

# Minimal modular flavor symmetry and lepton textures near fixed points

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An extension of the standard model with  $\Gamma_2 \simeq S_3$  modular flavor symmetry is presented. We consider the construction of the lepton sector, augmented by two right-handed neutrino states, in the vicinity of the fixed points  $\tau = i\infty$ ,  $\tau = i$ , and  $\tau = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ . Due to the residual symmetries at these points, and with the aid of nonholomorphic modular forms (which constitute representations of  $S_3$ ) and by assigning specific transformation properties to the fermion fields, highly economical models (without flavon fields) are constructed with interesting Yukawa textures. All presented models strongly prefer the inverted ordering for the neutrino masses.

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## I. INTRODUCTION

The hierarchies between quark and lepton masses and the suppressed values of the Cabibbo-Kobayashi-Maskawa matrix elements are unexplained within the standard model (SM). Besides these, the neutrino data [1] cannot be accommodated within the SM. The latter already indicate the presence of new physics, and the simplest extension is to add right-handed neutrino (RHN) states. On the other hand, for understanding the flavor puzzle, one can postulate some flavor symmetry. The simplest version of this is the Abelian  $U(1)_F$  flavor symmetry [2], which is desirable to gauge. Although there has been quite a bit of success in building models with gauged flavor  $U(1)_F$  [3–5], this approach is somewhat challenging because additional constraints from the requirement of anomaly cancellation need to be imposed.

Recently, models based on modular flavor symmetries have attracted considerable attention. This direction was initiated by a pioneering paper [6], after which the number of works in this area began to grow rapidly—a trend that continues to this day [6–24]. These constructions allow building simple models with modular finite non-Abelian flavor symmetries with a very economical scalar sector involving a single complex modulus field  $\tau$ . With the same  $\tau$ , the modular states are constructed, generating the effective Yukawa couplings and also responsible for the high flavor-symmetry breaking without introducing additional flavon states. The smallest modular finite group is  $\Gamma_2 \simeq S_3$ . As a flavor symmetry,  $S_3$  has a long history dating

back to the pioneering work of Ref. [25]. However, constructions with the modular flavor  $S_3$  symmetry have gained new insights [7–9,17–19], and because of its simplicity we aim to investigate it in the present work. In this paper, we focus only on the lepton sector, which will be enlarged by the RHNs.

In the next section we discuss some properties of the  $\Gamma_2 \simeq S_3$  modular symmetry and the construction of nonholomorphic modular forms, which will belong to representations of  $S_3$ . In Secs. III, IV, and V we present models near the  $\tau = i\infty$ ,  $\tau = i$ , and  $\tau = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$  fixed points, respectively. As we will see, the constructions considered can be very economical and allow for predictions. Specifically, due to residual discrete symmetries at these fixed points, all considered neutrino scenarios strongly suggest the inverted ordering of the neutrino masses. The paper includes three Appendixes. In Appendix A, some properties of the  $\Gamma_2 \simeq S_3$  symmetry, representations of nonholomorphic modular forms, and various useful expressions are derived. Also, the structures of modular representations at the considered fixed points are given. In Appendix B, the invariant kinetic couplings are presented, and simple ways of fixing the modulus field, with desirable values, are discussed. Appendix C provides the neutrino parametrization and some relations between the observables and the parameters of the specific model we are proposing.

## II. SM WITH $\Gamma_2 \simeq S_3$ MODULAR SYMMETRY

The model we propose is pretty simple. It extends the SM by incorporating right-handed neutrinos  $N_i$  (described below) along with the complex field  $\tau$ ,

$$\tau = x + iy, \quad (1)$$

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where both components  $x$  and  $y$  are functions of the four spacetime coordinates  $x_\mu$ . The field  $\tau$  is central to the flavor modular symmetry, under which  $\tau$  transforms as follows:

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \text{with } ad - cd = 1, \{a, b, c, d\} \in \mathbb{Z}. \quad (2)$$

The  $\tau$  gets values at the upper half of the complex plane. The transformation (2) allows us to consider the values of  $\tau$  picked from the fundamental domain

$$\mathcal{F} = \left\{ |\tau| \geq 1, -\frac{1}{2} \leq x < \frac{1}{2}, y > 0 \right\}. \quad (3)$$

Note that points with  $\text{Re}(\tau) = x = 1/2$  are excluded from the fundamental domain  $\mathcal{F}$  because the  $T$  transformation  $\tau \rightarrow \tau + 1$  [realized with  $a = b = d = 1, c = 0$  in (2)] maps points with  $\text{Re}(\tau) = -1/2$  onto the points with  $\text{Re}(\tau) = 1/2$ .

The boundary of  $\mathcal{F}$  includes the fixed points,

$$\text{Fixed points: } \tau = \left\{ i\infty, i, \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right\}. \quad (4)$$

[The fixed point  $1/2 + i\frac{\sqrt{3}}{2}$  is excluded from (4) because it is related to  $\omega$  via the  $T$  transformation, as noted following Eq. (3)].

The transformation properties of the fermionic and Higgs doublet fields will be addressed later. We begin our discussion with the modular forms, which are not independent fields but rather functions of the field  $\tau$ .

The modular form  $f^{(k)}$  of weight  $k$  is a function of  $\tau$  with the following transformation property:

$$f^{(k)}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f^{(k)}(\tau). \quad (5)$$

If one restricts attention to holomorphic forms, without further constraints on the integer numbers  $\{a, b, c, d\}$  [from Eq. (2)], the basis for the holomorphic even-weight forms is formed by the weight 4 and 6 forms  $E_4(\tau)$  and  $E_6(\tau)$ , respectively [26,27]. Thus, the holomorphic  $2p$  weight form can be expressed as the superposition  $\sum_{m,n} c_{mn} E_4^m E_6^n$ , where the non-negative integers  $(m, n)$  are all possible solutions to the equation  $4m + 6n = 2p$ . The modular forms  $E_{4,6}$  and their expansions are provided in Appendix A [see Eq. (A3)].

In this paper, we will focus on general (i.e., nonholomorphic) even-weighted modular forms. For their building, the minimal positive even-weighted form  $\tilde{E}_2$  (with  $k = 2$ ) will be used. The  $\tilde{E}_2$  is nonholomorphic and given in Eq. (A1). This form is constructed using the Eisenstein series  $E_2$  given in Eq. (A2). Besides this, the factor  $\kappa = (i\tau - i\bar{\tau})/2$  will be used, which under the modular transformation (2) transforms as

$$\kappa = \frac{1}{2}(i\tau - i\bar{\tau}) \rightarrow \frac{1}{2} \frac{(i\tau - i\bar{\tau})}{(c\tau + d)(c\bar{\tau} + d)}. \quad (6)$$

With the  $\kappa$ ,  $\tilde{E}_2$ , and  $E_{4,6}$  (and their conjugates), the modular form  $f^{(k)}$  can be constructed,

$$f^{(k)}(\tau) = \sum C_{lmnp}^{\bar{l}\bar{m}\bar{n}} \kappa^{2p} \tilde{E}_2^l E_4^m E_6^n (\tilde{E}_2^{\bar{l}} E_4^{\bar{m}} E_6^{\bar{n}})^*,$$

$$\text{with } k = 2(l + 2m + 3n - p), \quad \bar{l} + 2\bar{m} + 3\bar{n} = p, \quad (7)$$

which has a transformation property given in (5). Restricting to holomorphic forms, only even-weight forms with  $k \geq 4$  are nonzero [26,27]. Note that nonsupersymmetric constructions do not require holomorphicity,<sup>1</sup> so we should also consider nonholomorphic forms—such as  $\tilde{E}_2$  and the combinations in Eq. (7)—with arbitrary even weights (including negative ones) allowing positive powers of  $\tilde{E}_2$ ,  $E_{4,6}$ , and  $\kappa$ . During model building, when a large value of  $|\langle \kappa \rangle| = \langle y \rangle$  is required, the physical requirement is to use the low powers of  $\kappa$  in (7) to avoid large effective coupling. On the other hand, with  $\langle y \rangle \sim 1$ , no additional restriction is required for the construction of  $f^{(k)}$ . For the same reason, it is common not to use inverse powers of modular forms (also in supersymmetry (SUSY) constructions) to avoid poles or large values at cusp points or in their vicinity.

One may also apply the weight-raising and weight-lowering operators [29],  $\hat{\partial}$  and  $\hat{\delta}^*$  [see Eq. (A9) and the discussion therein]. When acting on a modular form of weight  $k$ ,  $\hat{\partial}$  and  $\hat{\delta}^*$  yield modular forms of weights  $k + 2$  and  $k - 2$ , respectively. In Appendix A we present simple examples to illustrate that the modular forms obtained in this way are contained in (7).

Denoted by  $\Gamma$ , the transformations (2) and (5) (without any additional constraints on integers  $\{a, b, c, d\}$ ) form a group. In conjunction with  $\Gamma$  one can also consider the group  $\Gamma(N)$  ( $N = 1, 2, \dots$ ),

$$\Gamma(N): \tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - cd = 1, \\ \{a, b, c, d\} \in \mathbb{Z} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_{2 \times 2} \pmod{N}. \quad (8)$$

While  $\Gamma$  and  $\Gamma(N)$  form infinite groups, the elements of the quotient group  $\Gamma_N = \bar{\Gamma}/\bar{\Gamma}(N)$  [where  $\bar{\Gamma} = \Gamma/\{\pm 1\}$  and  $\bar{\Gamma}(N) = \Gamma(N)/\{\pm 1\}$ ] are a finite modular group of level  $N$  [6]. Consequently, acting on the modular forms, the  $\Gamma_N$

<sup>1</sup>In the literature, the terms “holomorphic forms” and “entire modular forms” are used interchangeably. For the form  $\tilde{E}_2$  and the combinations in (7), which involve different powers of  $\kappa$ , the term “almost modular” is employed. For strict definitions see [26–29].

transform them as representations of the corresponding finite group [6,30].

The  $\Gamma_N$  groups, along with the singlets, possess non-trivial representations that enable the construction of compelling and intriguing modular flavor symmetric models [6–24]. Since the simplest and smallest case is  $\Gamma_2 \simeq S_3$ , this work will focus on this option and explore it in detail. Specifically, we will examine the lepton sector and demonstrate that, under certain conditions, the models we construct can be as successful and predictive as supersymmetric constructions.

The modular group  $\Gamma_2 \simeq S_3$  is a finite non-Abelian group with six elements  $g_i$  (where  $i = 1, \dots, 6$ ). This group possesses, in addition to the singlet representation  $\mathbf{1}$ , a pseudosinglet representation  $\mathbf{1}'$  and a doublet representation  $\mathbf{2}$ .

The modular forms  $\tilde{E}_2, E_{4,6}$ , and  $f^{(k)}(\tau)$  [given in Eq. (7)] are singlets under  $S_3$ . However, from these forms, one can construct the weight  $k$  doublet  $D^{(k)} = (D_1^{(k)}, D_2^{(k)}) \sim \mathbf{2}$  using the expressions in Eq. (A13) (for details, see Appendix A). One can also multiply modular forms belonging to different representations of  $S_3$ , and, by using the multiplication rules (e.g.,  $\mathbf{2} \times \mathbf{2} = \mathbf{2} + \mathbf{1} + \mathbf{1}'$ ) given in (A6)–(A8), construct modular forms in the required representations with the desired weights.

Thus far, we have discussed modular forms that are functions of the single field  $\tau$  and possess definite transformation properties under the modular group. However, the SM states and the RHNs are not modular forms, nonetheless, they will exhibit well-defined transformation properties under the modular group  $\Gamma_2 \simeq S_3$ .

