

Estimating CP violation in $B_c \rightarrow BK^0 + B\bar{K}^0 \rightarrow B\pi^\pm e^\mp \nu_e$ decays

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In this paper, we investigate the CP asymmetries $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays; both of them consist of three parts: the indirect CP violations in $K^0 - \bar{K}^0$ mixing $A_{CP,mix}^{pm}$ and $A_{CP,mix}^{mp}$, the direct CP violations in B_c decay $A_{CP,dir}^{pm}$ and $A_{CP,dir}^{mp}$, and the new CP violation effects $A_{CP,int}^{pm}$ and $A_{CP,int}^{mp}$, which originate from the interference between the amplitude of the $B_c^- \rightarrow B^- K^0 (\bar{K}^0) \rightarrow B^- \pi^- e^+ \nu_e (\pi^+ e^- \bar{\nu}_e)$ decay with the difference between the oscillating effect of $K^0 \rightarrow \bar{K}^0$ and that of $\bar{K}^0 \rightarrow K^0$. The strong phase differences of the direct CP asymmetries $A_{CP,dir}^{pm}$ and $A_{CP,dir}^{mp}$ arise from $K^0 - \bar{K}^0$ mixing parameters. $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ are of the order of $\mathcal{O}(10^{-4})$ and $\mathcal{O}(10^{-3})$, respectively. The new CP violation effect $\mathcal{A}_{CP,int}^{pm}$ plays a dominant role in $\mathcal{A}_{CP}^{pm}$; the CP asymmetry $\mathcal{A}_{CP}^{mp}$ is dominated by the indirect CP violation $\mathcal{A}_{CP,mix}^{mp}$, so the CP asymmetry $\mathcal{A}_{CP}^{pm}$ provides an ideal place to study the new CP violation effect. We derive another two asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$, which are dominated by $K^0 - \bar{K}^0$ mixing. The observables $\mathcal{A}_{CP}^{pm}$, $\mathcal{A}_{CP}^{mp}$, $\mathcal{A}_{CP}^{pp}$, and $\mathcal{A}_{CP}^{mm}$ are hopefully to be marginally observed at the LHC experiment and the HL-LHC experiment.

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I. INTRODUCTION

In the Standard Model of particle physics, CP violation originates from a nonzero phase parameter within the Cabibbo-Kobayashi-Mashawa (CKM) matrix [1,2]. In general, the CKM mechanism is very successful in describing all observed CP violation phenomena and is, very likely, the dominant source of CP violation in low energy flavor-changing processes [3,4]. However, the CKM mechanism fails to account for the astronomical observations suggested amount of matter-antimatter asymmetry, which presents a significant challenge to the Standard Model and hints at the presence of additional sources of CP violation [5]. Probing CP violation in various reactions and seeing the correlation between different processes may open a window to the discovery of additional sources of CP violation [6].

CP violation has been well established in both K and B systems; however, the only place where CP violation was discovered in charmed systems was the difference between the time-integrated asymmetries of $D^0 \rightarrow K^+ K^-$ and $D^0 \rightarrow \pi^+ \pi^-$ at 5.3σ level, which was reported by the LHCb collaboration in 2019 [7]. On the other hand, many theoretical efforts have so far been made to study CP violations in the D system and charm baryon during the past decade [8–30]. However, studies of CP -violating processes in the doubly heavy-flavored B_c decays that proceed via the c quark decays with the b quark as a spectator are very scarce [31].

The decays with final states including K^0 or $\bar{K}^0$ can be used to study several distinctive CP -violation effects, such as the indirect CP violation induced by the $K^0 - \bar{K}^0$ mixing [21,22,32–57]. Moreover, due to the $\Delta S = \Delta Q$ rule, the semileptonic K^0 and $\bar{K}^0$ transitions may have rich physics and provide a clean environment to study CP violation. In this paper, we calculate the CP violations in $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays. We predict the branching ratios of these decays and investigate the CP asymmetries, which have three kinds of CP violation effects. The paper is organized as follows. In Sec. II, we derive the branching ratios of the $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays. In Sec. III, we calculate the

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CP asymmetry observables for the $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays. The numerical results and discussions are presented in Sec. IV; we also investigate the sensitivity of these measurements in this section. Section V is the conclusion.

