

Revisiting the gluon density from the Balitsky-Kovchegov equation with kinematical constraints and large x terms

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We perform analysis of the small x nonlinear evolution equation formulated in momentum space supplemented by higher order terms. The equation is defined in the wide range of transverse momentum and longitudinal momentum fractions extending previous studies performed in Kutak and Kwiecinski [Screening effects in the ultrahigh-energy neutrino interactions, *Eur. Phys. J. C* **29**, 521 (2003).] and in Kutak and Stasto [Unintegrated gluon distribution from modified BK equation, *Eur. Phys. J. C* **41**, 343 (2005).]. The linear part of the equation is motivated by the renormalization group improved small x approach, which accounts for resummation of higher orders, and includes collinear splitting functions and kinematical constraints. The solution to the equation is then used to perform the fit to deep inelastic scattering reduced cross section data.

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I. INTRODUCTION

Present day high energy accelerators, like the Large Hadron Collider, allow for the exploration of the new kinematic regime in particle interactions. Tests of the Standard Model are thus possible with unprecedented accuracy in the region where the total center-of-mass energy is much higher than the masses of the incoming hadrons. For the precise description of the processes in these collisions, a detailed knowledge of the partonic structure of the incoming hadrons is necessary. The information is encoded in the *parton distribution functions* (PDFs), which in the usual collinear factorization framework are obtained from the solution to the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [1–4]. These equations allow for the evolution of *integrated* PDFs with some hard scale Q^2 and resum the large logarithms $\ln Q^2/Q_0^2$, where Q_0 is some reference scale. The splitting functions for DGLAP evolution are known up to Next-to-Next-to-Leading-Order [5,6] and recent intense theoretical efforts aim to push the accuracy beyond that level.

An alternative framework for the calculation of the processes in the high energy limit is given by the k_T or *high energy factorization* [7,8], where the structure of the hadron is encoded in the *unintegrated* gluon distribution function. This is usually referred to as the high-energy limit or the Regge limit, with $s \gg |t|, m^2$, where s is the center-of-mass energy squared, t is the momentum transfer squared, and m the typical masses of the produced particles. The corresponding evolution is governed by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution equation [9–11], which sums up the powers of logarithms $\ln s/s_0$ where s_0 is a reference scale.

For the case of the deep inelastic scattering (DIS) process of scattering electrons off protons, the energy scale s_0 is taken to be equal to the (negative) virtuality of the photon Q^2 . Therefore, the BFKL resums large logarithms of $\ln 1/x$, where $x \simeq Q^2/W^2$ is the Bjorken x variable with W^2 being the square of the photon-proton energy. The solution to the BFKL equation results in the powerlike growth of the gluon density at small x and thus of the corresponding cross sections. The BFKL evolution equation is known up to next-to-leading logarithmic (NLL) [12,13] accuracy in QCD and up to next-to-next-to-leading logarithmic level in $N = 4$ supersymmetric Yang-Mills theory [14–16]. The leading logarithmic (LL) solution gives the powerlike growth of the gluon density with the decreasing x as $x^{-\omega_P}$, with the LL intercept $\omega_P = 4 \ln 2 \frac{\alpha_s N_c}{\pi} \sim 0.5$. The solution at LL leads to the growth of structure functions that is too steep for the

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experimental data at Hadron-Electron Ring Accelerator (HERA), e.g., [17]. The NLL order corrections turned out to be large and negative leading to instabilities and thus triggered the need for the resummation of the higher order terms in order to stabilize the results.

The major ingredients of the resummation we implement in this paper are the kinematical constraints, DGLAP splitting function, and the running of the strong coupling. One of the first approaches, by Kwieciński-Martin-Stasto (KMS) [18], used the unified BFKL and DGLAP evolution. In this work the unified system of equations for the unintegrated gluon density was constructed, which was based on the BFKL evolution equation supplemented with the kinematical constraint and the nonsingular DGLAP evolution terms. Coupled with the equation for quarks, this system was shown to produce the gluon distribution which could be fitted to the structure function data from HERA, with only two free parameters. It contained both Q^2 and x evolutions on equal footing. Based on this system of unified equations, a resummation framework was developed by Ciafaloni-Colferai-Salam-Stasto (CCSS) in a series of works [19–24]. In that approach the KMS [18] formulation was supplemented by the NLL terms of the BFKL, with appropriate subtractions, as well as the anticollinear terms. Recently, the CCSS resummation was applied to the structure functions at HERA, and a good description of the data at small values of x was obtained [25].

In the region of very small x , there are additional effects that need to be taken into account. When the gluon density becomes very high, recombination of gluons will start to play an important role, leading to the taming of the rapid growth. This is referred to as *parton saturation* [26,27]. The effective theory which describes the limit of high energy and density QCD is the color glass condensate [28–30], with the corresponding Jalilian-Marian-Iancu-Weigert-Leonidov-Kovner (JIMWLK) evolution equations [31–36]. An equivalent approach is based on the operator expansion at high energy, together with the corresponding Balitsky hierarchy equations [37,38]. When the large color limit N_C is taken, this hierarchy allows for the decoupling of the nonlinear equation, the Balitsky-Kovchegov (BK) evolution equation [38–40] for the dipole scattering amplitude. It was later shown that the subleading- N_C corrections was very small, resulting in only 0.1% difference between the JIMWLK and BK simulations [41]. The BK equation contains the nonlinear term responsible for the gluon recombination, and the solution can be characterized by the presence of the dynamically generated energy dependent saturation scale $Q_s(x)$, which divides the region between the dilute and dense regimes.

