

Scenario for quark confinement from infrared safe Yang-Mills dynamics

Marcela Peláez¹, Urko Reinosa², Julien Serreau³, Matthieu Tissier⁴, and Nicolás Wschebor¹

¹*Instituto de Física, Facultad de Ingeniería, Universidad de la República,
Julio Herrera y Reissig 565, 11000 Montevideo, Uruguay*

²*Centre de Physique Théorique, CNRS, Ecole Polytechnique, IP Paris, F-91128 Palaiseau, France*

³*Université Paris Cité, CNRS, AstroParticule et Cosmologie, F-75013 Paris, France*

⁴*Sorbonne Université,*

CNRS, Laboratoire de Physique Théorique de la Matière Condensée, 75005 Paris, France



(Received 30 May 2024; revised 17 January 2025; accepted 4 September 2025; published 12 November 2025)

We revisit the non-Abelian dipole problem in the context of a simple semiclassical approach that incorporates some essential features of the infrared sector of Yang-Mills theories in the Landau gauge, in particular, the fact that both the running coupling and the gluon propagator remain finite at infrared scales and that the latter shows positivity violations that reflects the presence of massless modes. We obtain a simple flux tube solution in a controlled approximation scheme, which we compare to the results of lattice simulations.

DOI: [10.1103/rk6n-t661](https://doi.org/10.1103/rk6n-t661)

I. INTRODUCTION

Despite tremendous progress in unraveling the various aspects of quantum chromodynamics (QCD), a definite understanding of the basic mechanism underlying the phenomenon of confinement is still lacking. In pure gauge Yang-Mills (YM) theories, the confinement of static color sources has long been established in numerical simulations, notably through the area law for the Wilson loop or, equivalently, the linear interaction potential at large distances [1–4]. Various scenarios have been identified for the typical field configurations responsible for the latter [5], among which the formation of a chromoelectric flux tube between the sources, which has been observed in lattice simulations [6–8].

Effective semiclassical descriptions aiming at modeling the specific nonlinearities of the non-Abelian dipole problem have been proposed, notably, the popular dual superconductor model [9–14]. These typically appeal to effective scalar degrees of freedom and depart from a first principle calculation. Another line of development aims at obtaining flux tube solutions from the basic equations of motion of the color fields, including the effect of quantum fluctuations [15–17]. Those works, however, rely on standard perturbation theory in a regime of long distances, where the latter is

unreliable because of the diverging running coupling at infrared scales, the so-called Landau pole.

In the present work, we revisit the question of a semiclassical description of the flux tube formation with a fresh look, taking advantage of the fact, solidly established in the last two decades, that the Landau pole is an artifact of the standard perturbative description based on the Faddeev-Popov (FP) approach [18,19]. Extensive lattice simulations of YM theories in the Landau gauge have shown that the coupling (e.g., defined in the Taylor scheme) never diverges and even remains of moderate size [19]. This comes with another major observation, namely, the fact that the gluon field develops a nonzero screening mass of a few hundreds MeV together with positivity violations in the infrared [20,21]. Various approaches based on continuum field theory have been proposed to incorporate the relevant dynamics of the screening of the Landau pole [22–27] which have led to successful descriptions of a variety of nontrivial infrared phenomena, such as the confinement-deconfinement transition at nonzero temperature and baryonic chemical potential or the dynamical breaking of chiral symmetry in QCD [28–31]. We show how these “infrared safe” approaches allow for a controlled description of the non-Abelian dipole problem in YM theories with, in particular, simple semiclassical flux tube solutions.

II. SETUP

In continuum approaches, it often proves convenient to work with background field techniques [32]. Specifically, denoting A_μ^a as the gluon field, we introduce a background field $\bar{A}_\mu^a$ and we work in the Landau-DeWitt gauge defined

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

as $(\bar{D}_\mu a_\mu)^a = \partial_\mu a_\mu^a + g f^{abc} \bar{A}_\mu^b a_\mu^c = 0$, where $a_\mu^a = A_\mu^a - \bar{A}_\mu^a$. Here, g denotes the coupling constant and f^{abc} the $SU(N)$ structure constants. The corresponding FP gauge-fixed Euclidean Lagrangian reads

$$\mathcal{L}_{\text{FP}} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i h^a (\bar{D}_\mu a_\mu)^a + (\bar{D}_\mu \bar{c})^a (D_\mu c)^a, \quad (1)$$

with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ the field strength tensor, D_μ the covariant derivative, and h^a , c^a , and $\bar{c}^a$ the Nakanishi-Lautrup, ghost, and antighost fields, respectively. The gauge-fixed Lagrangian (1) possesses a formal invariance under gauge transformations of the dynamical fields together with the background field (which thus relate theories in different gauges). The same is true for the corresponding effective action: $\Gamma_{\bar{A}}[A] = \Gamma_{\bar{A}^U}[A^U]$, with A^U the gauge transformation of A under U . It is convenient to introduce the “background field effective action” $\tilde{\Gamma}[\bar{A}] = \Gamma_{\bar{A}}[\bar{A}]$: First, it is a gauge-invariant functional of its argument and, second, its extrema $\bar{A}_{\text{ext}}$ correspond to actual physical states, that is, to extrema of an actual effective action, namely, those of $\Gamma_{\bar{A}_{\text{ext}}}[A]$ [32,33].

In principle, we wish to extremize the background field effective action in the presence of $q\bar{q}$ sources. Aiming at a simple analytic description, we exploit the fact that, because the energy of a flux tube configuration grows linearly with the $q\bar{q}$ separation, it eventually comes to dominate the total energy. We therefore concentrate on the region of space between and sufficiently far from the quark and the antiquark, where we can exploit the cylindrical symmetry of the problem. We account for the presence of the sources by imposing a fixed value for the chromoelectric flux trough an infinite plane perpendicular to the $q\bar{q}$ axis, $\phi^a = \int d^2 x_\perp \langle E_z^a \rangle$.

