bar{z}^i})$ on T^6 have additional effects, and we can obtain their mass squared matrix $\mathcal{M}^2$ using $F_{z^i\bar{z}^j} = F_{i\bar{j}}$ and Δ

$$\mathcal{M}^2 = \Delta + 4 \begin{bmatrix} -F_{i\bar{j}} & O \\ O & F_{i\bar{j}} \end{bmatrix}. \quad (60)$$

Since we have found the eigenvalue of Δ , we can rewrite $\mathcal{M}^2$ in the diagonal basis

$$\text{diag} \mathcal{M}^2 = 2(\lambda_1 + \lambda_2 + \lambda_3) + 4 \cdot \text{diag}[-\lambda_1, -\lambda_2, -\lambda_3, +\lambda_1, +\lambda_2, +\lambda_3], \quad (61)$$

where λ_i denote the eigenvalues of $F_{i\bar{j}} = \pi[N^T(\text{Im} \Omega)^{-1}]_{ij}$, and note that $\text{tr} F_{i\bar{j}}$ is the lowest eigenvalue of Δ . From the above formula, the largest eigenvalue λ_I must be equal to summation of the remaining two eigenvalues λ_J, λ_K to realize the massless scalar in 4D effective field theory, namely

$$\lambda_I = \lambda_J + \lambda_K. \quad (62)$$

This corresponds to the D -term SUSY condition, that is, the condition to realize the D -flat solution [9,47]. On the other hand, the eigen equation of ψ_{---} for Δ is given by

$$\Delta \psi_{---} = -2 \left[\sum_{j=1}^3 F_{z^j\bar{z}^j} \right] \psi_{---}. \quad (63)$$

Next, we study the model, where two eigenvalues of $F_{i\bar{j}}$ are negative and the other one is positive. The eigenvalues of the mass squared matrix for 4D scalar $\mathcal{M}^2$ correspond to the 4D scalar masses squared, which are independent of the complex basis given by the $SO(3)$ rotation. Then, the 4D scalar masses squared $m_{4D,I}^2$ ($I = 1, 2, 3$), are written as

$$\text{diag}(m_{4D,1}^2, m_{4D,2}^2, m_{4D,3}^2) = D^{-1} \mathcal{M}_z^2 D = D'^{-1} \mathcal{M}_{z'}^2 D', \quad (64)$$

where $\mathcal{M}_z^2, \mathcal{M}_{z'}^2$ denote the mass squared matrices written in z, z' basis, respectively. Also, D, D' denote the diagonalization matrices of $\mathcal{M}_z^2, \mathcal{M}_{z'}^2$, respectively. We will describe $\mathcal{M}_{z'}^2$ using only ψ'_{+++} . From the previous review and the definition of z' , the spinor in z' , Ψ' can be written as

$$\Psi' = [0, 0, 0, \psi'_{+++}, 0, 0, 0, 0]^T. \quad (65)$$

Then, the Dirac equation can be described as $D_{z^j} \psi'_{+++} = 0$, ($j = 1, 2, 3$) and Eq. (65) is its solution.

We describe the Laplacian Δ' in the z' system

$$\Delta' \equiv -2 \sum_{j=1}^3 \{D_{z'^j}, \bar{D}_{\bar{z}'^j}\} \quad (66)$$

and introduce the eigen equation of Δ'

$$\Delta' \psi'_{+++} = 2 \left[\sum_{j=1}^3 F'_{z^j\bar{z}^j} \right] \psi'_{+++} = 2 \cdot (\text{tr} F'_{z^j\bar{z}^j}) \psi'_{+++}. \quad (67)$$

Therefore, we can write the mass squared matrix $\mathcal{M}_{z'}^2$ as

$$\mathcal{M}_{z'}^2 = \Delta' + 4 \begin{bmatrix} -F'_{i\bar{j}} & O \\ O & F'_{i\bar{j}} \end{bmatrix}, \quad (68)$$

where $F'_{i\bar{j}} = F'_{z^i\bar{z}^j}$. We finally obtain the following formula through the diagonalization

$$D'^{-1} \mathcal{M}_{z'}^2 D' = 2(\lambda'_1 + \lambda'_2 + \lambda'_3) + 4 \cdot \text{diag}[-\lambda'_1, -\lambda'_2, -\lambda'_3, +\lambda'_1, +\lambda'_2, +\lambda'_3], \quad (69)$$

where λ'_i denote the eigenvalues of $F'_{i\bar{j}}$. We therefore conclude that the D -term SUSY condition can be briefly described as follows

$$\lambda'_I = \lambda'_J + \lambda'_K, \quad (70)$$

where λ'_I denotes the largest eigenvalue of $F'_{i\bar{j}}$ and λ'_J, λ'_K denote the other two eigenvalues. In other words, we can write the D -term SUSY condition using the eigenvalues of $F'_{z^j\bar{z}^j}$, that is, ones with the largest absolute value λ'_I , the others λ'_J, λ'_K

$$|\lambda'_I| = |\lambda'_J| + |\lambda'_K|. \quad (71)$$

IV. WAVE FUNCTIONS UNDER $Sp(6, \mathbb{Z})$ MODULAR TRANSFORMATION

The T^6 compactification has the $Sp(6, \mathbb{Z})$ modular symmetry [21,46,48–55]. The modular transformations have been used for realizing $\mathbb{Z}_N$ twist, counting zero-mode numbers, and studying generation structure in magnetized orbifold models [13,34,35]. Also, the modular symmetry is important to control 4D low energy effective field theory.⁵ In what follows, we will consider behavior of the wave functions under the $Sp(6, \mathbb{Z})$ modular transformation.

⁵Recently, modular flavor symmetries have been studied intensively [56]. (See for early works Refs. [57–66] and for reviews Refs. [67,68].)

According to Ref. [35], the $Sp(6, \mathbb{Z})$ modular symmetry comes from 6D lattice spanned by the basis vectors e_i , $\Omega_{ij}e_j$, ($i, j = 1, 2, 3$).⁶ The generators are represented by the matrices S and T

$$S = \begin{bmatrix} O_3 & \mathbf{1}_3 \\ -\mathbf{1}_3 & O_3 \end{bmatrix}, \quad T = \begin{bmatrix} \mathbf{1}_3 & B \\ 0_3 & \mathbf{1}_3 \end{bmatrix}, \quad (72)$$

where the B matrix corresponds to a 3×3 integer symmetric matrix. Then, z and Ω are transformed as

$$S: (z, \Omega) \Rightarrow (-\Omega^{-1}z, -\Omega^{-1}), \quad T: (z, \Omega) \Rightarrow (z, \Omega + B), \quad (73)$$

under the S and T transformation. In what follows, we will describe how the wave functions with various chiralities in magnetized T^6 are transformed under the modular S and T transformation.

A. Modular T transformation

First, let us consider the T transformation under $Sp(6, \mathbb{Z})$ symmetry. When we consider the background magnetic fluxes F shown in Eqs. (11) and (34), we require the following condition

$$(NB)^T = NB \quad \text{where } (NB)_{ii} \in 2\mathbb{Z} \quad (74)$$

to be consistent with the relation between the wave functions and the T transformation.

The wave functions are transformed as

$$\psi_{M,N}^J(z, \Omega + B) = e^{\pi i J^T N^{-1} B J} \psi_{M,N}^J(z, \Omega) \quad (75)$$

under the T transformation. This can be also described as

$$\psi_{+++}^{IJ}(z', \Omega' + B) = e^{\pi i J^T N^{-1} B J} \psi_{+++}^{IJ}(z', \Omega') \quad (76)$$

in the z' system.

B. Modular S transformation

Next we consider the wave functions behaviors under the S transformation. When we consider the background magnetic fluxes, the integer fluxes N are transformed as N^T under the S transformation. In the following, we consider the wave functions' behaviors and suppose that both N and M are invariant under the S transformation, that is, the $R = N^{-1}M$ is also S invariant. We stress that since we have considered the wave functions with positive chirality $(+++)$ in z' under $R^2 = \mathbf{1}_3$ and $[\Omega, \bar{\Omega}] = 0$, the transformation of z' and Ω' corresponds to the

original S transformation

$$\begin{aligned} \Omega' &\Rightarrow -\Omega'^{-1} \quad \text{under } S: \Omega \rightarrow -\Omega^{-1}, \\ z' &\Rightarrow -\Omega'^{-1} z' \quad \text{under } S: z \rightarrow -\Omega^{-1} z. \end{aligned} \quad (77)$$

We conclude that the wave functions are transformed as

$$\begin{aligned} \psi_{+++}^{IJ}(-\Omega'^{-1} z', -\Omega'^{-1}) \\ = \frac{\sqrt{\det(-i\Omega')}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} \cdot \psi_{+++}^{IK}(z', \Omega'), \end{aligned} \quad (78)$$

under the S transformation in the z' system. When we analyze the zero-mode numbers in $T^{2g}/\mathbb{Z}_N$ orbifolds, we consider $(ST \cdots)$ -invariant moduli such that $\Omega = -(\Omega + B)^{-1}$. Thus we derive

$$\Omega' = -(\Omega' + B)^{-1} \quad (79)$$

and

$$\psi_{N,R}^J(\Omega' z', \Omega') = \rho_{JK}(ST \cdots) \psi_{N,R}^K(z', \Omega') \quad (80)$$

through $(ST \cdots)$ transformation, where $\rho_{JK}(ST \cdots)$ denotes a unitary representation derived from the modular transformations and the magnetic fluxes. We will make use of the above formula and the spinor representation matrix to count the number of zero modes in magnetized $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_{12}$ orbifolds in Sec. VI.

V. YUKAWA COUPLINGS IN MAGNETIZED T^{2g} MODEL

A. Calculation for Yukawa couplings

We consider the Yukawa couplings taking chirality into account. The Yukawa couplings can be described by the spinor wave functions in left-handed, right-handed, and Higgs sectors. They may have different magnetic fluxes and chiralities. We are assuming 4D $\mathcal{N} = 1$ SUSY. For the Higgs sector, we denote the chirality of Higgsino fields. We denote magnetic fluxes for left-handed and right-handed sectors by N_L and N_R , respectively. The magnetic flux appearing in the Higgs sector must satisfy $N_H = N_L + N_R$ because of gauge invariance. We also use the notation, M_L , M_R , M_H , where $M_i = N_i R_i$, ($i = L, R, H$). According to Ref. [69], since we have imposed $[N, \Omega] = 0$ derived from the F -term SUSY condition, we can generally describe N as linear combinations of the unit matrix $\mathbf{1}_3$, the moduli Ω , and the inverse Ω^{-1} ,

$$N = k_1 \mathbf{1}_3 + k_2 \Omega + k_3 \Omega^{-1}, \quad (81)$$

where k_i , ($i = 1, 2, 3$), denotes complex coefficients such that N can be the integer fluxes. Then, when we consider the integer fluxes in left-handed, right-handed sectors,

⁶The definition of the $Sp(2g, \mathbb{Z})$ symplectic modular group is described in Appendix D of Ref. [45].

or N_L, N_R , we can describe them as follows:

$$\begin{aligned} N_L &= k_{1L} \mathbf{1}_3 + k_{2L} \Omega + k_{3L} \Omega^{-1}, \\ N_R &= k_{1R} \mathbf{1}_3 + k_{2R} \Omega + k_{3R} \Omega^{-1}. \end{aligned} \quad (82)$$

We therefore obtain $[N_L, N_R] = 0$ under $[N, \Omega] = 0$, which means we can find different chiralities in each sector. Similarly, we can find $[M_L, M_R] = 0$ and $[N, M] = 0$ under $[N, \Omega] = [R, \Omega] = 0$.

When we consider the wave functions in left-handed, right-handed, and Higgs sectors on the compact space, an overlap integral constructed by products of these wave functions can introduce three-point couplings, namely Yukawa couplings. The Yukawa couplings among modes with the same chirality $(++)$ were studied in Ref. [10]. Here, we study the Yukawa couplings among other

combinations of chiralities. We denote wave functions of the left-handed matter field, the right-handed matter one, and the Higgs field by ψ_L^I , ($I \in \Lambda_{N_L}$), ψ_R^J , ($J \in \Lambda_{N_R}$), and ψ_H^K , ($K \in \Lambda_{N_H}$), where all N are integer matrices. The Yukawa couplings $Y^{I,J,K}$ are described as

$$Y^{I,J,K} \equiv \sigma_{abc} g' \int_{T^6} d^3 z d^3 \bar{z} \psi_L^I \cdot \psi_R^J \cdot (\psi_H^K)^*, \quad (83)$$

where g' and σ_{abc} denote the gauge coupling constant in higher dimensional theory and a sign defined by the product of differences of fluxes between D-branes, respectively [9]. Under Eq. (83), when we focus on the product of the wave functions with various chiralities, we can analyze the following integral,

$$\begin{aligned} C &\equiv \int_{T^6} d^3 z d^3 \bar{z} \psi_{N_L, M_L}^I(z_L, \Omega_L) \cdot \psi_{N_R, M_R}^J(z_R, \Omega_R) \cdot (\psi_{N_H, M_H}^K(z_H, \Omega_H))^* \\ &= |-2i \operatorname{Im} \Omega| \cdot \int_0^1 d^3 x \int_0^1 d^3 y \psi_{N_L, M_L}^I(z_L, \Omega_L) \cdot \psi_{N_R, M_R}^J(z_R, \Omega_R) \cdot (\psi_{N_L+N_R, M_H}^K(z_H, \Omega_H))^*, \end{aligned} \quad (84)$$

where we have made use of $N_H = N_L + N_R$ under the gauge symmetry. Also, we have introduced the following measure derived from the Jacobian

$$d^3 z d^3 \bar{z} = |-2i \operatorname{Im} \Omega| d^3 x d^3 y, \quad (85)$$

and the identifications, i.e., $x \sim x + e$ and $y \sim y + e$. Here, we describe f_{N_ℓ} and $\Theta_{N_\ell}^J(z'_\ell, \Omega'_\ell)$ derived from Eq. (44),

$$f_{N_\ell} \equiv e^{i\pi[N_\ell z'_\ell]^T (\operatorname{Im} \Omega'_\ell)^{-1} \operatorname{Im} z'_\ell}, \quad \Theta_{N_\ell}^J(z'_\ell, \Omega'_\ell) \equiv \theta \begin{bmatrix} J^T N_\ell^{-1} \\ 0 \end{bmatrix} (N_\ell z'_\ell, N_\ell \Omega'_\ell), \quad (86)$$

where we have defined $J \in \Lambda_{N_\ell}$, $z'_\ell = \operatorname{Re} z + i R_\ell \operatorname{Im} z$, $\Omega'_\ell = \operatorname{Re} \Omega + i R_\ell \operatorname{Im} \Omega$, and $R_\ell = N_\ell^{-1} M_\ell$, ($\ell = L, R, H$). We will use the following relation in the Higgs sector:

$$(\Theta_{N_H}^K(z'_H, \Omega'_H))^* = \Theta_{-N_H}^{-K}(\bar{z}'_H, \bar{\Omega}'_H). \quad (87)$$