II. BRANCHING FRACTIONS OF THE $B_c \rightarrow BK^0 + B\bar{K}^0 \rightarrow B\pi^\pm e^\mp \nu_e$ DECAYS

Within the Standard Model, the $B_c^+ \rightarrow B^+ \bar{K}^0$ and $B_c^+ \rightarrow B^+ K^0$ decays are generated by $c \rightarrow s u \bar{d}$ and $c \rightarrow d u \bar{s}$ transitions, respectively. Their charge conjugate decays $B_c^- \rightarrow B^- K^0$ and $B_c^- \rightarrow B^- \bar{K}^0$ can occur through $\bar{c} \rightarrow \bar{s} \bar{u} d$ and $\bar{c} \rightarrow \bar{d} \bar{u} s$ transitions, respectively. The Feynman diagrams of these decays are shown in Figs. 1 and 2. The effective Hamiltonian for the $B_c^\pm \rightarrow B^\pm K^0(\bar{K}^0)$ decays is

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} [V_{cs}^* V_{ud} \bar{s} \gamma^\mu (1 - \gamma_5) c \bar{u} \gamma_\mu (1 - \gamma_5) d + V_{cd}^* V_{us} \bar{d} \gamma^\mu (1 - \gamma_5) c \bar{u} \gamma_\mu (1 - \gamma_5) s] + \text{H.c.}, \quad (1)$$

where G_F is the Fermi coupling constant, and V_{ij} denotes the CKM matrix element. To calculate the hadronic matrix elements of the effective Hamiltonian, we need to reconstruct the effective Hamiltonian in a form suitable for the use of the factorization assumption by Fierz transformation [12,58–60]. By making use of the Fierz transformation, we obtain the factorized Hamiltonian,

$$\mathcal{H}_{\text{eff}}^{\text{fac}} = -\frac{G_F}{\sqrt{2} N_c} [V_{cs}^* V_{ud} \bar{s} \gamma^\mu (1 - \gamma_5) d \bar{u} \gamma_\mu (1 - \gamma_5) c + V_{cd}^* V_{us} \bar{d} \gamma^\mu (1 - \gamma_5) s \bar{u} \gamma_\mu (1 - \gamma_5) c] + \text{H.c.}, \quad (2)$$

with $N_c = 3$ is the number of colors. According to the factorized Hamiltonian, the hadronic matrix elements $\langle K^0(q_t) B^\pm(p_2) | \mathcal{H}_{\text{eff}}^{\text{fac}} | B_c^\pm(p_1) \rangle$ and $\langle \bar{K}^0(q_t) B^\pm(p_2) | \mathcal{H}_{\text{eff}}^{\text{fac}} | B_c^\pm(p_1) \rangle$ can be written as [31]

$$\begin{aligned} & \langle K^0(q_t) B^+(p_2) | \mathcal{H}_{\text{eff}}^{\text{fac}} | B_c^+(p_1) \rangle \\ &= \frac{G_F f_K}{3\sqrt{2}} V_{cd}^* V_{us} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2), \end{aligned} \quad (3)$$

$$\begin{aligned} & \langle \bar{K}^0(q_t) B^+(p_2) | \mathcal{H}_{\text{eff}}^{\text{fac}} | B_c^+(p_1) \rangle \\ &= \frac{G_F f_K}{3\sqrt{2}} V_{cs}^* V_{ud} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2), \end{aligned} \quad (4)$$

$$\begin{aligned} & \langle K^0(q_t) B^-(p_2) | \mathcal{H}_{\text{eff}}^{\text{fac}} | B_c^-(p_1) \rangle \\ &= \frac{G_F f_K}{3\sqrt{2}} V_{cs} V_{ud}^* e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2), \end{aligned} \quad (5)$$

$$\begin{aligned} & \langle \bar{K}^0(q_t) B^-(p_2) | \mathcal{H}_{\text{eff}}^{\text{fac}} | B_c^-(p_1) \rangle \\ &= \frac{G_F f_K}{3\sqrt{2}} V_{cd} V_{us}^* e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2), \end{aligned} \quad (6)$$

