

Quark mass effects in octet baryon magnetic polarizabilities via lattice QCD

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The quark mass dependence of octet baryon magnetic polarizabilities is examined at the level of individual quark-sector contributions in the uniform background-field approach of lattice QCD. The aim is to understand the direct impact of increasing the mass of a quark flavor on the magnetic polarizability and indirect or environmental effects associated with changing the mass of spectator quarks, insensitive to the background magnetic field. Noting the need to set the electric charge of some quark flavors to zero, a fractionally charged baryon formalism is introduced. We find that increasing the mass of the charged quark flavor directly causes its contribution to the magnetic polarizability to decrease. However, increasing the mass of the spectator quark flavor indirectly acts to increase the magnetic polarizability. To gain a deeper understanding of these effects, we evaluate the predictions of the constituent quark model in this context. While the model provides a compelling explanation for the environmental effect of varying the spectator quark mass, an explanation of the direct mass dependence is more complicated as competing factors combine in the final result. The lattice results indicate the key factor is a reduction in the constituent quark magnetic moment with increasing quark mass, as it governs the strength of the magnetic transition to the nearby decuplet baryon.

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I. INTRODUCTION

Understanding the manner in which individual quark sectors contribute to QCD observables provides a deep understanding of QCD dynamics. The isolation of quark sectors arises naturally in theoretical calculations as one considers the coupling of a probing current to each of the quarks in turn. By effectively setting the charge of all but one quark sector to zero, one can disclose the contribution of a single quark sector.

With a quark sector isolated, one can then unravel the relationship between direct effects and indirect effects due to changes in the environment in which the selected quark flavor resides. For example, consider the u -quark sector within the proton, isolated by setting the electric charge of the d -quark to zero. One can then probe the manner in which the u -sector contribution to a proton electromagnetic observable changes as the u -quark mass is changed. Herein, we will refer to such effects as a “direct” effect. However, “indirect” effects associated with changes in the d -quark mass are also observable. We refer to such indirect effects

as “environmental” effects, as the environment in which the u -quark resides changes.

Direct and indirect environmental effects in the electromagnetic form factors of octet baryons [1], decuplet baryons [2] and their transitions [3], were discovered long ago. For magnetic moments, the discovery of environmental effects was surprising as a basic tenet of the constituent quark model is that a constituent quark has an intrinsic magnetic moment governed solely by its mass. There is no scope for an environmental dependence. However, such a dependence could explain violations in magnetic moment sum rules [4], and it is central to early precise predictions of the strange quark contribution to the proton’s magnetic moment [5], predictions that have withstood the tests of modern determinations more than a decade later [6,7].

In this article we present the first examination of quark-sector contributions to octet baryon magnetic polarizabilities. We explore both direct and indirect environmental effects through the variation of the masses of the quark flavors under consideration. Noting the need to set the electric charge of some quark flavors to zero, a fractionally charged baryon formalism is introduced to the background-field formalism.

In the process of understanding the underlying mechanisms that may be responsible to the effects observed in our lattice QCD calculations, we refer to the constituent quark model [8], improved to approximate recent lattice QCD results for octet baryons [9].

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There the model highlights the importance of the u quark in generating a large magnetic polarizability and the impact of the octet-decuplet mass difference which suppresses the nucleon magnetic polarizabilities relative to the hyperons, especially the Ξ^0 . However, there are more subtle discrepancies that remain to be understood. For example, the subtle differences that order the p , n , and Σ^+ magnetic polarizabilities are not fully captured in the simple model.

In this work, we aim to understand the direct and indirect effects in quark mass changes, to elucidate the role of environmental effects in QCD [10]. We aim to understand the extent to which the current simple quark model contains environmental effects and explore a possible role for additional effects that are yet to be considered in the model.

Our approach is via lattice QCD, building upon the established background field formalism. We will build on the advanced lattice techniques established in Ref. [9], this time neutralizing quark sectors to access the contributions one flavor at a time. We will carefully examine the direct and indirect effects through both lattice QCD and the constituent quark model, allowing a determination of the capability of the constituent quark model and the opportunity to evaluate whether environmental effects are as relevant for the magnetic polarizability as they are for the magnetic moment.

In Sec. II we define a fractionally charged baryon formalism that enables an examination of direct and indirect environment effects. In Sec. III we briefly revisit the magnetic polarizability in the constituent quark model before we consider its predictions for fractionally charged baryons. In Sec. IV we review our implementation of the background field method. The extraction of the magnetic polarizability on the lattice is presented in Sec. V. In Sec. VI we discuss our fitting methodology and treatment of excited state contamination of the underlying correlation functions required to extract the magnetic polarizability in the background field method. The details of the lattice simulation are presented in Sec. VII, and the analysis of lattice QCD results is presented in Sec. VIII. In Sec. IX we present our results of the lattice simulation and evaluate the quark model's ability to describe them. In Sec. X we summarize our findings.

II. FRACTIONALLY CHARGED BARYONS

We begin by defining a formalism in which we can probe the direct and indirect environmental effects of quark mass changes in the magnetic polarizabilities of octet baryons. As in Ref. [9], we consider the outer members of the baryon octet to avoid issues with the mixing of Σ^0 and Λ baryons in the background field. As such, the quantum numbers of these baryons are carried by two quark flavors. We are specifically interested in two concepts.

Direct quark mass dependence: How the contribution of a quark sector changes as its mass changes between that of

TABLE I. The properties of the four quark flavors considered in isolating individual quark-sector contributions to baryon magnetic polarizabilities.

Quark	Symbol	Electric charge	Mass
Down	d	$-1/3$	m_d
Strange	s	$-1/3$	m_s
Light	l	0	m_d
Heavy	h	0	m_s

a down quark and that of a strange quark (e.g., $d \rightarrow s$ in $n \rightarrow \Xi^0$).

Indirect environmental dependence: How the contribution of a quark sector changes as the mass of the other sector changes between that of a down and strange quark (e.g., u in $p \rightarrow \Sigma^+$).

We determine these contributions by constructing baryons with a single electrically charged flavor and a neutrally charged flavor, thereby isolating the electrically charged flavor's contribution. To do so we utilize the four quark flavors defined in Table I and construct eight baryons with the flavors

$$\begin{array}{cccc} ddl, & lld, & ssl, & lls, \\ ddh, & hhd, & ssh, & hhs. \end{array}$$

These eight baryons allow us to examine the direct quark mass dependence

$$\begin{aligned} ddl &\rightarrow ssl, \\ ddh &\rightarrow ssh, \\ lld &\rightarrow lls, \\ hhd &\rightarrow hhs, \end{aligned} \tag{1}$$

and the indirect environment dependence

$$\begin{aligned} ddl &\rightarrow ddh, \\ ssl &\rightarrow ssh, \\ lld &\rightarrow hhd, \\ lls &\rightarrow hhs \end{aligned} \tag{2}$$

of all possible charge and environment sectors. To examine the direct and environmental dependence, we examine differences in the magnetic polarizability such as $\beta_{ssl} - \beta_{ddl}$ for the direct mass effect. In calculating the difference, we will take advantage of correlations between the quantities to accurately discern changes due to variation of the quark masses.

III. CONSTITUENT QUARK MODEL

In a simple constituent quark model, the constituent quark masses are tuned to reproduce the magnetic moments of the p , n , and Λ baryons [11]. Such a simple model relies on the assumption that the quarks are blind to their environment. For example, this assumption makes the contribution of the u quark the same in the $n(ddu)$ and the $\Xi^0(ssu)$, despite the change in the mass of its environment. However, experimental evidence of environmental effects was highlighted in Ref. [5] for the magnetic moments. Nonetheless, the constituent quark model retains significant value. Its ability to simply describe the qualitative pattern exhibited by the baryon magnetic moments is complemented by a remarkable ability to explain the broad behaviors of the magnetic polarizability of the octet baryons as presented in Ref. [9].

In that work we derived the following expression for the magnetic polarizability of an outer octet baryon:

$$\begin{aligned}\beta &= \frac{1}{2\pi} \frac{|\langle \mathcal{B} | \hat{\mu}_z | \mathcal{B}^* \rangle|^2}{m_{\mathcal{B}^*} - m_{\mathcal{B}}} - \sum_{f=1}^3 \frac{q_f^2 \alpha}{6m_f} \langle r^2 \rangle_f \\ &= \frac{4}{9\pi} \frac{|\mu_D - \mu_S|^2}{m_{\mathcal{B}^*} - m_{\mathcal{B}}} - \sum_{f=1}^3 \frac{q_f^2 \alpha}{6m_f} \langle r^2 \rangle_f \\ &\equiv \beta_1 - \beta_2,\end{aligned}\quad (3)$$

where $\alpha = e^2/4\pi$ is the fine structure constant. β_1 depends on the magnetic transition moment between octet and decuplet baryons for the outer members of the baryon octet

$$\langle \mathcal{B} | \hat{\mu}_z | \mathcal{B}^* \rangle = \frac{2\sqrt{2}}{3} (\mu_D - \mu_S), \quad (4)$$

expressed in terms of the constituent quark moments of the doubly, μ_D , and singly, μ_S , represented sectors for an outer octet baryon. In the constituent quark model, the constituent quark moments are given in terms of the constituent quark masses, m_f , for each flavor, f ,

$$\mu_f = \frac{q_f e}{2m_f}. \quad (5)$$

The contributions of opposing sign in Eq. (3) produce interesting complexity [9]. β_1 probes the transition to an intermediate excitation through virtual photon exchange, while β_2 probes the distribution of quarks within the hadron.