Working with an arbitrary background is, of course, too difficult and we shall restrict to a simple subspace designed to capture the desired physics. We introduce $x = (\tau, z, r, \varphi)$, with τ the Euclidean time, z the coordinate along the $q\bar{q}$ axis, and (r, φ) the polar coordinates in the transverse plane. Lattice simulations show that the chromomagnetic fields are essentially zero, whereas the chromoelectric field has a dominant longitudinal component and a somewhat suppressed transverse component [34]. With this in mind, we choose the ansatz

$$\bar{A}_\mu^a(x) = \frac{\kappa}{g} \delta_{\mu,z} \delta^{a,3} + i A(r) \delta_{\mu,\tau} \delta^{a,1}, \quad (2)$$

which consists of a (nonperturbative) longitudinal homogeneous contribution κ/g with a fixed color orientation¹ and

¹Not surprisingly, a nontrivial Wilson loop requires large field configurations $g\bar{A} \sim 1$. This is analogous to the large background field that encodes the nontrivial Polyakov loop and the associated physics of the confinement-deconfinement transition at nonzero temperature [33,35,36].

a (perturbative) temporal inhomogeneous contribution $A(r)$ in a perpendicular color direction. Both κ and $A(r)$ are real so that the z component of the average field is real and the τ component is imaginary, as required in the Euclidean formulation. The nonvanishing components of the background field strength tensor $\bar{F}_{\mu\nu}^a$ are, in terms of the (real) Minkowski chromoelectric field, $\bar{E}_z^2(r) = i\bar{F}_{\tau z}^2 = \kappa A(r)$ and $\bar{E}_r^1(r) = i\bar{F}_{\tau r}^1 = A'(r)$. At the background level, a purely non-Abelian longitudinal chromoelectric field exists for $\kappa \neq 0$ and a nontrivial flux tube profile necessarily comes with a nonvanishing radial component.

To make further progress, we expand the background field effective action in powers of the inhomogeneous component A , treating the κ dependence exactly. We have, up to an irrelevant constant,²

$$\tilde{\Gamma}[\bar{A}] = -\frac{1}{2} \int_{x,y} \tilde{\Gamma}_\kappa^{(2)}(x-y) A(r_x) A(r_y) + \mathcal{O}(A^4), \quad (3)$$

where $\int_x = \int d^4 x$. The negative sign in front of the second term on the right-hand side comes from the factor i in the definition (2) and relates to the fact that, for imaginary fields, one has to maximize rather than minimize the (Euclidean) effective action. In other words, we have to maximize in A but minimize in κ . The two-point function $\tilde{\Gamma}_\kappa^{(2)}(x-y)$ in the presence of the homogeneous background field κ is invariant under spacetime translations and can be written in terms of its Fourier transform as

$$\tilde{\Gamma}_\kappa^{(2)}(x) = \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot x} \hat{\Gamma}_\kappa^{(2)}(q) = \hat{\Gamma}_\kappa^{(2)}(i\partial_x) \delta^{(4)}(x). \quad (4)$$

Factorizing a volume $\int_0^T d\tau \int_0^R dz \int_0^{2\pi} d\varphi$, we write $\tilde{\Gamma}[\bar{A}] = 2\pi R T \tilde{\gamma}_\kappa[A]$, with the effective one-dimensional functional³

$$\tilde{\gamma}_\kappa[A] = -\frac{1}{2} \int_0^\infty r dr A(r) \hat{\Gamma}_\kappa^{(2)}(i\partial_r) A(r) + \mathcal{O}(A^4). \quad (5)$$

The extremization with respect to A yields

$$\hat{\Gamma}_\kappa^{(2)}(i\partial_r) A(r) = 0 \quad (6)$$

and the flux constraint reads, at leading order in a semiclassical expansion around the background,

$$\int_0^\infty r dr A(r) = \frac{\phi}{2\pi\kappa}. \quad (7)$$

²The background field gauge symmetry mentioned above guarantees that the A -independent term in $\tilde{\Gamma}[\bar{A}]$ cannot depend on the homogeneous background κ , which is a pure gauge configuration.

³With a slight abuse of notation, we write $\hat{\Gamma}_\kappa^{(2)}(0,0,i\partial_r,0) \equiv \hat{\Gamma}_\kappa^{(2)}(i\partial_r)$.

Flux tube boundary conditions require both $\bar{E}_z^2$ and $\bar{E}_r^1$ to be regular at $r = 0$ and cylindrical symmetry further implies $\bar{E}_r^1(0) = 0$. We thus impose the regularity conditions $A(0) = \text{const}$ and $A'(0) = 0$. The constraint (7) implies that $A(r)$ decreases fast enough at $r \rightarrow \infty$.

III. FLUX TUBE SOLUTION

We now come to the calculation of $\hat{\Gamma}_\kappa^{(2)}(q)$. Background field gauge invariance ensures that the nonzero background κ simply amounts to a shift $q_z \rightarrow q_z \pm \kappa$ for the color component relevant here. In fact, we have $\hat{\Gamma}_\kappa^{(2)}(q) = \sum_{\sigma=\pm} \hat{\Gamma}_{\kappa=0}^{(2)}(q_\tau, q_z + \sigma\kappa, \vec{q}_\perp)/2$, where $\hat{\Gamma}_{\kappa=0}^{(2)}(q) = \hat{\Gamma}^{(2)}(q^2)$ only depends on q^2 . For the case of interest here, $q_\tau = q_z = q_\varphi = 0$, we have, in Eq. (6), $\hat{\Gamma}_\kappa^{(2)}(i\partial_r) = \hat{\Gamma}^{(2)}(-\Delta_r + \kappa^2)$, with $\Delta_r = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$. The tree-level expression $\hat{\Gamma}^{(2)}(q^2) = q^2$ yields the equation

$$A'' + \frac{A'}{r} - \kappa^2 A = 0. \quad (8)$$

The solution that vanishes at $r \rightarrow \infty$ and satisfies the constraint (7) is

$$A(r) = \frac{\phi\kappa}{2\pi} K_0(\kappa r), \quad (9)$$

with $K_n(x)$ the modified Bessel function of the second kind of order n . The latter is, however, singular at $r \rightarrow 0$, which results in an infinite energy (see below) for $\kappa \neq 0$.

To obtain a regular solution, we take into account the one-loop corrections to the effective action (3). A straightforward calculation with the Lagrangian (1) gives (see Appendix A)

$$\hat{\Gamma}_{\text{FP}}^{(2)}(q^2) = q^2 \left(1 + \frac{11}{3} \lambda \ln \frac{q^2}{\mu^2} + c_0 \lambda \right), \quad (10)$$

where $\lambda = g^2 N / (16\pi^2)$, μ is an arbitrary renormalization scale, and where the number c_0 depends on the renormalization scheme. In addition to the trivial root $q^2 = 0$ (a consequence of the background field gauge invariance), the term in parentheses gives a second root. This is very much welcome since, as discussed below, it is then possible to obtain a solution regular enough at $r = 0$ to produce a finite string tension. The caveat here is that the second root is near the Landau pole of the FP theory, where the perturbative expression (10) is not reliable.