The wave functions above converge since $-M_H \operatorname{Im} \bar{\Omega} = M_H \operatorname{Im} \Omega > 0$. Thus C is proportional to

$$\int_{T^6} d^3 z d^3 \bar{z} [f_{N_L} \cdot \Theta_{N_L}^I(z'_L, \Omega'_L)] \cdot [f_{N_R} \cdot \Theta_{N_R}^J(z'_R, \Omega'_R)] \cdot [f_{N_H}^* \cdot \Theta_{-N_H}^{-K}(\bar{z}'_H, \bar{\Omega}'_H)]. \quad (88)$$

In order to evaluate Eq. (84), we need to calculate the following product

$$\psi_{N_L, M_L}^I(z'_L, \Omega'_L) \cdot \psi_{N_R, M_R}^J(z'_R, \Omega'_R), \quad (89)$$

where $I = N_L \vec{i} \in \Lambda_{N_L}$ and $J = N_R \vec{j} \in \Lambda_{N_R}$. In this Subsection, we will use $\vec{i}$ and $\vec{j}$ when calculating and we can generally discuss the product regardless of the dimension of the torus. By use of $z = \operatorname{Re} z + i \operatorname{Im} z = x + (\operatorname{Re} \Omega)y + i(\operatorname{Im} \Omega)y$, the product up to the normalization constant is calculated as follows:

$$\begin{aligned} \psi_{N_L, M_L}^I(z'_L, \Omega'_L) \cdot \psi_{N_R, M_R}^J(z'_R, \Omega'_R) &\propto e^{i\pi[N_L z'_L]^T (\operatorname{Im} \Omega)^{-1} \operatorname{Im} z} \cdot e^{i\pi[N_R z'_R]^T (\operatorname{Im} \Omega)^{-1} \operatorname{Im} z} \cdot \Theta_{N_L}^I(z'_L, \Omega'_L) \cdot \Theta_{N_R}^J(z'_R, \Omega'_R) \\ &= e^{i\pi[x^T(N_L+N_R)y + y^T[(N_L+N_R)\operatorname{Re} \Omega + i(M_L+M_R)\operatorname{Im} \Omega]y]} \\ &\quad \cdot \sum_{\vec{\ell}_1, \vec{\ell}_2 \in \mathbb{Z}^g} e^{i\pi(\vec{\ell}_1^T \hat{Q}_{N, M} \vec{\ell}_2)} \cdot e^{2\pi i[\vec{\ell}_1^T Q_N \vec{X}]} \cdot e^{2\pi i(\vec{\ell}_2^T \hat{Q}_{N, M} \vec{Y})}, \end{aligned} \quad (90)$$

where we have defined the $2g$ -dimensional real vectors,

$$\vec{L} = \begin{bmatrix} \vec{i} + \vec{\ell}_1 \\ \vec{j} + \vec{\ell}_2 \end{bmatrix}, \quad \vec{X} = \begin{bmatrix} \vec{x} \\ \vec{x} \end{bmatrix}, \quad \vec{Y} = \begin{bmatrix} \vec{y} \\ \vec{y} \end{bmatrix}, \quad (91)$$

and the following $2g \times 2g$ matrices:

$$\mathcal{Q}_N = \begin{bmatrix} N_L & O \\ O & N_R \end{bmatrix},$$

$$\hat{\mathcal{Q}}_{N,M} = \begin{bmatrix} M_L \text{Im} \Omega - iN_L \text{Re} \Omega & O \\ O & M_R \text{Im} \Omega - iN_R \text{Re} \Omega \end{bmatrix} \equiv \begin{bmatrix} M'_L & O \\ O & M'_R \end{bmatrix}. \quad (92)$$

Here, we have defined $M' = M_L \text{Im} \Omega - iN_L \text{Re} \Omega$. In what follows, we will focus on how to take the summation for indices $\vec{i}N_L, \vec{j}N_R, \vec{k}(N_L + N_R)$. In that case, note that M' are irrelevant to the index summation. According to Ref. [10], we can make use of the following transformation matrix T' under $[N_L, N_R] = [M_L, M_R] = 0$, defined as

$$T' \equiv \begin{bmatrix} 1_g & 1_g \\ \alpha N_L^{-1} & -\alpha N_R^{-1} \end{bmatrix}, \quad (T')^T = \begin{bmatrix} 1_g & N_L^{-1} \alpha^T \\ 1_g & -N_R^{-1} \alpha^T \end{bmatrix},$$

$$(T')^{-1} = (N_L^{-1} + N_R^{-1})^{-1} \begin{bmatrix} N_R^{-1} & \alpha^{-1} \\ N_L^{-1} & -\alpha^{-1} \end{bmatrix},$$

$$((T')^{-1})^T = (N_L^{-1} + N_R^{-1})^{-1} \begin{bmatrix} N_R^{-1} & N_L^{-1} \\ (\alpha^{-1})^T & -(\alpha^{-1})^T \end{bmatrix}, \quad (93)$$

with $g \times g$ matrix α , and $g \times g$ unit matrix 1_g . In the process of calculating series in the product $\psi_{N_L, M_L}^I(z'_L, \Omega'_L) \cdot \psi_{N_R, M_R}^I(z'_R, \Omega'_R)$, we take $\alpha = |N_L||N_R|1_g$ when $|N_L|$ and $|N_R|$ are coprime. This comes from the condition that $\alpha(N_L^{-1} + N_R^{-1})$ has integer values since both N_L and N_R are integer matrices. This condition is necessary to maintain consistency with the series. We can then choose α as the least common multiple of $|N_L|$ and $|N_R|$ when they are not coprime. In general, we can choose the following α

$$\alpha = \begin{cases} |N_L||N_R|1_g & \text{for } |N_L|, |N_R| \text{ coprime,} \\ \text{L.C.M.}(|N_L|, |N_R|)1_g & \text{for the others.} \end{cases} \quad (94)$$

After the calculation for the Yukawa couplings,⁷ when $\alpha = |N_L||N_R|1_g$ and $g = 3$, we can find the integral part of the Yukawa couplings in magnetized T^6 by using the Riemann-theta function with character,

$$C = |-2i \text{Im} \Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \cdot \int_0^1 d^3 y \left[e^{-\pi[y^T[M_L + M_R + M_H](\text{Im} \Omega)y]} \right. \\ \left. \cdot \sum_{\vec{\ell}_3, \vec{\ell}_4 \in \mathbb{Z}^3} \sum_{\vec{p}, \vec{\tilde{p}}} e^{\pi i(i)[\vec{K} + \vec{L}_1]^T \cdot \hat{\mathbf{Q}}_{N, M'}^{[N_L][N_R]} \cdot [\vec{K} + \vec{L}_1]} \cdot e^{2\pi i(i)[\vec{K} + \vec{L}_1]^T \vec{\mathbf{Y}}_{N, M'}^{[N_L][N_R]}} \right] \\ = |-2i \text{Im} \Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \cdot \sum_{\vec{p}, \vec{\tilde{p}}} \int_0^1 d^3 y \left[e^{-\pi[y^T[M_L + M_R + M_H](\text{Im} \Omega)y]} \cdot \theta \left[\begin{matrix} \vec{K} \\ 0 \end{matrix} \right] \left(i \vec{\mathbf{Y}}_{N, M'}^{[N_L][N_R]}, i \hat{\mathbf{Q}}_{N, M'}^{[N_L][N_R]} \right) \right], \quad (95)$$

⁷The detail of the calculation for the Yukawa couplings is described in Appendix E of Ref. [45].

where

$$\begin{aligned}\vec{\mathbf{L}}_1 &= \begin{bmatrix} \vec{\ell}_3 \\ \vec{\ell}_4 \end{bmatrix}, \quad \vec{\mathbf{K}} = \begin{bmatrix} \vec{k} \\ \frac{N_R(N_L+N_R)^{-1}N_L}{|N_L||N_R|}(\vec{i}-\vec{j}+\vec{m}') \end{bmatrix}, \\ \vec{\mathbf{Y}}_{N,M'}^{[N_L][N_R]} &\equiv \begin{bmatrix} [M'_L + M'_R + (M'_H)^*]y \\ [(|N_L||N_R|)(M'_L N_L^{-1} - M'_R N_R^{-1})]y \end{bmatrix} \\ &= \begin{bmatrix} [(M_L + M_R + M_H) \text{Im} \Omega]y \\ |N_L||N_R|[(R_L - R_R) \text{Im} \Omega]y \end{bmatrix},\end{aligned}\quad (96)$$

and

$$\hat{\mathbf{Q}}_{N,M'}^{[N_L][N_R]} \equiv \begin{bmatrix} M'_L + M'_R + (M'_H)^* & |N_L||N_R|(M'_L N_L^{-1} - M'_R N_R^{-1}) \\ |N_L||N_R|(N_L^{-1} M'_L - N_R^{-1} M'_R) & (|N_L||N_R|)^2(N_L^{-1} M'_L N_L^{-1} + N_R^{-1} M'_R N_R^{-1}) \end{bmatrix}. \quad (97)$$

Here, we take $\text{Re} \hat{\mathbf{Q}}_{N,M'}^{[N_L][N_R]} > 0$, $\alpha = |N_L||N_R|1_g$, $M'_L = M_L \text{Im} \Omega - iN_L \text{Re} \Omega$, $M'_R = M_R \text{Im} \Omega - iN_R \text{Re} \Omega$, and $(M'_H)^* = M_H \text{Im} \Omega + i(N_L + N_R) \text{Re} \Omega$. Note that when we assign the integer fluxes N , complex structure moduli Ω , and chirality, we can also apply $M_i = N_i R_i$, ($i = L, R, H$), to numerical computations. We have replaced $\vec{m}$ satisfying $\vec{m} N_L (N_L + N_R)^{-1} \in \mathbb{Z}$ with $\vec{m}' \equiv \vec{m} + \vec{p} |N_L| (N_L + N_R) N_L^{-1} + \vec{p} |N_R| (N_L + N_R) N_R^{-1}$, where $\vec{m}$ is the solution for $(N_L + N_R) \vec{k} = N_L \vec{i} + N_R \vec{j} + N_L \vec{m}$, and $\vec{p} \in \Lambda_{|N_R||N_L|}^{-1}$, $\vec{p} \in \Lambda_{|N_L||N_R|}^{-1}$. We have also used $[N_L, N_R] = 0$.

On the other hand, when we take $\alpha = \text{L.C.M.}(|N_L|, |N_R|)1_g \equiv L_R 1_g$ (L.C.M. refers to least common multiple) and $g = 3$ under the condition that $|N_L|$ and $|N_R|$ are not coprime, we obtain the integral part of the Yukawa couplings as follows

$$\begin{aligned}C &= |-2i \text{Im} \Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \cdot \int_0^1 d^3 y \left[e^{-\pi[y^T[M_L + M_R + M_H](\text{Im} \Omega)y]} \right. \\ &\quad \cdot \sum_{\vec{\ell}_3, \vec{\ell}_4 \in \mathbb{Z}^3} \sum_{\vec{p}, \vec{p}} e^{\pi i(i)[\vec{\mathbf{K}} + \vec{\mathbf{L}}_1]^T \cdot \hat{\mathbf{Q}}_{N,M'}^{L_R} \cdot [\vec{\mathbf{K}} + \vec{\mathbf{L}}_1]} \cdot e^{2\pi i(i)[\vec{\mathbf{K}} + \vec{\mathbf{L}}_1]^T \vec{\mathbf{Y}}_{N,M'}^{L_R}} \left. \right] \\ &= |-2i \text{Im} \Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \cdot \sum_{\vec{p}, \vec{p}} \int_0^1 d^3 y \left[e^{-\pi[y^T[M_L + M_R + M_H](\text{Im} \Omega)y]} \cdot \theta \begin{bmatrix} \vec{\mathbf{K}} \\ 0 \end{bmatrix} \left(i \vec{\mathbf{Y}}_{N,M'}^{L_R}, i \hat{\mathbf{Q}}_{N,M'}^{L_R} \right) \right],\end{aligned}\quad (98)$$

where

$$\vec{\mathbf{L}}_1 = \begin{bmatrix} \vec{\ell}_3 \\ \vec{\ell}_4 \end{bmatrix}, \quad \vec{\mathbf{K}} = \begin{bmatrix} \vec{k} \\ \frac{N_R(N_L+N_R)^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{m}') \end{bmatrix}, \quad (99)$$

$$\vec{\mathbf{Y}}_{N,M'}^{L_R} \equiv \begin{bmatrix} [M'_L + M'_R + (M'_H)^*]y \\ [L_R(M'_L N_L^{-1} - M'_R N_R^{-1})]y \end{bmatrix} = \begin{bmatrix} [(M_L + M_R + M_H) \text{Im} \Omega]y \\ L_R[(R_L - R_R) \text{Im} \Omega]y \end{bmatrix}, \quad (100)$$

and

$$\hat{\mathbf{Q}}_{N,M'}^{L_R} \equiv \begin{bmatrix} M'_L + M'_R + (M'_H)^* & L_R(M'_L N_L^{-1} - M'_R N_R^{-1}) \\ L_R(N_L^{-1} M'_L - N_R^{-1} M'_R) & (L_R)^2(N_L^{-1} M'_L N_L^{-1} + N_R^{-1} M'_R N_R^{-1}) \end{bmatrix}. \quad (101)$$

Here, we take $\text{Re} \hat{\mathbf{Q}}_{N,M'}^{L_R} > 0$, $M'_L = M_L \text{Im} \Omega - iN_L \text{Re} \Omega$, $M'_R = M_R \text{Im} \Omega - iN_R \text{Re} \Omega$, and $(M'_H)^* = M_H \text{Im} \Omega + i(N_L + N_R) \text{Re} \Omega$, similarly to the calculations under $\alpha = |N_L||N_R|1_g$. Also, we have replaced $\vec{m}$ satisfying

$\vec{m}N_L(N_L + N_R)^{-1} \in \mathbb{Z}$ with $\vec{m}' \equiv \vec{m} + \vec{p}(N_L + N_R)N_L^{-1} + \vec{p}'(N_L + N_R)N_R^{-1}$, where $\vec{m}$ is the solution for $(N_L + N_R)\vec{k} = N_L\vec{i} + N_R\vec{j} + N_L\vec{m}$, and $\vec{p} \in \Lambda_{L_R N_R^{-1}}$, $\vec{p}' \in \Lambda_{L_R N_L^{-1}}$.

