Timelike entanglement entropy revisited

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We present an operator-algebraic definition for timelike entanglement entropy in quantum field theory under a few mild postulates. This rigorously defined timelike entanglement entropy is real-valued due to the timelike tube theorem. We further demonstrate why the timelike entanglement entropy should be real-valued from both path integral argument and holography perspective.

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I. INTRODUCTION

As an intrinsic ingredient in quantum theory, entanglement entropy is increasingly playing a central role in quantum field theory (QFT) and quantum gravity. The standard definition of entanglement entropy in a pure state is the von Neumann entropy $S_{\mathrm{vN}} = -\mathrm{Tr}\rho_A \log \rho_A$ for a reduced density matrix $\rho_A = \mathrm{Tr}_{A^c}\rho$ which is obtained after tracing out the complement of A. Usually, the entanglement entropy in QFT is defined for a spacelike region. Recently, entanglement entropy for "timelike regions" has been introduced from different theoretical motivations [1–5]. Two seemingly conceptually distinct configurations of timelike entanglement are proposed. The configuration A proposed in [1–4] addresses the entanglement between an interval and the remaining segment on the temporal axis, as shown in Fig. 1. Configuration B, proposed in [5], investigates the entanglement between two timelike-separated spacelike regions. We focus on configuration A in the present work, and reserve the discussion of configuration B for future studies. An intrinsic definition of timelike entanglement entropy, however, is still unclear. In particular, two key questions remain unanswered; What exactly is the timelike subsystem in QFT? And how to define entanglement entropy of the timelike subsystem?

A local field operator $\phi(x)$ in QFT cannot be considered a true observable, as $\phi(x)$ takes states out of the Hilbert space $\mathcal{H}$ when acting on them. This is manifested by the singularity appears in the operator product expansion (OPE) of $\phi(x)\phi(y)$. To obtain a well-defined Hilbert space operator, one might smear $\phi(x)$ over an open region $\mathcal{O}$, yielding $\Phi_f = \int_{\mathcal{O}} \mathrm{d}\mu f(x)\phi(x)$, where f is a smooth test function. However, smearing field operators over a spatial region V does not always produce a true operator. For instance, in quantum chromodynamics, smearing in space fails due to the nonintegrable singularities in the OPE [6]. Instead, Borchers proved [7] that it suffices to smear out the field operator in *real time*,

$$\Phi_f(\vec{x}_0) = \int_{\mathcal{T}} \mathrm{d}t f(t)\phi(x), \tag{1}$$

where the test-function $f(t)$ is supported on a timelike interval,

$$\mathcal{T} = \left\{ x = (t, \vec{x}); |t| < \frac{T}{2}, \vec{x} = \vec{x}_0 \right\}. \tag{2}$$

Smearing in real time always converts $\phi(t, \vec{x})$ into true operators, enabling the definition of an algebra $\mathcal{A}$ generated by bounded functions of operators supported on $\mathcal{T}$. In the framework of algebraic QFT [8,9], each open region in Minkowski spacetime is associated with an algebra $\mathcal{A}$ of

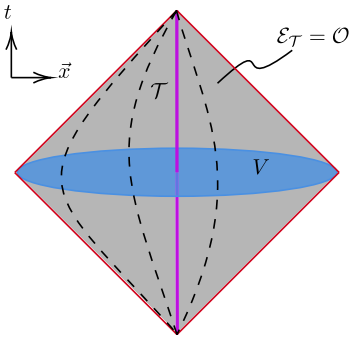


FIG. 1. The timelike envelope $\mathcal{E}_{\mathcal{T}}$ (the gray shaded region) consists of all points that can be reached by deforming the timelike interval $\mathcal{T}$ (the purple line) to a family of timelike curves (black dashed lines), which is equivalent to a causal diamond $\mathcal{O}$. All points at $t = 0$ constitutes a spacelike ball V (the blue region).