The BK equation is originally formulated for the dipole amplitude, and thus, it is written in the coordinate space [37,39,40]. Recently, it has been generalized and solved at

next-to-leading order (NLO) accuracy [42,43]. In papers [44–46] the BK evolution equation has been rederived in the momentum space, and in [47] this nonlinear term was included in the KMS [18] formulation. The gluon density obtained by solving this equation was used to successfully describe the F_2 structure function data and was used in numerous phenomenological applications [48–52]. One assumption of [18,47] was that the solution was truncated by an infrared cutoff and extrapolated “by hand” to lower transverse momentum k_T . On the other hand, it is customary to solve the BK evolution equation for large dipole sizes, when using its solutions to fit to the structure function data [47], where the nonperturbative contribution is usually taken automatically into account by integration over the large dipole sizes with the flat dipole cross section, see, e.g., [53–56].

In this paper we extend the Kutak-Sapeta (KS) approach of [47] and solve the nonlinear BK equation in momentum space including running coupling, kinematical constraint in the linear term, and the nonsingular part of the DGLAP splitting function for both collinear and anticollinear regions. The nonlinear term regulates the behavior at very low transverse momenta, and therefore, it allows for extending the solution to this region. Unlike other approaches for the resummed BK [57], this formulation in momentum space allows for the inclusion of the full DGLAP gluon splitting function in the evolution on an equal footing to the BFKL kernel. The initial condition is given by a form motivated by the momentum space Golec-Biernat and Wusthoff (GBW) model [58], and the fit is performed to the reduced cross section data from HERA [59], and a consistent description is achieved. This formulation can be easily extended to include the full CCSS resummation for the linear term, as it has been done in [25], and also for the case of nuclei, by rescaling the radius parameter by $A^{1/3}$ for the nonlinear term. Considering the numerical aspects, we iteratively solve an integro-differential evolution equation, Eq. (1), for consecutive values of x starting from $x = 1$. Our numerical algorithm is implemented in CUDA C, an extension to the C programming language, that allows usage of graphics processing units for accelerated general-purpose processing. We enabled parallel evaluation of the equation (1) with respect to k^2 . Moreover, the converged part of the solution for previous values of x is used in integration with respect to the next value of x ; parts of DGLAP integrals are precomputed and tabulated. The aforementioned optimizations allowed for a considerable speedup compared to a sequential code.

The structure of the paper is as follows: In Sec. II, we begin with the LL BK equation in momentum space and outline the higher-order resummation improvements implemented in the equation. Section III introduces the k_T factorization framework, which allows us to compute the structure functions. In Sec. IV, we present our fits to the HERA data and analyze the behavior of the resulting gluon density. Finally, we conclude in Sec. V.

II. LL BK IN MOMENTUM SPACE EQUATION AND RESUMMATIONS

A. LL BK equation in momentum space

The BK equation was originally derived in the coordinate space [38–40]. It can be reformulated in the momentum space [44,60]. The equation for unintegrated gluon density $\mathcal{F}(x, k^2)$ as a function of longitudinal momentum fraction x and transverse momentum k reads,

$$\mathcal{F}(x, k^2) = \mathcal{F}^{(0)}(x, k^2) + \int_x^1 \frac{dz}{z} \int dk'^2 \bar{\alpha}_s(k^2, k'^2) \left[\frac{\mathcal{F}\left(\frac{x}{z}, k'^2\right)}{|k^2 - k'^2|} - \frac{k^2}{k'^2} \frac{\mathcal{F}\left(\frac{x}{z}, k^2\right)}{|k^2 - k'^2|} + \frac{k^2}{k'^2} \frac{\mathcal{F}\left(\frac{x}{z}, k^2\right)}{\sqrt{k^4 + 4k'^4}} \right] - \frac{2\pi^2}{N_c^2 R^2} \int_x^1 \frac{dz}{z} \left\{ \left[\int_{k^2}^\infty \frac{dl^2}{l^2} \bar{\alpha}_s(k^2, l^2) \mathcal{F}\left(\frac{x}{z}, l^2\right) \right]^2 + \mathcal{F}\left(\frac{x}{z}, k^2\right) \int_{k^2}^\infty \frac{dl^2}{l^2} \bar{\alpha}_s^2(k^2, l^2) \ln\left(\frac{l^2}{k^2}\right) \mathcal{F}\left(\frac{x}{z}, l^2\right) \right\}, \quad (1)$$