A. Implementing the quark model

The quark model predictions require the model's input quantities be determined on the lattice and interpolated. The details of this interpolation are described in depth in Ref. [9]. To estimate the constituent quark masses away

from the physical point, we utilize a simple model:

$$\begin{aligned}m_{\text{oct}}^0 + \alpha_{\text{oct}}(3m_l) &= m_N, \\ m_{\text{oct}}^0 + \alpha_{\text{oct}}(m_l + 2m_h) &= m_{\Xi}.\end{aligned}\quad (6)$$

Here $m_l = m_d$ and $m_h = m_s$ are the light and heavy constituent quark masses already determined in the simple constituent quark model via baryon magnetic moments at the physical point, as summarized in Ref. [11]. m_{oct}^0 and α_{oct} are fit parameters, constrained by the physical nucleon and Ξ masses. With the parameters determined, one can then use Eq. (6) with baryon masses from our lattice QCD simulations to solve for the constituent quark masses, m_l and m_h , away from the physical point.

We linearly interpolate the octet baryon masses as a function of pion mass [12], allowing the determination of the constituent quark masses as a function of the pion mass.

In the spirit of the constituent quark model, we utilize these constituent quark masses and equations of the form Eq. (6) to determine the octet baryon masses, including the masses of fractionally charged baryons that would otherwise not exist. This includes the mass of an octet ssh or hhs with a mass of

$$m_{ssh} = m_{hhs} = m_{\text{oct}}^0 + \alpha_{\text{oct}}(3m_h). \quad (7)$$

The decuplet baryon masses are determined using an equivalent model with fit parameters determined in the same manner using the same physical point m_l and m_h constituent quark masses, and the physical Δ and Ω masses.

Finally, we interpolate the quark distribution radii determined on the PACS-CS ensembles by Stokes *et al.* [13]. The proton and neutron squared radii are characterized well with an interpolation that is linear in $\log(m_\pi)$. Quark distribution radii for single quark flavors of unit charge in the doubly (D), and singly (S) represented sectors are defined by

$$\langle r^2 \rangle_p = 2\frac{2}{3} \langle r^2 \rangle_D - \frac{1}{3} \langle r^2 \rangle_S, \quad (8)$$

$$\langle r^2 \rangle_n = -2\frac{1}{3} \langle r^2 \rangle_D + \frac{2}{3} \langle r^2 \rangle_S, \quad (9)$$

where charge and quark-counting factors are explicit. The resulting distribution radii for single quarks of unit charge are

$$\langle r^2 \rangle_D = \frac{1}{2} (2\langle r^2 \rangle_p + \langle r^2 \rangle_n), \quad (10)$$

$$\langle r^2 \rangle_S = \langle r^2 \rangle_p + 2\langle r^2 \rangle_n. \quad (11)$$

As only the proton and neutron were examined in Ref. [13], we approximate the doubly and singly represented strange-quark distribution radii to be that of the light quark

at the heaviest PACS-CS quark mass labeled by $m_\pi = 701$ MeV.

B. Constituent quark model predictions for the magnetic polarizability

Using the above implementation, the magnetic polarizability of the outer octet baryons is predicted in Ref. [9]. A simple magnitude correction is required:

$$\beta = a_1 \beta_1 - a_2 \beta_2, \quad (12)$$

where the fit parameters $a_1 = 0.401$, $a_2 = 0.532$ best reproduce the lattice QCD data. The phenomenology associated with this scaling is discussed in Ref. [9]. With this improvement, the quark model is able to capture the qualitative features of the lattice QCD determinations as illustrated in Fig. 1.

Specifically, it predicts the small nature of the magnetic polarizability of the Σ^- and Ξ^- baryons, which was otherwise not known. It also correctly predicts the large magnetic polarizability of the Ξ^0 . Finally, it predicts approximately the correct magnitude for the magnetic polarizability of the p , n , and Σ^+ though there is a lack of clarity about their ordering. The successes of the model are significant given its simplicity.

The model also highlights the importance of the u -quark in generating a large magnetic polarizability through the transition term β_1 with the factor $(\mu_D - \mu_S)^2$. Combinations of μ_u and μ_d or μ_s add and square to a large magnitude, whereas a combination of μ_d and μ_s approximately cancel and become small in the square. This latter point explains the small magnitude of the Σ^- and Ξ^- magnetic polarizabilities.

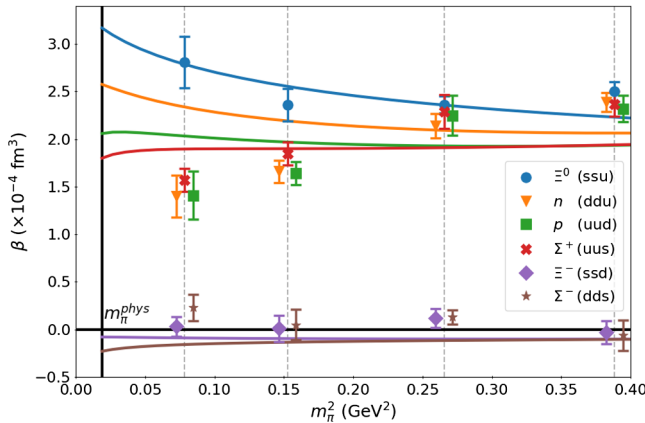


FIG. 1. Improved quark model predictions from Ref. [9] for the magnetic polarizabilities of octet baryons (curves) are compared with lattice QCD determinations (points) calculated on the PACS-CS ensembles considered herein. The legend is ordered to match the vertical ordering of the quark model curves at the physical pion mass. Dashed lines represent the pion masses of the PACS-CS ensembles.

The more subtle discrepancies between the model and the lattice simulation results may be associated with an environmental effect in the constituent quark moments as discussed in the Introduction. Thus, we turn our attention to the contributions of individual quark sectors with an aim to understanding direct and indirect environmental effects governing their contributions to the magnetic polarizability.

C. Fractionally charged baryon formalism

We first evaluate the quark model predictions when considering fractionally charged baryons. Consider the magnetic polarizability of Eq. (3) for the proton

$$\begin{aligned} \beta_{uud} &= \frac{4}{9\pi} \frac{|\mu_u - \mu_d|^2}{m_\Delta - m_N} - \frac{8}{9} \frac{\alpha}{6m_l} \langle r^2 \rangle_l - \frac{1}{9} \frac{\alpha}{6m_l} \langle r^2 \rangle_l \\ &= \frac{4}{9\pi} \frac{9\mu_d^2}{m_\Delta - m_N} - \frac{\alpha}{6m_l} \langle r^2 \rangle_l, \end{aligned} \quad (13)$$

where we have used the charge symmetric relation $\mu_u = -2\mu_d$. Now consider neutralizing the u quark sector for the lld baryon

$$\begin{aligned} \beta_{lld} &= \frac{4}{9\pi} \frac{|\mu_d|^2}{m_\Delta - m_N} - \frac{1}{9} \frac{\alpha}{6m_l} \langle r^2 \rangle_l \\ &= \frac{4}{9\pi} \frac{\mu_d^2}{m_\Delta - m_N} - \frac{1}{9} \frac{\alpha}{6m_l} \langle r^2 \rangle_l. \end{aligned} \quad (14)$$

We note a significant drop in the magnitude by a factor of 9. Our focus on the elemental d quark in the lattice QCD simulations means that this reduction by 1 order of magnitude will have significant implications that will be discussed in due course.

1. Environmental dependence in the quark model

Considering the environmental dependence in $lld \rightarrow hhd$, we note that the charged sector is unchanged. As a result, when taking the difference

$$\beta_{hhd} - \beta_{lld} = \frac{4}{9\pi} \left[\frac{\mu_d^2}{m_{\Xi^*} - m_{\Xi}} - \frac{\mu_d^2}{m_\Delta - m_N} \right], \quad (15)$$

the quark distribution term vanishes and we are left only with the difference in the octet-decuplet mass splittings. The quark model's prediction of a nontrivial environmental effect for the magnetic polarizability contrasts that for the magnetic moment.

The result of Eq. (15) can be easily extended to the other three environmental effects of Eq. (2). In every case, there are additional heavy quarks in the right-hand baryon of Eq. (2). As hyperfine splittings are inversely proportional to the product of the constituent quark masses, the octet-decuplet mass splitting is smaller for baryons with more heavy quarks. Thus, the decuplet-octet baryon mass

splitting is smaller for the leading term, and the sign of all environment effects are predicted to be positive.

2. Direct quark mass dependence in the quark model

In the case of the mass dependence, the mass of the charged sector is changing, and the expression becomes more complex. Considering $lld \rightarrow lls$

$$\beta_{lls} - \beta_{lld} = \frac{4}{9\pi} \left[\frac{\mu_s^2}{m_{\Sigma^*} - m_{\Sigma}} - \frac{\mu_d^2}{m_{\Delta} - m_N} \right] - \left[\frac{\alpha q_s^2}{6m_h} \langle r^2 \rangle_h - \frac{\alpha q_d^2}{6m_l} \langle r^2 \rangle_l \right], \quad (16)$$

we observe none of the previous cancellation. Instead we observe competing effects.

As $\mu_d^2 > \mu_s^2$ and $(m_{\Delta} - m_N)^{-1} < (m_{\Sigma^*} - m_{\Sigma})^{-1}$ the sign of the first term depends on the interplay between these two factors. However the sign of the second term is clear as $\langle r^2 \rangle_h / m_h < \langle r^2 \rangle_l / m_l$, and thus the contribution including the leading minus sign is positive. Therefore, if $\beta_{lls} - \beta_{lld} < 0$ then one can point to $\mu_d^2 > \mu_s^2$ as the dominant effect.