In particular, when the transformations (5) and (8) are applied, the fermions  $\psi$  will transform as follows:

$$\psi' = (c\tau + d)^{k_\psi} \rho \cdot \psi, \quad (9)$$

where the operator  $\rho$  corresponds to the representation of  $\psi$ . If the latter is the  $S_3$  doublet, then  $\rho$  is a  $2 \times 2$  matrix representing the group element in the doublet representation. Two generating elements,  $\rho(T)$  and  $\rho(S)$ , in the doublet representation are given in (A11). The remaining elements are expressed in terms of these via the combinations shown in (A4), following the rules outlined in

Eq. (A12). Clearly, if  $\psi$  is in the  $\mathbf{1}$  or  $\mathbf{1}'$  representation, then  $\rho = 1$  or  $\rho = -1$ , respectively. The  $k_\psi$  in (9) represents the corresponding weight. Here and henceforth, for fermions  $\psi$ , we will assume the use of a two-component Weyl spinor.

Similarly, the SM Higgs doublet  $\varphi$  transforms as follows:

$$\varphi' = \pm(c\tau + d)^{k_\varphi} \varphi. \quad (10)$$

The “+” sign in (10) applies when  $\varphi \sim \mathbf{1}$ , whereas if  $\varphi$  is a pseudosinglet,  $\varphi \sim \mathbf{1}'$ , the sign should be “−.”

The structure of the scalar potential and Yukawa interactions depends on the assignment of representations and weights, which will be discussed in the following sections. The invariant kinetic terms for states with specific weights are given in Appendix B.

We would like to emphasize that the modular forms are not independent, rather, they are functions of  $\tau$  and can simultaneously mimic the nontrivial representations of  $S_3$ . Therefore, to achieve the breaking of the  $S_3$ , we do not need to introduce additional flavon states. As a result, the extension of the SM’s scalar sector is limited to the field  $\tau$ . In the fermion sector, we will introduce RHN states alongside the SM fermions, as RHNs are vital for the generation of neutrino masses. Consequently, the proposed extension—based on the  $SU(3)_c \times SU(2)_L \times U(1)_Y \times S_3$  symmetry—can be regarded as a minimal extension with non-Abelian symmetry.

Although the extension we are considering is relatively economical, in general, nonsupersymmetric constructions that permit nonholomorphic operators typically lead to an increase in couplings.

Among the holomorphic modular forms, the weight-two doublet is  $Y = (Y_1, Y_2)$ . In addition to this, there exists another independent doublet, the weight-four doublet  $Y^{(4)}$ , defined as  $Y^{(4)} \propto (Y \cdot Y)_2 = (Y_2^2 - Y_1^2, 2Y_1Y_2)$ . No other independent holomorphic doublets exist. Doublets of higher weights can be expressed in terms of  $Y$ ,  $Y^{(4)}$ , and the holomorphic functions  $E_4$  and  $E_6$ . Since we are not considering supersymmetry, holomorphy is not necessary, and we will have five independent doublets, given in (18). From the latter one can build the five independent doublets, of any fixed even weight  $k$ , which are given as

$$\begin{aligned} \text{doublets of weight } k, \quad D^{(k)} = \{ & Y f^{(k-2)}(\tau), Y^* \kappa^2 f^{(k+2)}(\tau), (Y \cdot Y)_2 f^{(k-4)}(\tau), \\ & (Y \cdot Y^*)_2 \kappa^2 f^{(k)}(\tau), (Y^* \cdot Y^*)_2 \kappa^4 f^{(k+4)}(\tau) \} \end{aligned} \quad (11)$$

(see Appendix A for a discussion), where  $f(\tau)$ ’s are the singlet modular forms, with corresponding weights, given by Eq. (7). As an illustrative example, the five doublets of weight  $k = 2$  are constructed in (A19) using (11).

Through the modular doublets of different weights, using the multiplication rule (A6), the pseudosinglet modular form with weight  $k$  can be constructed,

$$(\mathbf{1}')^{(k)} = (D^{(k-n)} \cdot D^{(n)})_{1'} = D_1^{(k-n)} \cdot D_2^{(n)} - D_2^{(k-n)} \cdot D_1^{(n)}. \quad (12)$$

Having numerous independent doublets will lead to an increase in the number of invariants and unknown couplings, which in turn reduces the predictive power of the construction. However, there is a solution to this unpleasant situation. As is well known, the fundamental domain of the  $\Gamma_2 \simeq S_3$  includes so-called fixed points, where enhanced symmetries—specifically the discrete symmetries, which are subgroups of  $\Gamma_2 \simeq S_3$ —are present. At these fixed points, either some modular forms vanish or the doublets become aligned. This can significantly reduce the number of parameters, and as we will demonstrate, the model can be as predictive as supersymmetric constructions. In the following section, we will discuss the lepton sector in the vicinity of the  $\tau = i\infty$  fixed point, while in Secs. IV and V the models near the  $\tau = i$  and  $\tau = \omega$  fixed points, respectively, are presented.

### III. MODEL NEAR $\tau = i\infty$ FIXED POINT: “INTERMEDIATE” $Z_2^T$ SYMMETRY

As discussed in Appendix A 1, at  $\tau = i\infty$  fixed point the  $S_3$  doublet modular form of any weight  $D^{(k)}$  has the second component  $D_2^{(k)} = 0$ , and also any pseudosinglet  $(\mathbf{1}')^{(k)}$  modular form vanishes,

$$\text{at } \tau = i\infty: \quad D^{(k)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\mathbf{1}')^{(k)} = 0, \quad (13)$$

where for  $D^{(k)}$  we are omitting the prefactor, bearing in mind that the latter can be absorbed in the appropriate coupling constant of the Lagrangian. Thus, at  $\tau = i\infty$  the  $Z_2^T$  symmetry is unbroken, because action of  $\rho_{[2]}(T)$  on  $D^{(k)}(\tau)$  renders it, and also any singlet modular form, invariant,

$$\text{at } \tau = i\infty, \quad Z_2^T: \quad D^{(k)}(\tau + 1) = \rho_{[2]}(T)D^{(k)}(\tau), \quad (\mathbf{1})^{(k)} \rightarrow (\mathbf{1})^{(k)}, \quad (14)$$

where the  $\rho_{[2]}(T)$  is given in (A11). On the other hand the fields in the  $\mathbf{2}$ ,  $\mathbf{1}$ , and  $\mathbf{1}'$  representations of  $S_3$ , under the  $Z_2^T$  (which is subgroup of  $S_3$ ), transform as

$$Z_2^T: \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}, \quad \mathbf{1} \rightarrow \mathbf{1}, \quad \mathbf{1}' \rightarrow -\mathbf{1}'. \quad (15)$$

Thus, any texture zero that is obtained due to  $Z_2^T$  will be protected by the same symmetry.<sup>2</sup>

<sup>2</sup>Residual discrete symmetries, i.e., subgroups of larger finite symmetries, can significantly aid in preserving texture zeros at the required level [13–16,20,22,31].

With mild violation of  $Z_2^T$ , the modular forms slightly deviate from their original values. In practice, instead of  $\text{Im}(\tau) = y \rightarrow \infty$  it is sufficient to have  $y \gtrsim 2$  because the small parameter, which characterizes the  $Z_2^T$  symmetry breaking effects, is  $\epsilon = e^{-\pi y} \ll 1$ . Therefore, for the doublets and the pseudosinglets, we can use

$$\text{with } \text{Im}(\tau) \gtrsim 2: \quad D^{(k)} = \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}, \quad (\mathbf{1}')^{(k)} \sim \epsilon, \quad (16)$$

where for the  $\epsilon$  parameter ( $|\epsilon| \ll 1$ ) appearing in the doublets with expressions of Eq. (A20) we have (see Appendix A 1 for an explanation)

$$|\epsilon| \sim 8\sqrt{3}e^{-\pi y} \quad \text{or} \quad 16\sqrt{3}e^{-\pi y}. \quad (17)$$

Thus, the  $\epsilon$  will serve as a small expansion parameter. In Appendix B we discuss and give the potential, which ensures to fix the needed value of  $\langle \tau \rangle$  near the fixed point  $\tau = i\infty$ .

#### A. Charged lepton sector

Since we are focusing on the lepton sector, we begin by outlining the transformation properties of the lepton doublets  $l_{1,2,3} = (\nu, e^-)_{1,2,3}$ , the iso-singlet charged leptons  $e_{1,2,3}^c$ , and the Higgs field  $\varphi$ . The RHNs will be discussed subsequently, when constructing the neutrino sector. The leptons  $l_1$  and  $l_2$  will be embedded in the  $S_3$  doublet  $L = (l_1, l_2)^T$ , while the remaining leptons will be  $S_3$  pseudosinglets. The Higgs doublet will be the singlet of  $S_3$ . Therefore, these states are the following representations of the group  $S_3$ :

$$L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \sim \mathbf{2}, \quad l_3 \sim \mathbf{1}', \quad e_{1,2,3}^c \sim \mathbf{1}', \quad \varphi \sim \mathbf{1}. \quad (18)$$

With the transformations given in (18) and by the following weight assignments:

$$k_L = -2, \quad k_{l_3} = k_{e_1^c} = k_\varphi = 0, \quad (19)$$

without loss of any generality, Lagrangian couplings relevant for the charged lepton masses and invariant under all symmetries are

$$-\mathcal{L}_E = \gamma_0 \tilde{\varphi} l_3 e_3^c - \tilde{\varphi} (LD^{(2-m)})_1 \mathbf{1}^{(m)} (\beta_0 e_2^c + \beta_0' e_3^c) + \tilde{\varphi} (LD^{(2-m)})_1 (\mathbf{1}')^{(m)} (\alpha_0 e_1^c + \alpha_0' e_2^c + \alpha_0'' e_3^c). \quad (20)$$

Upon writing the Yukawa coupling in (20), we have chosen the basis in which  $e_3^c$  couples only with  $l_3$ . In the remaining terms, we have displayed only the relevant invariants.

Using in (20) the structure of doublets, singlets, and pseudosinglets near the  $\tau = i\infty$  fixed point [see Eq. (16)] and using canonical normalization of the states (details discussed in Appendix B), the charged lepton mass matrix in the  $(l_1, l_2, l_3)M_E(e_1^c, e_2^c, e_3^c)^T$  basis will be

$$M_E = \begin{pmatrix} \alpha\epsilon_e & \alpha'\epsilon_e & \alpha''\epsilon_e \\ 0 & \beta & \beta' \\ 0 & 0 & \gamma \end{pmatrix} v + \mathcal{O}(\epsilon_e^2), \quad (21)$$

where  $\epsilon_e \sim \epsilon$ . From the second term of Eq. (20), the couplings  $\sim v\epsilon_e l_1(\beta_0 e_2^c + \beta'_0 e_3^c)$  arise, which are included in the first row of matrix (21) [the constants  $\alpha'$  and  $\alpha''$  are linear combinations of the original couplings  $(\alpha'_0, \beta_0)$  and  $(\alpha''_0, \beta'_0)$ , respectively]. The properly suppressed  $\epsilon \sim e^{-\pi y}$  can be naturally obtained with  $y > 1$ . We see that in the limit  $\epsilon_e \rightarrow 0$  [i.e., with unbroken  $Z_2^T$ , which means configurations of Eq. (13)], only  $\tau$  and  $\mu$  leptons gain masses, while the electron remains massless. While in this setting one cannot explain the small value of the  $\lambda_\tau (\sim 10^{-2})$  and also the hierarchy  $m_\mu/m_\tau (\approx 1/17)$ , it is still satisfactory that

$m_e$  is generated due to the  $Z_2^T$  symmetry breaking and suppressed with the small parameter  $\epsilon_e \sim \lambda_e \sim 10^{-6}$ . Since  $S_3$ 's largest representation is the doublet and there is only  $Z_2^T$  “intermediate” symmetry, only one small parameter  $\epsilon_e$  is appearing within this setup. Perhaps, within larger finite modular groups, such as  $A_4$ ,  $S_4$ , etc., it is worthwhile to attempt [32] to explain the suppressed values of  $\lambda_\tau$  and of the ratio  $m_\mu/m_\tau$ , in the spirit of Refs. [14,15,20,22,24].