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observables, constituting a subsystem. We can thus view the algebra $\mathcal{A}(\mathcal{T})$, supported on the timelike interval $\mathcal{T}$, as a proper timelike subsystem in QFT across general dimensions. In this paper, the class of algebras we consider is sometimes referred to in the literature as the *additive algebra* [10,11]. By this, we mean that the theory contains no extended operators, such as nontrivial Wilson loops, that are not generated by local fields. Under this assumption, the algebra $\mathcal{A}(\mathcal{O})$ exhausts all local observables measurable within the region $\mathcal{O}$. In this sense, an algebraic timelike subsystem $\mathcal{A}(\mathcal{T})$ is physically meaningful, since it is generated by observables localized along an observer's worldline $\mathcal{T}$.

Defining the density matrix and von Neumann entropy in QFT is not straightforward. Unlike the Type I von Neumann algebra in quantum mechanics, the algebra $\mathcal{A}(\mathcal{O})$ associated with a causal diamond (or double cone) $\mathcal{O}$ in QFT is typically not isomorphic to $\mathfrak{B}(\mathcal{H}_{\mathcal{O}})$, the algebra of all bounded linear operators on some Hilbert space. For a causal diamond $\mathcal{O}$ and its causal complement $\mathcal{O}'$, the factorization of the vacuum Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{O}} \otimes \mathcal{H}_{\mathcal{O}'}$ does not exist. Notwithstanding these challenges, for two centric causal diamonds $\mathcal{O}$ and $\tilde{\mathcal{O}}$, the *split property*, originally conjectured by Borchers [9] and studied in detail by Doplicher and Longo [12], guarantees the existence of an intermediate Type I factor $\mathcal{N}$ such that

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{N} \subset \mathcal{A}(\tilde{\mathcal{O}}), \qquad \mathcal{O} \subset \tilde{\mathcal{O}}. \tag{3}$$

The vacuum Hilbert space $\mathcal{H}$ can then be factorized as $\mathcal{H} = \mathcal{H}_{\mathcal{N}} \otimes \mathcal{H}_{\mathcal{N}'}$, where $\mathcal{N}'$ is the commutant of $\mathcal{N}$, defined as the set of all bounded operators on $\mathcal{H}$ that commute with $\mathcal{N}$. It has been suggested [13] that the vacuum von Neumann entropy of $\mathcal{A}$ with respect to $\mathcal{O}$ and $\tilde{\mathcal{O}}$ could be defined as

$$S_{\mathcal{A}}(\mathcal{O}, \tilde{\mathcal{O}}) = -\mathrm{Tr}\rho_{\mathcal{N}} \log \rho_{\mathcal{N}}, \tag{4}$$

where $\mathcal{N}$ is chosen as a canonical intermediate Type I factor [12], and $\rho_{\mathcal{N}}$ is the vacuum density matrix corresponding to $\mathcal{N}$. We emphasize that the definition of entropic quantities in QFT via the split property is far from unique. A number of distinct approaches have appeared in the literature; see Refs. [14–20]. In our analysis, it is sufficient to consider the limit,

$$S(\mathcal{O}) \coloneqq \lim_{\tilde{\mathcal{O}} \to \mathcal{O}} S_{\mathcal{A}}(\mathcal{O}, \tilde{\mathcal{O}}). \tag{5}$$

Here, $S(\mathcal{O})$ would capture the universal UV divergence when $\tilde{\mathcal{O}}$ approaches to $\mathcal{O}$. Therefore, in QFT, $S(\mathcal{O})$ can be interpreted as the entanglement entropy for the algebra $\mathcal{A}(\mathcal{O})$ of operators in a causal diamond $\mathcal{O}$. The split property is often taken to characterize physically acceptable theories, but it is not guaranteed in a general quantum field theory. In this paper, we shall always assume that the QFTs of interest satisfy the split property. It should be noted that the split property can be derived from the Buchholz-Wichmann nuclearity condition, which constrains the admissible number of local degrees of freedom [21,22]. This notion of nuclearity has also been formulated in terms of the modular operator of local algebras [23,24]. For further discussions of the split property and nuclearity in conformal field theory, see Refs. [19,25–27]; for the corresponding developments in quantum field theory on curved spacetime, see Refs. [28–31].