where $\mathcal{F}^{(0)}$ is starting distribution. The linear part of the equation (the integral in the first line) corresponds to the angular averaged BFKL kernel whose real part represents emission of gluons and virtual corresponding to the reggeization. The nonlinear part (the second line) accounts for recombination of gluons and is given by triple Pomeron vertex, here angular averaged [45]. The strength of the nonlinear term is controlled by the effective radius of proton or nucleus R , which will be one of the fitting parameters. In the coordinate space based derivation, this parameter originates from the assumption that the target is a large homogeneous disk of radius R . After performing a Fourier transform from coordinate space to momentum space, integrating over the impact parameter, and normalizing the resulting gluon density so that $\int d^2b \mathcal{F}(x, k^2, b) = \mathcal{F}(x, k^2)$, the nonlinear term gains the $1/R^2$ factor.¹ In the above the rescaled coupling constant is $\bar{\alpha}_s(k^2, k'^2) = N_c \alpha_s(\max(k^2, k'^2))/\pi$ with number of colors $N_c = 3$. In the LO the strong coupling is fixed in the high energy limit; it starts to run at NLO order. The NLL calculation [12,13] suggests that the natural scale as the argument of the running coupling is the virtuality of the emitted gluon $q^2 = (k - k')^2$ (see Fig. 7), see also a discussion in [23]. Here we use a choice of $\max(k'^2, k^2)$, which is numerically close to the above choice and also symmetric with respect to k^2, k'^2 . In the present calculation we aim to solve Eq. (1) down to very low transverse momenta. Therefore, we need some form of regularization of the strong coupling. We shall take the parametrization [61,62]

$$\alpha_s(k^2) = \frac{12\pi}{33 - 2N_f} F(k, k_{fr}, \Lambda_{\text{QCD}}), \quad (2)$$

where

$$F(k, k_{fr}, \Lambda_{\text{QCD}}) = \left[\log \left(\frac{k^2 + k_{fr}^2}{\Lambda_{\text{QCD}}^2} \right) \right]^{-1} - \frac{\Lambda_{\text{QCD}}^2}{k^2 + k_{fr}^2 - \Lambda_{\text{QCD}}^2}, \quad (3)$$

with number of flavors $N_f = 4$ and parameter $k_{fr}^2 = 1.4 \text{ GeV}^2$ obtained by fitting the α_s from [63]. The formula follows from studies of analyticity properties of QCD coupling constant and was recently used in successful description of F_2 at small values of Q^2 [64]. Furthermore, it gives smooth extrapolation of the strong coupling to small transverse momenta.

In the following we shall illustrate basic properties and behavior of the solution to Eq. (1) at lowest order, including only the running coupling.

In Figs. 1 and 2 we present the solution of Eq. (1) using the initial condition given by input motivated by the GBW model,

$$\mathcal{F}^{(0)}(x, k^2) = (1 - x)^\alpha k^2 e^{-k^2/Q_s^2}. \quad (4)$$

In the above we set the saturation scale $Q_s^2 = 1 \text{ GeV}^2$ and $\alpha = 1$, where we take $R^2 \approx 2.55 \text{ GeV}^{-2}$ in Eq. (1), which is the result of the our later reduced cross section fit in Sec. IV. The factor $(1 - x)^\alpha$ proposed in [65] regulates the behavior

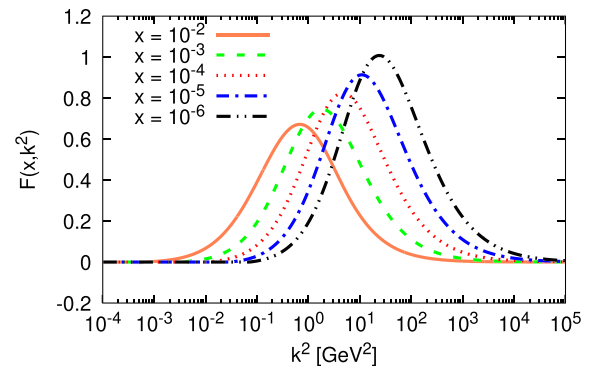


FIG. 1. Solution of BK equation from Eq. (1) as function of k^2 for various values of x .

¹For more details see recent review [52].

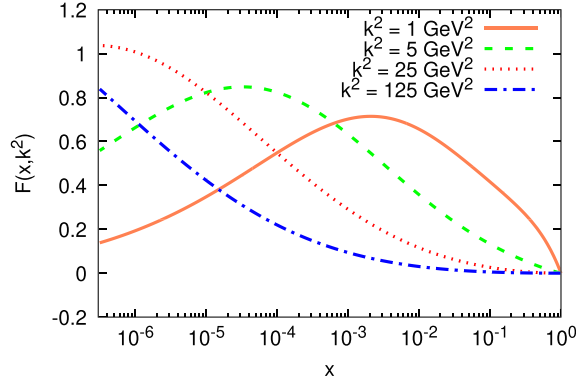


FIG. 2. Solution of BK equation from Eq. (1) as function of x for various values of k^2 .

of the starting distribution at $x = 1$. This is necessary especially in the case when DGLAP terms are included, as studied later in this section. In Fig. 1 the solution is shown as a function of the transverse momentum for several values of x . As x becomes lower, the typical value of transverse momentum characterized by the saturation scale becomes larger and larger. In Fig. 2, x dependence is illustrated and we observe that, for small transverse momentum, the gluon

density has a valencelike behavior at small x . As k^2 becomes larger than the saturation scale, the falling part of gluon density happens at lower x .