As explained above, a variety of infrared safe continuum approaches have been proposed that avoid this spurious artifact of standard perturbation theory. Here, we use one of the simplest such approaches, based on the Curci-Ferrari (CF) model [37], a simple infrared deformation of the gauge-fixed Lagrangian (1),

$$\mathcal{L}_{\text{CF}} = \mathcal{L}_{\text{FP}} + \frac{m^2}{2} a_\mu^a a_\mu^a. \quad (11)$$

Such a mass term does not spoil the background field gauge invariance [33] and leaves the ultraviolet sector of the theory unchanged, while successfully screening the FP Landau pole. The model possesses infrared safe renormalization group trajectories, where the coupling remains moderate, allowing for controlled perturbative calculations all the way down to deep infrared scales in line with lattice results [27,31]. Notice though that the presence of massless degrees of freedom—a generic feature of local formulations—still allows for nontrivial infrared physics [27,38], notably, the existence of a confining phase at low temperatures and of a confinement-deconfinement transition [33]. Here, we show that this also yields controlled semiclassical flux tube solutions.

The one-loop expression of the background field two-point function is presented in Appendix A. In addition to the root $q^2 = 0$, it presents a single other root in the Euclidean domain, $q^2 = M^2 > 0$. This is a direct consequence of the nonmonotonicity of the background field two-point function at low momentum, which, in the present calculation, originates from the loop of massless ghosts. We expect this to be a generic feature of any reliable infrared safe approach, directly related to the positivity violation measured in the lattice Landau gauge gluon correlator. Of course, the precise value of M depends on the details of the infrared safe approach—for instance, in the present CF model, it depends on the values of the parameters m and λ and on the renormalization scheme—but we expect its very existence and thus the scenario we propose here to be generic and largely independent of these details.

In terms of Eq. (6), the two roots $q^2 = 0$ and $q^2 = M^2$ correspond to taking linear combinations of solutions of the two differential equations $(-\Delta_r + \kappa^2)A = 0$ and $(-\Delta_r + \kappa^2)A = M^2 A$, that is,

$$A'' + \frac{A'}{r} - \kappa^2 A = 0 \quad \text{and} \quad A'' + \frac{A'}{r} - \tilde{\kappa}^2 A = 0, \quad (12)$$

where we defined $\tilde{\kappa} = \sqrt{\kappa^2 - M^2}$. There exists a unique⁴ linear combination that vanishes at $r \rightarrow \infty$ and is finite at $r = 0$, namely, after imposing the flux constraint (7),

$$A(r) = \frac{\phi}{2\pi\kappa} \frac{\kappa^2 \tilde{\kappa}^2}{M^2} [K_0(\tilde{\kappa} r) - K_0(\kappa r)]. \quad (13)$$

This is the central result of the present work: the nonmonotonicity of the background field two-point function, an expected fundamental feature of YM theories, guarantees the existence of a nontrivial flux tube profile. We shall

⁴There exists no solution compatible with the constraint (7) in the case $\kappa \leq M$.

see below that Eq. (13) describes well the existing lattice data for the latter.

As mentioned above, the precise value of M in Eq. (13) depends on the details of the infrared safe approach. However, the generic features of our scenario happen to be largely independent of these details and thus robust. To illustrate this in the spirit of the present perturbative calculation, it suffices to consider the low q^2 expansion (see Appendix A)

$$\frac{\hat{\Gamma}^{(2)}(q^2)}{q^2} = 1 + \frac{\lambda}{6} \left[\ln \frac{q^2}{m^2} + c_m + \mathcal{O}\left(\frac{q^2}{m^2} \ln \frac{q^2}{m^2}\right) \right]. \quad (14)$$

All the terms in brackets but $c_m = c(m/\mu)$ are independent of the renormalization scheme. This gives a root at the (model-dependent) infrared scale $M^2 = m^2 \exp(-6/\lambda - c_m)$. In contrast with the FP case (10), where the infrared and the ultraviolet logarithms $\ln q^2$ and $\ln \mu^2$ are intrinsically entangled, the $\ln(q^2/m^2)$ term in Eq. (14) is of purely infrared origin. The positive sign of the coefficient of that logarithm is the crucial point here, which guarantees the existence of the root $M^2 > 0$.

For small couplings, we typically have $M^2 \ll \kappa^2$ and Eq. (13) approximates to

$$A(r) \approx \frac{\phi\kappa}{4\pi} f(\kappa r) \quad \text{with} \quad f(x) = xK_1(x). \quad (15)$$

Remarkably, the solution (15) is independent of M and thus of the details of the infrared safe approach. Also, note that it survives the limit $\lambda \rightarrow 0$, although the one-loop correction to the effective action has been crucial to establish it. In this sense, it is the leading order of an approximation scheme with corrections that can be computed in a systematic expansion around Eqs. (3) and (14). For the purpose of illustrating our scenario in the simplest way, we shall use the expression (15) in our discussion below. We have also performed calculations with the complete expression (13) and the actual value of M^2 obtained from the complete one-loop expression of $\hat{\Gamma}^{(2)}(q^2)$ given in Appendix A. We mention how the results are affected when needed.

The energy of the field configuration (15) is

$$V_{q\bar{q}}(R) = \frac{1}{2} \int d^3x \langle E^2 + B^2 \rangle, \quad (16)$$

where the integral spans the whole space. Assuming that, for large $q\bar{q}$ separation R , the total energy is dominated by the region between the sources, we have⁵ $V_{q\bar{q}}(R) \approx \sigma R$

⁵At large (Euclidean) time T , the Wilson loop $\mathcal{W}(T, R)$ describing a $q\bar{q}$ pair separated by a distance R behaves as $\mathcal{W}(T, R) \propto \exp(-V_{q\bar{q}}(R)T)$ [39]. A linear behavior $V_{q\bar{q}}(R) \propto R$ is equivalent to the area law for the Wilson loop.

where the energy per unit length σ defines the string tension. At leading order, we obtain

$$\sigma = \pi \int_0^\infty r dr [(A')^2 + \kappa^2 A^2] \approx \frac{\phi^2 \kappa^2}{16\pi}. \quad (17)$$

Note that this expression is independent of the details of the infrared safe approach and thus generic.⁶

Before proceeding with the discussion, let us mention a caveat of our flux tube solution (13) or (15). The key point of the present scenario is that the loop contribution from the massless degrees of freedom to the background field two-point function in Eq. (3) smoothens the singularity of the tree-level solution (9) at $r \rightarrow 0$. However, we point out that this mechanism alone is not enough to produce a completely regular solution, for we have $A''(r) \propto \ln(\kappa r)$ for $r \rightarrow 0$. Such a singularity is, of course, spurious, as there is nothing special about the line $r = 0$ and a fully consistent flux tube solution must be completely regular there.