By the assumption

$$\beta, \beta' \sim \frac{\gamma}{10}, \quad \beta, \beta' \gg \epsilon_e, \quad (22)$$

we can obtain

$$m_\tau \simeq |\gamma|v, \quad m_\mu \simeq |\beta|v, \quad m_e \simeq |\alpha\epsilon_e|v. \quad (23)$$

Diagonalization of  $M_E$  is achieved by the biunitary transformation

$$U_l M_E U_{e^c} = M_E^{\text{Diag}},$$

$$\text{with } U_l \simeq \begin{pmatrix} 1 & -\frac{\alpha'\epsilon_e}{\beta} & 0 \\ \left(\frac{\alpha'\epsilon_e}{\beta}\right)^* & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{\alpha''\epsilon_e}{\gamma} \\ 0 & 1 & 0 \\ \left(\frac{\alpha''\epsilon_e}{\gamma}\right)^* & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\beta'}{\gamma} \\ 0 & \left(\frac{\beta'}{\gamma}\right)^* & 1 \end{pmatrix}. \quad (24)$$

This result, obtained upon the assumption (22), corresponds to the hierarchical structure of the mass matrix  $M_E$  (21).

## B. Neutrino sector

To implement the concept discussed in the previous section, we will investigate the generation of neutrino masses through the type-I seesaw mechanism. To do this, we introduce two right-handed neutrinos,  $N$  and  $N'$ , which are associated with the representations  $\mathbf{1}$  and  $\mathbf{1}'$  (respectively) of  $\Gamma_2 \simeq S_3$ ,

$$S_3: \quad N \sim \mathbf{1}, \quad N' \sim \mathbf{1}'. \quad (25)$$

With this extension and the weight assignments

$$k_N = k_{N'} = 0, \quad (26)$$

the relevant Lagrangian terms are

$$-\mathcal{L}_\nu = [(LD^{(n+2)})_1 N \mathbf{1}^{(-n)} + (LD^{(n+2)})_{1'} N' \mathbf{1}^{(-n)} + l_3 N'] \varphi - \frac{1}{2} [MNN + nMN'N' + 2M'NN'(\mathbf{1}')^{(0)}], \quad (27)$$

where in the Dirac-type couplings we have not included terms that would give the corrections of the order of  $\sim \epsilon_e$ . Thus, in the  $\tau \rightarrow i\infty$  limit we have

$$m_D^{(0)} = \begin{pmatrix} b_0 & 0 \\ 0 & a_0 \\ 0 & 1 \end{pmatrix} \lambda v, \quad M_N^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} M, \quad (28)$$

where the bases  $(\nu_e, \nu_\mu, \nu_\tau) m_D^{(0)}(N, N')^T$  and  $\frac{1}{2}(N, N') M_N^{(0)}(N, N')^T$  have been used. Structures in (28) exhibit residual  $Z_2^T$  symmetry  $\{l_1, N\} \rightarrow \{l_1, N\}$ ,  $\{l_2, l_3, N'\} \rightarrow -\{l_2, l_3, N'\}$ . When incorporating contributions that break this  $Z_2^T$  symmetry, we can safely ignore changes in  $m_D^{(0)}$  because with  $\alpha \sim (1/3 - 3)$ ,  $\gamma \simeq \lambda_\tau \sim 10^{-2}$ , the parameter  $\epsilon_e$  [see Eq. (23)],

$$\epsilon_e \simeq \frac{\gamma m_e}{\alpha m_\tau} = (0.1 - 1) \times 10^{-5}, \quad (29)$$

is sufficiently small. Conversely, due to the distinct nature of the Majorana mass terms related to their origins, the effects of  $Z_2^T$  breaking can be more significant in  $M_N$ . Therefore, for the neutrino Dirac and Majorana matrices we will consider the following forms:

$$m_D = U_l m_D^{(0)}, \quad M_N = \begin{pmatrix} 1 & -\epsilon_0 \\ -\epsilon_0 & n \end{pmatrix} M, \quad (30)$$

with  $\epsilon_0 \sim \frac{M'}{M} \epsilon_e$ . Since  $\epsilon_0$  can be much larger than  $\epsilon_e$  (provided that  $\frac{M'}{M} \gg 1$ ), we retain  $\epsilon_0$  entries in the  $M_N$ . The unitary matrix  $U_l$  in (30) arises from the choice of basis in which the charged lepton mass matrix is diagonal. It is important to note that the 2–3 rotation of  $U_l$  [the last multiplier matrix of  $U_l$  in Eq. (24)] does not alter the structure of  $m_D^{(0)}$  and can, therefore, be absorbed into the redefinition of  $\lambda$  and  $a_0$ . However, the 1–2 and 1–3 rotations of  $U_l$  do affect the structure of the Dirac neutrino matrix  $m_D$ . Using (23) and (24), for these rotation angles we have

$$\begin{aligned} \theta_{13}^l &\simeq \left| \frac{\alpha'' \epsilon_e}{\gamma} \right| = |\alpha''| (0.1 - 1) \times 10^{-3}, \\ \theta_{12}^l &\simeq \left| \frac{\alpha' \epsilon_e}{\beta} \right| \simeq \frac{1}{\lambda_\mu} |\alpha' \epsilon_e| = |\alpha'| (0.2 - 2) \times 10^{-2}. \end{aligned} \quad (31)$$

Assuming that the values of the couplings  $\alpha', \alpha''$  can be within the range  $1/5 - 5$ , the largest and most relevant contribution is expected to be due to the  $\theta_{12}^l$  (it can be as large as  $\sim 0.1$ ). Therefore, for  $U_l$  we will consider the approximation

$$U_l \simeq \begin{pmatrix} c_e & s_e e^{i\eta} & 0 \\ -s_e e^{-i\eta} & c_e & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } \cos \theta_e \equiv c_e, \quad \sin \theta_e \equiv s_e, \quad (32)$$

where we have introduced the angle  $\theta_e \sim \theta_{12}^l$  and  $\eta$  is some phase.

Since both  $\epsilon_0$  and  $\theta_e$  emerge by  $Z_2^T$  symmetry breaking and we assume that the latter symmetry is mildly violated (i.e., the value of  $e^{-\pi y} \ll 1$ ), we will require that  $\epsilon_0, \theta_e \ll 1$ . As we will see, this requirement and forms of  $m_D, M_N$ , and  $U_l$  given in (30) and (32) allow only for an inverted ordering (IO) scenario for the light neutrinos. Applying the seesaw formula  $M_\nu = -m_D M_N^{-1} m_D^T$ , we will get

$$M_\nu = \begin{pmatrix} b & a(\epsilon + \epsilon_1) & \epsilon \\ a(\epsilon + \epsilon_1) & a^2 \left( 1 + \frac{\epsilon_1^2}{b - \epsilon^2} \right) & a \\ \epsilon & a & 1 \end{pmatrix} \bar{m}, \quad (33)$$

where

$$\begin{aligned} \bar{m} &= -\frac{\lambda^2 v^2}{M(n - \epsilon_0^2)}, \\ a &= a_0 c_e - b_0 \epsilon_0 s_e e^{-i\eta}, \quad \epsilon = b_0 c_e \epsilon_0 + a_0 s_e e^{i\eta}, \\ b &= (b_0 c_e)^2 (n - \epsilon_0^2) + (b_0 c_e \epsilon_0 + a_0 s_e e^{i\eta})^2, \\ \epsilon_1 &= \frac{\epsilon^2 - b}{a} e^{-i\eta} \tan \theta_e. \end{aligned} \quad (34)$$

As noted, within our setup the texture (33) permits the IO neutrino masses. This can be demonstrated as follows. Having chosen the basis in which the charged lepton mass matrix is diagonal, the lepton mixing originates solely from  $M_\nu$ . Thus, we can write

$$M_\nu = P U^* P' M_\nu^{\text{Diag}} U^\dagger P, \quad (35)$$

where  $U$  represents the lepton mixing matrix in the standard parametrization, and  $P, P'$  are the diagonal phase matrices [see Eqs. (C1) and (C2) of Appendix C]. Utilizing (33) and (35), we can express  $\bar{m}, a, b, \epsilon$ , and  $\epsilon_1$  in terms of the entries of  $U, P, P'$  [see relations in Eq. (C4) and the discussion in Appendix C]. Subsequently, using (34), we can compute the original parameters. Specifically, for the parameter  $\theta_e$ , we find the following results for IO and normal ordering (NO) neutrino scenarios:

$$\begin{aligned} \text{For IO:} \quad \tan \theta_e &= \frac{s_{13}}{c_{13} s_{23}} \simeq 0.2, \\ \text{For NO:} \quad \tan \theta_e &= \frac{c_{12} c_{13}}{|s_{12} c_{23} + e^{i\delta} c_{12} s_{23} s_{13}|} \simeq 1.68 - 2.56. \end{aligned} \quad (36)$$

For the numerical estimates of (36), we used the central values of  $\theta_{ij}$  [1] and varied  $\delta$  within a range  $[0, 2\pi]$ . This analysis indicates that, for the normal ordering,  $\theta_e$  is  $\sim 1$ . This would correspond to the strong breaking of  $Z_2^T$ —a scenario we are not interested in exploring.

Therefore, with a focus on the IO case, we present the parameter choices that produce the desired fit.

*Fit for the IO case*

With the selection

$$\begin{aligned} m &= 0.028804 \text{ eV}, \\ \{a, b, \epsilon, \epsilon_1\} &= \{0.83518, 1.621, -0.002092 \\ &\quad + 0.16686i, -0.017033 + 0.395i\}, \end{aligned} \quad (37)$$

from (33) for the light neutrino masses and mixing angles, we obtain

$$\{m_1, m_2, m_3\} = \{0.04917, 0.04992, 0\} \text{ eV}, \quad (38)$$

$$\{\sin^2 \theta_{12}, \sin^2 \theta_{23}, \sin^2 \theta_{13}\} = \{0.3035, 0.57, 0.02235\}. \quad (39)$$

From (38) we get

$$\begin{aligned} \Delta m_{\text{sol}}^2 &= m_2^2 - m_1^2 = 7.39 \times 10^{-5} \text{ eV}^2, \\ \Delta m_{\text{atm}}^2 &= m_2^2 = 2.492 \times 10^{-3} \text{ eV}^2. \end{aligned} \quad (40)$$

Results of (39) and (40) correspond to the best fit values of the inverted ordering neutrino scenario [1]. Moreover, for the phases we get

$$\{\delta, \rho_1\} \simeq \{1.523\pi, 1.821\pi\}. \quad (41)$$

(Since  $m_3 = 0$ , the phase  $\rho_2$  is unphysical.) For the parameter choice given above, the neutrinoless double  $\beta$ -decay parameter  $m_{\beta\beta} = |\sum U_{ei}^2 m_i P_i^*|$  is  $m_{\beta\beta} = 0.0467 \text{ eV}$ .