B. Higher order improvements to the BK evolution

The equation discussed above is LL in $\alpha_s \ln 1/x$ and subject to large corrections [12,13,66] at higher orders. Below we will use the prescription developed by [21,23]. The equation with resummed higher order corrections assumes the schematic form

$$\mathcal{F}(x, k^2) = \mathcal{F}^{(0)}(x, k^2) + \mathcal{K}_{\text{res}} \otimes \mathcal{F}(x, k^2) - \mathcal{V} \otimes \mathcal{F}^2(x, k^2), \quad (5)$$

with

$$\begin{aligned} \mathcal{K}_{\text{res}} \otimes \mathcal{F} \equiv & \mathcal{K}_0^{\text{kc}}(z; k, k') \otimes \mathcal{F}\left(\frac{x}{z}, k'\right) \\ & + \mathcal{K}_0^{\text{coll}}(z; k, k') \otimes \mathcal{F}\left(\frac{x}{z}, k'\right). \end{aligned} \quad (6)$$

The first term in Eq. (6) is

$$\mathcal{K}_0^{\text{kc}}(z; k, k') \otimes \mathcal{F}\left(\frac{x}{z}, k'\right) = \int_x^1 \frac{dz}{z} \int dk'^2 \bar{\alpha}_s(k^2, k'^2) \left[\frac{\mathcal{F}\left(\frac{x}{z}, k'^2\right)}{|k^2 - k'^2|} \Theta\left(\frac{k^2}{z} - k'^2\right) - \frac{k^2}{k'^2} \frac{\mathcal{F}\left(\frac{x}{z}, k^2\right)}{|k^2 - k'^2|} + \frac{k^2}{k'^2} \frac{\mathcal{F}\left(\frac{x}{z}, k^2\right)}{\sqrt{k^4 + 4k'^4}} \right]. \quad (7)$$

The kinematical (or consistency) constraint $\Theta\left(\frac{k^2}{z} - k'^2\right)$ term is implemented onto the real emissions only. It is here asymmetric, which corresponds to the asymmetric scale choice suitable for the DIS problem we are considering. It is implemented as

$$k'^2 \leq \frac{k^2}{z}. \quad (8)$$

In Figs. 3 and 4 we show the solution of the BK equation with kinematical constraint. We see that the normalization of gluon density changed almost by a factor of 2 as compared to the running coupling BK gluon in Figs. 1 and 2. Growth with decreasing x is slowed down and the valence like behavior is observed for low values of k^2 only.²

The second contribution in (6) is

$$\begin{aligned} \mathcal{K}_0^{\text{coll}}(z; k, k') \otimes \mathcal{F}\left(\frac{x}{z}, k'\right) = & \int_x^1 \frac{dz}{z} \int_0^{k^2} \frac{dk'^2}{k^2} \bar{\alpha}_s(k^2) z \tilde{P}_{gg}(z) \mathcal{F}\left(\frac{x}{z}, k'\right) \\ & + \int_x^1 \frac{dz}{z} \int_{k^2}^{k^2/z} \frac{dk'^2}{k'^2} \bar{\alpha}_s(k'^2) z \frac{k'^2}{k^2} \tilde{P}_{gg}\left(z \frac{k'^2}{k^2}\right) \mathcal{F}\left(\frac{x}{z}, k'\right). \end{aligned} \quad (9)$$

It is the sum of the collinear and anticollinear parts with the nonsingular part of the splitting function

$$\tilde{P}_{gg}^{(0)} = P_{gg}^{(0)} - \frac{1}{z}, \quad (10)$$

where the $P_{gg}^{(0)}$ is the DGLAP gluon-gluon splitting function in LO given by

²For work addressing accounting for kinematical constraints in the coordinate space BK equation see [67], for pointing out relevance of kinematical constraints in the context of jet physics and NLO corrections, see [68].

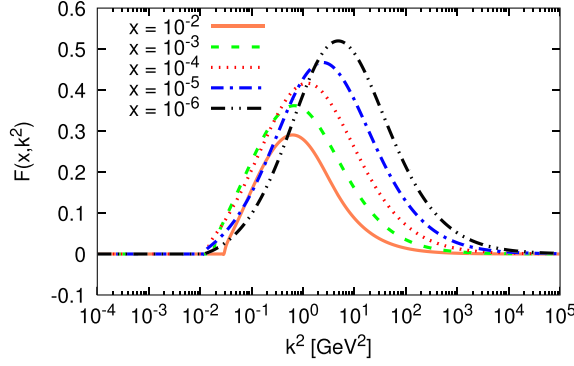


FIG. 3. Solution of BK equation supplemented with kinematical constraint as a function of k^2 for various values of x .

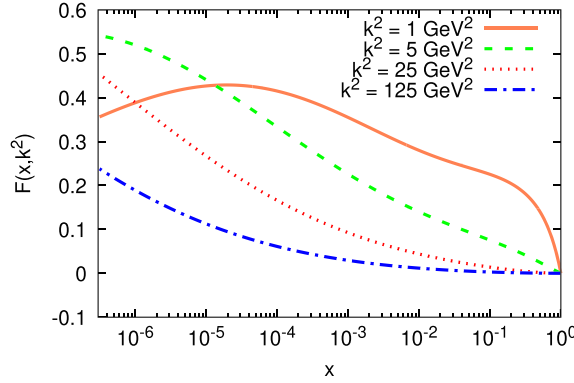


FIG. 4. Solution of BK equation supplemented with kinematical constraint as a function of x for various values of k^2 .