II. TIMELIKE TUBE THEOREM AND TIMELIKE ENTANGLEMENT ENTROPY

To define the entanglement entropy for the algebra $\mathcal{A}(\mathcal{T})$ of operators on a timelike interval $\mathcal{T}$, we first investigate the relationship between the algebra $\mathcal{A}(\mathcal{T})$ and the algebra $\mathcal{A}(\mathcal{O})$ associated with a causal diamond $\mathcal{O}$. In Minkowski spacetime, this question has been addressed by the timelike tube theorem [32,33]. Starting with the timelike interval $\mathcal{T}$, its timelike envelope $\mathcal{E}_{\mathcal{T}}$ consists of all points reachable by deforming $\mathcal{T}$ along a family of timelike curves while keeping it fixed near its endpoints. In the following discussion, we restrict our analysis to favorable cases where the timelike envelope $\mathcal{E}_{\mathcal{T}}$ coincides with a causal diamond $\mathcal{O}$, which is defined as the intersection of the past and future of $\mathcal{T}$,

$$\mathcal{E}_{\mathcal{T}} = \mathcal{O} = J^+(\mathcal{T}) \cap J^-(\mathcal{T}), \tag{6}$$

as illustrated in Fig. 1.

The timelike tube theorem asserts that the algebra of operators on a timelike interval $\mathcal{T}$ is identical to the algebra of operators in its timelike envelope $\mathcal{E}_{\mathcal{T}}$, i.e.,

$$\mathcal{A}(\mathcal{T}) = \mathcal{A}(\mathcal{E}_{\mathcal{T}}). \tag{7}$$

This relation indicates that the timelike entanglement entropy can be defined as

$$S(\mathcal{T}) \coloneqq S(\mathcal{E}_{\mathcal{T}}), \tag{8}$$

where $S(\mathcal{E}_{\mathcal{T}})$ is the von Neumann entropy defined in (5). This definition of timelike entanglement entropy relies on the principle that identical algebras should yield identical entanglement entropies. We emphasize that the timelike entanglement entropy depends only on the timelike interval $\mathcal{T}$, since the timelike envelope $\mathcal{E}_{\mathcal{T}}$ is uniquely determined by $\mathcal{T}$. Notice that $\mathcal{E}_{\mathcal{T}}$ is the domain of dependence of a spacelike ball V at $t = 0$, see Fig. 1.

Observable algebra in the QFT of our interest should be equipped with the so-called primitive causality [34], which assumes a dynamical equation, or in algebraic terms a time-slice property, such that the observables on a Cauchy slice determine those in the spacetime region causally dependent

on that slice. The reader could find more helpful discussions on the local algebras in the nice review [18]. Consequently, the algebra $\mathcal{A}(\mathcal{E}_\mathcal{T})$ is equivalent to the algebra $\mathcal{A}(V)$ of operators in an arbitrarily small spacetime neighborhood of V, up to unitary time evolution. Several known results for $S(V)$, or equivalently $S(\mathcal{E}_\mathcal{T})$, have been derived using the Euclidean path integral approach [35] and holography [36,37].

It is ready to calculate the timelike entanglement entropy in conformal field theory (CFT) for a timelike interval,

$$\mathcal{T} = \left\{ x = (t, \vec{x}); |t| < \frac{T}{2}, \vec{x} = \vec{x}_0 \right\}. \tag{9}$$