This difficulty is to be expected for linearized equations of motion and is thus a mere artifact of our quadratic approximation (3) for the effective action. Physically, we expect the latter to be valid far from the core of the flux tube where the field is small, but nonlinearities cannot be neglected well inside the flux tube. It is worth emphasizing though that the present approximation is sufficient to give a reasonable estimate of the shape of the flux tube and of the corresponding string tension as we discuss below. This indicates that the range of values of r where nonlinearities are important is small. We proceed with this analysis and leave the discussion of the regularity near $r \rightarrow 0$ and the role of the higher-than-quadratic terms in the action (3) for a separate work.

IV. THE HOMOGENEOUS BACKGROUND

The homogeneous background field component κ is dynamically determined as the minimum of the function $\tilde{\gamma}(\kappa)$, defined as the functional (5) evaluated at the extremum (13) or (15). However, our quadratic approximation trivially yields $\tilde{\gamma}(\kappa) = 0$ and there is no preferred value of κ . We lift this spurious degeneracy by including the next terms in the field expansion (5). To simplify matters, we treat the higher-than-quadratic terms in a local potential approximation, that is,

$$\tilde{\gamma}_\kappa[A] = \int_0^\infty r dr \left[-\frac{1}{2} A \hat{\Gamma}_\kappa^{(2)}(i\partial_r) A + \tilde{V}_\kappa(A) \right]. \quad (18)$$

⁶The complete expression (13), yields

$$\sigma = \frac{\phi^2 \kappa^2}{8\pi} \left(\frac{\kappa^2}{M^2} - 1 \right)^2 \left[\ln \left(1 - \frac{M^2}{\kappa^2} \right) + \frac{M^2}{\kappa^2 - M^2} \right].$$

Using the equation $\hat{\Gamma}_\kappa^{(2)}(i\partial_r)A_\kappa = \tilde{V}'_\kappa(A_\kappa)$, defining the extremum $A_\kappa(r)$, we obtain, for $\tilde{\gamma}(\kappa) = \tilde{\gamma}_\kappa[A_\kappa]$,

$$\tilde{\gamma}(\kappa) = \int_0^\infty r dr \left[-\frac{1}{2} A_\kappa \tilde{V}'_\kappa(A_\kappa) + \tilde{V}_\kappa(A_\kappa) \right]. \quad (19)$$

This expression being of the order of the degeneracy-lift potential $\tilde{V}_\kappa$, one can use expression (13) or (15) for A_κ to compute the minimum κ_0 at leading order.

As before, a controlled infrared safe approach is required to compute the potential $\tilde{V}_\kappa(A)$ which, due to infrared effects, is nonanalytic in A . Its field expansion reads, at one-loop order in the CF model,

$$\tilde{V}_\kappa(A) = a_\kappa A^4 \left(\ln \frac{A}{m} + b_\kappa \right) + \mathcal{O}(A^6 \ln A), \quad (20)$$

where the functions $a_\kappa = a(\kappa/m)$ and $b_\kappa = b(\kappa/m)$ are given in Appendix B for $N = 2$ and $N = 3$. Using the profile (15), Eqs. (19) and (20) yield

$$\tilde{\gamma}(\kappa) = -\frac{\phi^4 \kappa^2}{(4\pi)^4} a_\kappa \left[c_1 \left(\ln \frac{\phi \kappa}{4\pi m} + \frac{1}{2} + b_\kappa \right) + c_2 \right], \quad (21)$$

with the constants $c_1 = \int_0^\infty dx x f^4(x) \approx 0.24$ and $c_2 = \int_0^\infty dx x f^4(x) \ln f(x) \approx -0.08$. In Fig. 1, we show the function $\tilde{\gamma}(\kappa)$ for various values of ϕ , which exhibits a nontrivial minimum⁷ $\kappa_0 \neq 0$ that we plot as a function of ϕ in Fig. 2.

V. DISCUSSION

At this point, we are in a position to predict the string tension σ in terms solely of the flux ϕ . It is interesting to attempt a rough order-of-magnitude estimate from a simplistic perturbative picture, where the flux is directly given by the color charge of a static quark, that is, $\phi = g\sqrt{(N-1)/(2N)}$ in the fundamental representation of $SU(N)$. For our estimate, we use the values of the coupling and the mass obtained by fitting the Landau gauge gluon and ghost propagators measured in numerical simulations against the one-loop CF predictions [27]. For the $SU(2)$ data of Ref. [20], the best fit values are $g = 7.5$ ($\lambda = 0.71$) and $m = 0.68$ GeV at the scale $\mu = 1$ GeV. This gives

⁷We find that $\kappa_0(\phi)$ presents a discontinuous jump to $\kappa_0 = 0$ for large values of ϕ for both $N = 2$ and $N = 3$. However, such a limiting value of the flux appears to be an artifact of the approximation (15), whose validity becomes questionable at small κ , corresponding to large ϕ . We find a nontrivial minimum $\kappa_0 \neq 0$ for all values of ϕ when using the expression (13) for the flux tube profile.

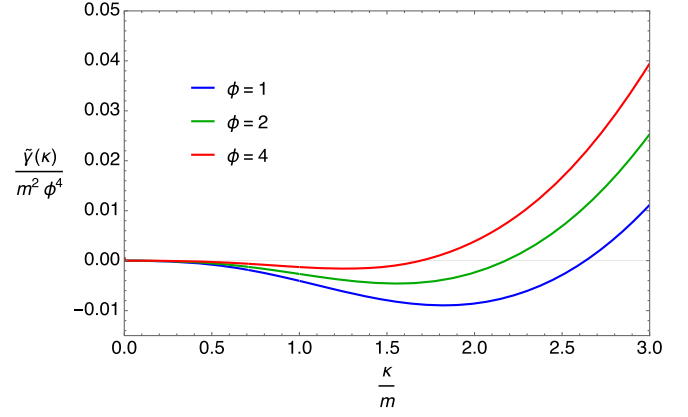


FIG. 1. The effective potential (21) (arbitrary normalization) for the background κ for various values of the flux ϕ for $N = 2$.

$\phi = g/2 = 3.75$, which corresponds to $\kappa_0/m = 1.3$, that is, $\kappa_0 = 0.87$ GeV, roughly consistent with the scale μ . We get $\sqrt{\sigma} = 463$ MeV, to be compared with $\sqrt{\sigma_{\text{lat}}} = 440$ MeV used in Ref. [20]. Similar fits of the $SU(3)$ data of Ref. [40] give $g = 4.9$ ($\lambda = 0.46$) and $m = 0.54$ GeV. We thus have $\phi = g/\sqrt{3} = 2.83$, for which we find $\kappa_0/m = 1.55$, that is, $\kappa_0 = 0.84$ GeV, resulting in $\sqrt{\sigma} = 334$ MeV, to be compared, again, with $\sqrt{\sigma_{\text{lat}}} = 440$ MeV [40].