where ϕ denotes the strong phase of the involved matrix elements, which are identical in the $B_c^\pm \rightarrow B^\pm K^0$ and $B_c^\pm \rightarrow B^\pm \bar{K}^0$ decays. The reason is as follows: both the $B_c^\pm \rightarrow B^\pm K^0$ and $B_c^\pm \rightarrow B^\pm \bar{K}^0$ decays are simple; they receive only the contribution of the color-suppressed tree-emission diagram, without the contributions of the color-allowed tree-emission diagram, the W-annihilation diagram, and the W-exchange diagram. In this case, the strong phases in the Cabibbo-favored decay (the $B_c^+ \rightarrow B^+ \bar{K}^0$ and $B_c^- \rightarrow B^- K^0$ decays) and the doubly Cabibbo-suppressed decay (the $B_c^- \rightarrow B^- \bar{K}^0$ and $B_c^+ \rightarrow B^+ K^0$ decays) are identical [24,31], so the presence of the strong phase ϕ in the involved matrix elements will not change the results of the branching ratios and the CP asymmetries of the $B_c^\pm \rightarrow B^\pm K^0$ and $B_c^\pm \rightarrow B^\pm \bar{K}^0$ decays, which are related to the products of the matrix elements in Eqs. (3)–(6) and the complex conjugates of them. f_K is the decay constant of K^0 and is defined by [61,62]

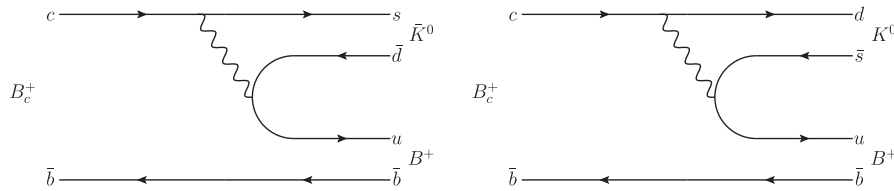


FIG. 1. Quark diagrams for the $B_c^+ \rightarrow B^+ \bar{K}^0$ and $B_c^+ \rightarrow B^+ K^0$ decays.

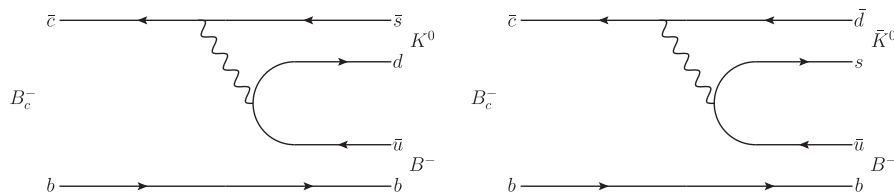


FIG. 2. Quark diagrams for the $B_c^- \rightarrow B^- K^0$ and $B_c^- \rightarrow B^- \bar{K}^0$ decays.

$$\langle K^0(q_t) | \bar{d} \gamma^\mu \gamma_5 s | 0 \rangle = i q_t^\mu f_K, \quad (7)$$

$$\langle \bar{K}^0(q_t) | \bar{s} \gamma^\mu \gamma_5 d | 0 \rangle = i q_t^\mu f_K. \quad (8)$$

$q_t = p_1 - p_2$ is the four-momentum transfer, $f_0(q_t^2)$ is one of two form factors that parametrize the matrix elements [63],

$$\begin{aligned} \langle B^+(p_2) | \bar{u} \gamma^\mu c | B_c^+(p_1) \rangle \\ = f_0(q_t^2) \left(\frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu \right) + f_+(q_t^2) \left(p_1^\mu + p_2^\mu - \frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu \right), \end{aligned} \quad (9)$$

$$\begin{aligned} \langle B^-(p_2) | \bar{c} \gamma^\mu u | B_c^-(p_1) \rangle \\ = -f_0(q_t^2) \left(\frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu \right) - f_+(q_t^2) \left(p_1^\mu + p_2^\mu - \frac{p_1^2 - p_2^2}{q_t^2} q_t^\mu \right). \end{aligned} \quad (10)$$