$$P_{gg}^{(0)} = \frac{1-z}{z} + z(1-z) + \frac{z}{(1-z)_+} + \frac{1}{2C_A} \delta(1-z) \frac{11C_A - 4N_f T_r}{6}. \quad (11)$$

Here $C_A = 3$, $T_r = 1/2$, and N_f is the number of quark flavors. We stress that one of the big advantages of the formulation of the evolution in the momentum space is the

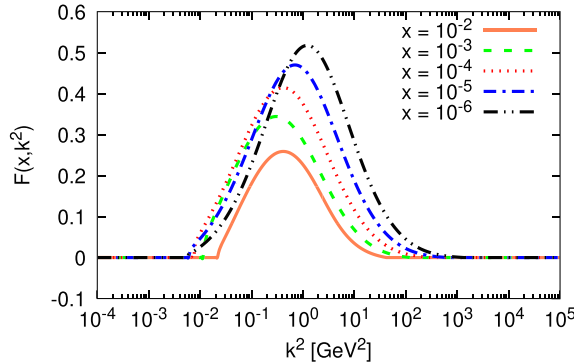


FIG. 5. Solution of BK equation supplemented with kinematical constraint and DGLAP as a function of k^2 for various values of x .

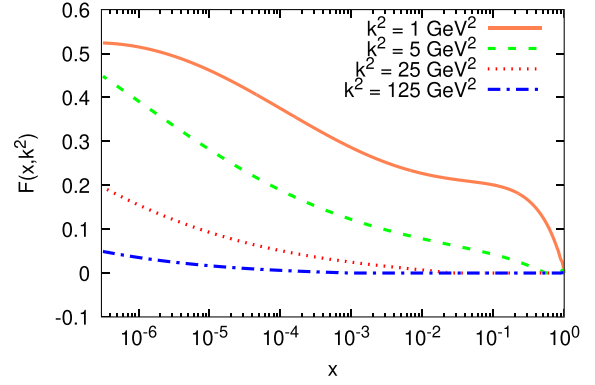


FIG. 6. Solution of BK equation supplemented with kinematical constraint and DGLAP as a function of x for various values of k^2 .

ability to include the full DGLAP terms. In Figs. 5 and 6 we present the solution of the BK supplemented with DGLAP correction and the kinematical constraint. Compared to the scenario that accounts for running coupling and the kinematical constraint, the inclusion of DGLAP further suppresses the unintegrated gluon density, leading to a flatter x spectrum.

III. STRUCTURE FUNCTION IN THE k_T FACTORIZATION

The DIS structure functions F_2 in the small x region can be expressed as a convolution of the off-shell photon-gluon partonic cross section with the unintegrated gluon density. The longitudinal structure function F_L , which is a necessary contribution to compute the reduced cross section, can be calculated in a same fashion, see e.g., [69]. This scheme, known as k_T factorization [7,8,70], is illustrated schematically in Fig. 7.

The perturbative contribution to the structure function F_2 from the k_T factorization is given by

$$F_2(x, Q^2) = \sum_q e_q^2 S_q(x, Q^2), \quad (12)$$

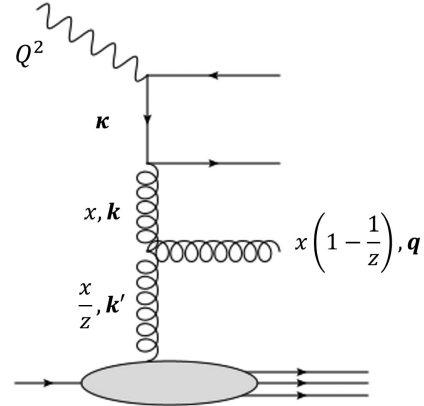


FIG. 7. DIS diagram in the high energy factorization, where the transverse momentum of the emitting gluon $q = k' - k$.

where the sum is over the quark flavors and $S_q(x, Q^2)$ can be factorized into the following form:

$$S_q(x, Q^2) = \int_x^1 \frac{dz}{z} \int dk^2 S_{\text{box}}^q(z, m_q^2, k^2, Q^2) \mathcal{F}\left(\frac{x}{z}, k^2\right). \quad (13)$$

$$S_q(x, Q^2) = \frac{Q^2}{4\pi^2} \int_{k_{\min}^2}^{\infty} \frac{dk^2}{k^2} \int_0^1 d\beta \int d\kappa' \alpha_s(\mu^2) \left\{ [\beta^2 + (1 - \beta^2)] \left(\frac{\kappa}{D_{1q}} - \frac{\kappa - k}{D_{2q}} \right)^2 + [m_q^2 + 4Q^2\beta^2(1 - \beta)^2] \right. \\ \left. \times \left(\frac{1}{D_{1q}} - \frac{1}{D_{2q}} \right)^2 \right\} \mathcal{F}\left(\frac{x}{z}, k^2\right) \Theta\left(1 - \frac{x}{z}\right), \quad (14)$$

where κ and k are quark and gluon transverse momenta, respectively, and β is the longitudinal momentum fraction of the photon carried by the quark variable defined in the of the quark momentum (see Sudakov decomposition in [71]). Instead of integrating on κ , it turns out to be convenient to work with the shifted quark transverse momentum $\kappa' = \kappa - (1 - \beta)k$. Masses of the quarks are denoted by m_q .