Similar arguments hold for the case of $hhd \rightarrow hhs$. If $\beta_{hhs} - \beta_{hhd} < 0$ then one can point to $\mu_d^2 > \mu_s^2$ as the dominant effect.

Now consider $ddl \rightarrow ssl$,

$$\beta_{ssl} - \beta_{ddl} = \frac{4}{9\pi} \left[\frac{\mu_s^2}{m_{\Xi^*} - m_{\Xi}} - \frac{\mu_d^2}{m_{\Delta} - m_N} \right] - 2 \left[\frac{\alpha q_s^2}{6m_h} \langle r^2 \rangle_h - \frac{\alpha q_d^2}{6m_l} \langle r^2 \rangle_l \right]. \quad (17)$$

This time the octet-decuplet mass splitting creates even greater tension between competing effects as does the charge distribution term. While $\mu_d^2 > \mu_s^2$ as before, $(m_{\Delta} - m_N)^{-1} \ll (m_{\Xi^*} - m_{\Xi})^{-1}$ and thus there is greater competition between the factors in setting the sign. The sign of the second term is as before, but with double the magnitude.

If $\beta_{ssl} - \beta_{ddl} < 0$ then one can once again point to $\mu_d^2 > \mu_s^2$ as the dominant effect. However, the quark model predicts a suppressed effect relative to $\beta_{lls} - \beta_{lld}$ due to a reduced difference in mass splittings and the magnitude of the quark distribution term.

Finally, similar arguments hold for the case of $ddh \rightarrow ssh$. If $\beta_{ssh} - \beta_{ddh} < 0$ then one can point to $\mu_d^2 > \mu_s^2$ as the dominant effect. Again the effect is predicted to be suppressed relative to $hhd \rightarrow hhs$.

To summarize, if the change in the constituent quark magnetic moment dominates the direct quark-mass effect, the case of $hhd \rightarrow hhs$ holds the greatest promise for illustrating an effect in the lattice QCD simulations. The prevalence of heavy quarks will aid in suppressing statistical uncertainties, and the tension between $\mu_d^2 > \mu_s^2$

and competing effects is lowest for $\beta_{hhs} - \beta_{hhd}$. If $\beta_{hhs} - \beta_{hhd} < 0$ then one has a simple intuitive picture of the physics driving the direct quark mass effect. It is the simple suppression of the constituent quark moment that suppresses transitions to nearby decuplet baryons and thus diminishes the polarizability.

We now proceed to the lattice QCD calculation to provide reference for these predictions.

IV. BACKGROUND FIELD IMPLEMENTATION

The background field method is a well established approach to calculating the magnetic polarizability in a lattice QCD calculation. The approach used in this work is identical to our approach in Ref. [9]. Full details are given there.

In the background field method, a minimal coupling to the discretized covariant derivative results in the application of a phase to the QCD gauge field. Through the requirement for periodicity at the boundary, we obtain a quantization condition for the magnetic field strength B given by

$$eB = \frac{2\pi}{N_x N_y a^2} k_d, \quad (18)$$

where k_d is an integer specifying the field strength in multiples of the minimum uniform field strength quantum. The condition is written in terms of the down quark charge $q_d e$, the number of spatial sites N_x, N_y and the lattice spacing a .

The magnetic field is defined such that the field corresponding to integer $k_d = 1$ is oriented in the negative $\hat{z}$ -direction. This work utilizes quark propagators and baryon correlation functions calculated at $k_d = 0, \pm 1, \pm 2$. We will refer to the strength of the magnetic field in terms of k_d throughout this work.

V. LATTICE QCD FORMALISM

In the presence of a uniform background magnetic field, the energy of a baryon changes as a function of magnetic field strength [14,15]:

$$E(B) = m + \vec{\mu} \cdot \vec{B} + \frac{|q_B e B|}{2m} (n + 1) - \frac{1}{2} 4\pi\beta B^2 + \mathcal{O}(B^3). \quad (19)$$

Here the mass of the baryon, m , is complemented by contributions from the magnetic moment $\vec{\mu}$, the Landau term proportional to $|q_B e B|$ where q_B is the charge of the baryon, and the magnetic polarizability β . As the baryons in this work are fractionally charged, we use a U(1) Landau-mode projection to select $n = 0$. This aspect of the calculation is detailed in Sec. VII D.

As we demonstrated in Ref. [9], the contribution from the magnetic moment may be subtracted by consideration

of the average of “spin-field aligned” and “spin-field antialigned” energy. The baryon mass may also be removed through the subtraction of the zero-field energy.

This process may be mirrored in a correlation function ratio which enables cancellation of correlated fluctuations. We define a “spin-field aligned” correlator

$$G_{\uparrow\uparrow}(B) = G(+s, +B) + G(-s, -B), \quad (20)$$

where the baryon’s spin is aligned with the magnetic field and a “spin-field antialigned” correlator

$$G_{\uparrow\downarrow}(B) = G(+s, -B) + G(-s, +B), \quad (21)$$

where the spin and field are opposed. These correlators are combined in the ratio

$$R(B, t) = \frac{G_{\uparrow\uparrow}(B, t)G_{\uparrow\downarrow}(B, t)}{G(0, t)^2}, \quad (22)$$

which aggregates the positive and negative field strengths together. Due to this aggregation, any reference to the magnetic field strength from this point refers to the aggregated positive field strength.

The effective energy of this correlation function ratio acts to subtract the magnetic moment term and baryon mass. As such, we define the magnetic polarizability energy shift $\delta E_\beta(B, t)$

$$\begin{aligned} \delta E_\beta(B, t) &= \frac{1}{2} \frac{1}{\delta t} \lim_{t \rightarrow \infty} \log \left(\frac{R(B, t)}{R(B, t + \delta t)} \right) \\ &= \frac{1}{2} [\delta E_{\uparrow\uparrow}(B) + \delta E_{\uparrow\downarrow}(B)] - \delta E(0) \\ &= \frac{|qeB|}{2m} - \frac{4\pi}{2} \beta |B|^2 + \mathcal{O}(B^3). \end{aligned} \quad (23)$$

However, we require more than the magnetic polarizability of a single baryon; instead we require the difference in magnetic polarizability between two baryons. We saw in Sec. II that fractionally charged baryons have magnetic polarizabilities 1 order of magnitude smaller than that of the physical baryons composed with a u quark. As such, extracting sufficient signal to take a meaningful difference is difficult. Extracting the magnetic polarizability of the fractionally charged baryons separately and determining a correlated difference often fails to produce meaningful insight, especially at lighter quark masses.

However, the highly correlated nature of the two baryons can be exploited by calculating the difference at the correlator level. As such, we construct the ratio $R(B, t)$ for both baryons in question and construct the following effective energy shift:

$$\begin{aligned} \delta E_{\beta_1 - \beta_2}(B, t) &= \frac{1}{2} \frac{1}{\delta t} \lim_{t \rightarrow \infty} \log \left(\frac{R_{\beta_1}(B, t)R_{\beta_2}(B, t + \delta t)}{R_{\beta_1}(B, t + \delta t)R_{\beta_2}(B, t)} \right) \\ &= \left(\frac{|q_{B_1}|}{m_{B_1}} - \frac{|q_{B_2}|}{m_{B_2}} \right) \frac{|eB|}{2} - \frac{4\pi}{2} (\beta_1 - \beta_2) |B|^2 \\ &\quad + \mathcal{O}(B^3). \end{aligned} \quad (24)$$

This allows for fitting of the magnetic polarizability difference $\beta_1 - \beta_2$ directly, providing maximal cancellation of correlated fluctuations. We note that the magnetic polarizability differences we are interested in here will always have $q_{B_1} = q_{B_2}$; hence the quark charge may be factored out of the difference in the Landau term.

VI. FITTING

To extract the magnetic polarizability difference $\beta_1 - \beta_2$ we fit Eq. (24) as a function of field strength. In practice, it is simpler to fit in terms of the field strength quanta k_d . Using the quantization condition of Eq. (18),

$$eB = \frac{2\pi}{N_x N_y a^2} \frac{1}{q_d} k_d, \quad (25)$$

we substitute for eB in Eq. (24) to rewrite $\delta E_{\beta_1 - \beta_2}$ in terms of the field strength quanta

$$\delta E_{\beta_1 - \beta_2}(k_d) = L(k_d, m_{B_1}, m_{B_2}) + C(\beta_1 - \beta_2) k_d^2, \quad (26)$$

where

$$L(k_d, m_{B_1}, m_{B_2}) = \frac{1}{2} \frac{2\pi}{N_x N_y a^2} \left(\frac{|q_{B_1}|}{m_{B_1}} - \frac{|q_{B_2}|}{m_{B_2}} \right) \left| \frac{1}{q_d} k_d \right|, \quad (27)$$

and

$$C = -\frac{1}{2\alpha} \left[\frac{2\pi}{N_x N_y a^2} \right]^2 \frac{1}{q_d^2}. \quad (28)$$

It is important to note that q_d is the down quark charge while q_{B_1}, q_{B_2} are the hadronic charges of the two baryons. For convenient future reference, we have defined $L(k_d, m_{B_1}, m_{B_2})$ as the combined Landau contribution and C the remaining coefficient to the magnetic polarizability difference.

We fit $\delta E_{\beta_1 - \beta_2}$ at two positive field strengths to extract the magnetic polarizability difference $\beta_1 - \beta_2$. The best estimate for $\delta E_{\beta_1 - \beta_2}(k_d)$ is obtained through a weighted average of fit windows which provides a systematic approach to plateau fitting [16]. The method is detailed fully in Ref. [9].