Repeating this analysis with the complete expression (13) for the flux tube profile gives similar results. Using the renormalization scheme of Ref. [27], for which the best fit values quoted above are obtained, we find, for the $SU(2)$ case, $M = 0.7$ GeV (see Appendix A). With this value and using $\phi = g/2 = 3.75$, the potential (19) computed with the profile (13) is minimized for $\kappa_0/m = 1.56$, that is, $\kappa_0 = 1.06$ GeV. The relevant expression of the string tension (see footnote 6) then gives $\sqrt{\sigma} = 459$ MeV. For the $SU(3)$ case, we obtain, with the parameters quoted above, $M = 0.58$, $\kappa_0/m = 1.8$, that is, $\kappa_0 = 0.972$ GeV, and $\sqrt{\sigma} = 333$ MeV.

These relatively good numbers must be taken with a grain of salt, first, because of the naive estimation of the flux and,

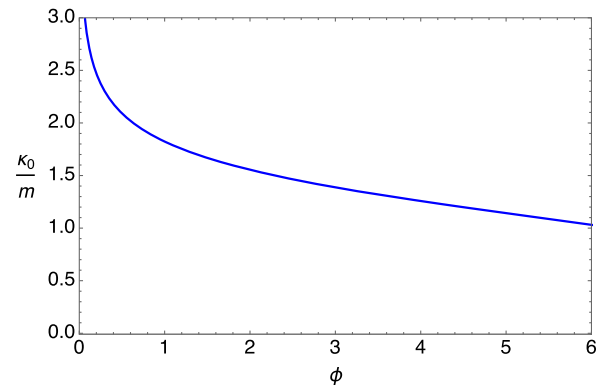


FIG. 2. The parameter κ_0/m as a function of ϕ for $N = 2$.

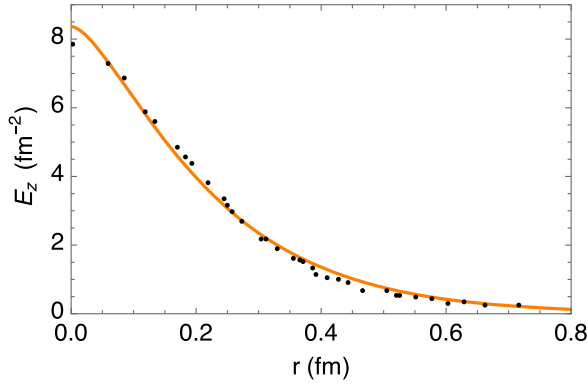


FIG. 3. The lattice data of Ref. [41] for the longitudinal chromoelectric field against the present leading-order result $\bar{E}_z^{a=2}(r) = \kappa A(r) = \phi \kappa^2 f(\kappa r)/(4\pi)$.

second, because of the expected precision of the present leading-order calculation. We have repeated the analysis using different renormalization schemes for the definition of the CF parameters m and λ , which roughly yields a 30% variability in the above estimations.⁸

These qualitative estimates are supported by actual fits of the flux tube profile obtained in lattice simulations as we now discuss. Figure 3 shows a fit of the SU(2) data of Ref. [41] for the longitudinal chromoelectric field against the present leading-order expression.⁹ The best fit values using the approximation¹⁰ (15) are $\phi = 2.38$ and $\kappa = 1.33$ GeV. This yields $\sqrt{\sigma} = 446$ MeV, remarkably close to the value $\sqrt{\sigma}_{\text{lat}} = 440$ MeV used in Ref. [41]. For this value of ϕ , we obtain $\kappa_0/m = 1.5$, which, when combined with the value of κ from the fit, gives a CF mass $m = 0.9$ GeV, in qualitative agreement with estimates from other sources [31].

It is interesting to make predictions independent of the details of the infrared safe approach, here, the CF mass. For instance, the critical temperature T_c of the confinement-deconfinement transition is well measured on the lattice [45]. The perturbative CF model yields, at leading order, $T_c/m = 0.34$ for $N = 2$ [33], from which we obtain

⁸For instance, in the so-called infrared safe scheme, we find the best fit parameters for SU(2): $g = 6.2$ ($\lambda = 0.49$) and $m = 0.5$ GeV at the scale $\mu = 1$ GeV and we obtain $M = 0.72$ GeV. The approximation (15) gives $\kappa_0 = 0.69$ GeV and $\sqrt{\sigma} = 300$ MeV, whereas the complete expression (13) yields $\kappa_0 = 0.93$ GeV and $\sqrt{\sigma} = 293$ MeV.

⁹We obtain equally good fits for the more recent data of Refs. [42,43] for both SU(2) and SU(3). However, the unconventional normalization of the operator used there as a proxy for the field does not allow for a direct estimate of the string tension [42,44].

¹⁰We have checked that either the complete expression (13) or the approximation (15) give fits of equally good quality with similar values of ϕ and κ and, thus, of σ . For instance, for the SU(2) case with $M = 0.7$ GeV, we obtain $\phi = 2.39$ and $\kappa = 1.42$ GeV, yielding $\sqrt{\sigma} = 437$ MeV.

$T_c/\sqrt{\sigma} = (T_c/m)(m/\kappa)(\kappa/\sqrt{\sigma}) = 0.68$, in remarkable agreement with the lattice result $(T_c/\sqrt{\sigma})_{\text{lat}} = 0.7$ [45].

Fits of the SU(3) flux tube profile of Ref. [46] are of as good quality as above, with $\phi = 2.8$ and $\kappa = 1.08$ GeV. This gives $\sqrt{\sigma} = 425$ MeV, where Ref. [46] measures $\sqrt{\sigma}_{\text{lat}} = 464$ MeV. We obtain $\kappa_0/m = 1.55$, giving $m = 0.7$ GeV, which, again, agrees qualitatively with other estimates [31]. The leading-order CF prediction for the transition temperature is $T_c/m = 0.36$ [33] and we obtain $T_c/\sqrt{\sigma} = 0.59$, where $(T_c/\sqrt{\sigma})_{\text{lat}} = 0.64$ [45].

VI. CONCLUSION

In conclusion, we have proposed a generic scenario for explaining the existence of a flux tube configuration solely in terms of basic infrared properties of YM dynamics in the Landau gauge, namely, the moderate infrared coupling, the nonzero gluon screening mass, and the associated positivity violations reflecting the remaining massless modes. This is a nontrivial extension of the observation that these infrared aspects alone suffice to explain the confinement of static quarks—as measured by a vanishing Polyakov loop—at low temperatures [36]. We have illustrated our point with the perturbative CF model, which, among other possible infrared safe approaches, allows for a simple and transparent analytical treatment. We have obtained consistent semiclassical flux tube solutions that describe well the existing lattice data and correctly predict the string tension for a static $q\bar{q}$ pair. Although obtained within a definite infrared safe approach, we have argued our results to be robust and to describe a fundamental aspect of the physics of confinement. It is of great interest to test the present scenario in other infrared safe approaches.