Here, the form factors can be represented as

$$f_0(q_t^2) = \frac{1}{1 - \frac{q_t^2}{m_{D^*}^2}} \left(b_0^{(0)} + b_0^{(1)} Z_p + b_0^{(2)} Z_p^2 + b_0^{(3)} Z_p^3 \right), \quad (11)$$

$$f_+(q_t^2) = \frac{1}{1 - \frac{q_t^2}{m_{D^*}^2}} \left(b_+^{(0)} + b_+^{(1)} Z_p + b_+^{(2)} Z_p^2 + b_+^{(3)} Z_p^3 \right). \quad (12)$$

m_{D^*} and m_{D^*} denote, respectively, the mass of the neutral scalar meson $D_0^*(2300)^0$ and the neutral vector meson $D^*(2007)^0$. The variable Z_p is defined as

$$Z_p = \left(\frac{\sqrt{t_+ - q_t^2} - \sqrt{t_+}}{\sqrt{t_+ - q_t^2} + \sqrt{t_+}} \right) \cdot \left| \frac{\sqrt{t_+ - m_{D^*}^2} + \sqrt{t_+}}{\sqrt{t_+ - m_{D^*}^2} - \sqrt{t_+}} \right|, \quad (13)$$

with $t_+ = (m_{B_c^+} + m_{B^+})^2$. The polynomial coefficients $b_{0,+}^{(i)}$ ($i = 0, 1, 2$, and 3) appearing in Eqs. (11) and (12) have been calculated in Ref. [63]:

$$\begin{aligned} b_0^{(0)} &= 0.548 \pm 0.023, & b_0^{(1)} &= -0.19 \pm 0.22, \\ b_0^{(2)} &= 0.05 \pm 0.74, & b_0^{(3)} &= 0, \end{aligned} \quad (14)$$

$$\begin{aligned} b_+^{(0)} &= 0.548 \pm 0.023, & b_+^{(1)} &= -0.48 \pm 0.21, \\ b_+^{(2)} &= 0.12 \pm 0.77, & b_+^{(3)} &= 0. \end{aligned} \quad (15)$$

Under the assumption that CPT is invariant, the time-evolved state of an initially ($t = 0$) pure K^0 or $\bar{K}^0$ is given as [64]

$$|K_{\text{phys}}^0(t)\rangle = g_+(t)|K^0\rangle - \frac{q}{p}g_-(t)|\bar{K}^0\rangle, \quad (16)$$

$$|\bar{K}_{\text{phys}}^0(t)\rangle = -\frac{p}{q}g_-(t)|K^0\rangle + g_+(t)|\bar{K}^0\rangle. \quad (17)$$

with

$$g_\pm(t) = \frac{1}{2} \left(e^{-im_L t - \frac{1}{2}\Gamma_L t} \pm e^{-im_S t - \frac{1}{2}\Gamma_S t} \right), \quad (18)$$

where m_L and m_S are the masses of the mass eigenstates K_L^0 and K_S^0 , respectively. Γ_L and Γ_S are the widths of the mass eigenstates K_L^0 and K_S^0 , respectively. The difference in masses of K_L^0 and K_S^0 and the average in widths of K_L^0 and K_S^0 are given by

$$\Delta m = m_L - m_S, \quad \Gamma = \frac{\Gamma_L + \Gamma_S}{2}. \quad (19)$$

p and q in Eqs. (16) and (17) are complex mixing parameters and can be parametrized as

$$p = \frac{1 + \epsilon}{\sqrt{2(1 + |\epsilon|^2)}}, \quad q = \frac{1 - \epsilon}{\sqrt{2(1 + |\epsilon|^2)}}, \quad (20)$$

where the complex parameter ϵ signifies deviation of the mass eigenstates from the CP eigenstates. Under the assumption that the $\Delta S = \Delta Q$ rule is valid, the amplitudes for the semileptonic decays $K^0 \rightarrow \pi^+ e^- \bar{\nu}_e$ and $\bar{K}^0 \rightarrow \pi^- e^+ \nu_e$ are given by

$$A(K^0 \rightarrow \pi^+ e^- \bar{\nu}_e) = 0, \quad (21)$$

$$A(\bar{K}^0 \rightarrow \pi^- e^+ \nu_e) = 0. \quad (22)$$

By combining Eqs. (16), (17), (21), and (22), we obtain the amplitudes for the $K_{\text{phys}}^0(t) \rightarrow \pi^\pm e^\mp \nu_e$ and $\bar{K}_{\text{phys}}^0(t) \rightarrow \pi^\pm e^\mp \nu_e$ decays as follows:

$$A(K_{\text{phys}}^0(t) \rightarrow \pi^+ e^- \bar{\nu}_e) = -\frac{q}{p}g_-(t)A(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e), \quad (23)$$

$$A(\bar{K}_{\text{phys}}^0(t) \rightarrow \pi^- e^+ \nu_e) = -\frac{p}{q}g_-(t)A(K^0 \rightarrow \pi^- e^+ \nu_e), \quad (24)$$

$$A(K_{\text{phys}}^0(t) \rightarrow \pi^- e^+ \nu_e) = g_+(t)A(K^0 \rightarrow \pi^- e^+ \nu_e), \quad (25)$$

$$A(\bar{K}_{\text{phys}}^0(t) \rightarrow \pi^+ e^- \bar{\nu}_e) = g_+(t)A(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e), \quad (26)$$

where t refers to duration time of $K^0(\bar{K}^0)$ in the rest frame of $K^0(\bar{K}^0)$, which relates to the $K^0(\bar{K}^0)$ flight length in the LHCb detector [25].