Once we have found the parameters giving the good fit, we can go back and see what are the values of the corresponding original parameters. (See the discussion in Appendix C.) Our interest is in those that correspond to the  $Z_2^T$  breaking. Using (37) in Eq. (C5), we get

$$\tan \theta_e \simeq 0.2, \quad |\epsilon_0| \simeq 0.0069 \times \sqrt{|n|}. \quad (42)$$

As we see, for  $|n| \sim 1$  both parameters are reasonably small, justifying our assumptions pointed out above. While  $\theta_e$  is uniquely determined by (36),  $|\epsilon_0|$  depends on  $\delta$  and  $\rho_1$ , fixing their correlation. Figure 1 displays the  $(\delta, \rho_1)$  values that yield  $|\frac{\epsilon_0}{\sqrt{|n|}}| \leq 0.2$ . With  $\epsilon_0 \sim \frac{M'}{M} \epsilon_e$ , and  $\epsilon_e \sim 10^{-5}$ , the value of  $\epsilon_0$  given in (42), with  $n = 1/2$ , can be obtained for

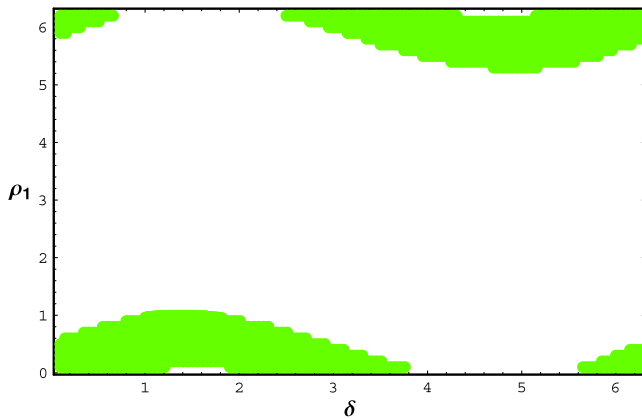


FIG. 1. Values of  $\delta$  and  $\rho_1$  (green shade) for IO neutrinos, giving  $|\frac{\epsilon_0}{\sqrt{|n|}}| \leq 0.2$  [see, e.g., Eq. (C5)]. We used the best fit values of the oscillation parameters [1].

the mass ratio  $\frac{M'}{M} \sim 5 \times 10^2$ . Finally, for  $|\lambda| = |n| = 1/2$  the selection of (37), for the heavy RHN masses, gives

$$\{M_1, M_2\} = \{2.63, 5.26\} \times 10^{14} \text{ GeV}. \quad (43)$$

#### IV. MODELS NEAR $\tau=i$ FIXED POINT: “INTERMEDIATE” $Z_4^S$ SYMMETRY

Since we are considering even-weighted modular forms  $f^{(2n)}(\tau)$ , as noted in [13], at  $\tau = i$  fixed point the  $S$  transformation, determined by (A12),  $f^{(2n)} \rightarrow (-1)^n \rho(S) f^{(2n)} = \pm \rho(S) f^{(2n)}$ , acts as  $Z_2^S$  symmetry. However, the fields  $\psi$  can have odd weight and, therefore, in the field space the  $S$  transformation can act as the  $Z_4^S$  symmetry [13,16].

As shown in Appendix A 2, at the fixed point  $\tau = i$  there are only two independent (and real) modular doublets with weights  $(4n+2)$  and  $4n$ , which possess the following structures:

$$\text{at } \tau = i: D^{(4n+2)} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad D^{(4n)} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}. \quad (44)$$

It is easy to check that  $S$  transformation leaves them invariant,

$$\begin{aligned} D^{(4n+2)}(i) &= i^2 \rho_{[2]}(S) D^{(4n+2)}(i), \\ D^{(4n)}(i) &= \rho_{[2]}(S) D^{(4n)}(i), \end{aligned} \quad (45)$$

where  $\rho_{[2]}(S)$  is given in (A11).

The  $k$  weight fermion state  $\psi^{(k)} = (\psi_1, \psi_2)^T$  in the doublet representation of  $S_3$ , under the  $Z_4^S$  symmetry, transforms as

$$\text{at } \tau = i: \psi^{(k)} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow -\frac{ik}{2} \begin{pmatrix} \psi_1 + \sqrt{3}\psi_2 \\ \sqrt{3}\psi_1 - \psi_2 \end{pmatrix}. \quad (46)$$

Besides these, at  $\tau = i$ , among the modular forms  $\tilde{E}_2$  and  $E_{4,6}$  only  $E_4$  has nonzero value (A21). Also, as discussed in Appendix A 2, any singlet modular form of weight  $(4n+2)$  and any pseudosinglet modular form of weight  $4n$  vanish at  $\tau = i$ ,

$$\text{at } \tau = i: \mathbf{1}^{(4n+2)} = 0, \quad (\mathbf{1}')^{(4n)} = 0. \quad (47)$$

A few additional useful properties observed at  $\tau = i$  arise from the structure presented in (44) and the multiplication rule in (A6). Specifically, for the doublet field  $\psi = (\psi_1, \psi_2)$  and the modular doublets, we have

$$\begin{aligned} \text{at } \tau = i: (\psi D^{(4n)})_1 &= \sqrt{3}(\psi D^{(4n+2)})_{1'}, \\ (\psi D^{(4n)})_{1'} &= -\sqrt{3}(\psi D^{(4n+2)})_1. \end{aligned} \quad (48)$$

These properties will be utilized in the upcoming model building. Below we present two models, referred to as Model A and Model B.

### A. Model A

For this model, the transformation properties of the SM leptons and the Higgs field under  $S_3$  are the same as those given in (18). As far as the RHN states are concerned, we still introduce two of them, but now embed them in the  $S_3$  doublet,

In this case, the weights are chosen as follows:

$$k_L = k_{e_i^c} = k_N = -1, \quad k_{l_3} = 1, \quad k_\phi = 0. \quad (50)$$

With these, the couplings relevant for the charged lepton masses will be

$$\begin{aligned} -\mathcal{L}_E = & \gamma \tilde{\phi} l_3 e_3^c - \tilde{\phi} \{ (LD^{(4n+2)})_{1'} \mathbf{1}^{(-4n)} + (LD^{(4n)})_1 (\mathbf{1}')^{(2-4n)} \} (\beta_0 e_2^c + \beta'_0 e_3^c) \\ & + \tilde{\phi} \{ (LD^{(4n+2)})_1 (\mathbf{1}')^{(-4n)} + (LD^{(4n)})_{1'} \mathbf{1}^{(2-4n)} \} (\alpha_0 e_1^c + \alpha'_0 e_2^c + \alpha''_0 e_3^c). \end{aligned} \quad (51)$$

Without loss of any generality one can choose the basis in which  $e_3^c$  couples only with  $l_3$ . In the second and third terms of Eq. (51) we have given only relevant invariants.

The couplings responsible for the neutrino sector are

$$\begin{aligned} -\mathcal{L}_\nu = & \phi l_3 \{ (ND^{(4n)})_{1'} \mathbf{1}^{(-4n)} + (ND^{(4n+2)})_1 \mathbf{1}'^{(-4n-2)} \} \\ & + \phi \{ (LD^{(4n+2)} N)_1 \mathbf{1}^{(-4n)} + (LD^{(4n)} N)_{1'} (\mathbf{1}')^{(2-4n)} + (LN)_{1'} (\mathbf{1}')^{(2)} \} \\ & + M \{ (ND^{(4n+2)} N)_1 \mathbf{1}^{(-4n)} + (ND^{(4n)} N)_{1'} (\mathbf{1}')^{(2-4n)} \} \\ & + M' \{ (NN)_1 \mathbf{1}^{(2)} + (ND^{(4n)} N)_1 \mathbf{1}^{(2-4n)} + (ND^{(4n+2)} N)_{1'} (\mathbf{1}')^{(-4n)} \}. \end{aligned} \quad (52)$$

In the Dirac-type couplings of (52) we have not included terms that give  $Z_4^S$  symmetry breaking effects. The coupling  $\gamma$  in (51) is responsible for the mass of the  $\tau$  lepton, while  $\beta_0$  is responsible for the generation of the muon mass. These occur even in the limit of unbroken  $Z_4^S$  symmetry. The mass of the electron will arise from the mild breaking of the  $Z_4^S$  symmetry, achieved through a slight deviation from the point  $\tau = i \rightarrow i + \epsilon_e$ . Consequently, the structures in (47), (44), and the relations in (48) will also shift by  $\sim \epsilon_e$ . In Appendix B we give the potential, with properly selected parameters, giving the needed value of  $\langle \tau \rangle$  near the fixed point  $\tau = i$ .

Taking into account all these and relations of (48), the Dirac and Majorana mass matrices obtained from (51) and (52) will be

$$M_E^{(0)} = \begin{pmatrix} \alpha \epsilon_e & -\frac{\beta}{\sqrt{3}} + \alpha' \epsilon_e & -\frac{\beta'}{\sqrt{3}} + \alpha'' \epsilon_e \\ \alpha \frac{\epsilon_e}{\sqrt{3}} & \beta + \alpha' \frac{\epsilon_e}{\sqrt{3}} & \beta' + \alpha'' \frac{\epsilon_e}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \gamma \end{pmatrix} \frac{\sqrt{3} v}{2} + \mathcal{O}(\epsilon_e^2), \quad (53)$$

$$m_D^{(0)} = \begin{pmatrix} -a & \frac{a}{\sqrt{3}} + b \\ \frac{a}{\sqrt{3}} - b & a \\ \sqrt{3} & 1 \end{pmatrix} \lambda v, \quad M_N = \begin{pmatrix} -1 + n\epsilon_0 & \frac{1}{\sqrt{3}} + \epsilon_0 \\ \frac{1}{\sqrt{3}} + \epsilon_0 & 1 + n\epsilon_0 \end{pmatrix} M. \quad (54)$$

[As noted, in the  $m_D^{(0)}$  the entries  $\sim \epsilon_e$  (i.e., the  $Z_4^S$  breaking effects) are neglected. The latter parameter will come out to be  $\epsilon_e \sim \lambda_e$ .] In  $M_N$ , since the parameter  $\epsilon_0 \sim \frac{M'}{M} \epsilon_e$  can be

much larger than  $\epsilon_e$  (provided that  $\frac{M'}{M} \gg 1$ ), we retain the corresponding entries. Furthermore, making the  $l_1 - l_2$  rotation, i.e., going to the basis  $l \rightarrow U_0^T l$ , where

$$U_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (55)$$

Applying the seesaw formula  $M_\nu = -m_D M_N^{-1} m_D^T$  and using forms of (54), (55), (32), and (57), for the neutrino mass matrix we obtain

the Dirac-type matrices will become

$$M_\nu = U_l \begin{pmatrix} \epsilon_2 & \bar{a} & 1 \\ \bar{a} & \bar{a}^2 \epsilon_1 & \bar{a} \epsilon_1 \\ 1 & \bar{a} \epsilon_1 & \epsilon_1 \end{pmatrix} U_l^T \bar{m}, \quad (58)$$

$$M_E = U_0 M_E^{(0)} = \begin{pmatrix} \alpha \epsilon_e & \alpha' \epsilon_e & \alpha'' \epsilon_e \\ 0 & \beta & \beta' \\ 0 & 0 & \gamma \end{pmatrix} v, \quad m_D^{(1)} = U_0 m_D^{(0)}. \quad (56)$$

where

$$\begin{aligned} \bar{a} &= \frac{a}{\sqrt{3}} - \frac{b}{2}, & \epsilon_1 &= \frac{6(\sqrt{3} - 2n)}{(2a + \sqrt{3}b)(4 + \sqrt{3}\epsilon_0)} \epsilon_0, \\ \epsilon_2 &= -\frac{(2a + \sqrt{3}b)(\sqrt{3} + 2n)}{2(4 + \sqrt{3}\epsilon_0)} \epsilon_0, \\ \bar{m} &= -\frac{(\lambda v)^2}{M} \frac{(2a + \sqrt{3}b)(4 + \sqrt{3}\epsilon_0)}{4 + 2\sqrt{3}\epsilon_0 + 3(1 - n^2)\epsilon_0^2}. \end{aligned} \quad (59)$$

Since the members  $\nu$  and  $e^-$  of the  $l$  doublets get rotated by the same unitary matrix  $U_0$ , the latter does not contribute to the lepton mixing matrix. The form of  $M_E$  coincides with the one given in Eq. (21) obtained and analyzed in the model presented in Sec. III A. Therefore, for the details of  $M_E$ 's study we refer to this section. As before, from the diagonalization of  $M_E$  we will take the form of (32) and make further transformation of the  $l$  states by this matrix. Therefore, in this basis the neutrino's Dirac matrix becomes

$$m_D = U_l m_D^{(1)} = U_l U_0 m_D^{(0)}. \quad (57)$$

In a limit  $\epsilon_0 \rightarrow 0, \theta_e \rightarrow 0$ , Eq. (58) gives bilarge neutrino mixing with the IO neutrinos, but with  $\Delta m_{\text{sol}}^2 = 0, \theta_{13} = 0$ , and  $\theta_{12} = \frac{\pi}{4}$ . Therefore, the  $Z_4^S$  breaking effects are important for the realistic model.<sup>3</sup> With the selection

$$\{\bar{a}, \epsilon_1, \epsilon_2, \theta_e, \eta\} = \{0.85266, 0.11718, -0.1824, 0.19775, \pi\}, \quad |\bar{m}| = 0.037307 \text{ eV}, \quad (60)$$

the texture (58) gives the IO neutrino masses, which coincides with those given in Eq. (38) with the best-fit values of the oscillation parameters of Eqs. (39) and (40). The neutrinoless double  $\beta$ -decay parameter is  $m_{\beta\beta} = 0.0187 \text{ eV}$ .