In general, when we impose

$$\begin{aligned} N^T &= N, M^T = M, \Omega^T = \Omega, \\ R &= N^{-1}M = PUP^{-1} = PUP^T = R^T = R^{-1}, \\ [N, \Omega] &= [N_L, N_R] = [R, \Omega] = [N, R] = [\Omega, \bar{\Omega}] = 0, \\ N_H &= N_L + N_R, (M \operatorname{Im} \Omega)^T = M \operatorname{Im} \Omega > 0, \\ N_L\vec{i} + N_R\vec{j} + N_L\vec{m} &= (N_L + N_R)\vec{k}, \\ \vec{p} &\in \Lambda_{L_R N_R^{-1}}, \vec{p}' \in \Lambda_{L_R N_L^{-1}}, \\ L_R &= \text{L.C.M.}(|N_L|, |N_R|), \mathcal{N}_i = |\det N|^{\frac{1}{4}} \cdot [\operatorname{Vol}(T^6)]^{-\frac{1}{2}}, \end{aligned} \quad (102)$$

we obtain the C

$$C = \frac{|-2i \operatorname{Im} \Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k}{\sqrt{|(M_L + M_R + M_H) \operatorname{Im} \Omega|}} \cdot \sum_{\vec{p}, \vec{p}'} \theta \left[\begin{array}{c} \frac{N_R(N_L + N_R)^{-1}N_L}{L_R} (\vec{i} - \vec{j} + \vec{m}') \\ \vec{0} \end{array} \right] (\vec{0}, \tilde{\Omega}_Y) \cdot \delta_{(N_L + N_R)\vec{k}, N_L\vec{i} + N_R\vec{j} + N_L\vec{m}}, \quad (103)$$

where

$$\tilde{\Omega}_Y \equiv (L_R)^2 [(N_L^{-1} + N_R^{-1}) \operatorname{Re} \Omega + i[(R_L N_L^{-1} + R_R N_R^{-1}) - (R_L - R_R)(M_L + M_R + M_H)^{-1}(R_L - R_R)] \operatorname{Im} \Omega], \quad (104)$$

and $\tilde{\Omega}_Y \in \mathcal{H}^3$. We prove Eq. (103) in Appendix E.2 of Ref. [45]. We can also see the modular T transformation of the C as follows:

$$T: C \Rightarrow e^{i\pi(\vec{i} - \vec{j} + \vec{m}')^T B N_R (N_L + N_R)^{-1} N_L (\vec{i} - \vec{j} + \vec{m}')} \cdot C, \quad (105)$$

where B satisfies $(L_R)^2 [(N_L^{-1} + N_R^{-1})B]_{ii} \in 2\mathbb{Z}$ under $T: \Omega \Rightarrow \Omega + B$. We therefore may realize mass hierarchy, mixing angle in both lepton and quark sectors through such Yukawa couplings. In the future, we would study numerical computations of the Yukawa couplings and verify whether realistic models can be realized or not.

VI. ZERO-MODE NUMBERS IN MAGNETIZED $T^{2g}/\mathbb{Z}_N$

Here, we study the number of zero modes in magnetized $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$ orbifold models. (See Refs. [39–44] for various 6D Lie lattices to construct T^6 and its orbifolds.)

A. Reduction from T^6 to T^4

We will analyze the zero-mode number of wave functions with negative chirality in magnetized $T^4/\mathbb{Z}_N$. We can discuss the number in T^4 from the discussion on

magnetized T^6 . We have represented the Dirac equation Eq. (31), the integrable conditions shown in Eq. (33), and the background magnetic fluxes $F_{z\bar{z}}^{(T^6)}$ in Eq. (34).

As we have discussed in Sec. III, we find the Dirac equation, the magnetic fluxes, and the integrable conditions on T^6 . In the following, we will introduce the magnetic fluxes on T^4 . In order to realize magnetized $T^4 \times T^2$, when we take $\tilde{N}'_{12} = 0$ and $q_2 = 0$, or $\theta_2 = 0$, we find the following formulas

$$\begin{aligned} F_{1\bar{1}}^{\text{part}} &= [F_{1\bar{1}} + F_{2\bar{2}}q_1^2 - 2F_{1\bar{2}}q_1] \cos^2 \theta_1 = \frac{\hat{N}'}{\cos^2 \theta_1}, \\ F_{2\bar{2}}^{\text{part}} &= [2F_{1\bar{2}}q_1 + F_{1\bar{1}}q_1^2 + F_{2\bar{2}}] \cos^2 \theta_1 = \frac{\tilde{N}'_{11}}{\cos^2 \theta_1}, \\ F_{3\bar{3}}^{\text{part}} &= F_{3\bar{3}} = \tilde{N}'_{22}, \\ F_{2\bar{3}}^{\text{part}} &= [F_{1\bar{3}}q_1 + F_{2\bar{3}}] \cos \theta_1 = \frac{\tilde{N}'_{12}}{\cos \theta_1} = 0. \end{aligned} \quad (106)$$

Note that $q_1 = -\tan \theta_1$ under $q_2 = 0$. Next we consider $F_{i\bar{j}} = P F_{ij}^{\text{part}} P^{-1}$ using Eq. (106). We can obtain the following fluxes

$$\begin{aligned}
F_{i\bar{j}} &= P F_{i\bar{j}}^{\text{part}} P^{-1} \\
&= \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\hat{N}'}{\cos^2 \theta_1} & 0 & 0 \\ 0 & \frac{\tilde{N}'_{11}}{\cos^2 \theta_1} & \frac{\tilde{N}'_{12}}{\cos \theta_1} \\ 0 & \frac{\tilde{N}'_{12}}{\cos \theta_1} & \tilde{N}'_{22} \end{bmatrix} \begin{bmatrix} \cos \theta_1 & \sin \theta_1 & 0 \\ -\sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \hat{N}' + \tilde{N}'_{11} \tan^2 \theta_1 & \hat{N}' \tan \theta_1 - \tilde{N}'_{11} \tan \theta_1 & 0 \\ \hat{N}' \tan \theta_1 - \tilde{N}'_{11} \tan \theta_1 & \hat{N}' \tan^2 \theta_1 + \tilde{N}'_{11} & 0 \\ 0 & 0 & \tilde{N}'_{22} \end{bmatrix} \\
&= \begin{bmatrix} \hat{N}' + \tilde{N}'_{11} q_1^2 & -\hat{N}' q_1 + \tilde{N}'_{11} q_1 & 0 \\ -\hat{N}' q_1 + \tilde{N}'_{11} q_1 & \hat{N}' q_1^2 + \tilde{N}'_{11} & 0 \\ 0 & 0 & \tilde{N}'_{22} \end{bmatrix}.
\end{aligned}$$

In particular, when we focus on (1,1), (1,2), (2,1), and (2,2) components of the above $F_{i\bar{j}}$, we can realize a reduction from T^6 to T^4 . That is, we can find the following magnetic fluxes on T^4 with $q_1 = -\tan \theta_1$, $q_2 = 0$, $\hat{N}'$, and $\tilde{N}'_{11}$:

$$F_{i\bar{j}}^{T^4} = \hat{N}' \begin{bmatrix} 1 & -q_1 \\ -q_1 & q_1^2 \end{bmatrix} + \tilde{N}'_{11} \begin{bmatrix} q_1^2 & q_1 \\ q_1 & 1 \end{bmatrix}. \quad (107)$$

According to the fluxes in Eq. (107), spinors ψ_{++} and ψ_{--} on magnetized T^4 are excited simultaneously when we consider the nonzero off-diagonal components of the fluxes. We will make use of the fluxes to analyze the number of wave functions with negative chirality on magnetized $T^4/\mathbb{Z}_N$ twisted orbifolds.

B. Magnetized $T^4/\mathbb{Z}_N$ orbifold

First we review the $T^4/\mathbb{Z}_N$ twisted orbifold and then analyze the number of zero modes with negative chirality $(+, -)$, $(-, +)$. In the T^4 torus, we can introduce complex structure moduli Ω and magnetic fluxes N as the following 2×2 symmetric complex or integer matrices, respectively

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix}, \quad N = \begin{bmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{bmatrix}. \quad (108)$$

We then introduce the coordinates $\vec{z} = \vec{x} + \Omega \vec{y}$ ($\vec{x}, \vec{y} \in \mathbb{R}^2$). By introducing specific complex structures Ω , we define the $T^4/\mathbb{Z}_N$ twisted orbifold. According to Ref. [34], the solutions to the Dirac equation are of the form

$$\Psi(\vec{z}, \vec{\bar{z}}) = \begin{pmatrix} \psi_{++} \\ \psi_{+-} \\ \psi_{-+} \\ \psi_{--} \end{pmatrix}, \quad (109)$$

where the four components are expressed as linear combinations in the following way

$$\psi = \sum_{\vec{j} \in \Lambda_N} a_{\vec{j}} \cdot \psi_{\vec{N}}^{\vec{j}}, \quad a_{\vec{j}} \in \mathbb{C}. \quad (110)$$

$T^4/\mathbb{Z}_N$ twisted orbifolds are identified by $\mathbb{Z}_N$ twist. In other words, we can introduce the following identification for $\vec{z}$ on T^4 to construct the $T^4/\mathbb{Z}_N$ orbifold,

$$\vec{z} \sim \Omega_{\text{twist}} \vec{z}, \quad (111)$$

where Ω_{twist}^N represents the $\mathbb{Z}_N$ twist. It is taken to be a 2×2 unitary matrix satisfying $\Omega_{\text{twist}}^N = \mathbf{1}_2$. We define the boundary condition for the spinor under a $\mathbb{Z}_N$ twist as

$$\Psi(\Omega_{\text{twist}} \vec{z}, \bar{\Omega}_{\text{twist}} \vec{\bar{z}}) = \mathcal{S} V(\vec{z}, \vec{\bar{z}}) \Psi(\vec{z}, \vec{\bar{z}}), \quad (112)$$

where $\mathcal{S}$ is the spinor representation of the twist and $V(\vec{z}, \vec{\bar{z}})$ is the transformation function. Since $\mathcal{S}$ is chosen to leave ψ_{++} invariant, the spinor representation is described as,

$$\mathcal{S} = \text{diag}(1, e^{2\pi i k_1/N}, e^{2\pi i k_2/N}, e^{-2\pi i (k_1 + k_2)/N}), \quad (113)$$

for the following factorizable twist with integers k_1, k_2

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{2\pi i k_1/N} z_1 \\ e^{2\pi i k_2/N} z_2 \end{pmatrix}. \quad (114)$$

The transformation function is written as

$$V(\vec{z}, \vec{\bar{z}}) = e^{2\pi i \beta}. \quad (115)$$

Since we consider the identification for a $\mathbb{Z}_N$ twist, we find the following condition

$$\beta N \equiv 0 \pmod{1}. \quad (116)$$

Taking $\beta = \frac{1}{N}$, we obtain $\mathbb{Z}_N$ sectors with the $\mathbb{Z}_N$ charges $e^{2\pi i \rho / N}$, ($\rho = 0, 1, \dots, N-1$). We can use the modular transformations to define the transformation matrix ρ and find the spectrum. Then, we will make use of the $Sp(4, \mathbb{Z})$ generators S and T_i , ($i = 1, 2, 3$) when we construct lattices of $T^4/\mathbb{Z}_N$, ($N = 3, 4, 6$).

C. Parity transformations

Under parity transformation, a two component spinor in Minkowski space transforms as,

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}, \quad (117)$$

which flips the chiralities. We would like to do the same for the 4D spinor on the torus. In order to flip chiralities, we need to parity transform one of the 2D spinors it is constructed from. However, since the flux is not diagonal, we must first diagonalize it. We express the flux in a new coordinate system defined as $\vec{w} = P\vec{z}$ as

$$\begin{aligned} F &= \pi [N^T (\text{Im } \Omega)^{-1}]_{i\bar{j}} (idz^i \wedge d\bar{z}^{\bar{j}}) \\ &= \pi \begin{pmatrix} x & z \\ z & y \end{pmatrix}_{i\bar{j}} (idz^i \wedge d\bar{z}^{\bar{j}}). \end{aligned} \quad (118)$$

When the negative chirality wave function is nonzero, the determinant is negative, so we have $xy - z^2 < 0$. Let the matrix P diagonalize the flux. The eigenvalues are

$$\lambda_1, \lambda_2 = \frac{1}{2} \left[x + y \pm \sqrt{(x+y)^2 - 4(xy - z^2)} \right]. \quad (119)$$

We know that the flux is not positive definite in this case, so there is at least one negative eigenvalue. Since the determinant is negative, the square root, which is positive, is larger than the trace. Thus, we conclude $\lambda_1 > 0$ and $\lambda_2 < 0$. This gives us a flux

$$F = \pi \begin{pmatrix} |\lambda_1| & \\ & -|\lambda_2| \end{pmatrix}_{i\bar{j}} (idw^i \wedge d\bar{w}^{\bar{j}}). \quad (120)$$

If we think of the T^4 spinor as a product of two spinors on T^2 , we can parity transform one of them to change the overall chirality. This is equivalent to changing to $w^2 \rightarrow \bar{w}^2$. The transformation matrix for the flux is thus

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (121)$$

and the flux in the new coordinate system w' is

$$F = \pi \begin{pmatrix} |\lambda_1| & \\ & |\lambda_2| \end{pmatrix}_{i\bar{j}} (idw'^i \wedge d\bar{w}'^{\bar{j}}). \quad (122)$$

We then rotate back into the z coordinates

$$F = \pi \left[P \begin{pmatrix} |\lambda_1| & \\ & |\lambda_2| \end{pmatrix} P^T \right]_{i\bar{j}} (idz'^i \wedge d\bar{z}'^{\bar{j}}). \quad (123)$$

The whole procedure is defined as by $R = PUP^T$, which gives the flux in the new coordinates as

$$F = \pi [N^T (R \text{Im } \Omega)^{-1}]_{i\bar{j}} (idz'^i \wedge d\bar{z}'^{\bar{j}}), \quad (124)$$

where

$$\begin{aligned} \vec{z}' &= \text{Re } \vec{z} + iR \text{Im } \vec{z}, \\ \Omega' &= \text{Re } \Omega + iR \text{Im } \Omega. \end{aligned} \quad (125)$$

Since both eigenvalues are positive, we can find $N \text{Im } \Omega' > 0$. This means we can describe the spinor in z' system as follows

$$\Psi' = \begin{pmatrix} \psi'_{++} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (126)$$

The wave functions on T^4 are then the same as the positive chirality case,

$$\psi^{\vec{J}} = \mathcal{N} e^{\pi i [N \vec{z}']^T (\text{Im } \Omega')^{-1} \text{Im } \vec{z}'} \theta \left[\begin{pmatrix} \vec{J}^T N^{-1} \\ 0 \end{pmatrix} \right] (N \vec{z}', N \Omega'), \quad (127)$$

if we change $\vec{z}, \Omega$ according to Eq. (125). We could also consider $\lambda_2 > 0$ and $\lambda_1 < 0$. This would mean $U \rightarrow -U$, and therefore $\Omega' \rightarrow \bar{\Omega}'$, $\vec{z}' \rightarrow \bar{\vec{z}}'$, which just exchanges the zero modes in each $\mathbb{Z}_N$ sector.