Now, we proceed to consider the time-dependent amplitude of the cascade decay processes $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\mp e^\pm \nu_e$, which

can be written as

$$\begin{aligned}
 & A(B_c^\pm \rightarrow B^\pm K^0(t) \rightarrow B^\pm \pi^+ e^- \bar{\nu}_e) \\
 &= \langle K^0(q_t) B^\pm(p_2) | \mathcal{H}_{\text{eff}} | B_c^\pm(p_1) \rangle \cdot A(K_{\text{phys}}^0(t) \rightarrow \pi^+ e^- \bar{\nu}_e) \\
 &+ \langle \bar{K}^0(q_t) B^\pm(p_2) | H_{\text{eff}} | B_c^\pm(p_1) \rangle \cdot A(\bar{K}_{\text{phys}}^0(t) \rightarrow \pi^+ e^- \bar{\nu}_e),
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 & A(B_c^\pm \rightarrow B^\pm K^0(t) \rightarrow B^\pm \pi^- e^+ \nu_e) \\
 &= \langle K^0(q_t) B^\pm(p_2) | \mathcal{H}_{\text{eff}} | B_c^\pm(p_1) \rangle \cdot A(K_{\text{phys}}^0(t) \rightarrow \pi^- e^+ \nu_e) \\
 &+ \langle \bar{K}^0(q_t) B^\pm(p_2) | H_{\text{eff}} | B_c^\pm(p_1) \rangle \cdot A(\bar{K}_{\text{phys}}^0(t) \rightarrow \pi^- e^+ \nu_e).
 \end{aligned} \tag{28}$$

Substituting Eqs. (3)–(6) and Eqs. (23)–(26) into Eqs. (27) and (28), we obtain

$$\begin{aligned}
 & A(B_c^+ \rightarrow B^+ K^0(t) \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) \\
 &= \frac{G_F f_K}{3\sqrt{2}} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2) A(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) \left[-V_{cd}^* V_{us} \cdot \frac{q}{p} g_-(t) + V_{cs}^* V_{ud} \cdot g_+(t) \right],
 \end{aligned} \tag{29}$$

$$\begin{aligned}
 & A(B_c^- \rightarrow B^- K^0(t) \rightarrow B^- \pi^- e^+ \nu_e) \\
 &= \frac{G_F f_K}{3\sqrt{2}} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2) \cdot \\
 &\quad \cdot A(K^0 \rightarrow \pi^- e^+ \nu_e) \left[-V_{cd} V_{us}^* \cdot \frac{q}{p} g_-(t) + V_{cs} V_{ud}^* \cdot g_+(t) + V_{cd} V_{us}^* \cdot g_-(t) \left(\frac{q}{p} - \frac{p}{q} \right) \right] \\
 &= \frac{G_F f_K}{3\sqrt{2}} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2) A(K^0 \rightarrow \pi^- e^+ \nu_e) \left[-V_{cd} V_{us}^* \cdot \frac{p}{q} g_-(t) + V_{cs} V_{ud}^* \cdot g_+(t) \right],
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 & A(B_c^+ \rightarrow B^+ K^0(t) \rightarrow B^+ \pi^- e^+ \nu_e) \\
 &= \frac{G_F f_K}{3\sqrt{2}} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2) A(K^0 \rightarrow \pi^- e^+ \nu_e) \left[V_{cd}^* V_{us} \cdot g_+(t) - V_{cs}^* V_{ud} \cdot \frac{p}{q} g_-(t) \right],
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 & A(B_c^- \rightarrow B^- K^0(t) \rightarrow B^- \pi^+ e^- \bar{\nu}_e) \\
 &= \frac{G_F f_K}{3\sqrt{2}} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2) \cdot \\
 &\quad \cdot A(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) \left[V_{cd} V_{us}^* \cdot g_+(t) - V_{cs} V_{ud}^* \cdot \frac{p}{q} g_-(t) + V_{cs} V_{ud}^* \cdot g_-(t) \left(\frac{p}{q} - \frac{q}{p} \right) \right] \\
 &= \frac{G_F f_K}{3\sqrt{2}} e^{i\phi} f_0(q_t^2) (p_1^2 - p_2^2) A(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) \left[V_{cd} V_{us}^* \cdot g_+(t) - V_{cs} V_{ud}^* \cdot \frac{q}{p} g_-(t) \right].
 \end{aligned} \tag{32}$$