The choice (60), with the relations of Eq. (59) for the original parameters, gives the natural selection

$$\{a, b, n, \epsilon_0\} = \{1.2472e^{-0.4864i}, -0.8e^i, 1.187e^{0.71586i}, 0.14484e^{0.63142i}\}. \quad (61)$$

With these, for  $|\lambda| \simeq 0.31$  we obtain  $|M| \simeq 2 \times 10^{14} \text{ GeV}$ , which for the heavy RHN masses gives

$$\{M_1, M_2\} = \{2.29, 2.62\} \times 10^{14} \text{ GeV}. \quad (62)$$

## B. Model B

Within this model, two RHNs,  $N$  and  $N'$ , are still introduced. The transformation properties of the states under  $S_3$  are the same as those given in (18) and (25), while the weights are chosen as follows:

$$k_L = -2, \quad k_{l_3} = k_{e_3^c} = k_N = k_{N'} = k_\varphi = 0. \quad (63)$$

With these, the couplings relevant for the charged lepton masses will be

$$\begin{aligned} -\mathcal{L}_E &= \gamma \tilde{\varphi} l_3 e_3^c - \tilde{\varphi} \{ (LD^{(4n+2)})_{1'} \mathbf{1}^{(-4n)} + (LD^{(4n)})_1 (\mathbf{1}')^{(2-4n)} \} (\beta_0 e_2^c + \beta'_0 e_3^c) \\ &\quad + \tilde{\varphi} \{ (LD^{(4n+2)})_1 (\mathbf{1}')^{(-4n)} + (LD^{(4n)})_{1'} \mathbf{1}^{(2-4n)} \} (\alpha_0 e_1^c + \alpha'_0 e_2^c + \alpha''_0 e_3^c), \end{aligned} \quad (64)$$

<sup>3</sup>In [33], a similar texture was obtained with a  $U(1)$  flavor symmetry, in which symmetry breaking effects played an important role in realistic neutrino oscillations.

while the couplings in the neutrino sector are

$$\begin{aligned}
-\mathcal{L}_\nu = & \varphi l_3 \{N' \mathbf{1}^{(0)} + N(\mathbf{1}')^{(0)}\} \\
& + \varphi \{ (LD^{(4n+2)})_1 \mathbf{1}^{(-4n)} + (LD^{(4n)})_{1'} (\mathbf{1}')^{(2-4n)} + (LD^{(4n)})_1 (\mathbf{1})^{(2-4n)} + (LD^{(4n+2)})_{1'} (\mathbf{1}')^{(-4n)} \} N \\
& + \varphi \{ (LD^{(4n+2)})_{1'} \mathbf{1}^{(-4n)} + (LD^{(4n)})_1 (\mathbf{1}')^{(2-4n)} + (LD^{(4n)})_{1'} (\mathbf{1})^{(2-4n)} + (LD^{(4n+2)})_1 (\mathbf{1}')^{(-4n)} \} N' \\
& + \frac{1}{2} M \{ N^2 \mathbf{1}^{(0)} + (N')^2 \mathbf{1}^{(0)} + 2NN'(\mathbf{1}')^{(0)} \}.
\end{aligned} \tag{65}$$

Using in (64) and (65) the structures of (47) and (44), and the relations (48) (shifted by  $\sim \epsilon_e$  due to  $Z_4^S$  breaking effects), and at the same time disregarding the effects of  $Z_4^S$  breaking in the neutrino Dirac-type couplings (as in Model A, discussed in Sec. IV A), the Dirac and Majorana mass matrices will be

$$M_E^{(0)} = \begin{pmatrix} \alpha \epsilon_e & -\frac{\beta}{\sqrt{3}} + \alpha' \epsilon_e & -\frac{\beta'}{\sqrt{3}} + \alpha'' \epsilon_e \\ \alpha \frac{\epsilon_e}{\sqrt{3}} & \beta + \alpha' \frac{\epsilon_e}{\sqrt{3}} & \beta' + \alpha'' \frac{\epsilon_e}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \gamma \end{pmatrix} \frac{\sqrt{3}v}{2} + \mathcal{O}(\epsilon^2), \tag{66}$$

$$m_D^{(0)} = \begin{pmatrix} \sqrt{3}b & -a \\ b & \sqrt{3}a \\ 0 & 2 \end{pmatrix} \frac{\lambda v}{2}, \quad M_N = \begin{pmatrix} 1 & -\epsilon_0 \\ -\epsilon_0 & n \end{pmatrix} M. \tag{67}$$

(As noted, in  $m_D$  are neglected entries  $\sim \epsilon_e$ . The latter parameter will come out to be  $\epsilon_e \sim \lambda_e$ .) Furthermore, as we have done for Model A (of Sec. IV A), making the  $l_1 - l_2$  rotation, i.e., going to the basis  $l \rightarrow U_0^T l$  [where  $U_0$  is given in Eq. (55)], the Dirac-type matrices will become

$$M_E = U_0 M_E^{(0)} = \begin{pmatrix} \alpha \epsilon_e & \alpha' \epsilon_e & \alpha'' \epsilon_e \\ 0 & \beta & \beta' \\ 0 & 0 & \gamma \end{pmatrix} v, \quad m_D = U_0 m_D^{(0)} = \begin{pmatrix} b & 0 \\ 0 & a \\ 0 & 1 \end{pmatrix} \lambda v. \tag{68}$$

Since  $U_0$  transforms the entire lepton doublet states  $l_{1,2,3}$ , the lepton mixing matrix is determined by the diagonalization of the matrices  $M_E$  and  $M_\nu$ . As we see, the matrices in Eq. (68) and  $M_N$  are the same as those in (21) and (30) (obtained near the  $\tau = i\infty$  fixed point). Therefore, the results (e.g., charged lepton masses and the neutrino fit) and conclusions will be essentially the same as those previously discussed in Sec. III. Thus, the IO neutrino scenario remains the preferred choice within this model as well.

## V. MODEL NEAR $\tau = \omega$ FIXED POINT: “INTERMEDIATE” $Z_3^{ST}$ SYMMETRY

At  $\tau = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$  fixed point we have residual  $Z_3^{ST}$  symmetry [10–12,16] realized by the  $ST$  transformation. Under this transformation, at this point, the modular form  $f^{(k)}$  transforms as  $f^{(k)} \xrightarrow{ST} \omega^k \rho(ST) f^{(k)}$ . Thus, the elements of  $Z_3^{ST}$  are  $g_i(Z_3^{ST}) = \{1, ST, (ST)^2\}$ .

As shown in Appendix A 3, at  $\tau = \omega$  the modular doublets with weights  $(6n + p)$  (where  $p = 0, 2, 4$ )

possess the following structures:

$$\begin{aligned}
\text{at } \tau = \omega: \quad D^{(6n)} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D^{(6n+2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \\
D^{(6n+4)} &= \begin{pmatrix} 1 \\ i \end{pmatrix}.
\end{aligned} \tag{69}$$

It is easy to check that  $ST$  transformation leaves them invariant,

$$D^{(4n+p)}(\omega) = \omega^p \rho_{[2]}(S) \rho_{[2]}(T) D^{(4n+p)}(\omega), \quad p = 0, 2, 4, \tag{70}$$

where  $\rho_{[2]}(S)$  and  $\rho_{[2]}(T)$  are given in (A11).

Besides these, at  $\tau = \omega$ , among the modular forms  $\tilde{E}_2$  and  $E_{4,6}$  only  $E_6$  has nonzero value (A26). Also, as discussed in Appendix A 3, at  $\tau = \omega$  the singlet and pseudosinglet modular forms of different weights are

$$\begin{aligned} \text{at } \tau = \omega: \quad \mathbf{1}^{(6n)} &\neq 0, & \mathbf{1}^{(6n+2)} &= \mathbf{1}^{(6n+4)} = 0, \\ (\mathbf{1}')^{(6n)} &\neq 0, & (\mathbf{1}')^{(6n+2)} &= (\mathbf{1}')^{(6n+4)} = 0. \end{aligned} \quad (71)$$

Additional properties useful for model building are provided by the following relations, which involve the doublet field  $\psi = (\psi_1, \psi_2)$  and the modular doublets:

$$\text{at } \tau = \omega: (\psi D^{(6n+2)})_1 = i(\psi D^{(6n+2)})_{1'}, (\psi D^{(6n+4)})_{1'} = i(\psi D^{(6n+4)})_1. \quad (72)$$

The following subsection details Model C in the neighborhood of the  $\tau = \omega$  fixed point.

### A. Model C

For this model, the transformation properties of the SM leptons and the Higgs field under  $S_3$  are the same as those given in (18). Two RHN states are embedded in the  $S_3$  doublet, as given in (49), and the weights are chosen as follows:

$$k_L = k_{e_i^c} = -1, \quad k_{l_3} = k_N = 1, \quad k_\varphi = 0. \quad (73)$$

With these, the couplings relevant for the charged lepton masses will be

$$\begin{aligned} -\mathcal{L}_E &= \gamma \tilde{\varphi} l_3 e_3^c + \tilde{\varphi} \{ (LD^{(6n+2)})_{1'} \mathbf{1}^{(-6n)} + (LD^{(6n+2)})_1 (\mathbf{1}')^{(-6n)} \} (\beta_0 e_2^c + \beta'_0 e_3^c) \\ &\quad + \tilde{\varphi} \{ (LD^{(6n+4)})_1 (\mathbf{1}')^{(-6n-2)} + (LD^{(6n+4)})_{1'} \mathbf{1}^{(-6n-2)} \} (\alpha_0 e_1^c + \alpha'_0 e_2^c + \alpha''_0 e_3^c). \end{aligned} \quad (74)$$

Invariants  $(LD^{(6n)})_{1'} \mathbf{1}^{(2-6n)}$  and  $(LD^{(6n)})_1 (\mathbf{1}')^{(2-6n)}$  are not included, as their contributions lead to negligible couplings of order  $\epsilon_e^2$ .

The couplings relevant for the neutrino sector are

$$\begin{aligned} -\mathcal{L}_\nu &= \varphi l_3 \{ (ND^{(6n-2)})_{1'} \mathbf{1}^{(-6n)} + (ND^{(6n-2)})_1 (\mathbf{1}')^{(-6n)} \} + \varphi \{ (LN)_1 \mathbf{1}^{(0)} + (LN)_{1'} (\mathbf{1}')^{(0)} \} \\ &\quad + M \{ (ND^{(6n-2)} N)_1 \mathbf{1}^{(-6n)} + (ND^{(6n-2)} N)_{1'} (\mathbf{1}')^{(-6n)} \} \\ &\quad + M' \{ (NN)_1 \mathbf{1}^{(-2)} + (ND^{(6n+2)} N)_1 \mathbf{1}^{(-6n-4)} + (ND^{(6n+2)} N)_{1'} (\mathbf{1}')^{(-6n-4)} \}. \end{aligned} \quad (75)$$

In Eq. (75), we have not included  $Z_3^{ST}$  symmetry breaking terms within the Dirac-type couplings. In addition, the invariants  $(ND^{(6n)} N)_1 \mathbf{1}^{(-6n-2)}$  and  $(ND^{(6n)} N)_{1'} (\mathbf{1}')^{(-6n-2)}$  have been omitted, as they provide corrections of order  $\sim \epsilon_e^2$ .