D. Magnetized $T^4/\mathbb{Z}_2$

We begin by analyzing the $\mathbb{Z}_2$ twist defined by the following identification

$$\vec{z} \sim -\vec{z}. \quad (128)$$

The transformation function is given by

$$V(\vec{z}, \vec{z}) = e^{2\pi i \rho / 2}, \quad \rho = (0, 1). \quad (129)$$

Thus, we have two sectors with eigenvalues ± 1 . Under the $\mathbb{Z}_2$ twist

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} -z_1 \\ -z_2 \end{pmatrix}, \quad (130)$$

the spinor representation $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \quad (131)$$

The wave functions are transformed as

$$\psi(-z) = (-1)^{\rho+1} \psi(z). \quad (132)$$

In $T^4/\mathbb{Z}_2$, the complex structure moduli Ω and the fluxes N are given by

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{pmatrix}, \quad N = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix}, \quad n_1, n_2, m \in \mathbb{Z} \quad (133)$$

satisfying the F -term SUSY condition $(N\Omega)^T = N\Omega$. To compute the zero-mode spectrum, let us find how the wave functions on T^4 transform under $\mathbb{Z}_2$. We define the even and odd modes,

$$\begin{aligned} \psi_{\text{even}}(-z) &= \psi_{\text{even}}(z), \\ \psi_{\text{odd}}(-z) &= -\psi_{\text{odd}}(z), \end{aligned} \quad (134)$$

for the positive chirality. These are constructed as a linear combination of the wave functions on T^4

$$\begin{aligned} \psi_{\text{even}} &\propto \psi(z) + \psi(-z), \\ \psi_{\text{odd}} &\propto \psi(z) - \psi(-z). \end{aligned} \quad (135)$$

We take $\rho = 0$ for even modes and $\rho = 1$ for odd modes. However, when we focus on the negative chirality, we have to add a factor $s = -1$ derived from the $\mathcal{S}$

$$\begin{aligned} \psi_{\text{even}}(-z) &= s^{-1} \psi_{\text{even}}(z) = -\psi_{\text{even}}(z), \\ \psi_{\text{odd}}(-z) &= -s^{-1} \psi_{\text{odd}}(z) = \psi_{\text{odd}}(z). \end{aligned} \quad (136)$$

We can therefore obtain the spectrum for negative chirality from the positive one by switching the odd and even sectors. The even and odd modes with negative chirality are transformed as

$$\begin{aligned} \psi_{\text{even}}(-z) &= -\psi_{\text{even}}(z), \\ \psi_{\text{odd}}(-z) &= \psi_{\text{odd}}(z), \end{aligned} \quad (137)$$

which means the spectrum is interchanged comparing with the positive chirality.

E. Magnetized $T^4/\mathbb{Z}_4$

We begin with magnetized $T^4/\mathbb{Z}_4$. (See Ref. [34].) We can construct this orbifold by the algebraic relation $S^4 = \mathbf{1}_4$. The S -invariant Ω_S is given by

$$\Omega_S = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \quad (138)$$

We get the following identification:

$$\vec{z} \sim \Omega_S \vec{z}. \quad (139)$$

The transformation function is

$$V(\vec{z}, \vec{z}) = e^{2\pi i k/4}, \quad k = (0, 1, 2, 3), \quad (140)$$

according to the $\mathbb{Z}_N$ -twist boundary condition. Under the S , the wave functions are transformed as

$$\psi_N^{\vec{J}}(\Omega_S \vec{z}, \Omega_S) = \frac{1}{\sqrt{\det N}} \sum_{\vec{K} \in \Lambda_N} e^{2\pi i \vec{J}^T N^{-1} \vec{K}} \psi_N^{\vec{K}}(\vec{z}, \Omega_S), \quad (141)$$

where the N is given by

$$N = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix}, \quad n_1, n_2, m \in \mathbb{Z}. \quad (142)$$

The trace of the transformation matrix from Eq. (141) is

$$\text{tr} \rho(S) = \frac{1}{\sqrt{\det N}} \sum_{\vec{K} \in \Lambda_N} e^{2\pi i \vec{K}^T N^{-1} \vec{K}}. \quad (143)$$

We can make use of the $2g$ -dimensional Landsberg-Schaar relation⁸ given by,

$$\frac{1}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{\pi i K^T N^{-1} B K} = \frac{e^{\pi i (\frac{q}{4} + \frac{n_N^N}{2} - \frac{n_B^B}{2})}}{\sqrt{|\det B|}} \sum_{K \in \Lambda_B} e^{-\pi i K^T N B^{-1} K}, \quad (144)$$

where n_N^N , n_B^B denote the number of negative eigenvalues of N , B , respectively. When we analyze the degenerated number of the wave functions in each $\mathbb{Z}_N$ sector, D_k , the spectrum can be determined by the following formulas:

⁸The detail of the $2g$ -dimensional Landsberg-Schaar is written in Ref. [21].

$$\begin{aligned}
\det N &= D_{+1} + D_{-1} + D_{+i} + D_{-i}, \\
\text{tr} \rho &= D_{+1} - D_{-1} + iD_{+i} - iD_{-i}, \\
N_{Z_2+} - N_{Z_2-} &= D_{+1} + D_{-1} - D_{+i} - D_{-i} \\
&= \begin{cases} 1; & \text{if } \det N \equiv 1 \pmod{2}, \\ 2; & \text{if } \det N \equiv 0 \pmod{2}, \text{ gcd}(n_i, m) \equiv 1 \pmod{2} \text{ for } i = 1 \text{ or } 2, \\ 4; & \text{gcd}(n_i, m) \equiv 0 \pmod{2} \text{ for } i = 1, 2, \end{cases} \quad (145)
\end{aligned}$$

and for negative chirality

$$\begin{aligned}
|\det N| &= D_{+1} + D_{-1} + D_{+i} + D_{-i}, \\
\text{tr} \rho &= D_{+1} - D_{-1} + iD_{+i} - iD_{-i}, \\
-(N_{Z_2+} - N_{Z_2-}) &= D_{+1} - D_{+i} + D_{-1} - D_{-i}, \\
&= \begin{cases} -1; & \text{if } \det N \equiv 1 \pmod{2}, \\ -2; & \text{if } \det N \equiv 0 \pmod{2}, \text{ gcd}(n_i, m) \equiv 1 \pmod{2} \text{ for } i = 1 \text{ or } 2, \\ -4; & \text{gcd}(n_i, m) \equiv 0 \pmod{2} \text{ for } i = 1, 2. \end{cases} \quad (146)
\end{aligned}$$

Let $\text{tr} \rho = a + ib$, where $a, b \in \mathbb{Z}$. Then, we can express the four eigenvalue degeneracies as

$$\begin{aligned}
D_{+1} &= \frac{|\det N| + \xi + 2a}{4}, \\
D_{+i} &= \frac{|\det N| + \xi - 2a}{4}, \\
D_{-i} &= \frac{|\det N| - \xi + 2a}{4}, \\
D_{-1} &= \frac{|\det N| - \xi - 2a}{4}, \quad (147)
\end{aligned}$$

where ξ is the difference between the degeneracy number in Z_2 . In other words, $\xi = 1, 2, 4$ and $\xi = -1, -2, -4$ for positive and negative chirality, respectively. We can find three-generation models in the following ranges. For positive chirality: $8 \leq \det N \leq 20$, however, since $\det N = 20$ is not possible, the upper limit is 16. For negative chirality: $-16 \leq \det N \leq -4$.

1. Positive chirality

We begin with the ψ_{++} spinor, where we require the following positive-definite condition:

$$\det N > 0, \quad \text{tr} N > 0. \quad (148)$$

After we substitute $B = 2I$, $\det B = 4$, $g = 2$, $n_-^B = 0$, and $n_-^N = 0$, into Eq. (144), we find

$$\begin{aligned}
\text{tr} \rho_+(S) &= \frac{i}{2} \sum_{\vec{K} \in \Lambda_B} e^{-\frac{g}{2} \vec{K}^T N \vec{K}} \\
&= \frac{i}{2} [1 + (-i)^{n_1} + (-i)^{n_2} + (-1)^m (-i)^{n_1+n_2}]. \quad (149)
\end{aligned}$$

2. Negative chirality

The S transformation matrix for ψ_{+-} is

$$\rho(S) = s^{-1} \sqrt{\det(N^{-1} \Omega' / i)} \sum_{\vec{K} \in \Lambda_N} \exp \{2\pi i \vec{J}^T N^{-1} \vec{K}\}, \quad (150)$$

where the spinor representation factor is denoted by s , and $\det R = -1$. In the z' system, the corresponding spinor representation can be written as (see Appendixes A and B),

$$V \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & \sin^2 \phi_0 - \cos^2 \phi_0 & -\sin \phi_0 \cos \phi_0 & 0 \\ 0 & -\sin \phi_0 \cos \phi_0 & \cos^2 \phi_0 - \sin^2 \phi_0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \quad (151)$$

We obtain

$$\text{tr} \rho_-(S) = \frac{1}{s \sqrt{-\det N}} \sum_{\vec{K} \in \Lambda_N} e^{2\pi i \vec{K}^T N^{-1} \vec{K}}. \quad (152)$$

To calculate the trace, we use the Landsberg-Schaar relation in Eq. (144) with $B = 2I$, $\det B = 4$, $g = 2$, and $n_+^B = 0$. Since we have considered the negative chirality $(+-)$, $(-+)$, we have $n_-^N = 1$ and get

$$\begin{aligned}
\text{tr} \rho_-(S) &= \frac{-1}{s \sqrt{\det B}} \sum_{\vec{K} \in \Lambda_B} e^{-\frac{g}{2} \vec{K}^T N \vec{K}} \\
&= (is^{-1}) \text{tr} \rho_+(S) = \text{tr} \rho_+(S). \quad (153)
\end{aligned}$$

3. Trace classification

The traces are classified by the mod 4 and 2 structure of the flux components n_i, m , respectively, as shown in Table I.

Table II is found by checking the mod 4 condition on the flux components in the following way:

$$\begin{aligned} n_1 n_2 - m^2 &\equiv 0 \pmod{4}, \\ n_1 n_2 &\equiv m^2 \pmod{4}, \\ n_1 n_2 &\equiv \begin{cases} 0 \pmod{4}; & \text{if } m \equiv 0 \pmod{2}, \\ 1 \pmod{4}; & \text{if } m \equiv 1 \pmod{2}, \end{cases} \end{aligned} \quad (154)$$

for $\det N \equiv 0 \pmod{4}$,

$$n_1 n_2 \equiv \begin{cases} 1 \pmod{4}; & \text{if } m \equiv 0 \pmod{2}, \\ 2 \pmod{4}; & \text{if } m \equiv 1 \pmod{2}, \end{cases} \quad (155)$$

for $\det N \equiv 1 \pmod{4}$,

$$n_1 n_2 \equiv \begin{cases} 2 \pmod{4}; & \text{if } m \equiv 0 \pmod{2}, \\ 3 \pmod{4}; & \text{if } m \equiv 1 \pmod{2}, \end{cases} \quad (156)$$

for $\det N \equiv 2 \pmod{4}$,

$$n_1 n_2 \equiv \begin{cases} 3 \pmod{4}; & \text{if } m \equiv 0 \pmod{2}, \\ 0 \pmod{4}; & \text{if } m \equiv 1 \pmod{2}, \end{cases} \quad (157)$$

for $\det N \equiv 3 \pmod{4}$.

There is further restriction on the existence of determinants. The conditions are shown in Tables III and IV. The whole spectrum is shown in Tables V and VI. We only show determinants up to 16 since that is the highest modulo condition—the spectrum pattern repeats after that except

TABLE I. All possible traces categorized by the mod 4 structure of the flux components.

n_1	n_2			
	0	1	2	3
(a) Traces for $m \equiv 0 \pmod{2}$.				
0	$2i$	$1+i$	0	$-1+i$
1	$1+i$	1	0	i
2	0	0	0	0
3	$-1+i$	i	0	-1
n_1	n_2			
	0	1	2	3
(b) Traces for $m \equiv 1 \pmod{2}$.				
0	i	i	i	i
1	i	$1+i$	1	0
2	i	1	$-i$	-1
3	i	0	-1	$-1+i$

TABLE II. Possible traces for different $\det N \pmod{4}$ structures.

n_1	n_2			
	0	1	2	3
(a) Traces for $\det N \equiv 0 \pmod{4}$ with $m \equiv 0 \pmod{2}$.				
0	$2i$	$1+i$	0	$-1+i$
1		$\times$	$\times$	$\times$
2			0	$\times$
3				$\times$
n_1	n_2			
	0	1	2	3
(b) Traces for $\det N \equiv 0 \pmod{4}$ with $m \equiv 1 \pmod{2}$.				
0	$\times$	$\times$	$\times$	$\times$
1		$1+i$	$\times$	$\times$
2			$\times$	$\times$
3				$-1+i$
n_1	n_2			
	0	1	2	3
(c) Traces for $\det N \equiv 1 \pmod{4}$ with $m \equiv 0 \pmod{2}$.				
0	$\times$	$\times$	$\times$	$\times$
1		1	$\times$	$\times$
2			$\times$	$\times$
3				-1
n_1	n_2			
	0	1	2	3
(d) Traces for $\det N \equiv 1 \pmod{4}$ with $m \equiv 1 \pmod{2}$.				
0	$\times$	$\times$	$\times$	$\times$
1		$\times$	1	$\times$
2			$\times$	-1
3				$\times$
n_1	n_2			
	0	1	2	3
(e) Traces for $\det N \equiv 2 \pmod{4}$ with $m \equiv 0 \pmod{2}$.				
0	$\times$	$\times$	$\times$	$\times$
1		$\times$	0	$\times$
2			$\times$	0
3				$\times$
n_1	n_2			
	0	1	2	3
(f) Traces for $\det N \equiv 2 \pmod{4}$ with $m \equiv 1 \pmod{2}$.				
0	$\times$	$\times$	$\times$	$\times$
1		$\times$	$\times$	0
2			$\times$	$\times$
3				$\times$
n_1	n_2			
	0	1	2	3
(g) Traces for $\det N \equiv 3 \pmod{4}$ with $m \equiv 0 \pmod{2}$.				
0	$\times$	$\times$	$\times$	$\times$
1		$\times$	$\times$	i

(Table continued)

TABLE II. (*Continued*)

n_1	n_2			
	0	1	2	3
2			$\times$	$\times$
3				$\times$

n_1	n_2			
	0	1	2	3
(h) Traces for $\det N \equiv 3 \pmod{4}$ with $m \equiv 1 \pmod{2}$.				
0	i	i	i	i
1		$\times$	$\times$	$\times$
2			$-i$	$\times$
3				$\times$

TABLE III. Traces for $\det N > 0$.