In the method, the i th eligible window of N candidates is assigned a weight according to

$$w_i = \frac{1}{Z} \frac{p_i}{(\delta E_i)^2}, \quad (29)$$

where

$$\mathcal{Z} = \sum_{i=1}^N \frac{p_i}{(\delta E_i)^2}. \quad (30)$$

Here, δE_i is the uncertainty of the i th fit, and p_i is the p value of the fit. The p value is the probability of a given fit occurring in a χ^2 distribution. With these weights, the average effective energy is

$$E = \sum_i^n w_i E_i, \quad (31)$$

and the statistical error is

$$(\delta E)^2 = \sum_i^n w_i (\delta E_i)^2. \quad (32)$$

Eligible windows must satisfy a few criteria. To avoid fitting noise, we define a $t_{\max}$, the last time slice before the signal is clearly lost to noise. We define a corresponding $t_{\min}$, the first time slice at which single state isolation has been reached for all underlying (spin-field aligned, anti-aligned, zero-field) correlation functions. We also require the fit window to be of minimum length three and $t_f = t_{\max}$. This fixing of the endpoint ensures that early time slices are not disproportionately favored.

In our examination of the octet baryons in Ref. [9] we found it vital to impose the condition of single state isolation on all underlying correlation functions when determining $t_{\min}$. In constructing the correlation function ratio Eq. (22) (which enables the extraction of the magnetic polarizability) the resulting correlator can display plateau-like behavior before the underlying correlation functions have reached single state isolation.

As we did in that work, we utilize the $\chi^2_{d.o.f}$ of a linear fit to $\log G$ as a metric for excited state contamination. An easy interface to the large number of $\chi^2_{d.o.f}$ to be checked is the set of heat maps we construct for each particle. The heat maps for the representative ssl , ddh , and ddl baryons are presented in Sec. VIII.

Having determined values for $\delta E_{\beta_1-\beta_2}(k_d)$ at $k_d = 1, 2$, we turn our attention to the Landau term $L(k_d, m_1, m_2)$ of Eq. (26), which depends on the baryon masses. These masses are also determined through the weighted average approach. Subtraction of this known term allows the magnetic polarizability difference to be extracted as a single parameter in the fit of

$$\frac{\delta E_{\beta_1-\beta_2}(k_d) - L(k_d, m_{B_1}, m_{B_2})}{C} = (\beta_1 - \beta_2) k_d^2. \quad (33)$$

The quality of the fits is similar to that presented in our previous work [9], and is illustrated in Sec. VIII.

To determine the uncertainty in the polarizability difference, a jackknife error estimate is performed. A second-order jackknife is used to obtain uncertainties on the correlation functions and correlation function ratios. Uncertainties for fit values such as the magnetic polarizability energy shift for each field strength are obtained from individual first-order jackknife subensembles. Finally, the fit of the energy shifts as a function of field strength is repeated on each jackknife subensemble and the error on the ensemble average calculated as the jackknife error.

VII. SIMULATION DETAILS

The details of the lattice simulation mirror those of our preceding work [9]. We reiterate the important details here.

A. Lattice actions

The four gauge ensembles used in this work are the four heaviest $2 + 1$ -flavor dynamical gauge configurations provided by the PACS-CS Collaboration[17] through the International Lattice Data Grid (ILDG) [18]. The configurations have a range of degenerate up and down quark masses while the strange quark mass is fixed. The Iwasaki gauge action and the clover fermion action with $C_{SW} = 1.715$ are used in generating the configurations. Details of the ensembles are summarized in Table II.

The ensembles were generated in the absence of a background magnetic field. As a result, the sea quarks are blind to the magnetic field and the ensembles may be regarded as electroquenched. On the fifth, lightest ensemble, we encounter uncertainties which do not respond to increased statistics, hinting at an exceptional configuration problem associated with the electroquenching of the light sea-quark sector. For this reason, it is omitted here. However, we do emphasize that there is no evidence of exceptional configuration issues in the analysis presented here.

Our valence fermion action matches the PACS-CS QCD action and includes the background field corrected clover term [20]. Periodic boundary conditions are used in the spatial directions. To avoid signal contamination from the backward propagating states, we use fixed boundary conditions in the temporal direction. The source is then

TABLE II. Details of the $32^3 \times 64$ PACS-CS ensembles used in this work. The lattice spacing of each ensemble is set using the Sommer scale with $r_0 = 0.4921(64)(+74)(-2)$ fm. In all cases $\kappa_s^{\text{sea}} = 0.13640$ and $\kappa_s^{\text{val}} = 0.13665$ [19]. N_{con} describes the number of configurations.

m_π^{PACS} (MeV)	$\kappa_{u,d}$	a (fm)	N_{con}
701	0.13700	0.1022(15)	399
570	0.13727	0.1009(15)	397
411	0.13754	0.0961(13)	449
296	0.13770	0.0951(13)	399

placed at $t = 16$, one quarter of the total lattice time extent such that one is always well away from the fixed boundary.

B. Baryon interpolating fields

The commonly used proton interpolating field in lattice QCD is given by [1]

$$\chi_p(x) = \epsilon^{abc} [u^{aT} C \gamma_5 d^b(x)] u^c(x), \quad (34)$$

where C is the charge-conjugation matrix. We construct the fractionally charged baryons described in this work through use of this interpolating field, but provide the appropriate quark propagators to produce the desired particle. For example, in the construction of the ddh baryon, we provide a light quark propagator determined with the background field included in place of the doubly represented up quark and a heavy quark propagator determined at zero field in place of the singly represented down quark.

Such interpolating fields, utilized purely with traditional gauge-covariant Gaussian smearing are ineffective at isolating the baryon ground state in a uniform background field [15,21–23]. The uniform background field breaks the spatial symmetry, and Landau-mode physics is present at both the quark and hadronic levels. This must be accommodated.

C. Quark operators

As detailed in Ref. [9], we utilize asymmetric source and sink operators to maximize overlap of the lowest energy eigenstates of our baryons in magnetic fields. The operators we utilize also act to accommodate for the spatial symmetry and Landau-mode issues mentioned above.

Source and sink construction utilizes links which are smeared using stout link smearing [24]. Ten sweeps with isotropic smearing parameter $\alpha_{\text{stout}} = 0.1$ are applied to spatially oriented links.

We construct the quark source using three-dimensional, gauge-invariant Gaussian smearing [25]. At all quark masses $\alpha = 0.7$ is used while the number of gauge invariant Gaussian smearing sweeps considered is quark mass dependent. Smearing is tuned to optimize the onset of early effective-mass plateaus with smaller numbers of sweeps associated with heavier quark masses. We utilized 150 to 350 sweeps.

The source construction is designed to provide a representation of the QCD interactions with the intent of isolating the QCD ground state. To encapsulate the quark-level physics of the electromagnetic and QCD interaction we utilize the eigenmode projection techniques demonstrated in Ref. [22], where a comprehensive explanation of the mechanisms may be found. A succinct summary of the approach is provided in Ref. [9].

The eigenmodes of the two-dimensional lattice Laplacian calculated on the $SU(3) \times U(1)$ gauge links are used to produce a projection operator applied to the propagator at the sink. It is shown in Ref. [22], that including too few modes

results in a noisy hadron correlation function in much the same manner as traditional sink smearing. As such, the number of modes is chosen to be large enough to minimize the noise of the correlation function, but small enough to retain the focus on the aforementioned low-energy physics. The work of Ref. [22] found that $n = 96$ modes provides balance to these two effects, and we use this here.

D. Hadronic projection

The inclusion of the background magnetic field induces a change to the wave function of a charged baryon [26]. The quark level electromagnetic physics is highlighted by the eigenmode projection at the sink. However, we must also ensure that our operator has the appropriate electromagnetic characteristics on the hadronic level. By projecting final-state fractionally charged baryons to the state corresponding to the lowest-lying Landau level, we ensure $n = 0$ in Eq. (19). The details of this projection are presented in Ref. [9].

Due to the color singlet nature of the baryon, we need only project the eigenmodes of the $U(1)$ Laplacian rather than the full lattice Laplacian used for the quark-propagator sink. These eigenmodes carry a degeneracy n in the finite volume of the lattice. For a baryon with charge q_B the degeneracy for the field strength k_d is

$$n = \left\lfloor k_d \frac{q_B}{q_d} \right\rfloor. \quad (35)$$

In this work, we consider fractionally charged baryons with hadronic charge $q_B = -\frac{1}{3}$ and $-\frac{2}{3}$. At $k_d = \pm 1$ their degeneracies will be 1 and 2 respectively, while at $k_d = \pm 2$ their degeneracies will be 2 and 4 respectively. We utilize a linear combination of the degenerate eigenmodes which provides optimal overlap with the fermion source. The probability densities of these hadronic-level projection modes are shown in Fig. 2.

E. Statistics

As periodic boundary conditions are used in all four dimensions for the gauge field generation, one can exploit the associated translational invariance of the gauge fields. A quark source can be placed at any position on the lattice and then cycled with the gauge field to the standard source position of $(x, y, z, t) = (1, 1, 1, N_t/4)$, well away from any fermion action boundary conditions. This enables additional sampling of the full gauge field.