Similar to the models presented in previous sections, the electron mass in this case arises from the mild breaking of the  $Z_3^{ST}$  symmetry, induced by a tiny deviation from the fixed point  $\tau = \omega \rightarrow \omega + \epsilon_e$ . Consequently, the structures

in (71) and (69), as well as the relations in (72), will also be shifted by  $\sim \epsilon_e$ . How the needed value of  $\langle \tau \rangle$  is obtained near the fixed point  $\tau = \omega$  is addressed in Appendix B. The couplings  $\gamma$  and  $\beta_0$  in (74) are responsible for the masses of the  $\tau$  lepton and the muon, respectively.

Taking into account all the above, the Dirac and Majorana mass matrices obtained from (74) and (75) will be

$$M_E^{(0)} = \begin{pmatrix} \alpha \epsilon_e & \beta + \alpha' \epsilon_e & \beta' + \alpha'' \epsilon_e \\ i \alpha \epsilon_e & -i \beta + i \alpha' \epsilon_e & -i \beta' + i \alpha'' \epsilon_e \\ 0 & 0 & \sqrt{2} \gamma \end{pmatrix} \frac{v}{\sqrt{2}} + \mathcal{O}(\epsilon_e^2), \quad (76)$$

$$m_D^{(0)} = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \\ 1 & i \end{pmatrix} \lambda v, \quad M_N = \begin{pmatrix} -1 + \epsilon_0 & i + n_0 \epsilon_0 \\ i + n_0 \epsilon_0 & 1 + \epsilon_0 \end{pmatrix} M. \quad (77)$$

The parameter  $\epsilon_e$  will come out to be  $\epsilon_e \sim \lambda_e$ . With  $\frac{M'}{M} \gg 1$ , in  $M_N$  the parameter  $\epsilon_0 \sim \frac{M'}{M} \epsilon_e$  can be much larger than  $\epsilon_e$ .

Now, going to the basis  $l \rightarrow U_0^T l$ , with

$$U_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (78)$$

(i.e., performing the  $l_1 - l_2$  rotation), the Dirac-type matrices will become

$$M_E = U_0 M_E^{(0)} = \begin{pmatrix} \alpha \epsilon_e & \alpha' \epsilon_e & \alpha'' \epsilon_e \\ 0 & \beta & \beta' \\ 0 & 0 & \gamma \end{pmatrix} v, \quad m_D^{(1)} = U_0 m_D^{(0)}. \quad (79)$$

The form of  $M_E$  coincides with the one given in Eq. (21), obtained and analyzed in the model presented in Sec. III A. Thus, for the details we refer to this section. As before, from the diagonalization of  $M_E$  we will take form of (32) and perform further transformation of the  $l$  states by this matrix. Thus, in this basis the neutrino's Dirac matrix becomes

$$m_D = U_l m_D^{(1)} = U_l U_0 m_D^{(0)}. \quad (80)$$

Using the forms of (77), (78), (32), and (80) in the seesaw formula  $M_\nu = -m_D M_N^{-1} m_D^T$ , for the neutrino mass matrix we obtain

$$M_\nu = \begin{pmatrix} n\epsilon & a(\epsilon + \epsilon_1) & \epsilon \\ a(\epsilon + \epsilon_1) & a^2 \left(1 + \frac{\epsilon_1^2}{n\epsilon - \epsilon^2}\right) & a \\ \epsilon & a & 1 \end{pmatrix} \bar{m}, \quad (81)$$

where the parameters of  $M_\nu$  are expressed in terms of the original parameters as follows:

$$\begin{aligned} a &= c_e a_1 - e^{-i\eta} s_e \xi, & n &= \frac{e^{2i\eta} s_e^2 a_1^2 + c_e^2 \xi n_1 + 2e^{i\eta} c_e s_e a_1 \xi}{e^{i\eta} s_e a_1 + c_e \xi}, \\ \epsilon &= e^{i\eta} s_e a_1 + c_e \xi, & \epsilon_1 &= \frac{c_e s_e (n_1 - \xi)}{s_e \xi - e^{i\eta} c_e a_1}, & \bar{m} &= \frac{(\lambda v)^2}{M} \frac{2(in_0 \epsilon_0 - 2)}{\epsilon_0^2 (1 - n_0^2) - 2in_0 \epsilon_0}, \\ \text{with } a_1 &= \frac{1}{\sqrt{2}} (a_0 - ib_0), & \xi &= \frac{1}{\sqrt{2}} \frac{a_0 + ib_0}{2 - in_0 \epsilon_0} \epsilon_0, & n_1 &= \frac{i}{\sqrt{2}} (a_0 + ib_0) n_0. \end{aligned} \quad (82)$$

The texture (81) matches the form of (33) (near the  $\tau = i\infty$  fixed point) by replacing  $n\epsilon$  with  $b$ . As a result, the relations from Eq. (36) still hold, favoring the IO neutrino mass hierarchy under the condition that  $\theta_e$  remains small.

We present a selection of natural parameter values that provide a consistent fit. In particular, the choice

$$\begin{aligned} \{a_0, b_0, \theta_e, \eta\} &= \{2.3394, 2.9732e^{-i1.9616}, 0.19765, -0.033185\}, \\ \{\lambda, n_0, \epsilon_0\} &= \{0.3, 1.2403e^{i2.345}, 0.2006e^{i0.94271}\}, & M &= 7.306 \times 10^{14} \text{ GeV}, \end{aligned} \quad (83)$$

yields oscillation parameters in Eqs. (38)–(40), which are in excellent agreement with the best-fit values for the IO scenarios. Note that a slightly larger selection for the values of  $a_0$  and  $b_0$  does not pose any problems, as the physical Yukawa couplings  $a_0 \lambda, b_0 \lambda \lesssim 1$  remain perturbative. The selection in (83) yields  $\{\delta, \rho_1\} = \{6.25, 5.55\}$ ,

$m_{\beta\beta} = 0.0456$  eV, and results in heavy RHN masses of  $\{M_1, M_2\} \simeq \{1.95, 14.6\} \times 10^{14}$  GeV.

In summary, we have investigated the SM extension with the minimal modular flavor symmetry  $\Gamma_2 \simeq S_3$  and built the lepton sector near the  $\tau = i\infty$ ,  $\tau = i$ , and  $\tau = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$  fixed points. In our construction, the scalar sector is extended

solely by a single complex modulus,  $\tau$ , which serves as the fundamental variable of the modular forms. These forms transform under  $S_3$  representations, allowing us to omit traditional flavon scalars from the theory. The fermion sector is similarly minimal, featuring an extension of only two right-handed neutrinos,  $N_{1,2}$ . The residual symmetries at the fixed points greatly reduce the number of independent invariants (in which nonholomorphic modular forms participate) and also allow us to make some predictions. In the same spirit, it would be interesting to examine larger modular groups, such as  $\Gamma_{3,4,5,\dots} \simeq A_4, S_4, A_5, \dots$ , and to include the quark sector as well. While the minimal  $S_3$  modular group is highly constrained, larger modular groups can possess several small expansion parameters, allowing for a successful explanation of the hierarchies among the three generations. These and related issues remain to be pursued elsewhere [32].

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### DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the author upon reasonable request.

### APPENDIX A: MODULAR FORMS AND $\Gamma_2 \simeq S_3$ REPRESENTATIONS

For building the modular forms, the Eisenstein series play a central role [26–28]. The modular form  $\tilde{E}_2(\tau)$  with the transformation property (5) (with  $k = 2$ ) is

$$\tilde{E}_2(\tau) = E_2(\tau) - \frac{6i}{\pi(\tau - \bar{\tau})}, \quad (\text{A1})$$

where  $E_2(\tau)$  is the lowest Eisenstein series,

$$\begin{aligned} E_2(\tau) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \\ &= 1 - 24(q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \dots), \end{aligned} \quad (\text{A2})$$

where  $\sigma_1(n)$  denotes the sum of all divisors of  $n$ , and  $q = e^{2\pi i \tau}$ . While  $\tilde{E}_2$  is not holomorphic, the  $E_4(\tau)$  and  $E_6(\tau)$  Eisenstein series, of weights 4 and 6, respectively, are holomorphic modular forms and are given by

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\ E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \end{aligned} \quad (\text{A3})$$

where  $\sigma_p(n)$  denotes the sum of the  $p$ th powers of all divisors of  $n$  [e.g.,  $\sigma_3(4) = 1 + 2^3 + 4^3$ ,  $\sigma_5(6) = 1 + 2^5 + 3^5 + 6^5$ , etc.].

*Group  $\Gamma_2 \simeq S_3$  and It's Properties.*

The finite group  $\Gamma_2 \simeq S_3$  has six elements  $g_i$  ( $i = 1 \div 6$ ), which are expressed by two generating elements  $S$  and  $T$ ,

$$g(S_3) = \{1, T, S, TS, ST, STS\}, \quad (\text{A4})$$

where  $S$  and  $T$  satisfy

$$T^2 = S^2 = 1, \quad (ST)^3 = 1. \quad (\text{A5})$$

The group  $S_3$  possesses, in addition to the singlet representation **1**, a pseudosinglet representation **1'** and a doublet representation **2**. According to the multiplication rule for doublets  $\mathbf{2} \times \mathbf{2} = \mathbf{2} + \mathbf{1} + \mathbf{1}'$ , we have

$$\begin{aligned} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_2 \times \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}_2 &= \begin{pmatrix} A_2 B_2 - A_1 B_1 \\ A_1 B_2 + A_2 B_1 \end{pmatrix}_2 + (A_1 B_1 + A_2 B_2)_1 \\ &\quad + (A_1 B_2 - A_2 B_1)_{1'}. \end{aligned} \quad (\text{A6})$$

In addition to (A6), the multiplication rule

$$C_{1'} \times (A_1, A_2)_2 = (-CA_2, CA_1) \quad (\text{A7})$$

may be useful for model building. Moreover, the following multiplication rules are straightforward:

$$\mathbf{1} \times \mathbf{1}' = \mathbf{1}' \quad \text{and} \quad \mathbf{1}' \times \mathbf{1}' = \mathbf{1}. \quad (\text{A8})$$

Using these, one can obtain representations **2**, **1**, and **1'** of any even weights by multiplying modular forms with a given weight and belonging to the appropriate representations of the  $S_3$ .

An  $S_3$  singlet modular form of any even weight is given by the general expression (7). In fact, starting from some modular form of weight  $k$  and applying the weight-raising operator  $\hat{\partial}$  and the weight-lowering operator  $\hat{\delta}^*$  [27,29], one can construct modular forms of weights  $k + 2$  and  $k - 2$ , respectively. These operators are defined by [27,29]

$$\begin{aligned} \hat{\partial} &:= \frac{1}{2\pi i} \left( \frac{\partial}{\partial \tau} + \frac{k}{\tau - \bar{\tau}} \right) = \frac{1}{4\pi i} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} - i \frac{k}{y} \right), \\ \hat{\delta}^* &:= 2\pi i (\tau - \bar{\tau})^2 \frac{\partial}{\partial \bar{\tau}} = -4\pi i y^2 \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \end{aligned} \quad (\text{A9})$$

By applying these operators in sequence, one can obtain modular forms of any desired weight. For example, taking the weight-0 form  $\kappa^2|\tilde{E}_2|^2$ , and applying these operators, one obtains modular forms of weights 2 and  $-2$ ,

$$\begin{aligned}\hat{\partial}(\kappa^2|\tilde{E}_2|^2) &= \frac{3}{4\pi^2}\tilde{E}_2 + \frac{1}{12}\kappa^2(\tilde{E}_2^2 - E_4)\tilde{E}_2^*, \\ \hat{\delta}^*(\kappa^2|\tilde{E}_2|^2) &= 12\kappa^2\tilde{E}_2^* + \frac{4\pi^2}{3}\kappa^4(\tilde{E}_2^{*2} - E_4^*)\tilde{E}_2,\end{aligned}\quad (\text{A10})$$

where the derivatives of modular forms have been evaluated employing Ramanujan's formulae (see Eq. (3) in [29]). Both resulting modular forms of Eq. (10) are contained in (7).