We have mentioned that the mechanism proposed here is not enough to produce a completely regular solution at $r \rightarrow 0$ and it is important to resolve this issue. As we have argued, this requires one to go beyond the quadratic approximation (3) and the corresponding simple analytical approximation (13). A first study in this direction which consists in solving the nonlinear equations of motion for the flux tube profile stemming from the local potential approximation (18) with the quartic potential (20). A preliminary study indicates that this solves the regularity issue and provides fully regular profiles. Remarkably, the nonanalytic contribution $\propto A^4 \ln(A/m)$ from the massless degrees of freedom of the infrared safe approach appears essential in producing regular flux tube solutions, thereby extending the scenario proposed in this article. We will present this analysis elsewhere.

The present work opens many interesting directions of research, among which, studying the string tension for sources in higher representations [47], computing the corrections to the present semiclassical result, and studying the finite $q\bar{q}$ distance effects and the Lüscher term [48],

including dynamical quarks [46], as well as nonzero temperature effects [49,50].

ACKNOWLEDGMENTS

We thank B. Delamotte, D. Dudal, and G. Hernández for useful discussions. M. P. and N. W. are thankful for the support of the Programa de Desarrollo de las Ciencias Básicas (PEDECIBA). This work received support from the French-Uruguayan Institute of Physics Project (IFUP) and from the Agencia Nacional de Investigación e Innovación (Uruguay), Grant No. FCE-1-2021-1-166479.

APPENDIX A: TWO-POINT FUNCTION

The background field two-point function $\hat{\Gamma}^{(2)}(q^2)$ can be computed at one-loop order in the CF model by straightforward diagrammatic techniques. Equivalently, it can be obtained from the field expansion of the background field effective action $\tilde{\Gamma}[\bar{A}]$ evaluated for homogeneous fields κ and A . The coefficient of the $A^2/2$ term is $\hat{\Gamma}_\kappa^{(2)}(q=0) = \hat{\Gamma}^{(2)}(\kappa^2)$, as explained in the text. We have performed both calculations as a cross-check. The result is ultraviolet divergent and requires renormalization. This is achieved in terms of the background field renormalization factor $Z_{\bar{A}}$, which relates the bare and renormalized background fields as $\bar{A} = \sqrt{Z_{\bar{A}}} \bar{A}_R$. The gauge invariance of the background field action guarantees that the product $g\bar{A}$ is finite. We can thus choose the renormalization condition $Z_{g^2} Z_{\bar{A}} = 1$. Using dimensional regularization with $d = 4 - 2\epsilon$ and defining the dimensionless renormalized coupling g_R through $g^2 = Z_{g^2} g_R^2 \mu^{2\epsilon}$, with μ the renormalization scale, we have, at one-loop order,

$$Z_{g^2} = 1 - \frac{11\lambda}{3\epsilon} + \lambda \delta z_f, \quad (\text{A1})$$

where $\lambda = g_R^2 N / (16\pi^2)$. The finite part δz_f is a function of the dimensionless ratio m/μ with a regular limit $m/\mu \rightarrow 0$. We obtain, writing $\bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma}$, with γ the Euler-Mascheroni constant,

$$\hat{\Gamma}^{(2)}(q^2) = q^2 \left(1 - \lambda \delta z_f + \frac{11}{3} \lambda \ln \frac{m^2}{\bar{\mu}^2} \right) + \lambda m^2 \mathcal{F} \left(\frac{q^2}{m^2} \right), \quad (\text{A2})$$

where

$$\begin{aligned} \mathcal{F}(x) = & 4 - \frac{223}{36}x - \frac{x}{12}(x^2 - 20x + 12) \left(\frac{x+4}{x} \right)^{3/2} \\ & \times \tanh^{-1} \sqrt{\frac{x}{x+4}} + \frac{x+1}{12}(x^2 - 10x + 1) \ln(1+x) \\ & - \frac{x}{24}(x^2 - 4x - 4) \ln x. \end{aligned} \quad (\text{A3})$$

The corresponding expression (10) for the FP theory is obtained by taking the limit $m \rightarrow 0$ and using the large- x expansion

$$\mathcal{F}(x) = \frac{11}{3}x \ln x - \frac{205}{36}x + \mathcal{O}(\ln x). \quad (\text{A4})$$

One checks that $\hat{\Gamma}^{(2)}(q^2=0) = 0$, as required by the background gauge invariance. Moreover, thanks to the last term $\propto \ln x$ in Eq. (A3), which originates from the remaining massless degrees of freedom of the (infrared regulated) theory, there always exists—independent of the choice of renormalization condition—a unique root $\hat{\Gamma}^{(2)}(q^2) = 0$ at $q^2 \neq 0$. As long as the coefficient of the term in the parentheses $\propto q^2$ on the right-hand side of Eq. (A3) is positive, this root lies at $x < 1$. In fact, in the spirit of the perturbative expansion, where the mentioned q^2 -term is $1 + \mathcal{O}(\lambda)$, this root lies at $x \ll 1$. In this limit, we have

$$\mathcal{F}(x) = \frac{1}{6}x \ln x - \frac{7}{9}x + \mathcal{O}(x^2 \ln x). \quad (\text{A5})$$

For definiteness, we give the expression of the counter-term δz_f in the so-called vanishing momentum scheme, used in Ref. [27] to fit the lattice data for the ghost and the gluon correlators,

$$\begin{aligned} \delta z_f = & \frac{11}{3} \ln \frac{\mu^2}{\bar{\mu}^2} - \frac{211}{36} + \frac{33}{8y} - \frac{1}{12y^2} \\ & - \frac{y^2 - 20y + 12}{12} \left(\frac{y+4}{y} \right)^{3/2} \tanh^{-1} \sqrt{\frac{y}{y+4}} \\ & + \frac{y^2 - 4y + 1}{12} \left(\frac{y+1}{y} \right)^3 \ln(1+y) \\ & - \frac{y^2 + 12y + 86}{24} \ln y, \end{aligned} \quad (\text{A6})$$

where $y = \mu^2/m^2$. We plot the corresponding background field two-point vertex function (A2) in Fig. 4 for the case $N = 2$.

APPENDIX B: QUARTIC POTENTIAL

Next, we obtain the potential (20) by evaluating the background field effective action for homogeneous fields κ and A at one-loop order and expanding in powers of A . Because of the massless degrees of freedom, this expansion is nonanalytic and there appears a $A^4 \ln A$ term. The calculation is straightforward but cumbersome as it involves two homogenous backgrounds in different space-time and color directions. We shall present it in detail elsewhere and we simply give the resulting expressions for $N = 2$ and $N = 3$ here.