The energy denominators are defined as

$$D_{1q} = \kappa^2 + \beta(1 - \beta)Q^2 + m_q^2, \quad (15)$$

$$D_{2q} = (\kappa - k)^2 + \beta(1 - \beta)Q^2 + m_q^2. \quad (16)$$

In addition, the argument of the unintegrated gluon density from the resummed BK equation is x/z , where

$$z = \left[1 + \frac{\kappa'^2 + m_q^2}{\beta(1 - \beta)Q^2} + \frac{k^2}{Q^2} \right]^{-1}. \quad (17)$$

This prescription above arises from a rigorous treatment of the photon-gluon fusion process, incorporating its exact kinematic [71]. In the k_T factorization framework, such a precise kinematic description elevates the photon impact factor beyond LL approximations [72,73] yielding significant phenomenological consequences, as demonstrated in [69]. Notably, recent advances have enabled collinear resummation to the photon impact factor [74], which retain consistency with the exact kinematic formulation in the collinear limit.³

We take the argument of the strong coupling α_s to be $\mu^2 = k^2 + \kappa'^2 + m_q^2$, while the masses of quarks are $m_u = m_d = m_s = 0$ and $m_c = 1.4$ GeV.

In principle, the k_T factorization approach requires integrating over the transverse momentum κ down to zero in Eq. (14), which extends into the nonperturbative region. Meanwhile, although the nonlinear BK evolution allows

Here, S_{box}^q is the off-shell photon-gluon partonic cross section (*box* named after the calculation of the amplitude of Fig. 7 with its conjugate).

The explicit expression of Eq. (13) after the Sudakov decomposition is given by [18,71]

the unintegrated gluon density $\mathcal{F}(x, k^2)$ to be evolved into the low gluon transverse momentum k region, it does not alter the infrared behavior of the quark transverse momentum κ . Therefore, following [18], we include a separate nonperturbative contribution. This follows from the well-known fact, the structure function F_2 receives large soft contribution. As proposed in [76], this soft or nonperturbative contribution has been simply parametrized as the constant background term in addition to the perturbative contribution. In the approaches within the dipole model, the nonperturbative contribution corresponds to integration over the large dipole sizes with the flat dipole cross section, see, for example, discussion in [53,54]. For small values of κ , this nonperturbative region corresponds to the so-called *soft Pomeron exchange* contribution [77], which can be parametrized phenomenologically as

$$S^{(a)} = S_u^{\text{IP}} + S_d^{\text{IP}} + S_s^{\text{IP}}, \quad (18)$$

for light quarks and

$$S_u^{\text{IP}} = S_d^{\text{IP}} = 2S_s^{\text{IP}} = C_{\text{IP}} x^{-\lambda} (1 - x)^8, \quad (19)$$

where coefficient C_{IP} is a free parameter independent of Q^2 and λ is the soft Pomeron power to be fitted from experimental data. We assume that the charm contribution is generated purely perturbatively, and therefore, there is no soft Pomeron contribution corresponding to it, i.e., $S_c^{\text{IP}} = 0$.

By noting the perturbative contribution to the structure function in Eq. (14) as S^{pert} , the total structure function F_2 is therefore simply

$$F_2 = \sum_q e_q^2 \left(S_q^{\text{IP}} + S^{\text{pert}} \right). \quad (20)$$

IV. FIT RESULTS

In this section, we present the fit to HERA data [59] and the corresponding unintegrated gluon density, i.e., the solution of Eq. (5). We choose the following initial

³For recent work on evaluation of F_2 in coordinate space at NLO accuracy see [75].