In the doublet representation, two generating group elements  $\rho_{[2]}(T)$  and  $\rho_{[2]}(S)$  are

$$\rho_{[2]}(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_{[2]}(S) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad (\text{A11})$$

and the  $S_3$  doublet of weight  $k$ — $D^{(k)}(\tau) = (D_1^{(k)}(\tau), D_2^{(k)}(\tau))$ —has the following transformation properties:

$$\begin{aligned}T: \quad D^{(k)}(\tau+1) &= \rho_{[2]}(T)D^{(k)}(\tau), \\ S: \quad D^{(k)}\left(-\frac{1}{\tau}\right) &= \tau^k \rho_{[2]}(S)D^{(k)}(\tau).\end{aligned}\quad (\text{A12})$$

The  $S_3$  doublet of weight  $k$ — $D^{(k)}(\tau) = (D_1^{(k)}(\tau), D_2^{(k)}(\tau))$ —can be constructed from the  $f^{(k)}(\tau)$  modular form of weight  $k$  [see (5)] via the functions  $f^{(k)}(\frac{\tau}{2})$ ,  $f^{(k)}(\frac{\tau+1}{2})$ , and  $f^{(k)}(2\tau)$ , as follows:

$$\begin{aligned}D_1^{(k)}(\tau) &= -a_k \left[ f^{(k)}\left(\frac{\tau}{2}\right) + f^{(k)}\left(\frac{\tau+1}{2}\right) - 2^{k+1}f^{(k)}(2\tau) \right], \\ D_2^{(k)}(\tau) &= -a_k \sqrt{3} \left[ f^{(k)}\left(\frac{\tau}{2}\right) - f^{(k)}\left(\frac{\tau+1}{2}\right) \right],\end{aligned}\quad (\text{A13})$$

where  $a_k$  are some normalization factors and can be selected by convenience. Using (5), (A11), and (A13) one can easily verify that the transformations (A12) are indeed acquired.

Using this result, the weight-2 doublet—denoted by  $Y = (Y_1, Y_2)$ —can be built from  $\tilde{E}_2$ . Using in (A13)  $f^{(2)} = \tilde{E}_2$  and  $a_2 = 1/48$ , we will have

$$\begin{aligned}Y_1(\tau) &= 1/8 + 3q + 3q^2 + 12q^3 + 3q^4 + 18q^5 + 12q^6 + 24q^7 + 3q^8 + \dots, \\ Y_2(\tau) &= \sqrt{3}e^{i\pi\tau}(1 + 4q + 6q^2 + 8q^3 + 13q^4 + 12q^5 + 14q^6 + 24q^7 + 18q^8 + \dots).\end{aligned}\quad (\text{A14})$$

Note that in  $Y_{1,2}$  the nonholomorphic parts cancel out and, therefore, the doublet  $Y$  is holomorphic. The weight-4 holomorphic doublet— $Y^{(4)} = (Y_1^{(4)}, Y_2^{(4)})$ —can be built by taking  $f^{(4)} = E_4$  and using it in (A13) with  $k = 4$ . As a result (by selecting  $a_4 = -1/480$ ), we obtain

$$\begin{aligned}Y_1^{(4)}(\tau) &= -1/16 + 9q + 57q^2 + 252q^3 + 441q^4 + 1134q^5 + 1596q^6 \dots, \\ Y_2^{(4)}(\tau) &= \sqrt{3}e^{i\pi\tau}(1 + 28q + 126q^2 + 344q^3 + 757q^4 + 1332q^5 + 2198q^6 \dots).\end{aligned}\quad (\text{A15})$$

We could obtain these results by extracting the doublet component from the product  $(Y \cdot Y)_2$ ,

$$(Y \cdot Y)_2 = (Y_2^2 - Y_1^2, 2Y_1Y_2), \quad (\text{A16})$$

and then by applying in (A16) the expansions given in (14).

Since we have only two independent holomorphic doublets  $Y$  and  $(Y \cdot Y)_2$ , the holomorphic doublet of weight  $k = 2n$  is given by

$$(Y^n)_2 = a^{(2n-2)}Y + b^{(2n-4)}(Y \cdot Y)_2, \quad (\text{A17})$$

with  $a^{(2n-2)}$  and  $b^{(2n-4)}$  being the holomorphic forms of weights  $(2n-2)$  and  $(2n-4)$ , respectively. They are given by the polynomials  $\sum_{m,n} c_{mn} E_4^m E_6^n$  of non-negative powers of  $E_4$  and  $E_6$ . Thus, with only holomorphic forms remaining, we have only two independent doublets. That is why SUSY model building is very constrained and allows us to have some predictions.

Without sticking on holomorphy, from the  $Y$  we can have five independent doublets,

$$\begin{aligned} Y &= (Y_1, Y_2), & Y^* &= (Y_1^*, Y_2^*), & (Y \cdot Y)_2 &= (Y_2^2 - Y_1^2, 2Y_1Y_2), \\ (Y \cdot Y^*)_2 &= (|Y_2|^2 - |Y_1|^2, Y_1Y_2^* + Y_1^*Y_2), & (Y^* \cdot Y^*)_2 &= (Y_2^{*2} - Y_1^{*2}, 2Y_1^*Y_2^*). \end{aligned} \quad (\text{A18})$$

All these have right transformation properties under  $S_3$ , because the matrices  $\rho_{[2]}$  [given in Eq. (A11)] are real. Doublets built from the powers of  $Y, Y^*$  higher than 2 will not be independent. For example, using the rules (A6) and (A7), one can verify that  $((Y \cdot Y)_2 Y^*)_2 = (Y \cdot Y^*)_1 Y \oplus ((Y \cdot Y^*)_1 Y)_2$ . From these, and with the help of the modular forms  $\tilde{E}_2, E_{4,6}$ , and  $\kappa = (i\tau - i\bar{\tau})$  [given in Eq. (6)], one can construct five doublets of any fixed even weight  $k$  and they are given in (11), where  $f(\tau)$ 's are the corresponding singlet modular forms given in (7).

Applying (11) we can write down the five doublets with weight  $k = 2$ ,

$$\begin{aligned} k = 2 \text{ weight doublets: } & Y, & \kappa^2(\tilde{E}_2^2 + c_1 E_4) Y^*, & \kappa^2 \tilde{E}_2 (Y \cdot Y)_2, \\ & \kappa^2 \tilde{E}_2 (Y \cdot Y^*)_2, & \kappa^4(\tilde{E}_2^3 + c_2 \tilde{E}_2 E_4 + c_3 E_6)(Y^* \cdot Y^*)_2, \end{aligned} \quad (\text{A19})$$

where  $c_{1,2,3}$  are couplings used to construct the given weight form from specific superpositions. In (19) we kept minimal possible powers of the factor  $\kappa$  [given in Eq. (6)].

Having constructed the doublets of different weights, using the multiplication rule (A6), we can construct the pseudosinglet modular form with weight  $k$ , which is given in (12).

Before moving to the next subsection, several comments are in order. While Eq. (7) provides even-weight modular forms with the transformation properties given in (5), one can obtain a harmonic Maass form of weight  $k$  [34] by imposing the additional constraint  $\Delta_k f^{(k)}(\tau) = 0$ . Here,  $\Delta_k = -4y^2 \frac{\partial}{\partial \tau} y^k \frac{\partial}{\partial \bar{\tau}}$  represents the weight- $k$  hyperbolic Laplacian. The form  $\tilde{E}_2$  is the weight-2 Maass form captured by Eq. (7). Furthermore, because any holomorphic modular form is a harmonic Maass form, those with weights  $k > 2$  are simply holomorphic forms. Given that combinations such as  $E_4^m E_6^n$  are included in Eq. (7), it follows that all even-weight harmonic Maass forms for  $k \geq 2$  are subsets of (7). If we allow negative powers of  $\kappa, \tilde{E}_2$ , and  $E_{4,6}$ , even-weight Maass forms with  $k < 2$  (including negative weights) may emerge from Eq. (7), provided the couplings  $C_{lmnp}^{l\bar{m}\bar{n}}$  are properly selected. Indeed, the holomorphic parts of these negative even-weight Maass forms are  $\sim 1/(E_4^m E_6^n)$  [35]. Since the present work focuses on models in the vicinity of fixed points, we have excluded negative powers of the forms  $\tilde{E}_2$  and  $E_{4,6}$  to avoid introducing poles into the modular forms. Furthermore, beyond modular symmetry itself, we find no compelling motivation for imposing the additional harmonic constraint  $\Delta_k f^{(k)}(\tau) = 0$  in our construction. While models utilizing harmonic Maass forms are interesting and were constructed in [23], such frameworks would require extra caution when building models near fixed points due to the potential emergence of poles.

### 1. Representations at $\tau = i\infty$ fixed point

In this case we can take the limit  $\langle y \rangle \rightarrow \infty$  and examine the behavior of the modular forms. From (A1)–(A3) we see that  $\tilde{E}_2 = E_4 = E_6 = 1$ . Also, with (14) we see the  $Y_2 \rightarrow 0$ , and therefore, through (11), one can verify that the second component of any modular doublet vanishes,  $D_2^{(k)}(i\infty) = 0$ . On the other hand, using (12), one can see that any pseudosinglet modular form is also zero. Therefore, for the doublet and the pseudosinglet modular forms we will have the structures given in (13).

For model building, instead of taking  $\langle y \rangle$  to be infinite, one can obtain quite good approximations even with  $\langle y \rangle \geq 2$ . This situation can be referred to as the vicinity of the fixed point  $\tau = i\infty$ . Since all doublet modular forms (11) are built either from one or two powers of  $Y$ , with the help of Eqs. (14) and (15), it is easy to check that, with a good approximation, one can obtain

$$\text{for } y \gtrsim 2: \quad \left| \frac{\langle D_2 \rangle}{\langle D_1 \rangle} \right| \simeq 8\sqrt{3}e^{-\pi y} \quad \text{or} \quad 16\sqrt{3}e^{-\pi y}. \quad (\text{A20})$$

As a consequence, in the vicinity of the fixed point  $\tau = i\infty$ , the modular forms can be parametrized as in Eq. (16).

### 2. Representations at $\tau = i$ fixed point

At the fixed point  $\tau = i$ , among the three modular forms  $\tilde{E}_2, E_4$ , and  $E_6$ , only  $E_4$  has a nonvanishing value,

$$\text{at } \tau = i: \quad \tilde{E}_2 = E_6 = 0, \quad E_4 \simeq 1.4558. \quad (\text{A21})$$

Therefore, in the construction of the singlet modular forms (7), only  $E_4$  and  $\kappa$  will participate, and we will have

$$\text{at } \tau = i: \quad f^{(k)} = \sum C_m^{\bar{m}} \kappa^{2p} E_4^m (E_4^{\bar{m}})^*,$$

$$\text{with } p = 2\bar{m}, \quad k = 4(m - \bar{m}), \quad (\text{A22})$$

from which we can see that nonvanishing singlet modular forms have weights  $4n$ , while singlets with weights  $(4n + 2)$  vanish,

$$\text{at } \tau = i: \quad \mathbf{1}^{(4n)} \neq 0, \quad \mathbf{1}^{(4n+2)} = 0, \quad (\text{A23})$$

as also shown in Eq. (47).