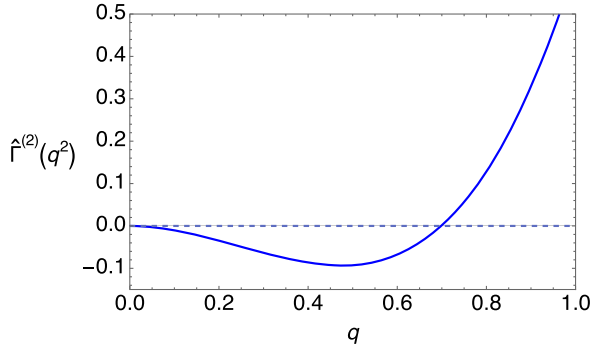


FIG. 4. The function (A2) with the counterterm (A6) for the SU(2) parameters $m = 0.68$ GeV and $g = 7.5$ at $\mu = 1$ GeV. The nonmonotonous behavior at low momentum is related to the positivity violation of the gluon propagator in the Landau gauge—corresponding to a vanishing background—and is responsible for the nontrivial root at $q = M = 0.7$ GeV.

Writing the coefficients of Eq. (20) as

$$a_\kappa = \frac{g^4}{128\pi^2} \tilde{a}_N(\kappa^2/m^2), \quad (\text{B1})$$

$$a_\kappa b_\kappa = \frac{g^4}{960\pi^2} \tilde{c}_N(\kappa^2/m^2), \quad (\text{B2})$$

we have, for $N = 2$,

$$\tilde{a}_2(x) = 6 \left[\frac{1}{(1+x)^2} - 2 \right] \quad (\text{B3})$$

and

$$\begin{aligned} \tilde{c}_2(x) = & \frac{p_1(x)}{16x(1+x)^2} + \frac{p_2(x)}{2(1-x^2)^2} + \frac{p_3(x)}{8(1+x-2x^2)^2} \ln x \\ & + \frac{p_4(x)}{4x(1-5x^2+4x^4)^2} \ln(1+x) \\ & + \frac{p_5(x)}{64(1-4x^2)^2} \ln(1+4x) \\ & + \frac{p_6(x)}{x^2(1+x)^2} \sqrt{\frac{1+x}{x}} \tanh^{-1} \sqrt{\frac{x}{1+x}} \\ & + \frac{p_7(x)}{4x^2(1-2x)^2} \sqrt{\frac{4+x}{x}} \tanh^{-1} \sqrt{\frac{x}{4+x}}, \end{aligned} \quad (\text{B4})$$

where

$$\begin{aligned} p_1(x) = & 1728 + 3276x + 4144x^2 + 4567x^3 + 3190x^4 \\ & + 1035x^5 - 112x^6 + 12x^7, \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} p_2(x) = & (-31 - 62x + 71x^2 + 184x^3 - 108x^4 - 60x^5 \\ & + 70x^6) \ln 2, \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} p_3(x) = & 180 + 800x - 280x^2 - 2680x^3 + 1599x^4 \\ & + 1987x^5 - 2365x^6 + 1093x^7 + 54x^8 \\ & - 112x^9 + 12x^{10}, \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} p_4(x) = & -4 - 250x + 541x^2 + 3425x^3 - 3222x^4 \\ & - 15799x^5 + 6189x^6 + 39091x^7 + 1628x^8 \\ & - 51877x^9 - 68x^{10} + 19754x^{11} - 1180x^{12} \\ & + 268x^{13} + 400x^{14} - 48x^{15}, \end{aligned} \quad (\text{B8})$$

$$p_5(x) = -5x(1+4x)^3(103 - 168x + 112x^2), \quad (\text{B9})$$

$$\begin{aligned} p_6(x) = & 144 + 232x - 35x^2 - 310x^3 - 177x^4 + 30x^5 \\ & + 35x^6, \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} p_7(x) = & -576 + 1952x - 432x^2 - 3112x^3 + 452x^4 \\ & + 3948x^5 - 2083x^6 - 1917x^7 + 362x^8 - 136x^9 \\ & + 12x^{10}. \end{aligned} \quad (\text{B11})$$

The corresponding expressions for $N = 3$ can be written $\tilde{a}_3(x) = \tilde{a}_2(x) + \delta\tilde{a}(x)$ and $\tilde{c}_3(x) = \tilde{c}_2(x) + \delta\tilde{c}(x)$, with

$$\delta\tilde{a}(x) = \frac{1}{(1+x)^2} - 2 \quad (\text{B12})$$

and

$$\begin{aligned} \delta\tilde{c}(x) = & \frac{\delta p_1(x)}{16x(1+x)^2} + \frac{\delta p_2(x)}{2(1+x)^2} + \frac{\delta p_3(x)}{64} \ln x \\ & + \frac{\delta p_4(x)}{32x(1+x)^2} \ln(1+x) \\ & + \frac{\delta p_7(x)}{32x} \sqrt{\frac{4+x}{x}} \tanh^{-1} \sqrt{\frac{x}{4+x}}, \end{aligned} \quad (\text{B13})$$

where

$$\begin{aligned} \delta p_1(x) = & 720 + 1018x + 592x^2 + 971x^3 + 964x^4 \\ & + 286x^5 - 28x^6 + 3x^7, \end{aligned} \quad (\text{B14})$$

$$\delta p_2(x) = 15(1+4x+2x^2) \ln 2, \quad (\text{B15})$$

$$\delta p_3(x) = 240 + 800x - 1040x^2 + 520x^3 - 9x^4 - 50x^5 + 6x^6, \quad (\text{B16})$$

$$\delta p_4(x) = -16 - 792x - 342x^2 + 3483x^3 + 6334x^4 + 2924x^5 + 58x^6 + 103x^7 + 38x^8 - 6x^9, \quad (\text{B17})$$

$$\delta p_7(x) = -1440 + 880x + 80x^2 - 2464x^3 - 902x^4 + 127x^5 - 62x^6 + 6x^7. \quad (\text{B18})$$

We emphasize that expression (20) is ill defined in the FP theory, hence the need for an infrared safe approach. For instance, for $N = 2$, we have, in the limit $m \rightarrow 0$, $\tilde{a}_2(x) =$

$-12 + \mathcal{O}(x^{-2})$ and $\tilde{c}_2(x) = -\frac{2565}{16} \ln x + \mathcal{O}(x^0)$, which gives

$$\tilde{V}_\kappa(A) = -\frac{3g^4}{32\pi^2} A^4 \left(\ln \frac{A}{m} + \frac{57}{16} \ln \frac{\kappa}{m} \right) + \mathcal{O}(g^4 A^4). \quad (\text{B19})$$

Finally, in the weak coupling regime, where $\phi \ll 1$ and $\kappa_0/m \gg 1$, we can approximate the function (21) as

$$\tilde{\gamma}(\kappa) = \frac{3g^4}{32\pi^2} c_1 \frac{\phi^4 \kappa^2}{(4\pi)^4} \left(\ln \phi + \frac{73}{16} \ln \frac{\kappa}{m} + C \right), \quad (\text{B20})$$

with C a calculable constant. We get $\kappa_0/m \propto \phi^{-16/73}$.