$\text{tr}\rho$	$\det N > 0$ conditions	ξ
i	3 (mod 4)	1
$-i$	3 (mod 8)	1
0	2 (mod 4)	2
1	1 (mod 4), except $\det N = 1$	1
-1	1 (mod 4)	1
$2i$	0, 12 (mod 16)	4
0	0 (mod 4)	4
$1+i$	0 (mod 4)	2
$-1+i$	except $\det N = 4$, $\det N = 4(4x)^2$, $x \in \mathbb{Z}$	2

TABLE IV. Traces for $\det N < 0$.

$\text{tr}\rho$	$\det N < 0$ conditions	ξ
i	3 (mod 4)	-1
$-i$	3 (mod 8)	-1
0	2 (mod 4)	-2
1	1 (mod 4)	-1
-1	1 (mod 4)	-1
$2i$	0, 12 (mod 16)	-4
0	0 (mod 4)	-4
$1+i$	0 (mod 4)	-2
$-1+i$	0 (mod 4)	-2

for $\text{tr}\rho = -1 + i$, which needs to be checked case by case. We note that Table IV agrees with Ref. [34] as we expect.

F. Magnetized $T^4/\mathbb{Z}_3$

We consider the algebraic relation $(ST_1T_2)^3 = I$. This makes the following complex structure

TABLE V. $T^4/\mathbb{Z}_4$ spectrum for $\det N > 0$.

$\det N$	Trace	D_{+1}	D_{+i}	D_{-1}	D_{-i}
1	1	1	0	0	0
2	0	1	0	1	0
3	i	1	1	1	0
3	$-i$	1	0	1	1
4	0	2	0	2	0
4	$1+i$	2	1	1	0
5	1	2	1	1	1
5	-1	1	1	2	1
6	0	2	1	2	1
7	i	2	2	2	1
8	0	$\boxed{3}$	1	$\boxed{3}$	1
8	$-1+i$	2	2	$\boxed{3}$	1
8	$1+i$	$\boxed{3}$	2	2	1
9	1	$\boxed{3}$	2	2	2
9	-1	2	2	$\boxed{3}$	2
10	0	$\boxed{3}$	2	$\boxed{3}$	2
11	i	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	2
11	$-i$	$\boxed{3}$	0	$\boxed{3}$	$\boxed{3}$
12	$2i$	4	$\boxed{3}$	4	1
12	0	4	2	4	2
12	$-1+i$	$\boxed{3}$	$\boxed{3}$	4	2
12	$1+i$	4	$\boxed{3}$	$\boxed{3}$	2
13	1	4	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$
13	-1	$\boxed{3}$	$\boxed{3}$	4	$\boxed{3}$
14	0	4	$\boxed{3}$	4	$\boxed{3}$
15	i	4	4	4	$\boxed{3}$
16	$2i$	5	4	5	2
16	0	5	$\boxed{3}$	5	$\boxed{3}$
16	$1+i$	5	4	4	$\boxed{3}$

$$\Omega_{(ST_1T_2)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega = e^{2\pi i/3}. \quad (158)$$

The identification is

$$\vec{z} \sim \Omega_{(ST_1T_2)} \vec{z}, \quad (159)$$

which leads to the transformation function

$$V(\vec{z}, \vec{z}) = e^{2\pi i \rho/3}, \quad \rho = (0, 1, 2). \quad (160)$$

The $\text{tr}\rho_+(ST_1T_2)$ is given by

$$\text{tr}\rho_+(ST_1T_2) = \sqrt{\det(iN^{-1}\Omega_{(ST_1T_2)}^{-1})} \sum_{\vec{K} \in \Lambda_N} e^{3\pi i \vec{K}^T N^{-1} \vec{K}}. \quad (161)$$

TABLE VI. $T^4/\mathbb{Z}_4$ spectrum for $\det N < 0$.

$\det N$	Trace	D_{+1}	D_{+i}	D_{-1}	D_{-i}
-1	i	0	1	0	0
-2	0	0	1	0	1
-3	1	1	1	0	1
-3	-1	0	1	1	1
-4	$2i$	0	$\boxed{3}$	0	1
-4	0	0	2	0	2
-4	$1+i$	1	2	0	1
-4	$-1+i$	0	2	1	1
-5	i	1	2	1	1
-5	$-i$	1	1	1	2
-6	0	1	2	1	2
-7	1	2	2	1	2
-7	-1	1	2	2	2
-8	0	1	$\boxed{3}$	1	$\boxed{3}$
-8	$1+i$	2	$\boxed{3}$	1	2
-8	$-1+i$	1	$\boxed{3}$	2	2
-9	i	2	$\boxed{3}$	2	2
-10	0	2	$\boxed{3}$	2	$\boxed{3}$
-11	1	3	$\boxed{3}$	2	$\boxed{3}$
-11	-1	2	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$
-12	0	2	4	2	4
-12	$1+i$	$\boxed{3}$	4	2	$\boxed{3}$
-12	$-1+i$	2	4	$\boxed{3}$	$\boxed{3}$
-13	i	$\boxed{3}$	4	$\boxed{3}$	$\boxed{3}$
-13	$-i$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	4
-14	0	$\boxed{3}$	4	$\boxed{3}$	4
-15	1	4	4	$\boxed{3}$	4
-15	-1	$\boxed{3}$	4	4	4
-16	$2i$	$\boxed{3}$	6	$\boxed{3}$	4
-16	0	$\boxed{3}$	5	$\boxed{3}$	5
-16	$1+i$	4	5	$\boxed{3}$	4
-16	$-1+i$	$\boxed{3}$	5	4	4

According to the T_i symmetry, we should define the fluxes N so that the diagonal elements of N can be even

$$N = \begin{pmatrix} 2n_1 & m \\ m & 2n_2 \end{pmatrix}, \quad n_1, n_2, m \in \mathbb{Z}. \quad (162)$$

1. Positive chirality

When we consider the ψ_{++} spinor, i.e.,

$$\det N > 0, \quad \text{tr} N > 0, \quad (163)$$

the trace becomes

$$\text{tr} \rho_+(ST_1 T_2) = \frac{i\bar{\omega}}{\sqrt{\det N}} \sum_{\vec{K} \in \Lambda_N} e^{3\pi i \vec{K}^T N^{-1} \vec{K}}, \quad (164)$$

where we take the positive square root. By use of Eq. (144), when we substitute $B = 3I$, $\det B = 9$, $g = 2$, $n_-^B = 0$, and $n_-^N = 0$, the trace is simplified to

$$\text{tr} \rho_+(ST_1 T_2) = \frac{-\bar{\omega}}{3} [1 + 2(\bar{\omega}^{n_1} + \bar{\omega}^{n_2} + \bar{\omega}^{n_1+n_2}(\bar{\omega}^m + \bar{\omega}^{2m}))]. \quad (165)$$

We check possible $\det N$ numerically. The first few solutions for $4n_1 n_2 - m^2 \in \mathbb{Z}_+$ are,

$$\det N = 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24, 27, 28. \quad (166)$$

2. Negative chirality

By use of Eq. (161), we find

$$\text{tr} \rho_-(ST_1 T_2) = s^{-1} \sqrt{\det(iN^{-1} \Omega_{ST_1 T_2}^{-1})} \sum_{\vec{K} \in \Lambda_N} e^{3\pi i \vec{K}^T N^{-1} \vec{K}}. \quad (167)$$

Substituting back, we obtain the entire transformation function as

$$\text{tr} \rho_-(ST_1 T_2) = \frac{s^{-1}}{\sqrt{-\det N}} \sum_{\vec{K} \in \Lambda_N} e^{3\pi i \vec{K}^T N^{-1} \vec{K}}. \quad (168)$$

The formulas for the spectrum calculation become

$$D_1 + D_\omega + D_{\bar{\omega}} = |\det N|, \quad (169)$$

$$D_1 + \omega D_\omega + \bar{\omega} D_{\bar{\omega}} = \text{tr} \rho(ST_1 T_2). \quad (170)$$

We can simplify Eq. (168) using Eq. (144) with $g = 2$. Since we take $B = 3\mathbf{1}_2$, $n_-^B = 0$. Also, we find $n_-^N = 1$ and

$$\begin{aligned} \text{tr} \rho_-(ST_1 T_2) &= -\frac{s^{-1}}{3} [1 + 2(\bar{\omega}^{n_1} + \bar{\omega}^{n_2} + \bar{\omega}^{n_1+n_2}(\bar{\omega}^m + \bar{\omega}^{2m}))] \\ &= (s\bar{\omega})^{-1} \text{tr} \rho_+. \end{aligned} \quad (171)$$

The spinor representation in z' coordinates becomes

$$V \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \sin^2 \phi_0 + \bar{\omega} \cos^2 \phi_0 & (\bar{\omega} - 1) \sin \phi_0 \cos \phi_0 & 0 \\ 0 & (\bar{\omega} - 1) \sin \phi_0 \cos \phi_0 & \cos^2 \phi_0 + \bar{\omega} \sin^2 \phi_0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}. \quad (172)$$

Thus, we find $\text{tr} \rho_- = \text{tr} \rho_+$.

3. Trace classification

There are only 18 possible cases. We list them all below:

- (1) $m \equiv 0 \pmod{3} \rightarrow -\frac{\bar{\omega}}{3}[1 + 2(\bar{\omega}^{n_1} + \bar{\omega}^{n_2} + 2\bar{\omega}^{n_1+n_2})]$,
 - (a) $n_1 \equiv 0 \pmod{3} \rightarrow -\bar{\omega}(1 + 2\bar{\omega}^{n_2})$,
 - (i) $n_2 \equiv 0 \pmod{3} \rightarrow 3(1 + \omega)$,
 - (ii) $n_2 \equiv 1 \pmod{3} \rightarrow 2 + \bar{\omega}$,
 - (iii) $n_2 \equiv 2 \pmod{3} \rightarrow \bar{\omega} + 2\omega$,
- (b) $n_1 \equiv 1 \pmod{3} \rightarrow -\frac{\bar{\omega}}{3}[1 + 2(\bar{\omega} + \bar{\omega}^{n_2}(1 + 2\bar{\omega}))]$,
 - (i) $n_2 \equiv 0 \pmod{3} \rightarrow 1 - \omega$,
 - (ii) $n_2 \equiv 1 \pmod{3} \rightarrow \bar{\omega}$,
 - (iii) $n_2 \equiv 2 \pmod{3} \rightarrow 1 + \omega$,
- (c) $n_1 \equiv 2 \pmod{3} \rightarrow -\frac{\bar{\omega}}{3}[1 + 2(\omega + \bar{\omega}^{n_2}(1 + 2\omega))]$,
 - (i) $n_2 \equiv 0 \pmod{3} \rightarrow \bar{\omega} + 2\omega$,
 - (ii) $n_2 \equiv 1 \pmod{3} \rightarrow 1 + \omega$,
 - (iii) $n_2 \equiv 2 \pmod{3} \rightarrow \bar{\omega}$,
- (2) $m \not\equiv 0 \pmod{3} \rightarrow -\frac{\bar{\omega}}{3}[1 + 2(\bar{\omega}^{n_1} + \bar{\omega}^{n_2} - \bar{\omega}^{n_1+n_2})]$,
 - (a) $n_1 \equiv 0 \pmod{3} \rightarrow 1 + \omega$,
 - (b) $n_1 \equiv 1 \pmod{3} \rightarrow -\frac{\bar{\omega}}{3}[1 + 2(\bar{\omega} + \bar{\omega}^{n_2}(1 - \bar{\omega}))]$,
 - (i) $n_2 \equiv 0 \pmod{3} \rightarrow 1 + \omega$,
 - (ii) $n_2 \equiv 1 \pmod{3} \rightarrow 1 - \omega$,
 - (iii) $n_2 \equiv 2 \pmod{3} \rightarrow \bar{\omega}$,
 - (c) $n_1 \equiv 2 \pmod{3} \rightarrow -\frac{\bar{\omega}}{3}[1 + 2(\omega + \bar{\omega}^{n_2}(1 - \omega))]$,
 - (i) $n_2 \equiv 0 \pmod{3} \rightarrow 1 + \omega$,
 - (ii) $n_2 \equiv 1 \pmod{3} \rightarrow \bar{\omega}$, and
 - (iii) $n_2 \equiv 2 \pmod{3} \rightarrow \omega - 1$.

We find the traces shown in Table VII. Using these, we can express the degeneracy of each sector as in Table VIII. The degeneracy also gives us a condition on the determinant, since it has to be an integer. From Table VIII, we conclude three-generation models appear in the range of $6 \leq \det N \leq 15$.

Similarly to the $\mathbb{Z}_4$ case, we therefore find which traces are possible for different determinants mod 3.

For $\det N \equiv 1 \pmod{3}$, we express the determinant as,

$$\begin{aligned} 4n_1n_2 - m^2 &\equiv 1 \pmod{3}, \\ 4n_1n_2 &\equiv 1 + m^2 \pmod{3}, \\ 4n_1n_2 &\equiv \begin{cases} 1 \pmod{3}; & \text{if } m \equiv 0 \pmod{3}, \\ 2 \pmod{3}; & \text{if } m \equiv 1 \pmod{3}, \end{cases} \end{aligned} \quad (173)$$

since if $m \equiv 2 \pmod{3}$, we have $m^2 \equiv 1 \pmod{3}$.

TABLE VII. All possible traces categorized by the mod 3 structure of the flux components.

n_1	n_2		
	0	1	2
(a) Traces for $m \not\equiv 0 \pmod{3}$			
0	$3 + 3\omega$	$2 + \bar{\omega}$	$2\omega + \bar{\omega}$
1	$2 + \bar{\omega}$	$\bar{\omega}$	$1 + \omega$
2	$2\omega + \bar{\omega}$	$1 + \omega$	$\bar{\omega}$
n_1	n_2		
	0	1	2
(b) Traces for $m \equiv 0 \pmod{3}$			
0	$1 + \omega$	$1 + \omega$	$1 + \omega$
1	$1 + \omega$	$2 + \bar{\omega}$	$\bar{\omega}$
2	$1 + \omega$	$\bar{\omega}$	$2\omega + \bar{\omega}$

TABLE VIII. Degeneracy for different traces.