Here, we note that the interferences between the first two terms inside the square bracket in the second line of Eqs. (29)–(32) are related to the direct CP violation. The strong phase differences in $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e (\pi^\mp e^\pm \bar{\nu}_e)$ decays are provided by the difference between $g_+(t)$ and $-\frac{q}{p} g_-(t) (-\frac{p}{q} g_-(t))$; i.e., the strong phase differences in $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\pm e^\mp \nu_e (\pi^\mp e^\pm \bar{\nu}_e)$ decays arise from $K^0 - \bar{K}^0$ mixing, and thus prevent pollution from strong dynamics [65]. Meanwhile, the interference between the second and third terms inside the square bracket in the second line of Eq. (30) and the interference between the first and third terms inside the square bracket in the second line of Eq. (32) are related to a new CP violation effect, which will be discussed in the next section.

Making use of Eq. (29) and performing integration over phase space and over $t \in [0, +\infty)$, we can obtain the time-independent decay width for the $B_c^+ \rightarrow B^+ K^0 + B^+ \bar{K}^0 \rightarrow B^+ \pi^+ e^- \bar{\nu}_e$ (hereafter for brevity referred to as $B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e$) decay,

$$\begin{aligned}
 \Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) &= \frac{G_F^2 f_K^2 (f_0(m_{K^0}^2))^2}{288\pi m_{B_c^+}^3} f(m_{B_c^+}, m_{B^+}, m_{K^0}) (m_{B_c^+}^2 - m_{B^+}^2)^2 |V_{cs}|^2 |V_{ud}|^2 \\
 &\quad \cdot \Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) \int_0^\infty |g_+(t)|^2 dt \left[r_{wf}^2 r_g \left| \frac{q}{p} \right|^2 + r_{wf} r_{sf} e^{i(\delta+\phi)} \frac{q}{p} + r_{wf} r_{sf} e^{-i(\delta+\phi)} \frac{q^*}{p^*} + 1 \right].
 \end{aligned} \tag{33}$$

In the above equation, the following substitutions are made for convenience:

$$r_{wf}e^{i\phi} = -\frac{V_{cd}^*V_{us}}{V_{cs}^*V_{ud}}, \quad r_g = \frac{\int_0^\infty |g_-(t)|^2 dt}{\int_0^\infty |g_+(t)|^2 dt},$$

$$r_{sf}e^{i\delta} = \frac{\int_0^\infty g_-(t)(g_+(t))^* dt}{\int_0^\infty |g_+(t)|^2 dt}, \quad (34)$$

with

$$\int_0^\infty |g_+(t)|^2 dt = \frac{1}{4} \left[\frac{\Gamma_L + \Gamma_S}{\Gamma_L \Gamma_S} + \frac{2\Gamma}{\Gamma^2 + (\Delta m)^2} \right], \quad (35)$$

$$\int_0^\infty |g_-(t)|^2 dt = \frac{1}{4} \left[\frac{\Gamma_L + \Gamma_S}{\Gamma_L \Gamma_S} - \frac{2\Gamma}{\Gamma^2 + (\Delta m)^2} \right], \quad (36)$$

$$\int_0^\infty (g_-(t))^* g_+(t) dt = \frac{1}{4} \left[\frac{\Gamma_S - \Gamma_L}{\Gamma_L \Gamma_S} + \frac{2i\Delta m}{\Gamma^2 + (\Delta m)^2} \right], \quad (37)$$

where r_{sf} , r_{wf} , and r_g are positive numbers, and ϕ is the weak phase difference. m_r with $r = B_c^\pm, B^\pm, K^0$ in Eq. (33) denotes the mass of the resonance; the function $f(x, y, z)$ in Eq. (33) is defined as

$$f(x, y, z) = \sqrt{x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2}. \quad (38)$$