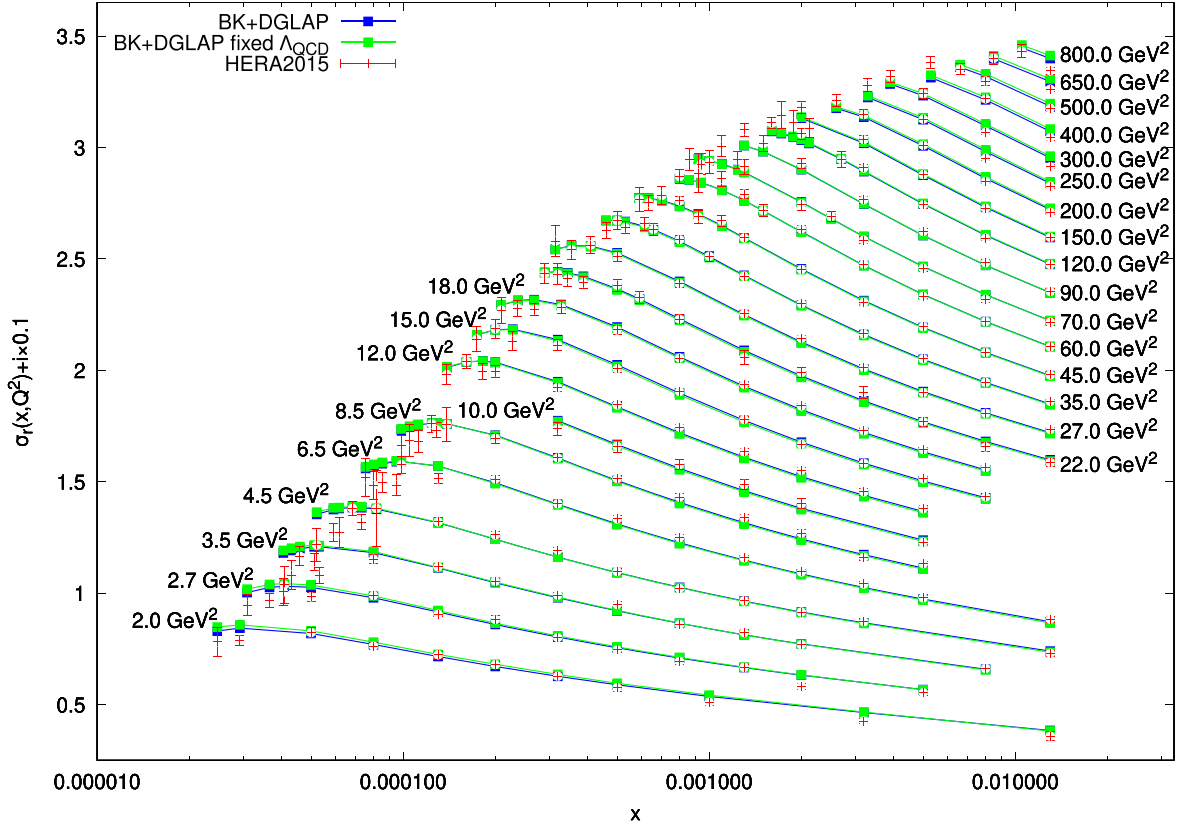


FIG. 8. Comparison between our reduced cross section fit with the HERA data. Each curve is separated with an additional offset 0.1 to better present a wider Q^2 spectrum. The blue curve is for fitted Λ_{QCD} and the green curve for fixed $\Lambda_{\text{QCD}} = 0.289$ GeV.

condition inspired by the GBW model:

$$\mathcal{F}^{(0)}(x, k) = A\alpha_s(k^2)(1-x)^\alpha x^\beta (k^2)^\gamma e^{-B^2 k^2}. \quad (21)$$

The parameters are fitted to the reduced cross section σ_r , which is given by the combination of the two structure functions F_2 and F_L ,

$$\sigma_r(y, x, Q^2) = F_2(x, Q^2) - \frac{y^2}{1 + (1-y)^2} F_L(x, Q^2). \quad (22)$$

Here, the inelasticity $y = Q^2/(sx)$ and $\sqrt{s}$ is the electron-proton collision energy. We set the flavor number $N_f = 4$.

In Fig. 8 we present our fit compared with HERA data. By selecting all low x data ($x < 0.013$), we fit a total of 239 data points. We propose two fit scenarios: one with fixed $\Lambda_{\text{QCD}} = 0.289$ GeV, matching the choice in [25], and the other where Λ_{QCD} is treated as a free fit parameter. Both fits

provide a consistent description of the HERA data, and the obtained fit parameter values are summarized in Table I.

The solutions with fitted Λ_{QCD} (solid curve) and fixed Λ_{QCD} (dashed curve) are presented in Figs. 9 and 10. The cusp appearing for lowest values of x around $k^2 \sim 1$ GeV² in Fig. 9 is due to the cut we enforced in the solver to prevent slightly negative values of the unintegrated gluon density. In Fig. 9, both transverse momentum spectrum of the fitted and fixed Λ_{QCD} fit resembles very much the characteristic scaling behavior of the saturation scales, i.e., as x becomes smaller the saturation scale grows and gluon momenta are pushed toward the perturbative region.

In Fig. 10, we observe that the gluon densities in two scenarios are generally close, except for the curves with $k^2 = 1$ GeV², where our negativity limiter has a more significant impact. We comment that the underlying negative contributions stem from the interplay of the negative nonlinear term in the BK equation and the collinear

TABLE I. χ^2 and fit parameters in the fitted and fixed Λ_{QCD} scenarios, respectively.

	α	β	A	B^2	γ	C_{IP}	λ	R^2	Λ_{QCD}
$\chi^2/d.o.f. = 1.6$	0.80662	-0.42651	0.74038	0.64239	1.09987	0.6163	-0.02361	2.50003	0.499
$\chi^2/d.o.f. = 2.0$	0.67356	-0.41625	0.65917	0.47992	1.05338	0.572	-0.00223	2.54913	fixed

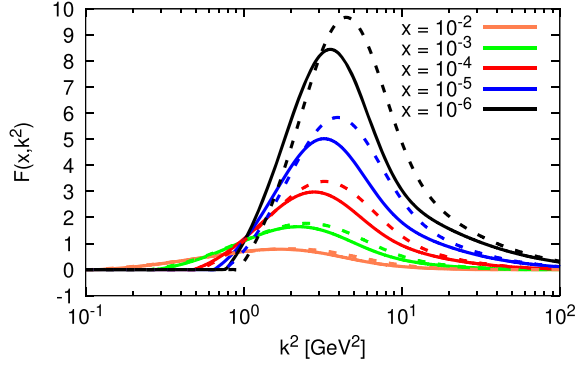


FIG. 9. Plots of fitted unintegrated gluon density as a function of k^2 for various values of x . Solid and dashed curves denote fitted and fixed Λ_{QCD} .