$\text{tr} \rho$	D_1	D_ω	$D_{\bar{\omega}}$	$\det N$
$3 + 3\omega$	$\frac{ \det N }{3} + 1$	$\frac{ \det N }{3} + 1$	$\frac{ \det N }{3} - 2$	$0 \pmod{3}$
$\bar{\omega}$	$\frac{ \det N - 1}{3}$	$\frac{ \det N - 1}{3}$	$\frac{ \det N + 2}{3}$	$1 \pmod{3}$
$1 + \omega$	$\frac{ \det N + 1}{3}$	$\frac{ \det N + 1}{3}$	$\frac{ \det N - 2}{3}$	$2 \pmod{3}$
$2 + \bar{\omega}$	$\frac{ \det N }{3} + 1$	$\frac{ \det N }{3} - 1$	$\frac{ \det N }{3}$	$0 \pmod{3}$
$2\omega + \bar{\omega}$	$\frac{ \det N }{3} - 1$	$\frac{ \det N }{3} + 1$	$\frac{ \det N }{3}$	$0 \pmod{3}$

For $\det N \equiv 2 \pmod{3}$, we express the determinant as,

$$4n_1n_2 \equiv \begin{cases} 2 \pmod{3}; & \text{if } m \equiv 0 \pmod{3}, \\ 0 \pmod{3}; & \text{if } m \equiv 1 \pmod{3}. \end{cases} \quad (174)$$

For $\det N \equiv 0 \pmod{3}$, we express the determinant as,

$$4n_1n_2 \equiv \begin{cases} 0 \pmod{3}; & \text{if } m \equiv 0 \pmod{3}, \\ 1 \pmod{3}; & \text{if } m \equiv 1 \pmod{3}. \end{cases} \quad (175)$$

The result is summarized in Table IX.

The fluxes are those with $\det N \equiv 0, 3 \pmod{12}$ for $\text{tr} \rho = 2 + \bar{\omega}$ and $\text{tr} \rho = 2\omega + \bar{\omega}$. For $\text{tr} \rho = 3 + 3\omega$, the possible determinants are $\det N \equiv 0, 9 \pmod{36}$. We can list the whole spectrum as shown in Tables X and XI. Table X agrees with Ref. [34].

G. Magnetized $T^4/\mathbb{Z}_6$

To construct $T^4/\mathbb{Z}_6$, we focus on the algebraic relation $((T_1 T_2)^{-1} S)^6 = \mathbf{1}_4$. According to Ref. [34], the complex structure moduli are fixed as

TABLE IX. All possible traces categorized by the mod 3 structure of the flux components.

		n_2		
n_1		0	1	2
(a) Traces for $\det N \equiv 1 \pmod{3}$ with $m \equiv 0 \pmod{3}$.				
0	$\times$	$\times$	$\times$	$\times$
1			$\bar{\omega}$	$\times$
2				$\bar{\omega}$
		n_2		
n_1		0	1	2
(b) Traces for $\det N \equiv 1 \pmod{3}$ with $m \not\equiv 0 \pmod{3}$.				
0	$\times$	$\times$	$\times$	$\times$
1			$\times$	$\bar{\omega}$
2				$\times$
		n_2		
n_1		0	1	2
(c) Traces for $\det N \equiv 2 \pmod{3}$ with $m \equiv 0 \pmod{3}$.				
0	$\times$	$\times$	$\times$	$\times$
1			$\times$	$1 + \omega$
2				$\times$
		n_2		
n_1		0	1	2
(d) Traces for $\det N \equiv 2 \pmod{3}$ with $m \not\equiv 0 \pmod{3}$.				
0	$1 + \omega$	$1 + \omega$	$1 + \omega$	$1 + \omega$
1			$\times$	$\times$
2				$\times$
		n_2		
n_1		0	1	2
(e) Traces for $\det N \equiv 0 \pmod{3}$ with $m \equiv 0 \pmod{3}$.				
0	$3 + 3\omega$	$2 + \bar{\omega}$	$2\omega + \bar{\omega}$	$2\omega + \bar{\omega}$
1		$\times$	$\times$	$\times$
2				$\times$
		n_2		
n_1		0	1	2
(f) Traces for $\det N \equiv 0 \pmod{3}$ with $m \not\equiv 0 \pmod{3}$.				
0	$\times$	$\times$	$\times$	$\times$
1			$2 + \bar{\omega}$	$\times$
2				$2\omega + \bar{\omega}$

$$\Omega_{(T_1 T_2)^{-1} S} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \quad (176)$$

The following identification corresponds to the $\mathbb{Z}_6$ twist:

TABLE X. $T^4/\mathbb{Z}_3$ spectrum for $\det N > 0$.

$\det N$	$\text{tr} \rho$	D_1	D_ω	$D_{\bar{\omega}}$
3	$2 + \bar{\omega}$	2	0	1
4	$\bar{\omega}$	1	1	2
7	$\bar{\omega}$	2	2	$\boxed{3}$
8	$1 + \omega$	$\boxed{3}$	$\boxed{3}$	2
11	$1 + \omega$	4	4	$\boxed{3}$
12	$2 + \bar{\omega}$	5	$\boxed{3}$	4
12	$2\omega + \bar{\omega}$	$\boxed{3}$	5	4
15	$2 + \bar{\omega}$	6	4	5
15	$2\omega + \bar{\omega}$	4	6	5
16	$\bar{\omega}$	5	5	6
19	$\bar{\omega}$	6	6	7
20	$1 + \omega$	7	7	6
23	$1 + \omega$	8	8	7
24	$2 + \bar{\omega}$	9	7	8
24	$2\omega + \bar{\omega}$	7	9	8
27	$3 + 3\omega$	10	10	7
27	$2 + \bar{\omega}$	10	8	9
28	$\bar{\omega}$	9	9	10

TABLE XI. $T^4/\mathbb{Z}_3$ spectrum for $\det N < 0$.

$\det N$	$\text{tr} \rho$	D_1	D_ω	$D_{\bar{\omega}}$
-9	$3 + 3\omega$	4	4	1
-9	$2 + \bar{\omega}$	4	2	$\boxed{3}$
-9	$2\omega + \bar{\omega}$	2	4	$\boxed{3}$
-12	$2 + \bar{\omega}$	5	$\boxed{3}$	4
-12	$2\omega + \bar{\omega}$	$\boxed{3}$	5	4
-21	$2 + \bar{\omega}$	8	6	7
-21	$2\omega + \bar{\omega}$	6	8	7
-24	$2 + \bar{\omega}$	9	7	8
-24	$2\omega + \bar{\omega}$	7	9	8
-33	$2 + \bar{\omega}$	12	10	11
-33	$2\omega + \bar{\omega}$	10	12	11

$$\vec{z} \sim \kappa \vec{z}, \quad \kappa = e^{\frac{\pi i}{3}}. \quad (177)$$

The transformation function and the modular transformation are given by

$$V(\vec{z}, \vec{\bar{z}}) = e^{\pi i \rho / 3}, \quad \rho = (0, 1, 2, 3, 4, 5),$$

$$\text{tr} \rho_+ = \sqrt{\det(iN^{-1} \Omega_{(T_1 T_2)^{-1} S})} \sum_{\vec{K} \in \Lambda_N} e^{\pi i \vec{K}^T N^{-1} \vec{K}}, \quad (178)$$

where we have used $\vec{J} = \vec{K} \in \Lambda_N$.

1. Positive chirality

The trace becomes

$$\text{tr}\rho_+ = \frac{e^{\pi i/6}}{\sqrt{\det N}} \sum_{\vec{K} \in \Lambda_N} e^{\pi i \vec{K}^T N^{-1} \vec{K}} = \omega e^{-2\pi i(n_1 K_1^2 + n_2 K_2^2 + m K_1 K_2)}. \quad (179)$$

The flux is the same as in the Z_3 case. We use Eq. (144) with $B = I$, $n_-^B = 0$, $n_-^N = 0$, and $g = 2$. Therefore we find $\text{tr}\rho = \omega = \kappa^2$. Also, we have the following formulas

$$\begin{aligned} D_1 + D_\kappa + D_{\kappa^2} + D_{\kappa^3} + D_{\kappa^4} + D_{\kappa^5} &= \det N, \\ D_1 + \kappa D_\kappa + \kappa^2 D_{\kappa^2} + \kappa^3 D_{\kappa^3} + \kappa^4 D_{\kappa^4} + \kappa^5 D_{\kappa^5} &= \text{tr}\rho_+ = \omega. \end{aligned} \quad (180)$$

Since $Z_6 = Z_2 \otimes Z_3$, we have the following Z_2 and Z_3 properties for some of the eigenvalues

$$\begin{aligned} D_1 + D_{\kappa^2} + D_{\kappa^4} &= N_{Z_2,+}, \\ D_1 + D_{\kappa^3} &= N_{Z_3,1}, \\ D_\kappa + D_{\kappa^4} &= N_{Z_3,\omega}, \\ D_{\kappa^2} + D_{\kappa^5} &= N_{Z_3,\bar{\omega}}. \end{aligned} \quad (181)$$

We find the following relation

$$D_1 - D_{\kappa^3} = D_{\kappa^4} - D_\kappa = D_{\kappa^2} - D_{\kappa^5} - 1, \quad (182)$$

where we have made use of the relations

$$\kappa^3 = -1, \quad \kappa^4 = -\kappa = \bar{\omega}, \quad \kappa^5 = -\kappa^2 = -\omega \quad (183)$$

to rewrite the trace as

$$(D_1 - D_{\kappa^3}) + \omega(D_{\kappa^2} - D_{\kappa^5}) + \bar{\omega}(D_{\kappa^4} - D_\kappa) = \omega. \quad (184)$$

Since the spectrum must correspond to that of Z_3 , and there is no additional conditions on the fluxes, we will have the same $\det N$ values available. The calculation is straightforward by rewriting

$$(2D_1 - N_{Z_3,1}) + \omega(2D_{\kappa^2} - N_{Z_3,\bar{\omega}}) + \bar{\omega}(2D_{\kappa^4} - N_{Z_3,\omega}) = \omega,$$

and using $1 + \omega + \bar{\omega} = 1$ as well as the Z_2 spectrum constraint.

2. Negative chirality

The trace is now

$$\begin{aligned} \text{tr}\rho_-((T_1 T_2)^{-1} S) &= s^{-1} \sqrt{\det(iN^{-1} \Omega'^{-1}_{(T_1 T_2)^{-1} S})} \\ &\times \sum_{\vec{K} \in \Lambda_N} e^{\pi i \vec{K}^T N^{-1} \vec{K}}. \end{aligned} \quad (185)$$

Then, the trace becomes

$$\text{tr}\rho_-((T_1 T_2)^{-1} S) = \frac{-s^{-1}}{\sqrt{-\det N}} \sum_{\vec{K} \in \Lambda_N} e^{\pi i \vec{K}^T N^{-1} \vec{K}}. \quad (186)$$

Using Eq. (144) with $B = I$, $n_-^B = 0$, $n_-^N = 1$ and $g = 2$, we get

$$\text{tr}\rho_-((T_1 T_2)^{-1} S) = s^{-1} e^{-2\pi i(n_1 K_1^2 + n_2 K_2^2 + m K_1 K_2)} = s^{-1}. \quad (187)$$

The spinor representation in z' becomes

$$V \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \sin^2 \phi_0 + \kappa^2 \cos^2 \phi_0 & (\kappa^2 - 1) \sin \phi_0 \cos \phi_0 & 0 \\ 0 & (\kappa^2 - 1) \sin \phi_0 \cos \phi_0 & \cos^2 \phi_0 + \kappa^2 \sin^2 \phi_0 & 0 \\ 0 & 0 & 0 & \kappa \end{pmatrix}. \quad (188)$$

Thus, we find $\text{tr}\rho_- = \kappa^5 = \omega$. We can calculate the spectrum using the following formulas:

$$\begin{aligned} D_1 + D_\kappa + D_{\kappa^2} + D_{\kappa^3} + D_{\kappa^4} + D_{\kappa^5} &= |\det N|, \\ D_1 + \kappa D_\kappa + \kappa^2 D_{\kappa^2} + \kappa^3 D_{\kappa^3} + \kappa^4 D_{\kappa^4} + \kappa^5 D_{\kappa^5} &= \text{tr}\rho_- = \omega, \end{aligned} \quad (189)$$

as well as

$$\begin{aligned} D_1 + D_{\kappa^2} + D_{\kappa^4} &= N_{Z_2,+}, \\ D_1 + D_{\kappa^3} &= N_{Z_3,1}, \\ D_\kappa + D_{\kappa^4} &= N_{Z_3,\omega}, \\ D_{\kappa^2} + D_{\kappa^5} &= N_{Z_3,\bar{\omega}}, \end{aligned} \quad (190)$$

or

$$(2D_1 - N_{Z_3,1}) + \omega(2D_{\kappa^2} - N_{Z_3,\bar{\omega}}) + \bar{\omega}(2D_{\kappa^4} - N_{Z_3,\omega}) = \omega, \quad (191)$$

where $N_{Z_2,+}$ is the even spectrum for negative chirality (odd sector for positive chirality).

TABLE XII. $T^4/\mathbb{Z}_6$ spectrum for $\det N > 0$.

$\det N$	D_1	D_κ	D_{κ^2}	D_{κ^3}	D_{κ^4}	D_{κ^5}
3	1	0	1	1	0	0
4	1	0	2	0	1	0
7	1	1	2	1	1	1
8	2	1	2	1	2	0
11	2	2	2	2	2	1
12	$\boxed{3}$	1	$\boxed{3}$	2	2	1
12	2	2	$\boxed{3}$	1	$\boxed{3}$	1
15	$\boxed{3}$	2	$\boxed{3}$	$\boxed{3}$	2	2
15	2	$\boxed{3}$	$\boxed{3}$	2	$\boxed{3}$	2
16	$\boxed{3}$	2	4	2	$\boxed{3}$	2
19	$\boxed{3}$	$\boxed{3}$	4	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$
20	4	$\boxed{3}$	4	$\boxed{3}$	4	4
23	4	4	4	4	4	$\boxed{3}$
24	5	$\boxed{3}$	5	4	4	$\boxed{3}$
24	4	4	5	$\boxed{3}$	5	$\boxed{3}$
27	5	5	4	5	5	$\boxed{3}$
27	5	4	5	5	4	4
28	5	4	6	4	5	6

3. Spectrum

The spectra are given in Tables XII and XIII (in agreement with Ref. [34]).

H. Magnetized $T^6/\mathbb{Z}_{12}$

1. Diagonalization of $T^6/\mathbb{Z}_{12}$

We review magnetized $T^6/\mathbb{Z}_{12}$ in Ref. [35]. $T^6/\mathbb{Z}_{12}$ is introduced by the following algebraic relation

$$(ST_1T_2T_3^{-1}T_5T_6)^{12} = \mathbf{1}_6, \quad (192)$$

and then we can describe the $\mathbb{Z}_{12}$ twist by $ST_1T_2T_3^{-1}T_5T_6$. Note that since $T^6/\mathbb{Z}_{12}$ has E_6 root lattice, all of the scale

TABLE XIII. $T^4/\mathbb{Z}_6$ spectrum for $\det N < 0$.