Further, the two-dimensional nature of the lattice Laplacian operator allows the eigenmodes for the sink projection to be reused when the gauge field is cycled solely in the time direction. Hence, we increase our statistics on the PACS-CS ensembles by considering four random spatial sources at $t = 16$. The gauge field is then cycled in the temporal direction by an eighth of the lattice time extent (eight slices in our case) for each random

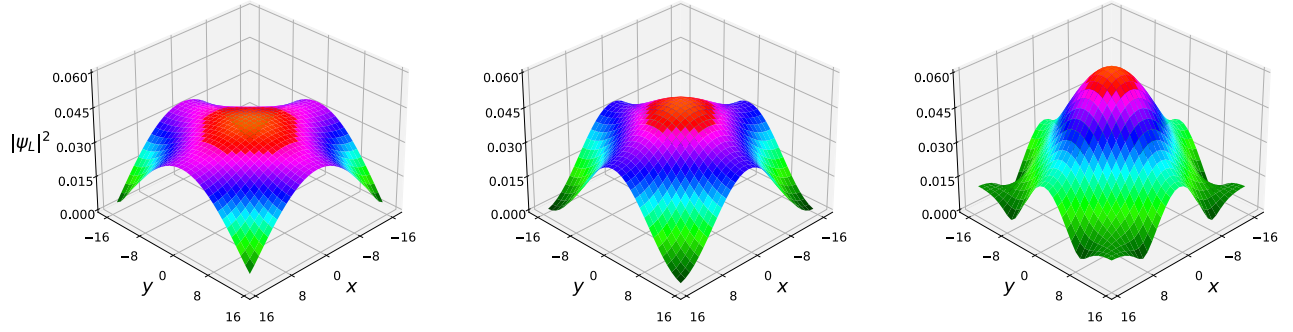


FIG. 2. Landau mode probability densities for fractional charge $q_B = -\frac{1}{3}$ and $-\frac{2}{3}$ baryons at our two lowest field strengths. Recalling $n = |k_d q_B / q_d|$, the $n = 1$ (left), $n = 2$ (center) and $n = 4$ (right) probability densities are illustrated for field strengths $|k_d| = 1$ and 2 . The degeneracy of each mode is n .

source. This results in a further increase in statistics by a factor of 8. Together, random sources and time-direction cycles increase our statistics by a factor of 32. These multiple samples are binned and averaged as a single configuration estimate in the error analysis.

VIII. LATTICE QCD ANALYSES

A. $\chi^2_{d.o.f}$ heat maps and fit windows

The $\chi^2_{d.o.f}$ heat maps for the representative *ssl*, *ddh*, and *ddl* baryons at $\kappa = 0.13770$, $m_\pi = 296$ MeV are shown in Figs. 3, 4, and 5. We consider a correlator to have reached

single state isolation when the first three windows have $\chi^2_{d.o.f} < 2.5$ which corresponds to the three leftmost boxes being colored blue, green or yellow. We can see this criteria has been fulfilled by all underlying correlation functions by $t = 22$ for the *ssl* baryon, $t = 22$ for the *ddh* baryon, and by $t = 24$ for the *ddl* baryon. In determination of the polarizability difference we consider the heat maps of both baryons in question and set $t_{\min}$ to the larger of the earliest times.

As shown in Sec. III C the magnetic polarizability of the fractionally charged baryons is 1 order of magnitude smaller than that of the physical baryons. Further, we are

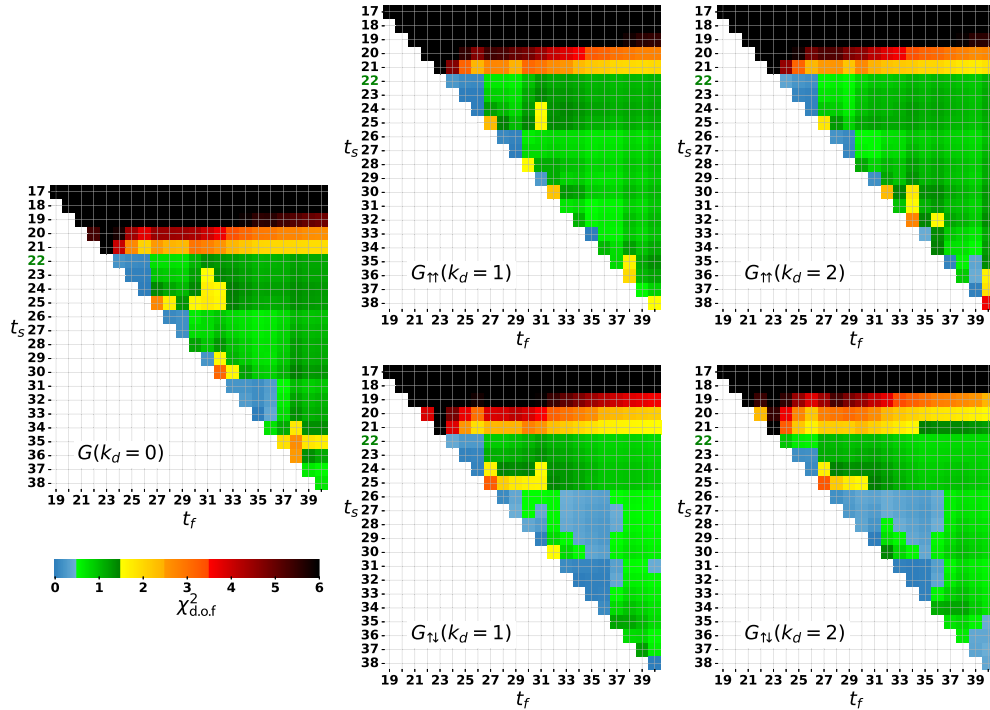


FIG. 3. $\chi^2_{d.o.f}$ heat maps for fits to $\log G$ for all fit windows from $t_s \rightarrow t_f$ of the aligned ($\uparrow\uparrow$) and antialigned ($\uparrow\downarrow$) correlators at each field strength of interest. The numbers in parentheses denote the field strength quanta governed by k_d . This example is a *ssl* baryon at $\kappa = 0.13770$, $m_\pi = 296$ MeV. The first time slice at which single-state isolation has been reached for each correlator is highlighted in green.

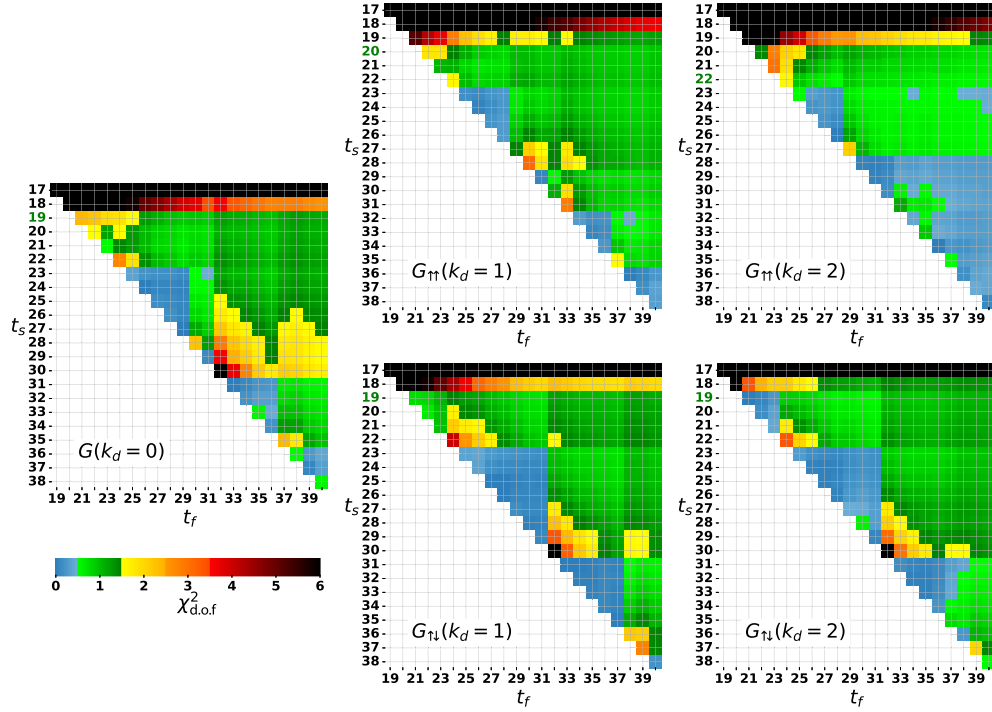


FIG. 4. $\chi^2_{d.o.f}$ heat maps for fits to $\log G$ as in Fig. 3 for a ddh baryon at $\kappa = 0.13770$, $m_\pi = 296$ MeV.

interested in the magnetic polarizability difference. The difference is a very sensitive quantity and extracting a meaningful signal is difficult, especially at light quark masses.

As a result, we will extract two sets of data in this work, one in which the underlying correlation functions are explicitly checked for single state isolation. This dataset we denote “corr.” to reflect the correlator level examination

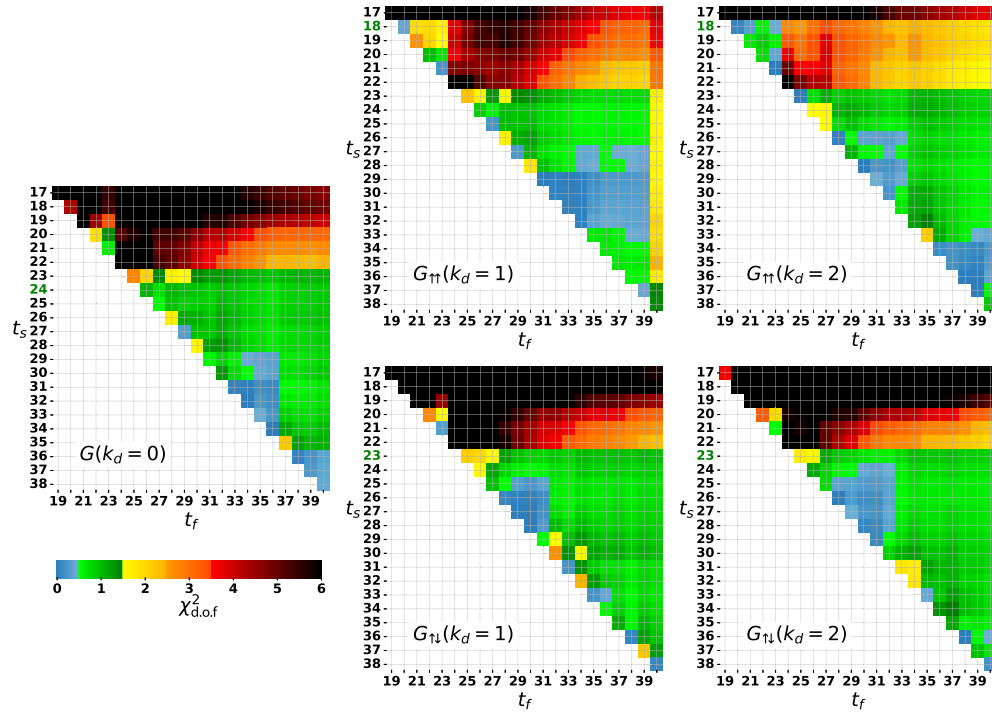


FIG. 5. $\chi^2_{d.o.f}$ heat maps for fits to $\log G$ as in Fig. 3 for a ddl baryon at $\kappa = 0.13770$, $m_\pi = 296$ MeV.