For the modular doublets, it turns out that for  $Y$  and  $(Y \cdot Y)_2 \propto Y^{(4)}$ ,

$$\text{at } \tau = i: \quad Y \propto \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (Y \cdot Y)_2 \propto \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \quad (\text{A24})$$

both are real. Using (A23) and (A24) in (11), the modular doublets with the weights  $(4n + 2)$  and  $4n$  yield the structures shown in Eq. (44).

Finally, with the forms of the doublets in (44) and using (12), the pseudosinglet modular forms of weights  $(4n + 2)$  and  $4n$ , respectively, yield

$$\text{at } \tau = i: \quad (\mathbf{1}')^{(4n+2)} \neq 0, \quad (\mathbf{1}')^{(4n)} = 0, \quad (\text{A25})$$

which is also indicated in Eq. (47).

### 3. Representations at $\tau = \omega$ fixed point

At the fixed point  $\tau = \omega$ , the modular forms  $\tilde{E}_2, E_4$ , and  $E_6$  take values

$$\text{at } \tau = \omega: \quad \tilde{E}_2 = E_4 = 0, \quad E_6 \simeq 2.8815. \quad (\text{A26})$$

Therefore, in the construction of the singlet modular forms (7), only  $E_6$  and  $\kappa$  will participate as

$$\text{at } \tau = \omega: \quad f^{(k)} = \sum C_m^{\bar{m}} \kappa^{2p} E_6^m (E_6^{\bar{m}})^*.$$

$$\text{with } p = 3\bar{m}, \quad k = 6(m - \bar{m}). \quad (\text{A27})$$

From (A27) we can see that nonvanishing singlet modular forms have weights  $6n$ , while singlets with weights  $(6n + 2)$  and  $(6n + 4)$  vanish, as given in Eq. (71).

For the modular doublets, it turns out that  $Y$  and  $(Y \cdot Y)_2 \propto Y^{(4)}$ ,

$$\text{at } \tau = \omega: \quad Y \propto \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad (Y \cdot Y)_2 \propto \begin{pmatrix} 1 \\ i \end{pmatrix} \propto Y^*, \quad (\text{A28})$$

have structures of the conjugate of each other. Using (A28) and (71) in (11), the modular doublets with the weights

$(6n + p)$  (where  $p = 0, 2, 4$ ) have the structures given in Eq. (69).

Finally, with the forms of the modular doublets in (69) and using (12), at  $\tau = \omega$  we can obtain the different weight pseudosinglet modular forms, which are given in Eq. (71).

## APPENDIX B: KINETIC COUPLINGS AND FIXING $\langle \tau \rangle$

For the fermionic state  $\psi$ , with the weight  $k_\psi$ , using the two-component Weyl spinor, the kinetic Lagrangian invariant under the modular transformation

$$\psi' = (c\tau + d)^{k_\psi} \psi, \quad \bar{\psi}' = (c\bar{\tau} + d)^{k_\psi} \bar{\psi},$$

has the form

$$\begin{aligned} \mathcal{L}_{\text{kin}}(\psi) = & \frac{i}{2} (\partial_\mu \bar{\psi} \bar{\sigma}^\mu \psi - \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi) (i\bar{\tau} - i\tau)^{k_\psi} \\ & - \frac{k_\psi}{2} (\partial_\mu \bar{\tau} + \partial_\mu \tau) (i\bar{\tau} - i\tau)^{k_\psi-1} \bar{\psi} \bar{\sigma}^\mu \psi. \end{aligned} \quad (\text{B1})$$

Note that, we are limiting ourselves by focusing solely on the minimal kinetic couplings, without accounting for kinetic mixings between different flavors. With the vacuum expectation value of the  $\langle \tau \rangle$ , the canonically normalized fermion state  $\psi_c$  will be related to  $\psi$  as follows:

$$\psi = \frac{1}{(i\langle \bar{\tau} \rangle - i\langle \tau \rangle)^{k_\psi/2}} \psi_c. \quad (\text{B2})$$

The kinetic term for the field  $\tau$  is

$$\mathcal{L}_{\text{kin}}(\tau) = \frac{\Lambda^2}{(i\bar{\tau} - i\tau)^2} \partial_\mu \bar{\tau} \partial^\mu \tau, \quad (\text{B3})$$

which is invariant under (2) transformation. The  $\Lambda$  is some mass scale. The canonically normalized state  $\tau_0 = \frac{1}{\sqrt{2}}(x_0 + iy_0)$  is related to  $\tau$  as

$$\tau = \langle \tau \rangle + i \frac{\langle \bar{\tau} \rangle - \langle \tau \rangle}{\sqrt{2}\Lambda} (x_0 + iy_0) + \dots, \quad (\text{B4})$$

where “...” stand for the higher powers of the fields, which are not relevant to us.

Near the  $\tau = i\infty$  and  $\tau = i$  fixed points, the desired values of  $\langle \tau \rangle$  can be obtained in a rather simple way via the two invariants,

$$X_1 = \kappa^2 |\tilde{E}_2|^2, \quad X_2 = \kappa^4 (\tilde{E}_2^2 E_4^* + \tilde{E}_2^{*2} E_4), \quad (\text{B5})$$

through which the potential for the  $\tau$  can be constructed,

$$V(\tau) = \frac{\lambda_1}{2} \Lambda^4 (X_1 - a_1)^2 + \frac{\lambda_2}{2} \Lambda^4 (X_2 - a_2)^2. \quad (\text{B6})$$

With the proper choice of the parameters  $\lambda_{1,2}$  and  $a_{1,2}$  we get the desirable value of the  $\langle\tau\rangle$ . Below, for the models near the  $\tau = i\infty$  and  $\tau = i$  fixed points, we give the selections and obtained values of  $\langle\tau\rangle$ , which correspond to the minimum of the potential  $V(\tau)$ :

$$\text{Near } \tau = i\infty: \quad \langle\tau\rangle = 4.5i + 0.01, \quad \{\lambda_1, \lambda_2, a_1, a_2\} = \{10^{-5}, 10^{-7}, 12.568, 508.99\},$$

$$\{M_{x_0}, M_{y_0}\} \simeq \Lambda\{8.4 \times 10^{-12}, 1\}, \quad (\text{B7})$$

$$\text{Near } \tau = i: \quad \langle\tau\rangle = i(1 + 10^{-5}) - 5 \times 10^{-7}, \quad \{\lambda_1, \lambda_2, a_1, a_2\} = \{1, 0.02, 1.54 \times 10^{-10}, 4.47 \times 10^{-10}\},$$

$$\{M_{x_0}, M_{y_0}\} \simeq \Lambda\{8.7 \times 10^{-8}, 4.7 \times 10^{-5}\}. \quad (\text{B8})$$

In (B7) and (B8) we also indicated the masses of the canonically normalized states  $x_0, y_0$  [defined in (B4)]. As we see, for the  $y = 4.5$  [case of Eq. (B7)] with  $\Lambda \simeq M_{\text{Pl}} = 1.22 \times 10^{19}$  GeV, the mass of  $x_0$  is  $\simeq 10^8$  GeV. This value is completely sufficient for suppressing the flavor-violating processes because the couplings of  $x_0$  with the matter have additional suppression factor  $\sim e^{-\pi\langle y \rangle} (\sim 10^{-6}$  for  $\langle y \rangle = 4.5$ ). For the selection given in (B8), corresponding to the model near the  $\tau = i$  fixed point, both states  $x_0$  and  $y_0$  have sufficiently large masses ( $\gtrsim 10^{12}$  GeV for  $\Lambda \simeq M_{\text{Pl}}$ ).

Near the  $\tau = \omega$  fixed point, the desired values of  $\langle\tau\rangle$  can be obtained via the potential

$$V_\omega(\tau) = \frac{\lambda_3}{2} \Lambda^4 (X_3 - a_3)^2 + \frac{\lambda_4}{2} \Lambda^4 (X_4 - a_4)^2,$$

$$\text{with: } X_3 = \kappa^6 |E_6|^2, \quad X_4 = \kappa^6 (\tilde{E}_2 E_4 E_6^* + \tilde{E}_2^* E_4^* E_6). \quad (\text{B9})$$

The minimum of the potential  $V_\omega(\tau)$  near the  $\tau = \omega$  fixed point is obtained for the selection,

$$\text{Near } \tau = \omega: \quad \langle\tau\rangle = \omega + (0.5196 + i1.093) \times 10^{-5}, \quad \{\lambda_3, \lambda_4, a_3, a_4\} = \{0.1304, 0.02, 3.503, 1.368 \times 10^{-9}\},$$

$$\{M_{x_0}, M_{y_0}\} \simeq \Lambda\{1.15 \times 10^{-6}, 8.46 \times 10^{-5}\}. \quad (\text{B10})$$

We see that for  $\Lambda \simeq M_{\text{Pl}}$  both states  $x_0$  and  $y_0$  are pretty heavy ( $\approx 1.4 \times 10^{13}$  GeV and  $10^{15}$  GeV, respectively).

### APPENDIX C: NEUTRINO PARAMETRIZATION

In a standard parametrization [36], the lepton mixing matrix  $U$  has the following form:

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \quad (\text{C1})$$

with  $s_{ij} = \sin \theta_{ij}$  and  $c_{ij} = \cos \theta_{ij}$ . The phase matrices  $P$  and  $P'$  are given by

$$P = \text{Diag}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3}), \quad P' = \text{Diag}(1, e^{i\rho_1}, e^{i\rho_2}), \quad (\text{C2})$$

where  $\omega_{1,2,3}$  are some phases and  $\rho_{1,2}$  are Majorana phases appearing in the neutrinoless double- $\beta$  decay amplitude.

The relation (35) allows us to express the entries of the  $M_\nu$  in terms of the  $\omega_i$  phases and  $\mathcal{A}_{ij}$ , where

$$\mathcal{A}_{ij} = (U^* P' M_\nu^{\text{Diag}} U^\dagger)_{ij}. \quad (\text{C3})$$

For instance, for the neutrino scenario, presented in Sec. III B, using the mass matrix (33), we can express  $\bar{m}, a, b, \epsilon$ , and  $\epsilon_1$  as follows:

$$\bar{m} = e^{2i\omega_3} \mathcal{A}_{33}, \quad a = e^{i(\omega_2 - \omega_3)} \frac{\mathcal{A}_{23}}{\mathcal{A}_{33}}, \quad b = e^{2i(\omega_1 - \omega_3)} \frac{\mathcal{A}_{11}}{\mathcal{A}_{33}},$$

$$\epsilon = e^{i(\omega_1 - \omega_3)} \frac{\mathcal{A}_{13}}{\mathcal{A}_{33}}, \quad \epsilon_1 = e^{i(\omega_1 - \omega_3)} \left( \frac{\mathcal{A}_{12}}{\mathcal{A}_{23}} - \frac{\mathcal{A}_{13}}{\mathcal{A}_{33}} \right). \quad (\text{C4})$$

On the other hand, from (34) we can express the original parameters as

$$\begin{aligned} \tan \theta_e &= \left| \frac{a\epsilon_1}{b - \epsilon^2} \right|, \quad \eta = -\text{Arg} \left( \frac{a\epsilon_1}{\epsilon^2 - b} \right), \quad a_0 = a\epsilon_e + \epsilon s_e e^{-i\eta}, \\ b_0 &= \pm \frac{1}{c_e \sqrt{n}} [b - \epsilon^2 + c_e^2 (\epsilon c_e - a s_e e^{i\eta})^2]^{1/2}, \quad \epsilon_0 = \pm c_e \sqrt{n} \frac{\epsilon c_e - a s_e e^{i\eta}}{[b - \epsilon^2 + c_e^2 (\epsilon c_e - a s_e e^{i\eta})^2]^{1/2}}. \end{aligned} \quad (\text{C5})$$

Using (C4) in (C5) for the parameter  $\theta_e$ , for IO and NO cases, respectively, we find the results given in Eq. (36).

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