$\det N$	D_1	D_κ	D_{κ^2}	D_{κ^3}	D_{κ^4}	D_{κ^5}
-9	2	2	1	2	2	0
-9	2	1	2	2	1	1
-9	2	1	2	2	1	2
-12	2	2	2	3	1	2
-12	1	$\boxed{3}$	2	2	2	2
-21	4	$\boxed{3}$	4	4	$\boxed{3}$	$\boxed{3}$
-21	$\boxed{3}$	4	4	$\boxed{3}$	4	$\boxed{3}$
-24	4	4	4	5	$\boxed{3}$	4
-24	$\boxed{3}$	5	4	4	4	4
-33	6	6	6	7	5	6
-33	5	7	6	6	6	6

factors $2\pi r_i$ become 1. The integer fluxes N and complex structure moduli Ω_{12} are given by

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{11} & n_{13} \\ n_{13} & n_{13} & n_{11} + n_{12} - 2n_{13} \end{bmatrix},$$

$$\Omega_{12} = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{6}i & \frac{\sqrt{3}}{3}i & -\frac{1}{2} + \frac{\sqrt{3}}{6}i \\ \frac{\sqrt{3}}{3}i & -\frac{1}{2} - \frac{\sqrt{3}}{6}i & -\frac{1}{2} + \frac{\sqrt{3}}{6}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{6}i & -\frac{1}{2} + \frac{\sqrt{3}}{6}i & \frac{1}{2} - \frac{\sqrt{3}}{6}i \end{bmatrix}, \quad (193)$$

where z is identified to $\Omega_{12}z$. We can verify that the lattice corresponds to E_6 root lattice. Here, the integer fluxes N satisfy the F -term SUSY condition $(N\Omega_{12})^T = N\Omega_{12}$, and we impose $n_{11} \equiv n_{12} \equiv n_{13} \pmod{2}$ because of the modular T symmetry. When we consider the diagonalization of $F_{ij} = \pi[N^T(\text{Im } \Omega_{12})^{-1}]_{ij}$, we can find the following diagonalization matrix P that corresponds to the $SO(3)$ rotation

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{3}-\sqrt{3}} & \frac{1+\sqrt{3}}{2\sqrt{3}+\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{3}-\sqrt{3}} & \frac{1+\sqrt{3}}{2\sqrt{3}+\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}-\sqrt{3}} & \frac{1}{\sqrt{3}+\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_3 & -\sin \theta_3 \\ 0 & \sin \theta_3 & \cos \theta_3 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta_1 \cos \theta_2 & \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_2 \cos \theta_3 & \sin \theta_1 \cos \theta_2 \sin \theta_3 + \sin \theta_2 \cos \theta_3 \\ \sin \theta_1 & \cos \theta_1 \cos \theta_3 & -\cos \theta_1 \sin \theta_3 \\ -\cos \theta_1 \sin \theta_2 & \sin \theta_1 \sin \theta_2 \cos \theta_3 + \cos \theta_2 \sin \theta_3 & \cos \theta_2 \cos \theta_3 - \sin \theta_1 \sin \theta_2 \sin \theta_3 \end{bmatrix}, \quad (194)$$

where $\det P = 1$ and note that the P does not depend on the values of the background magnetic fluxes. Then Ω_{12} are diagonalized as

$$P^{-1}\Omega_{12}P = \text{diag}(e^{-2\pi i/3}, e^{-\pi i/6}, e^{5\pi i/6}). \quad (195)$$

From Eq. (194), we can assign $\theta_1 = \frac{3}{4}\pi$, $\theta_2 = 0$ for simplicity, so we find $q_1 = 1$ and $q_2 = 0$. We can just obtain the wave functions $\psi_{+--} = \psi_{-+-}$ and $\psi_{--+} = 0$ in magnetized $T^6/\mathbb{Z}_{12}$. That is, we can describe the following spinor generated by the fluxes on $T^6/\mathbb{Z}_{12}$

$$\Psi_+ = \begin{bmatrix} \psi_{+--} \\ \psi_{-+-} \\ \psi_{--+} \end{bmatrix} = \alpha_1 \begin{bmatrix} \psi \\ \psi \\ 0 \end{bmatrix}. \quad (196)$$

Meanwhile, we have defined the spinor in z' system, namely Ψ' with only ψ'_{+++} . In what follows, we will make use of Ψ' to obtain the number of zero-modes in magnetized $T^6/\mathbb{Z}_{12}$. Also, z' is also transformed as $\Omega'z'$ under the transformation $z \rightarrow \Omega z$. In the z' system, ψ'^J_{+++} , ($J \in \Lambda_N$) is transformed under $ST_1T_2T_3^{-1}T_5T_6$ as

$$\psi'^J_{+++}(\Omega'_{\text{twist}}z', \Omega'_{\text{twist}}) = \frac{\sqrt{\det[-i(\Omega'_{\text{twist}} + B)]}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'^K_{+++}(z', \Omega'_{\text{twist}}), \quad (197)$$

where $B = B_1 + B_2 - B_3 + B_5 + B_6$ and $\Omega'_{\text{twist}} = \text{Re}\Omega_{12} + iR \text{Im}\Omega_{12}$. In the z' system, the $\mathbb{Z}_N$ -twist boundary condition of spinor $\Psi'(z')$ is given by

$$\Psi'(\Omega'z') = S'V'\Psi'(z'), \quad (198)$$

where S' , V' denote the spinor representation of the $\mathbb{Z}_N$ -twist and transformation function in the z' system, respectively. Note that since we can also identify z' to $\Omega'z'$ where $\Omega'^N = \mathbf{1}_3$, and only ψ'_{+++} is excited in the z' system, ψ'_{+++} is invariant under the S' if we define the following V'

$$V' = e^{2\pi i \beta'}, \quad \text{where } \beta' N \equiv 0 \pmod{1} \quad \text{and} \quad \beta' \neq \beta \text{ in general.} \quad (199)$$

This is consistent with the Dirac equation under the $\mathbb{Z}_N$ twist in the z' system. Then, by use of the wave functions with positive chirality ($+++$) and the $\mathbb{Z}_N$ -twist boundary condition of the spinor in the z' system, we can find the number of zero modes in each $\mathbb{Z}_N$ sector when we take $U_1 = \text{diag}(+, -, -)$, $U_2 = \text{diag}(-, +, -)$, $U_3 = \text{diag}(-, -, +)$.

2. The number of zero modes in magnetized $T^6/\mathbb{Z}_{12}$

We analyze three-generation models satisfying both the F -term and the D -term SUSY conditions in magnetized $T^6/\mathbb{Z}_{12}$. We can find some three-generation models when we make use of the F -term and D -term SUSY conditions in z' , the $\mathbb{Z}_N$ -twist boundary condition of ψ'_{+++} , the modular transformation, and the U_i , ($i = 1, 2, 3$). For example, when we take $U_1 = \text{diag}(+, -, -)$, we can find the following modular transformation

$$\begin{aligned} \psi'^J_{+++}(\Omega'_{\text{twist}}z', \Omega'_{\text{twist}}) &= \frac{e^{\frac{5}{12}\pi i}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'^K_{+++}(z', \Omega'_{\text{twist}}). \end{aligned} \quad (200)$$

Then, we obtain the consistent fluxes N and the M

$$N = \begin{bmatrix} -13 & 5 & -3 \\ 5 & -13 & -3 \\ -3 & -3 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} -5 & 13 & 3 \\ 13 & -5 & 3 \\ 3 & 3 & 2 \end{bmatrix}, \quad (201)$$

where $\det N = 36$ and the zero-modes number in the $\mathbb{Z}_N$ sectors n_i , ($i = 0, 1, \dots, 11$) is

$$\begin{aligned} [n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] \\ = [9, 0, 4, 0, 9, 0, 2, 0, 9, 0, \boxed{3}, 0]. \end{aligned} \quad (202)$$

The others are

$$N = \begin{bmatrix} -3 & 3 & -1 \\ 3 & -3 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} -3 & 3 & 1 \\ 3 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \quad (203)$$

where $\det N = 12$ and each zero-modes number is

$$\begin{aligned} [n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] \\ = [\boxed{3}, 0, 2, 0, \boxed{3}, 0, 0, 0, \boxed{3}, 0, 1, 0]. \end{aligned} \quad (204)$$

When we take $U_2 = \text{diag}(-, +, -)$, we can find the modular transformation

$$\begin{aligned} \psi'_{+++}(\Omega'_{\text{twist}} z', \Omega'_{\text{twist}}) \\ = \frac{e^{-\frac{\pi i}{12}}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'_{+++}(z', \Omega'_{\text{twist}}). \end{aligned} \quad (205)$$

Then, we obtain the consistent fluxes N and the M

$$N = \begin{bmatrix} 1 & -5 & -1 \\ -5 & 1 & -1 \\ -1 & -1 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} \sqrt{3}-3 & \sqrt{3}+3 & \sqrt{3} \\ \sqrt{3}+3 & \sqrt{3}-3 & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & 0 \end{bmatrix}, \quad (206)$$

where $\det N = 36$ and each zero-modes number is

$$\begin{aligned} [n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] \\ = [5, 1, \boxed{3}, \boxed{3}, 4, 0, 5, \boxed{3}, \boxed{3}, 2, 4, \boxed{3}]. \end{aligned} \quad (207)$$

When we take $U_3 = \text{diag}(-, -, +)$, we find the modular transformation

$$\begin{aligned} \psi'_{+++}(\Omega'_{\text{twist}} z', \Omega'_{\text{twist}}) \\ = \frac{e^{-\frac{\pi i}{12}}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'_{+++}(z', \Omega'_{\text{twist}}). \end{aligned} \quad (208)$$

Then, we obtain the consistent fluxes N and the M

$$N = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} \sqrt{3}-3 & \sqrt{3}+3 & \sqrt{3} \\ \sqrt{3}+3 & \sqrt{3}-3 & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & 0 \end{bmatrix}, \quad (209)$$

where $\det N = 36$ and each zero-modes number is

$$\begin{aligned} [n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] \\ = [5, \boxed{3}, \boxed{3}, 2, 4, \boxed{3}, 5, 1, \boxed{3}, \boxed{3}, 4, 0]. \end{aligned} \quad (210)$$

In conclusion, we have analyzed zero-modes counting in magnetized $T^{2g}/\mathbb{Z}_N$ models. We have found some three-generation models in both $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_{12}$ orbifolds using the wave functions with various chiralities.

VII. CONCLUSION

We have studied and constructed the magnetized T^6 model taking the chirality in spinor components into account. We have discovered the spinor wave functions

with various chiralities under nonfactorizable background magnetic fluxes $\pi N^T (\text{Im } \Omega)^{-1}$. When we define M and impose the condition $N \text{Re } \Omega + i M \text{Im } \Omega \in \mathcal{H}^3$, we find that the wave functions converge and satisfy both the Dirac equation and the boundary conditions in magnetized T^6 . By use of the $R = N^{-1} M$ derived from the diagonalization for the magnetic fluxes, we define z' system where the background magnetic fluxes become positive definite. In the z' system, we find the Laplacian, its eigen equation, and the D -term SUSY condition. We also analyze the modular transformations of the wave functions under $Sp(6, \mathbb{Z})$ modular symmetry in the z' system. They play a crucial role in the analysis on the zero-mode number of the wave functions in magnetized $T^{2g}/\mathbb{Z}_N$. On the other hands, we have also discussed how to derive the Yukawa couplings taking both the chirality and the $SO(6)$ symmetry into account. We find that each chirality in different sectors can be changed by linear combinations of the fluxes if they are commutative. In other words, we can introduce the wave functions with different chiralities in each sector through the fluxes. Thus, we can calculate the Yukawa couplings in T^6 and describe them specifically. Lastly, we have researched the number of the zero modes in magnetized $T^{2g}/\mathbb{Z}_N$ to verify whether we can realize three-generation models or not. We have three-generation models when we consider the wave functions with various chiralities in magnetized $T^4/\mathbb{Z}_N$, ($N = 2, 3, 4, 6$), and $T^6/\mathbb{Z}_{12}$. It would be important to verify the consistency with the Atiyah-Singer index theorem on orbifolds [36–38], e.g., through the blowup with localized flux and curvature [70–72]. We would study it elsewhere. It is also important to apply our results to realize quark and lepton masses and their mixing angles.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: SPINOR REPRESENTATION UNDER PARITY TRANSFORMATION

We need to define the spinor representation of the parity transformation. Define the R transformation acting on a spinor in the following way: $\Psi' = \Lambda_{\frac{1}{2}} \Psi$.

The boundary condition in the parity transformed coordinates is expressed as

$$\Psi' \rightarrow S' V' \Psi' = \Lambda_{\frac{1}{2}} S V \Lambda_{\frac{1}{2}}^{-1} \Psi. \quad (\text{A1})$$

We can find $\Lambda_{\frac{1}{2}}$ by considering the parity invariance of the Dirac equation. First, we note that since the R transformation is defined as $R = P^T U P$, we can express the corresponding spinor representation as

$$\Lambda_{\frac{1}{2}} = P_R^{-1} U P_R, \quad (\text{A2})$$

where P_R is the spinor representation of rotation. To find the parity transformation matrix, we first transform the Dirac equation to the $\vec{w} = P\vec{z}$ coordinate system. Note that $\Gamma^w = \Gamma^z$. Since it must be invariant, we find,

$$(\Gamma^{z^i} D_{z^i} + \Gamma^{\bar{z}^i} D_{\bar{z}^i}) \Psi = 0 \rightarrow (\Gamma^{z^i} D_{w^i} + \Gamma^{\bar{z}^i} D_{\bar{w}^i}) P_R \Psi = 0. \quad (\text{A3})$$

We denote $P_R \Psi = \Psi'$. Since we are in the $\vec{w}$ system now, we can directly find the parity transformation matrix. We note that under parity $w^1 \rightarrow w^1$ and $w^2 \rightarrow \bar{w}^2$. Then the Dirac equation transforms like

$$\begin{aligned} &(\Gamma^{z^1} D_{w^1} + \Gamma^{\bar{z}^1} D_{\bar{w}^1} + \Gamma^{z^2} D_{w^2} + \Gamma^{\bar{z}^2} D_{\bar{w}^2}) \Psi' = 0, \\ &\rightarrow U_S (\Gamma^{z^1} D_{w^1} + \Gamma^{\bar{z}^1} D_{\bar{w}^1} + \Gamma^{z^2} D_{\bar{w}^2} + \Gamma^{\bar{z}^2} D_{w^2}) \Psi' = 0. \end{aligned} \quad (\text{A4})$$

To make this invariant, we need to find a matrix U_S that satisfies the following conditions:

$$\begin{aligned} \{U_S, \Gamma^{z^1}\} &= 0, \\ \{U_S, \Gamma^{\bar{z}^1}\} &= 0, \\ U_S \Gamma^{z^2} &= -\Gamma^{\bar{z}^2} U_S, \\ U_S \Gamma^{\bar{z}^2} &= -\Gamma^{z^2} U_S. \end{aligned} \quad (\text{A5})$$

We find

$$U_S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (\text{A6})$$

This gives us

$$-(\Gamma^{z^1} D_{w^1} + \Gamma^{\bar{z}^1} D_{\bar{w}^1} + \Gamma^{z^2} D_{\bar{w}^2} + \Gamma^{\bar{z}^2} D_{w^2}) U_S \Psi' = 0. \quad (\text{A7})$$

Rotating back to the $\vec{z}$ system, we find the full transformation of the spinor to be

$$\Psi \xrightarrow{R} P_R^{-1} U_S P_R \Psi. \quad (\text{A8})$$

We can then express the boundary conditions as,

$$S' V' = -P_R^{-1} U_S P_R S V P_R^{-1} U_S P_R, \quad (\text{A9})$$

where we have used $U_S^{-1} = -U_S$. We could equivalently redefine the gamma matrices $\Gamma^{z^2} \leftrightarrow \Gamma^{\bar{z}^2}$, which would change the chirality matrix but keep the order of spinor components. Specifically, the chirality matrix can be written as

$$\Gamma^5 = \text{diag}(-1, +1, +1, -1). \quad (\text{A10})$$

The final spectrum is the same with both approaches.