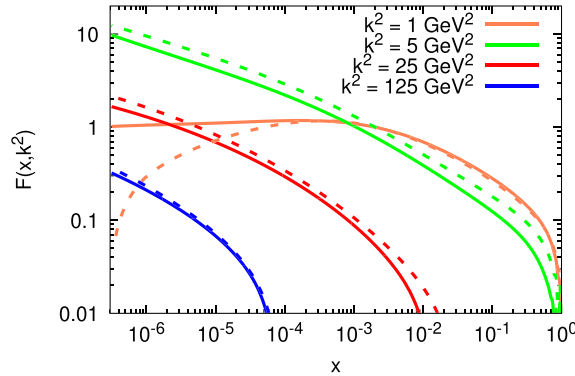


FIG. 10. Plots of fitted unintegrated gluon density as a function of x for various values of k^2 . Solid and dashed curves denote fitted and fixed Λ_{QCD} .

resummations in the linear part of the equation. Notably, our current resummed BK equation incorporates the kinematical constraint only in the linear term, making negative values more likely to appear. A possible improvement would be to introduce the kinematical constraint also in the nonlinear term. At present, such a constraint on the nonlinear term has been implemented in the BK equation only within the dipole model and in position space [67,78,79]. We plan to further investigate its effects within our formalism in future work.

In Figs. 11 and 12 we show a comparison of obtained gluon density to KS linear, KS nonlinear [47], and Li-Stasto (LS) [25]. The main difference between our gluon density and the other distributions is that we account for perturbative evolution down to low k of gluon. In practice, we set the lower bound $k_{\text{min}}^2 = 10^{-6} \text{ GeV}^2$. This is possible because the nonlinear BK term regulates the infrared behavior of the linear BFKL evolution. The consequence of that is that the integral with the box Eq. (13) can be evaluated directly with Eq. (14) without splitting it into perturbative and nonperturbative parts, where a separated collinear approximation is typically applied [25,47,80].

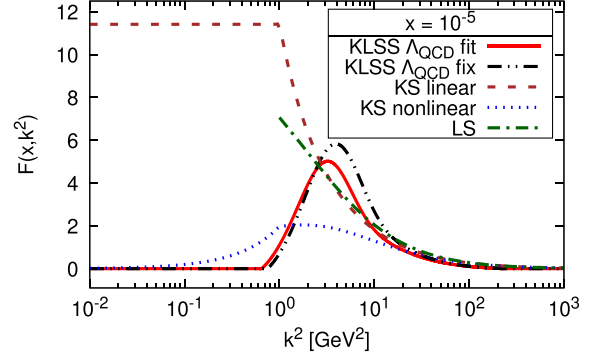


FIG. 11. Plots of fitted unintegrated gluon density (with fixed and fitted Λ_{QCD}) as a function of k^2 for $x = 10^{-5}$ compared to KS linear, KS nonlinear, and LS linear gluon densities.

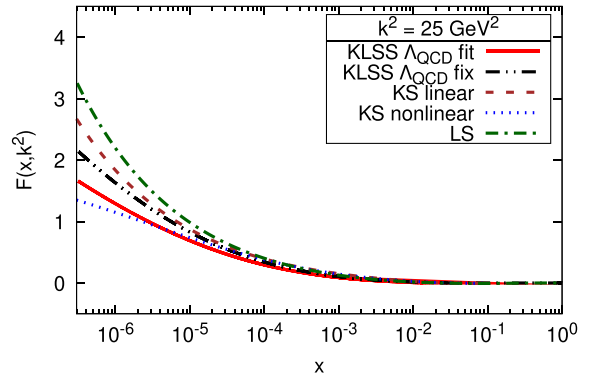


FIG. 12. Plots of the unintegrated gluon densities (with fixed and fitted Λ_{QCD}) as a function of x for $k^2 = 25 \text{ GeV}^2$ compared to KS linear, KS nonlinear, and LS linear gluon densities.

This leads to larger gluon density as compared to other gluon densities. We note that in the KS nonlinear, the method of solving the equation did not allow to go to transverse momentum smaller than 1 GeV^2 , and extrapolation was used in this region. The F_2 was calculated in a similar manner as in KS linear and LS.

V. CONCLUSIONS

In this paper we investigated the momentum space BK equation with running coupling, kinematical constraints, and DGLAP terms both in collinear and anticollinear limits. We first studied solutions for particular cases, including only the effects of the kinematical constraint and the kinematical constraint together with DGLAP terms. We demonstrated that each of these contributions have a substantial impact on the resulting solution and, as it changes the overall normalization and shape of gluon density. This is in agreement with previous findings by [81]. However, the present formulation allows for extending the solution to small transverse momentum scales. This procedure impacts the resulting gluon density in such way

that in the nonsaturated region it is larger than the KS [47] gluon density, while it is smaller in the saturated region. This may have important phenomenological consequences, e.g., for jet physics [47,49,52,82–85].

We used the obtained dipole gluon density together with the impact factor with exact kinematics to calculate the DIS reduced cross section. The form of the starting distribution was motivated by the Golec-Biernat-Wuesthoff model, but included additional modifications and parameters as required by the fit. Finally, we would like to add that although there is some modeling involved the equation that we proposed is in line of the CCSS resummation; in order to complete it one needs to account for remaining NLO contributions as it has been done in the linear case in [25]. We leave the study of equations with these modifications for future research.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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