-
- [1] K. G. Wilson, *Phys. Rev. D* **10**, 2445 (1974).
 - [2] M. Creutz, *Phys. Rev. D* **21**, 2308 (1980).
 - [3] G. S. Bali and K. Schilling, *Phys. Rev. D* **46**, 2636 (1992).
 - [4] M. Luscher and P. Weisz, *J. High Energy Phys.* **09** (2001) 010; **07** (2002) 049.
 - [5] J. Greensite, *Lect. Notes Phys.* **821**, 1 (2011).
 - [6] G. S. Bali, K. Schilling, and C. Schlichter, *Phys. Rev. D* **51**, 5165 (1995).
 - [7] R. W. Haymaker, V. Singh, Y. C. Peng, and J. Wosiek, *Phys. Rev. D* **53**, 389 (1996).
 - [8] P. Cea and L. Cosmai, *Phys. Rev. D* **52**, 5152 (1995).
 - [9] H. B. Nielsen and P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).
 - [10] S. Mandelstam, *Phys. Rep.* **23**, 245 (1976).
 - [11] G. 't Hooft, *Nucl. Phys.* **B79**, 276 (1974).
 - [12] J. B. Kogut and L. Susskind, *Phys. Rev. D* **9**, 3501 (1974).
 - [13] A. M. Polyakov, *Phys. Lett.* **59B**, 82 (1975); *Nucl. Phys.* **B120**, 429 (1977).
 - [14] G. Ripka, *Lect. Notes Phys.* **639**, 1 (2004).
 - [15] G. K. Savvidy, *Phys. Lett.* **71B**, 133 (1977).
 - [16] S. L. Adler, *Phys. Rev. D* **23**, 2905 (1981); **24**, 1063(E) (1981).
 - [17] S. L. Adler and T. Piran, *Phys. Lett. B* **113**, 405 (1982); **121**, 455(E) (1983).
 - [18] A. Deur, S. J. Brodsky, and C. D. Roberts, *Prog. Part. Nucl. Phys.* **134**, 104081 (2024).
 - [19] P. Boucaud, J. P. Leroy, A. L. Yaouanc, J. Micheli, O. Pene, and J. Rodriguez-Quintero, *Few Body Syst.* **53**, 387 (2012).
 - [20] A. Cucchieri and T. Mendes, *Phys. Rev. Lett.* **100**, 241601 (2008); *Proc. Sci. QCD-TNT09* (2009) 026.
 - [21] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, *Phys. Lett. B* **676**, 69 (2009).
 - [22] R. Alkofer and L. von Smekal, *Phys. Rep.* **353**, 281 (2001).
 - [23] C. S. Fischer, A. Maas, and J. M. Pawłowski, *Ann. Phys. (Amsterdam)* **324**, 2408 (2009).
 - [24] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, and H. Verschelde, *Phys. Rev. D* **78**, 065047 (2008).
 - [25] N. Vandersickel and D. Zwanziger, *Phys. Rep.* **520**, 175 (2012).
 - [26] A. C. Aguilar, D. Binosi, D. Ibañez, and J. Papavassiliou, *Phys. Rev. D* **89**, 085008 (2014).
 - [27] M. Tissier and N. Wschebor, *Phys. Rev. D* **82**, 101701 (2010); **84**, 045018 (2011).
 - [28] J. Braun, L. Fister, J. M. Pawłowski, and F. Rennecke, *Phys. Rev. D* **94**, 034016 (2016).
 - [29] C. S. Fischer, *Prog. Part. Nucl. Phys.* **105**, 1 (2019).
 - [30] W. j. Fu, J. M. Pawłowski, and F. Rennecke, *Phys. Rev. D* **101**, 054032 (2020).
 - [31] M. Peláez, U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, *Rep. Prog. Phys.* **84**, 124202 (2021).
 - [32] L. F. Abbott, *Nucl. Phys.* **B185**, 189 (1981).
 - [33] U. Reinosa, J. Serreau, M. Tissier, and N. Wschebor, *Phys. Lett. B* **742**, 61 (2015).
 - [34] M. Baker, P. Cea, V. Chelnokov, L. Cosmai, F. Cuteri, and A. Papa, *Eur. Phys. J. C* **79**, 478 (2019); **80**, 514 (2020).
 - [35] N. Weiss, *Phys. Rev. D* **24**, 475 (1981).
 - [36] J. Braun, H. Gies, and J. M. Pawłowski, *Phys. Lett. B* **684**, 262 (2010).
 - [37] G. Curci and R. Ferrari, *Nuovo Cimento A* **32**, 151 (1976).
 - [38] N. Barrios, M. Peláez, and U. Reinosa, *Phys. Rev. D* **106**, 114039 (2022).
 - [39] C. Michael, *Nucl. Phys.* **B280**, 13 (1987).
 - [40] D. Dudal, O. Oliveira, and N. Vandersickel, *Phys. Rev. D* **81**, 074505 (2010).
 - [41] Y. Koma, M. Koma, E. M. Ilgenfritz, and T. Suzuki, *Phys. Rev. D* **68**, 114504 (2003).
 - [42] P. Cea, L. Cosmai, and A. Papa, *Phys. Rev. D* **86**, 054501 (2012).
 - [43] P. Cea, L. Cosmai, F. Cuteri, and A. Papa, *Phys. Rev. D* **89**, 094505 (2014).
 - [44] N. Battelli and C. Bonati, *Phys. Rev. D* **99**, 114501 (2019).
 - [45] B. Lucini and M. Panero, *Phys. Rep.* **526**, 93 (2013).

-
- [46] V. G. Bornyakov *et al.* (DIK Collaboration), *Phys. Rev. D* **70**, 074511 (2004).
[47] G. S. Bali, *Phys. Rev. D* **62**, 114503 (2000).
[48] M. Luscher, *Nucl. Phys.* **B180**, 317 (1981).
[49] A. Di Giacomo, M. Maggiore, and S. Olejnik, *Nucl. Phys.* **B347**, 441 (1990).
[50] P. Cea, L. Cosmai, F. Cuteri, and A. Papa, *J. High Energy Phys.* **06** (2016) 033.