APPENDIX B: T^4/Z_N WITH DIAGONAL TWISTS

We would like to derive the spinor representation of the R transformation. The spinor representation of $\text{SO}(4)$ in real space is

$$S = e^{\{\frac{1}{2} i \omega_{ij} \Sigma^{ij}\}}. \quad (\text{B1})$$

Let us denote the axes $\text{Re } z^1, \text{Im } z^1, \text{Re } z^2, \text{Im } z^2$ by $i = 1, 2, 3, 4$. Then only the 13 and 24 components are nonzero. We want the commutators

$$\Sigma^{ij} = \frac{1}{4} i [\Gamma^i, \Gamma^j], \quad (\text{B2})$$

where the gamma matrices in real space are

$$\begin{aligned} \Gamma^1 &= \frac{1}{2} (\Gamma^{z^1} + \Gamma^{\bar{z}^1}) = \sigma^1 \otimes \sigma^3, \\ \Gamma^2 &= \frac{1}{2i} (\Gamma^{z^1} - \Gamma^{\bar{z}^1}) = \sigma^2 \otimes \sigma^3, \\ \Gamma^3 &= \frac{1}{2} (\Gamma^{z^2} + \Gamma^{\bar{z}^2}) = I \otimes \sigma^1, \\ \Gamma^4 &= \frac{1}{2i} (\Gamma^{z^2} - \Gamma^{\bar{z}^2}) = I \otimes \sigma^2. \end{aligned} \quad (\text{B3})$$

Then we find the nonzero components of Σ^{ij}

$$\begin{aligned} \Sigma^{13} &= -\frac{1}{2} \sigma^1 \otimes \sigma^2, \\ \Sigma^{24} &= \frac{1}{2} \sigma^2 \otimes \sigma^1. \end{aligned} \quad (\text{B4})$$

Since $\omega_{13} = -\omega_{31} = \phi_0$ and $\omega_{24} = -\omega_{42} = \phi_0$ we find

$$P_R = e^{\{i\phi_0(\sigma^2 \otimes \sigma^1 - \sigma^1 \otimes \sigma^2)\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi_0 & -\sin \phi_0 & 0 \\ 0 & \sin \phi_0 & \cos \phi_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B5})$$

Thus the spinor representation of the R transformation becomes

$$\Lambda_{\frac{1}{2}} = P_R^T U P_R = \begin{pmatrix} 0 & -\sin \phi_0 & \cos \phi_0 & 0 \\ \sin \phi_0 & 0 & 0 & \cos \phi_0 \\ -\cos \phi_0 & 0 & 0 & \sin \phi_0 \\ 0 & -\cos \phi_0 & -\sin \phi_0 & 0 \end{pmatrix}. \quad (\text{B6})$$

The full boundary condition term is

$$S'V' = -\Lambda_{\frac{1}{2}}SV\Lambda_{\frac{1}{2}} = V \begin{pmatrix} \alpha_1 s_0^2 + \alpha_2 c_0^2 & 0 & 0 & (\alpha_1 - \alpha_2)s_0 c_0 \\ 0 & s_0^2 + \beta c_0^2 & (\beta - 1)s_0 c_0 & 0 \\ 0 & (\beta - 1)s_0 c_0 & c_0^2 + \beta s_0^2 & 0 \\ (\alpha_1 - \alpha_2)s_0 c_0 & 0 & 0 & \alpha_1 c_0^2 + \alpha_2 s_0^2 \end{pmatrix}, \quad (\text{B7})$$

where $\alpha_1 = e^{2\pi i k_1/N}$, $\alpha_2 = e^{2\pi i k_2/N}$, $\beta = e^{2\pi i (k_1 + k_2)/N}$, $s_0 = \sin \phi_0$, $c_0 = \cos \phi_0$. These correspond to the transformation of coordinates under twists

$$\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 z_1 \\ \alpha_2 z_2 \end{pmatrix}. \quad (\text{B8})$$

Due to positive flux eigenvalues, the Dirac spinor in z' coordinates is

$$\Psi' = \begin{pmatrix} \psi'_{++} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (\text{B9})$$

This makes the boundary conditions

$$\Psi' \rightarrow V \begin{pmatrix} (\alpha_1 \sin^2 \phi_0 + \alpha_2 \cos^2 \phi_0) \psi'_{++} \\ 0 \\ 0 \\ (\alpha_1 - \alpha_2) \sin \phi_0 \cos \phi_0 \psi'_{++} \end{pmatrix}. \quad (\text{B10})$$

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- [1] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, *Nucl. Phys.* **B261**, 678 (1985).
[2] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, *Nucl. Phys.* **B274**, 285 (1986).
[3] E. Witten, *Phys. Lett.* **B149**, 351 (1984).
[4] C. Bachas, [arXiv:hep-th/9503030](https://arxiv.org/abs/hep-th/9503030).
[5] M. Berkooz, M. R. Douglas, and R. G. Leigh, *Nucl. Phys.* **B480**, 265 (1996).
[6] R. Blumenhagen, L. Goerlich, B. Kors, and D. Lust, *J. High Energy Phys.* **10** (2000) 006.
[7] C. Angelantonj, I. Antoniadis, E. Dudas, and A. Sagnotti, *Phys. Lett. B* **489**, 223 (2000).
[8] R. Blumenhagen, B. Kors, and D. Lust, *J. High Energy Phys.* **02** (2001) 030.
[9] D. Cremades, L. E. Ibanez, and F. Marchesano, *J. High Energy Phys.* **05** (2004) 079.

- [10] I. Antoniadis, A. Kumar, and B. Panda, *Nucl. Phys.* **B823**, 116 (2009).
- [11] L. De Angelis, R. Marotta, F. Pezzella, and R. Troise, *Proc. Sci. Corfu2012* (**2013**) 108 [arXiv:1304.0978].
- [12] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, *J. High Energy Phys.* **06** (2009) 080.
- [13] T. Kobayashi and S. Nagamoto, *Phys. Rev. D* **96**, 096011 (2017).
- [14] S. Kikuchi, T. Kobayashi, and H. Uchida, *Phys. Rev. D* **104**, 065008 (2021).
- [15] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, *Phys. Rev. D* **97**, 116002 (2018).
- [16] S. Kikuchi, T. Kobayashi, S. Takada, T. H. Tatsuishi, and H. Uchida, *Phys. Rev. D* **102**, 105010 (2020).
- [17] T. Kobayashi and S. Tamba, *Phys. Rev. D* **99**, 046001 (2019).
- [18] H. Ohki, S. Uemura, and R. Watanabe, *Phys. Rev. D* **102**, 085008 (2020).
- [19] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, and H. Uchida, *J. High Energy Phys.* **11** (2020) 101.
- [20] Y. Almumin, M. C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz, and S. Shukla, *J. High Energy Phys.* **05** (2021) 078.
- [21] S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, and H. Uchida, *Phys. Rev. D* **109**, 065011 (2024).
- [22] T. Kobayashi, K. Nasu, R. Nishida, H. Otsuka, and S. Takada, *J. High Energy Phys.* **12** (2024) 128.
- [23] H. Abe, T. Kobayashi, and H. Ohki, *J. High Energy Phys.* **09** (2008) 043.
- [24] T. H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, *J. High Energy Phys.* **01** (2014) 065.
- [25] H. Abe, K. S. Choi, T. Kobayashi, and H. Ohki, *Nucl. Phys.* **B814**, 265 (2009).
- [26] T. H. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, *Nucl. Phys.* **B894**, 374 (2015).
- [27] H. Abe, T. Kobayashi, H. Ohki, A. Oikawa, and K. Sumita, *Nucl. Phys.* **B870**, 30 (2013).
- [28] H. Abe, T. Kobayashi, K. Sumita, and Y. Tatsuta, *Phys. Rev. D* **90**, 105006 (2014).
- [29] Y. Fujimoto, T. Kobayashi, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, *Phys. Rev. D* **94**, 035031 (2016).
- [30] W. Buchmuller and J. Schweizer, *Phys. Rev. D* **95**, 075024 (2017).
- [31] W. Buchmuller and K. M. Patel, *Phys. Rev. D* **97**, 075019 (2018).
- [32] K. Hoshiya, S. Kikuchi, T. Kobayashi, and H. Uchida, *Phys. Rev. D* **106**, 115003 (2022).
- [33] S. Kikuchi, T. Kobayashi, K. Nasu, and H. Uchida, *J. High Energy Phys.* **08** (2022) 256.
- [34] S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, and H. Uchida, *J. High Energy Phys.* **06** (2023) 013.
- [35] S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, and H. Uchida, *Phys. Rev. D* **108**, 036005 (2023).
- [36] M. Sakamoto, M. Takeuchi, and Y. Tatsuta, *Phys. Rev. D* **103**, 025009 (2021).
- [37] T. Kobayashi, H. Otsuka, M. Sakamoto, M. Takeuchi, Y. Tatsuta, and H. Uchida, *Phys. Rev. D* **107**, 075032 (2023).
- [38] S. Aoki and M. Takeuchi, *Phys. Rev. D* **110**, 105013 (2024).
- [39] D. G. Markushevich, M. A. Olshanetsky, and A. M. Perelomov, *Commun. Math. Phys.* **111**, 247 (1987).
- [40] L. E. Ibanez, J. Mas, H. P. Nilles, and F. Quevedo, *Nucl. Phys.* **B301**, 157 (1988).
- [41] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono, and K. Tanioka, *Nucl. Phys.* **B341**, 611 (1990).
- [42] T. Kobayashi and N. Ohtsubo, *Int. J. Mod. Phys. A* **09**, 87 (1994).
- [43] D. Lust, S. Reffert, W. Schulgin, and S. Stieberger, *Nucl. Phys.* **B766**, 68 (2007).
- [44] D. Lust, S. Reffert, E. Scheidegger, W. Schulgin, and S. Stieberger, *Nucl. Phys.* **B766**, 178 (2007).
- [45] T. Jeric, T. Kobayashi, K. Nasu, and S. Takada, arXiv:2507.05645.
- [46] C. L. Siegel, *Am. J. Math.* **65**, 1 (1943).
- [47] H. Abe, T. Kobayashi, H. Ohki, and K. Sumita, *Nucl. Phys.* **B863**, 1 (2012).
- [48] G. J. Ding, F. Feruglio, and X. G. Liu, *J. High Energy Phys.* **01** (2021) 037.
- [49] J. Igusa, *Theta Functions*, Grundlehren der Mathematischen Wissenschaften 194 (Springer-Verlag, Berlin, Heidelberg, New York, 1972).
- [50] E. Freitag, *Siegelsche Modulformen*, Grundlehren der Mathematischen Wissenschaften 254 (Springer-Verlag, Berlin, Heidelberg, New York, 1983).
- [51] E. Freitag, *Singular Modular Forms and Theta Relations*, Lecture Notes in Mathematics Vol. 1487 (Springer-Verlag, Berlin, 1991).
- [52] H. Klingen, *Introductory Lecture on Siegel Modular Forms*, Cambridge Stud. Adv. Math., Vol. 20 (Cambridge University Press, 1990).
- [53] J. H. Bruinier, G. V. D. Geer, G. Harder, and D. Zagier, *The 1-2-3 of Modular Forms*, Universitext (Springer, Berlin, Heidelberg, 2008).
- [54] J. D. Fay, *Theta Functions on Riemann Surfaces*, Lecture Notes in Mathematics Vol. 352 (Springer-Verlag, Berlin, Heidelberg, New York, 1973).
- [55] D. Mumford, *Tata Lecture on Theta I* (Birkhäuser, 1984).
- [56] F. Feruglio, in *From My Vast Repertoire ...: Guido Altarelli's Legacy*, edited by A. Levy, S. Forte, and G. Ridolfi (World Scientific, Singapore, 2019), pp. 227–266.
- [57] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, *Phys. Rev. D* **98**, 016004 (2018).
- [58] J. T. Penedo and S. T. Petcov, *Nucl. Phys.* **B939**, 292 (2019).
- [59] J. C. Criado and F. Feruglio, *SciPost Phys.* **5**, 042 (2018).
- [60] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, *J. High Energy Phys.* **11** (2018) 196.
- [61] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* **04** (2019) 005.
- [62] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, *J. High Energy Phys.* **04** (2019) 174.
- [63] F. J. de Anda, S. F. King, and E. Perdomo, *Phys. Rev. D* **101**, 015028 (2020).
- [64] H. Okada and M. Tanimoto, *Phys. Lett. B* **791**, 54 (2019).

- [65] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, [Phys. Lett. B](#) **794**, 114 (2019).
- [66] P. P. Novichkov, S. T. Petcov, and M. Tanimoto, [Phys. Lett. B](#) **793**, 247 (2019).
- [67] T. Kobayashi and M. Tanimoto, [Int. J. Mod. Phys. A](#) **39**, 2441012 (2024).
- [68] G. J. Ding and S. F. King, [Rep. Prog. Phys.](#) **87**, 084201 (2024).
- [69] S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, and H. Uchida, [J. High Energy Phys.](#) **04** (2024) 045.
- [70] T. Kobayashi, H. Otsuka, and H. Uchida, [J. High Energy Phys.](#) **08** (2019) 046.
- [71] T. Kobayashi, H. Otsuka, and H. Uchida, [J. High Energy Phys.](#) **03** (2020) 042.
- [72] S. Kikuchi, T. Kobayashi, K. Nasu, H. Otsuka, S. Takada, and H. Uchida, [J. High Energy Phys.](#) **04** (2023) 003.