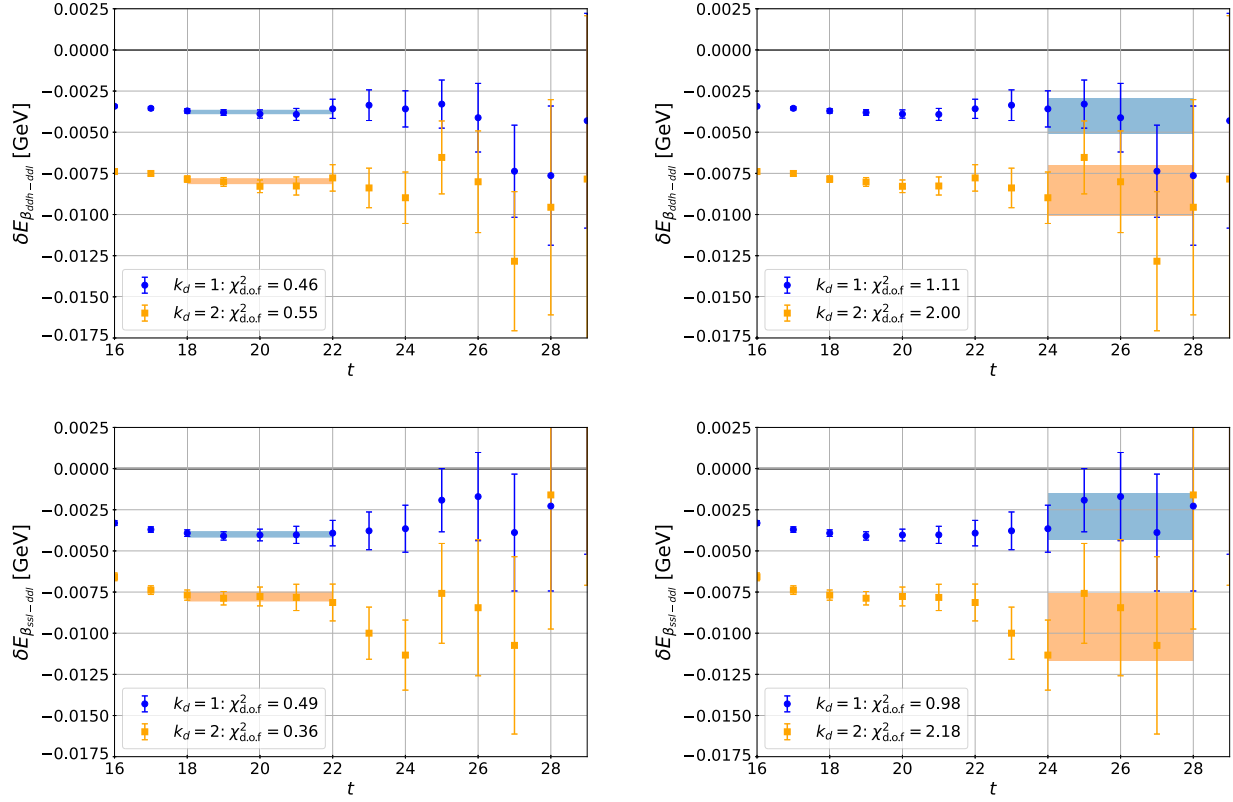


FIG. 6. Effective energies for $\delta E_{\beta_1 - \beta_2}(B, t)$ of Eq. (24) for $ddh - ddl$ (top) illustrating doubly represented environment sensitivity and $ssl - ddl$ (bottom) for the direct quark mass effect. The left column illustrates the highest weighted fits satisfying the ratio criteria, while the right-hand column illustrates the highest weighted fits satisfying the more restrictive corr. analysis criteria. Here $\kappa = 0.13770$ corresponds to $m_\pi = 296$ MeV, the lightest mass ensemble considered. These correlators are examples of the most challenging effective energies to fit due to the presence of the triply light ddl correlation function.

of the $\chi^2_{d.o.f}$. We will also produce a dataset where this criteria is relaxed and we examine the correlated $\chi^2_{d.o.f}$ of the energy shift ratio to produce appropriate sampling of the possible fit windows. We denote this dataset “ratio.” In this case we set $t_{\min} = t_{\text{source}} + 2$. In both cases a weighted average of fit windows is considered. However, in the more restrictive individual correlator case, the result is to exclude fits at early times as described above. As we will see, the two datasets agree at 1σ .

B. Correlation function fits

Figure 6 presents some of the most challenging effective energies for $\delta E_{\beta_1 - \beta_2}(B, t)$ of Eq. (24) to fit. These correlators involve the ddl correlation function. The presence of three light quarks means the signal in these correlators diminishes more rapidly than that for the other correlators under consideration.

Here one can observe the problem with the restrictive corr. analysis criteria. The ddl correlator heat maps introduce a cut that prevents fits from commencing earlier than $t = 24$, the very end of what might be regarded as robust signal as illustrated in Fig. 6. Fit windows in the corr. case will extend into the region where signal loss is

becoming manifest. This is particularly concerning for the $ssl - ddl$ correlators. Here, central values can fluctuate wildly and are unlikely to be representative of future next-generation fits.

The fits illustrated in Fig. 6 are example fits carrying a large weight in the weighted fit analysis described in Sec. VI. The left-hand column shows fits satisfying the ratio analysis criteria, whereas the right-hand column shows fits satisfying the corr. analysis criteria. While the effective energy fits in the left-hand column may have some excited-state contamination which largely cancels in the ratio of correlation functions, these results do not suffer wild fluctuations associated with a loss of signal.

C. Polarizability difference field-strength fits

Figure 7 presents the single parameter fits describing the magnetic field strength dependence of the Landau-term subtracted energy shift of Eq. (33). Here, the energy shifts have been determined via the ratio analysis criteria, where the uncertainties are smaller and present a more interesting examination of the fit quality. Both indirect environmental effects (top four plots) and direct quark mass effects (bottom four plots) are presented. A survey of the quark