In particular for a $(1+1)$-dimensional zero-temperature CFT on the spacetime $\mathbb{R}^2$, the spacelike ball V reduces to a spacelike interval of the length T. The entanglement entropy of V is known as [38,39]

$$S(V) = \frac{c}{3} \log \frac{T}{\epsilon}, \tag{10}$$

where c is the central charge and ϵ is the UV cutoff. Following the definition (8), the timelike entanglement entropy is then given by

$$S(\mathcal{T}) = S(V) = \frac{c}{3} \log \frac{T}{\epsilon}, \tag{11}$$

in agreement with earlier results in [2]. This result can be straightforwardly extended to finite temperatures,

$$S(\mathcal{T}) = \frac{c}{3} \log \left(\frac{\beta}{\pi\epsilon} \sinh \frac{\pi T}{\beta} \right), \tag{12}$$

where β is the inverse temperature. For a finite-size CFT_2 on $\mathbb{R} \times S^1$, the timelike entanglement entropy becomes

$$S(\mathcal{T}) = \frac{c}{3} \log \left(\frac{L}{\pi\epsilon} \sin \frac{\pi T}{L} \right), \tag{13}$$

with restriction $T < L$, where L is the total length of the spatial circle S^1. However, caution is required when $T \geq L$, as the timelike envelope $\mathcal{E}_\mathcal{T}$ may not always correspond to a causal diamond. More generally, for QFT_{d+1}, V is a d-dimensional spacelike ball with radius $R = T/2$, we still have

$$S(\mathcal{T}) = S(V), \tag{14}$$

since the timelike tube theorem (7) holds in general dimensions.

III. PATH INTEGRAL AND HOLOGRAPHY

Recent literature has predominantly considered a complex-valued timelike entanglement entropy, whereas our proposal yields a real-valued result. It is therefore important to resolve this contradiction. We work this out from both path integral argument and holography perspective. We stress again that we are going to focus on configuration A as depicted in Fig. 1.

A. Path integral argument

Let us consider the density matrix ρ_{AB} in a two-dimensional CFT on an annulus, first studied in detail by Cardy [40], where A and B are two disjoint cuts in the path integral, as shown in Fig. 2. Strictly speaking, the von Neumann entropy of entanglement between A and B should contain a nonuniversal contribution arising from operator insertions, which is encoded in the annulus partition function [41]. However, this partition function is dominated by the vacuum sector in the two limits of interest to us; the adjacent limit, in which A and B become neighboring intervals [42], and the large-central-charge limit associated to the holography [43]. In both cases, the vacuum contribution provides the leading behavior. To this end, the vacuum entanglement entropy, namely the contribution in the absence of operator insertions, between two disjoint symmetric spacelike intervals A and B can be written as [44–47]

$$S_{\mathrm{vN}}(A:B) = \frac{c}{6} \log \frac{R_2}{R_1}. \tag{15}$$

The entanglement entropy between C and D are also given in [44,46] after applying conformal transformations,

$$S_{\mathrm{vN}}(C:D) = \frac{c}{3} \log \frac{\sqrt{R}_2 + \sqrt{R}_1}{\sqrt{R}_2 - \sqrt{R}_1}. \tag{16}$$

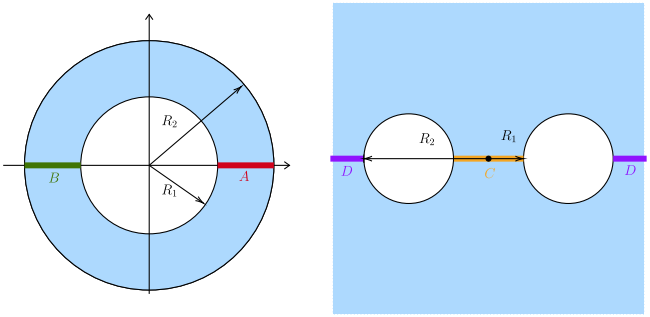


FIG. 2. The density matrix for disjoint symmetric segments $A(C)$ and $B(D)$, both of which are reflection-symmetric. The usual adjacent configurations with divergent entanglement entropies $c/3 \log \ell/\epsilon$ or $c/6 \log \xi/\epsilon$ are obtained by taking simple limits.