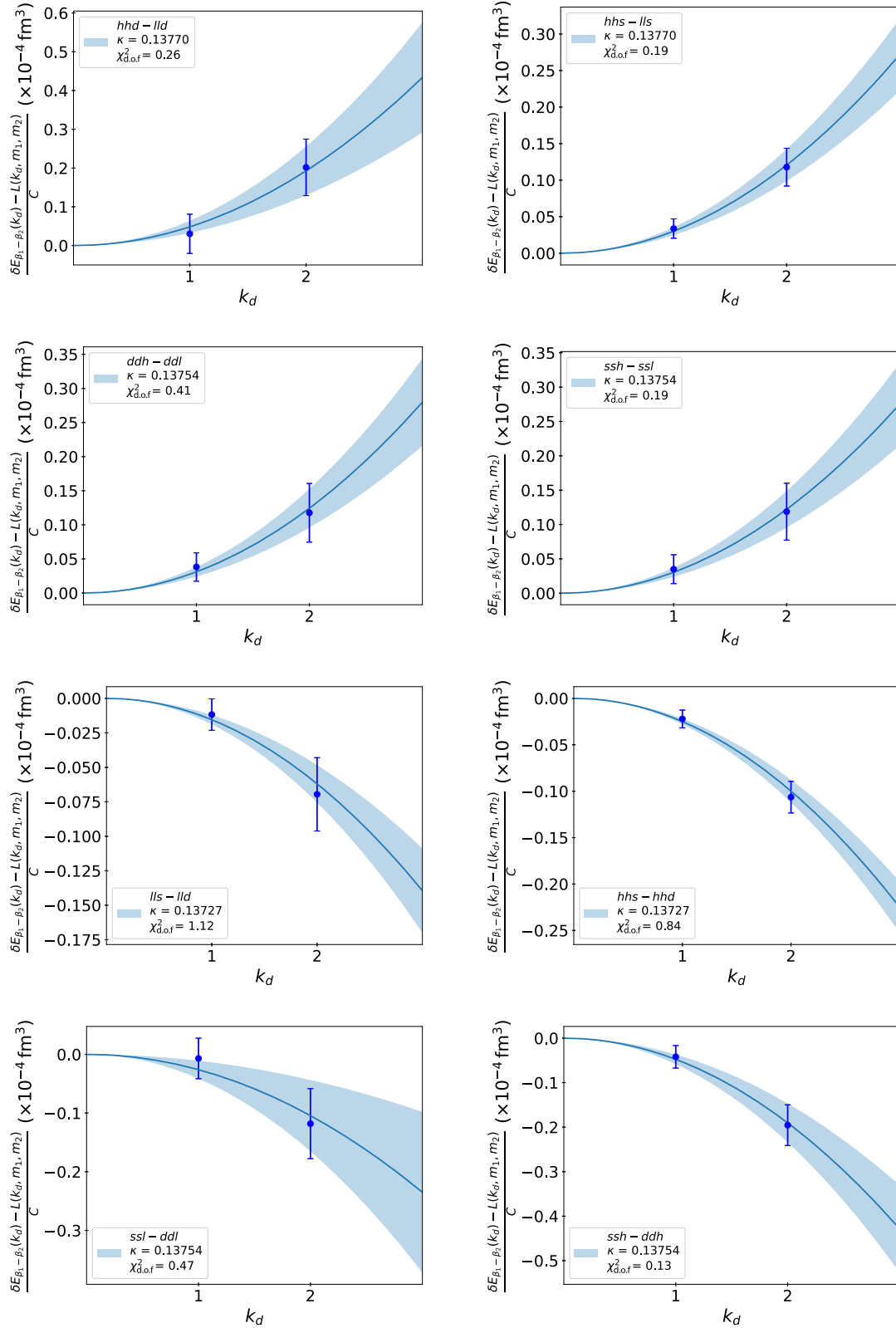


FIG. 7. Single parameter fits describing the magnetic field strength dependence of the Landau-term subtracted energy shift as in Eq. (33). As k_d describes the field strength in terms of a dimensionless integer, the magnetic polarizability difference can be inferred from the fit at $k_d = 1$. The top four plots illustrate indirect environmental effects while the bottom four plots illustrate the direct quark mass effects. A survey of the three lightest quark masses considered is presented.

mass dependence of the fits is provided over the three lightest quark masses considered.

Even though the results are determined via the ratio analysis criteria, the field strength dependence of the results is in accord with the expectations of Eq. (33) with $\chi^2_{d.o.f} \sim 1$ for most fits. For the corr. results the large uncertainties lead to very small $\chi^2_{d.o.f}$ values for almost all the fits.

IX. QUARK MASS EFFECTS IN BARYON POLARIZABILITIES

In Figs. 8 and 9 we show the main results of this investigation. The indirect environment driven changes in the magnetic polarizability are illustrated in Fig. 8. From a quark model perspective this is the simple case where most effects cancel and changes are driven by changes in the octet-decuplet baryon mass splitting as in Eq. (15). The direct quark mass driven changes in the magnetic polarizability are illustrated in Fig. 9. In all cases, we have taken the difference with the lighter baryon subtracted from the heavier baryon.

In these figures, both of our datasets are illustrated. Recall, the points labeled “ratio” are from the conventional analysis approach where reference to the correlation function ratio is used to identify fit plateaus. These results are complemented by the more rigorous but restrictive analysis techniques (offset for clarity) labeled “corr.” which are derived from strict requirements for single-state

isolation in each of the underlying correlation functions involved in the ratio. The trends observed in both datasets agree. However, in some cases a loss of signal gives rise to very large fluctuations in the subtle polarizability changes for the corr. method.

Referring back to Fig. 5 providing $\chi^2_{d.o.f}$ heat maps for the ddl correlators, we see that the restriction to set the lower limit of possible Euclidean time fit windows to $t_s = 24$ forces the analysis to the edge of available signal as illustrated in Fig. 6 for the $ddh - ddl$ indirect environment effect correlators and the $ssl - ddl$ direct effect correlators. It is apparent that fit windows commencing at $t = 24$ and beyond extend into the region where signal loss is manifest.

While the increased uncertainties that arise from the restrictions of the corr. analysis criteria are not problematic, the position of the central values can vary wildly in the signal loss region such that the results may not be representative of future next-generation calculations. For example, the lightest quark-mass point of the $ssh - ddh$ plot of Fig. 9 is off scale by more than a factor of 2.

This contrasts the results obtained from the ratio analysis criteria. While the ratio method may have some excited-state contamination which largely cancels in the ratio of correlation functions, these results are likely to be more indicative of next generation results. Thus where the uncertainties of the corr. analysis results are large, we defer to the results determined in the ratio method, as they are likely to be more representative.

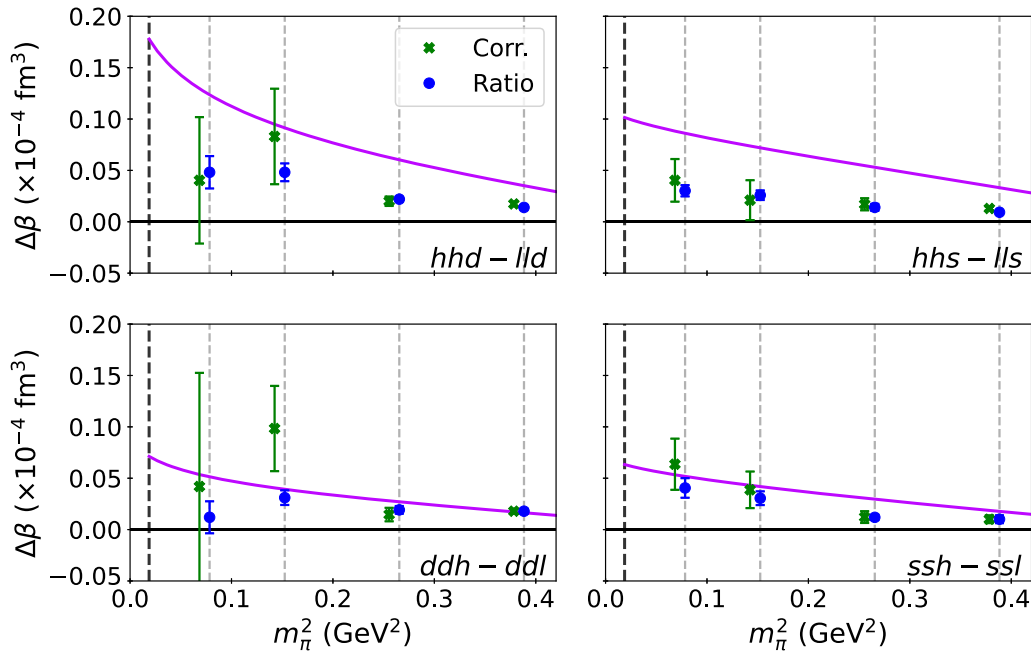


FIG. 8. Indirect environment driven changes in quark-sector contributions to octet baryon magnetic polarizabilities, $\Delta\beta$, for singly (upper) and doubly (lower) represented quark sectors. Results are plotted as a function of the pion mass governing the mass of the light d and l quarks. The labels in the lower-right of the panels indicate the magnetic polarizability difference illustrated. For example the upper left panel reports $\Delta\beta = \beta_{hhd} - \beta_{lld}$. The dataset labels “Ratio” and “Corr.” are described in the text. The purple curve is the quark model prediction for the environment dependence as discussed in Sec. III C 1.

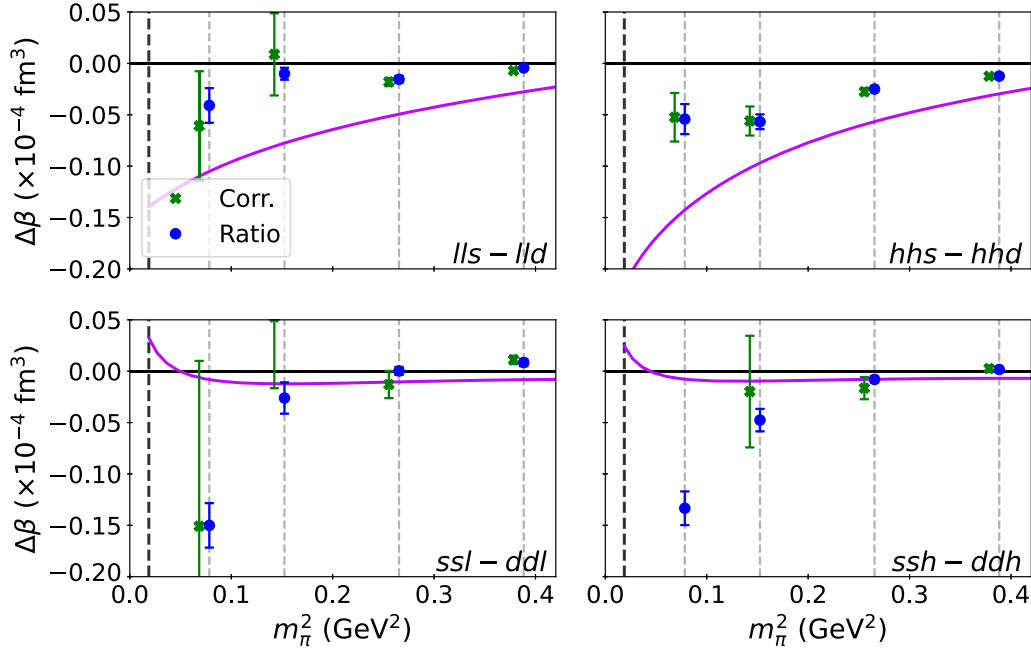


FIG. 9. Direct quark-mass driven changes in quark-sector contributions to octet baryon magnetic polarizabilities, $\Delta\beta$, for singly (upper) and doubly (lower) represented quark sectors. The presentation of the results is as in Fig. 8. Note, the missing data point at the lightest mass in the Corr. dataset for $ssh - ddh$ is outside the scale presented at $\Delta\beta = -0.41(14) \times 10^{-4} \text{ fm}^3$.

This situation for analyses involving the light-quark ddl and ddh correlators contrasts the heavy-quark dominated correlators such as the direct effect $ssh - ssl$ and even the $hhs - lls$ correlators. There the signal persists to $t \simeq 29$, and the corr. analysis restrictions are not problematic.

Let us commence with the general trends observed in the results. As the pion mass increases, all differences converge toward zero. This is to be expected as the strange quark mass is fixed for all PACS-CS ensembles. As the light d and l flavors approach the s -quark mass, the polarizability differences must vanish.

A. Environmental dependence in lattice QCD

Turning our attention to the environment driven changes in Fig. 8, we observe a clear environmental effect. We observe that increasing the mass of the spectator quark results in an increase of the magnetic polarizability. In Sec. III C 1, we showed that the quark model predicts such an effect for all four environmental considerations of Eq. (2). For example, consider Eq. (15):

$$\beta_{hhd} - \beta_{lld} = \frac{4}{9\pi} \left[\frac{\mu_d^2}{m_{\Xi^*} - m_{\Xi}} - \frac{\mu_d^2}{m_{\Delta} - m_N} \right]. \quad (36)$$

Here, the environment effect is driven by the octet-decuplet mass splittings which govern the strength of the transition term. The larger magnetic polarizability of the heavier baryon seen in $\Delta\beta$ is due to the smaller mass splitting for heavier baryons. In octet (decuplet) baryons the hyperfine

attraction (repulsion) between the quarks is inversely proportional to the product of the constituent quark masses. As a result, the order of the octet-decuplet mass splittings is $N > \Sigma > \Xi > hhh$. While the idea that heavier spectator quarks should give rise to larger magnetic polarizabilities may seem a little counterintuitive at first, the simple quark model provides a compelling explanation of the complex interactions of lattice QCD. It is satisfying that these early ideas founded in perturbation theory are manifest in the full nonperturbative formalism of lattice quantum field theory.

The quark model's suggestion of a simple effect based only on the octet-decuplet mass splitting does an impressive job of explaining the dependence, particularly for the doubly represented quark sector. The overestimate of the effect in the singly represented sector is interesting. It has long been known that the contributions of the singly represented quark flavor to baryon magnetic moments in lattice QCD are suppressed relative to constituent quark model predictions [1,27]. However, this suppression was not observed in the effective quark moments of the $\gamma N \rightarrow \Delta$ electromagnetic transition [3]. It would be interesting to revisit these early ideas drawing on modern lattice QCD techniques.

B. Direct quark mass dependence in lattice QCD

In the case of the direct quark mass effect, we now know that increasing the mass of the charged quark of interest results in a decrease in the magnetic polarizability. Intuitively this is a comfortable result. However, the quark model presents a more complex competition of contributions.

Recall the discussion around Eq. (16):

$$\beta_{lls} - \beta_{lld} = \frac{4}{9\pi} \left[\frac{\mu_s^2}{m_{\Sigma^*} - m_{\Sigma}} - \frac{\mu_d^2}{m_{\Delta} - m_N} \right] - \left[\frac{\alpha q_s^2}{6m_h} \langle r^2 \rangle_h - \frac{\alpha q_d^2}{6m_l} \langle r^2 \rangle_l \right], \quad (37)$$

in Sec. III C 2. As the constituent quark moments [Eq. (5)] are inversely proportional to quark mass, increasing the quark mass decreases their contributions, which are squared in the transition term of the magnetic polarizability. In contrast, octet-decuplet mass splittings act in the opposite manner to increase the magnitude of the transition term, as it did in the environmental case above. Moreover the sign of the quark distribution term contributions also acts to enhance the magnetic polarizability. However, the lattice QCD results now indicate the magnetic polarizability is suppressed as the quark mass of the active sector is increased, making the decrease in the constituent quark magnetic moment the dominant effect.

With regard to the singly represented quark sector results in the upper panel of Fig. 9, we once again observe an overestimate of the effect in the quark model. This is of a magnitude similar to that observed for the singly represented quark sector in the environmental case.

The tension between these effects increases for the doubly represented quark sector where the size of the quark distribution term doubles as in Eq. (37), but the sign of the magnetic polarizability difference in lattice QCD continues to show the same trend. However, the quark model predictions are significantly impacted. This leads to significant discrepancies for the direct quark mass driven differences in the doubly represented sector at the lightest quark masses considered where the enhanced quark distribution term β_2 is now able to dominate the contributions of the mass splitting and quark moments in the quark model.

Clearly, there is present behavior that the quark model fails to capture. An increase in the transition term, β_1 , relative to the distribution term, β_2 , could drive the model prediction back toward the lattice QCD results.

And given that the singly represented sector overestimates the direct quark mass effect, one needs to suppress the single quark transition moment. These effects are in accord with the results of lattice QCD calculations for octet-baryon magnetic moments [1,27]. Given that the model draws on these for the transition matrix element, there is qualitative agreement on the manner in which to improve the quark model further.

Finally, we connect the implications of these results to the physical octet baryons in Table III. Here we list the physical states in which the differences studied herein can manifest. The first four rows of the table correspond to indirect environment effects which cause an increase in the magnetic polarizability when the mass of a spectator quark

TABLE III. Changes in the physical octet baryon magnetic polarizabilities due to an indirect environmental effect (top four) and a direct quark mass effect (bottom four). Physical baryon states where the effect will come into play are indicated. The u transitions column is obtained by replacing the active d quark with u quark. As there is no octet state composed of three heavy quarks, there is no corresponding change.

Difference	d quark	u quark	Sign
$hhd - lld$	$\Xi^- - p$	$\Xi^0 - n$	+
$hhs - lls$			+
$ddh - ddl$	$\Sigma^- - n$	$\Sigma^+ - p$	+
$ssh - ssl$			+
$lls - lld$	$\Sigma - p$		-
$hhs - hhd$			-
$ssl - ddl$	$\Xi^0 - n$		-
$ssh - ddh$			-

is increased. The bottom four rows correspond to the direct quark mass effect which causes a decrease in the magnetic polarizability as the active quark mass is increased.

The most interesting environment effects for understanding the ordering of n , p , and Σ^+ baryons are the $hhd - lld$ and $ddh - ddl$ which correspond to the differences in the contribution of the up quark in $\Xi^0 - n$ and $\Sigma^+ - p$ due to a change of environment. In both cases, the effect enhances the magnetic polarizability of the hyperon relative to the nucleon. Both differences also contain a direct mass effect on the other sector given by $ssl - ddl$ and $lls - lld$ respectively. In this case, the magnetic polarizability of the hyperon is suppressed. The opposing signs of the indirect environment effect and the direct mass effect highlight the complexity in understanding the magnetic polarizabilities of octet baryons. We now appreciate that the quark model underestimates the direct mass effect in the doubly represented sector, leaving the neutron polarizability higher than it should be. Thus we have found the origin of the chief discrepancy depicted in Fig. 1.

X. CONCLUSION

Introducing fractionally charged baryons to the background field formalism of lattice QCD, we have disclosed the individual quark sector contributions to the magnetic polarizabilities of the outer octet baryons. We have exposed a direct quark mass effect that acts to suppress the magnetic polarizability as the quark mass increases. We have also observed an indirect environmental effect where the magnetic polarizability is enhanced as the mass of the spectator quark(s) is increased.

To gain an understanding of the underlying dynamics, we explored these direct and indirect effects in the context of a simple constituent quark model summarized in Eq. (3). The environmental effect appears as a relatively simple effect and may be explained by the change in the

octet-decuplet mass splitting in the transition term, β_1 . The behavior of the quark mass dependence is predicted to be more complex in the model with competition between the intrinsic constituent quark moment and the octet-decuplet baryon mass splitting in the transition term, β_1 , and further contributions from the quark distribution term, β_2 .

We now understand that it is the transition term of the polarizability, β_1 of Eq. (3), that contains the physics that describes our lattice QCD observations. The direct quark mass effect observed in the lattice results can be associated with the reduction of the intrinsic constituent quark moment for increasing quark mass, and this moment is squared in the numerator of β_1 . The indirect effect observed in the lattice results can be associated with the reduction of the octet-decuplet mass splitting in the denominator of β_1 , acting to enhance the magnetic polarizability as the quark mass increases and hyperfine effects separating the octet and decuplet baryon masses are suppressed.

We were not able to identify an equally important role for the quark distribution term of β_2 . In fact its enhancement in the direct quark mass effect in the double-represented quark sector is problematic and hides the important effect of the transition term. It is the distribution term, β_2 , that leaves the neutron magnetic polarizability high relative to the p and Σ^+ polarizabilities and the lattice QCD results.

It would be interesting to perform a modern lattice QCD study of the effective quark moments of octet and decuplet baryons and their electromagnetic transitions to gain more precise insight into this physics driving direct and indirect environmental effects in baryon magnetic polarizabilities. Similarly, it will be advantageous to address the exceptional configuration problem encountered at light quark masses, opening the opportunity to explore the role of chiral physics. A first step would be to perform a filtering of the exceptional configurations to establish that the problem

is understood. Longer term approaches could involve the use of chiral fermion actions to avoid problematic additive mass renormalizations. Alternatively, one could employ more efficient fermion actions and include the electric charges of the dynamical sea quarks such that they may correctly respond to the presence of a background magnetic field.

The results presented herein are founded on the PACS-CS ensembles of Ref. [17]. The ensembles are created by the PACS-CS Collaboration, Aoki *et al.* and are available from [28]. The ensembles in question are part of the April 2009 release.

Baryon correlation functions were constructed using the COLA software library, developed at the University of Adelaide [29].

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

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