We emphasize that the quantity considered here is not the entanglement entropy of a union of disconnected regions AB, which may be viewed as a subsystem of a larger state such as ρ_{ABCD}. Instead, after the subtraction procedure [44], one obtains a pure state ρ_{AB} in the annulus CFT. The entropic quantity we study is therefore the von Neumann entropy of the subsystem A in the pure state ρ_{AB}. In the vacuum-dominated limits relevant for our discussion, this quantity is governed by the annulus partition function and gives rise to the expressions above. Our construction is thus different from the standard problem of the entanglement entropy of two generic disjoint intervals in a CFT vacuum state. For this reason, the familiar nonuniversal dependence of the Rényi entropies for generic disjoint intervals on the full operator content does not apply here in the same direct way [48,49]. The usual divergent entanglement entropies between adjacent regions are obtained by taking simple adjacent limits of this disjoint system as follows:

(i) Under the limit $R_2 - R_1 = \epsilon \to 0$

$$S_{\mathrm{vN}}(C:D) = \frac{c}{3}\log\frac{\ell}{\epsilon} + \text{finite terms}, \tag{17}$$

where $\ell = 2R_1$ is the length of the entangling interval C;

(ii) Under the limit $R_1 = \epsilon \to 0$ and $R_2 = \xi \to \infty$,

$$S_{\mathrm{vN}}(A:B) = \frac{c}{6}\log\frac{\xi}{\epsilon} + \text{finite terms}, \tag{18}$$

which represents the half-space entanglement entropy.

It is evident that the path integrals on this single annulus Z_1 and replica n-folded annulus Z_n are both independent of the locations of A and B on the annulus. Consequently, the entanglement entropy between any two disjoint segments is always given by either Eq. (15) or Eq. (16). Asymmetric configurations can be generated via conformal transformations, which do not introduce complex numbers.

We now consider the case that A and B are both timelike. This is achieved by performing a Wick rotation on both R_2 and R_1. We of course still get the real-valued entanglement entropies given in Eqs. (15) and (16). For the adjacent limits, it is clear that Wick rotations must be applied simultaneously to both ℓ and ϵ in Eq. (17); and to both ξ and ϵ in Eq. (18), which surely leads to real-valued results.

However, if we Wick rotate only ℓ in Eq. (17), or only ξ in Eq. (18), while keeping ϵ fixed in both scenarios, spurious complex numbers emerge. This is precisely the approach adopted in previous works, where the UV cutoff ϵ is introduced *ad hoc* and arbitrarily set to a real value.

B. Holography perspective

Without loss of generality, consider the entanglement entropy of a single timelike interval $\mathcal{T}_1 = \{0 < t < 1, x = 0\}$ in zero-temperature CFT_2. This interval can be mapped onto the temporal half-axis $\mathcal{T}_{\mathrm{h}} = \{t > 0, x = 0\}$ via a special conformal transformation,

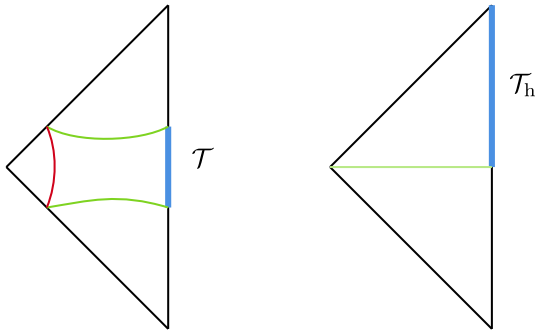


FIG. 3. Penrose diagrams of Poincaré patch of AdS_3 spacetime and holographic duals of $S(\mathcal{T})$, $S(\mathcal{T}_{\mathrm{h}})$. In the left panel, the blue line represents a finite timelike interval $\mathcal{T}$ on the conformal boundary of AdS_3, the red line denotes a timelike geodesic, and the two green lines denote spacelike geodesics. In the right panel, the blue line represents the temporal half-axis $\mathcal{T}_{\mathrm{h}}$, and the green line denotes a spacelike geodesic.

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2(b\cdot x) + b^2 x^2}, \tag{19}$$

with a timelike vector $b^\mu = (1, 0)$. Therefore, the entanglement entropy of any finite timelike interval $\mathcal{T}$ equals that of the temporal half-axis $\mathcal{T}_{\mathrm{h}}$ due to conformal symmetry,

$$S(\mathcal{T}) = S(\mathcal{T}_{\mathrm{h}}). \tag{20}$$

It has been conjectured that the holographic dual of $S(\mathcal{T})$ consists of two spacelike geodesics and one timelike geodesic connecting them in AdS_3 spacetime, with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu}{z^2}, \tag{21}$$

as illustrated in Fig. 3. The timelike geodesic is argued to contribute an imaginary part to $S(\mathcal{T})$.

However, this conjecture fails for $S(\mathcal{T}_{\mathrm{h}})$, whose holographic dual is a single spacelike geodesic from

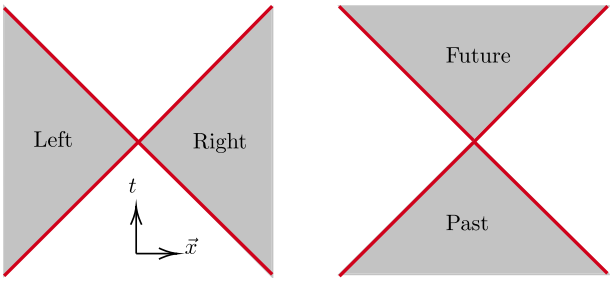


FIG. 4. The left/right Rindler wedges and the future/past light cones are illustrated, respectively.

$(z, x^\mu) = (0, 0)$ to $(z, x^\mu) = (+\infty, 0)$, containing no timelike geodesics. Thus, from the holographic perspective, $S(\mathcal{T}_h)$ is purely real-valued, and by the equivalence in (20), $S(\mathcal{T})$ must also be real-valued. Moreover, one can easily see that the real-valued $S(\mathcal{T}_h)$ represents the entanglement entropy between $t \in (0, \infty)$ and $t \in (-\infty, 0)$.

IV. ENTANGLEMENT ACROSS TIME

It is natural to ask whether the timelike entanglement entropy represents entanglement across time. The answer appears to be affirmative. A fundamental example of entanglement in QFT is the case of the left and right Rindler wedges, originally analyzed by Bisognano and Wichmann [50]. In fact, for an even-dimensional massless and noninteracting QFT, it has been shown [51,52] that entanglement exists between the operators within the future and past light-cones, analogous to the entanglement between the left and right Rindler wedges as shown in Fig. 4. This arises because, in such theories, every influence propagates along only lightlike intervals, leading to both spacelike commutativity and timelike commutativity. However, this is not generally true in massive, interacting QFTs. Nevertheless, we can conclude that entanglement across time does exist in certain special cases.

By the timelike tube theorem, the future and past light cones are associated with distinct segments of an observer's worldline. This provides a more physical interpretation of entanglement across time; if an observer measures only massless, noninteracting quantum fields, the observables measured in their future will be entangled with those measured in their past. To explore entanglement across time in the real world, a promising candidate is the pure Maxwell field, which is closely related to Wheeler's delayed-choice experiment with photons.

V. DISCUSSIONS

In this letter, using known results in algebraic QFT, we proposed a rigorous definition for timelike entanglement entropy in QFT of general dimensions. The key insight is that timelike entanglement entropy depends only on the timelike interval and is real-valued. However, our proposal may not apply to quantum field theories with defects or to generalized free field theories, where additivity or primitive causality is absent.

The algebra of observables has attracted significant attention recently in quantum gravity and holography [53–56]. In particular, the algebra of an observer's worldline plays a central role in recent developments in quantum gravity [6,57,58]. The timelike tube theorem has been generalized to QFTs in curved spacetime [59–61], which, in principle, allows us to define the timelike entanglement entropy in the presence of gravity.

ACKNOWLEDGMENTS

We deeply appreciate Edward Witten for pointing out that the strong subadditivity does not hold on the timelike direction by the noncommutativity of the algebras. This fails the simple proof of F-theorem included in the first version of this paper. This work is supported by NSFC (Grant No. 12275184).

DATA AVAILABILITY

No data were created or analyzed in this study.

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