In Eqs. (33), (35), (36), and (37), the upper limit of the integration over the duration time t of $K^0(\bar{K}^0)$ (hereafter for brevity referred to as t_{up}) is infinity, which is based on the following considerations:

- (1) The $K^0(\bar{K}^0)$ flight length is measured in the laboratory frame, whereas the duration time t of $K^0(\bar{K}^0)$ is measured in the rest frame of $K^0(\bar{K}^0)$. It would become larger during the transformation from the rest frame of $K^0(\bar{K}^0)$ to the lab frame; the $K^0(\bar{K}^0)$ flight length not only relates to the duration time t of $K^0(\bar{K}^0)$, but also relates to the energy and momentum of $B_c^\pm$ mesons in the laboratory frame and that of $K^0(\bar{K}^0)$ in the rest frame of $B_c^\pm$ mesons.
- (2) The $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ decays could be measured at LHCb [66]. There are three particularly important features in these decays. First, the neutrino cannot be detected by the LHCb detector, so a good candidate for the $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ decays must have the missing momentum. Second, there are a charged pion and a charged lepton ($e^\pm$) in the final states of these decays, so the $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ candidate can be selected by requiring a well-identified charged pion and a high quality electron; moreover, the tracks of the charged pion and the electron are constrained to originate from a common vertex in the event selection. Third, the invariant mass distribution of $\pi^\pm$ and $e^\mp$ originating from the $K^0(\bar{K}^0)$ decays must have a

specific range and line shape, which are determined by the Monte Carlo (MC) simulation, so a good candidate for the $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ decays requires the invariant mass distribution of $\pi^\pm$ and $e^\mp$ must lie in the specific range and have the specific line shape. Meanwhile, there are still additional experimental features which have not been included above. By considering these experimental features, the search for the $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ decays can be well performed by the experimental physicists at LHCb.

- (3) Because the invariant mass and line shape requirements of the $\pi^\pm e^\mp$ system can reject the backgrounds from the random combinations of $\pi^\pm$ and $e^\mp$ that do not originate from the $K^0(\bar{K}^0)$ decays, the dominant backgrounds are from the $K^0(\bar{K}^0)$ decays involving $\pi^\pm$ and $e^\mp$ in the final states, such as the $K_{S,L}^0 \rightarrow \pi^\pm \mu^\mp \nu_\mu \rightarrow \pi^\pm e^\mp \nu_e \bar{\nu}_\mu \nu_\mu$, $K_{S,L}^0 \rightarrow \pi^+ \pi^- e^+ e^-$, $K_L^0 \rightarrow \pi^0 \pi^\pm e^\mp \nu_e$, $K_L^0 \rightarrow \gamma \pi^\pm e^\mp \nu_e$, and $K_L^0 \rightarrow \gamma \pi^\pm \mu^\mp \nu_\mu \rightarrow \gamma \pi^\pm e^\mp \nu_e \bar{\nu}_\mu \nu_\mu$ decays [64]. The branching ratios of the $K_{S,L}^0 \rightarrow \pi^+ \pi^- e^+ e^-$, $K_L^0 \rightarrow \pi^0 \pi^\pm e^\mp \nu_e$, $K_L^0 \rightarrow \gamma \pi^\pm e^\mp \nu_e$, and $K_L^0 \rightarrow \gamma \pi^\pm \mu^\mp \nu_\mu \rightarrow \gamma \pi^\pm e^\mp \nu_e \bar{\nu}_\mu \nu_\mu$ decays much less than that of the $K_{S,L}^0 \rightarrow \pi^\pm e^\mp \nu_e$ decays, so these backgrounds can be safely neglected. The branching ratios of the $K_{S,L}^0 \rightarrow \pi^\pm \mu^\mp \nu_\mu \rightarrow \pi^\pm e^\mp \nu_e \bar{\nu}_\mu \nu_\mu$ decays have the same order of magnitude as the signal processes $K_{S,L}^0 \rightarrow \pi^\pm e^\mp \nu_e$; however, due to the mean life of the muon is of the order of 10^{-6} s. The probability of the decays $\mu^\pm \rightarrow e^\pm \nu_e \nu_\mu$ occur in the vertex detector (VELO) and the Tracker Turicensis (TT) is small; however, the tracks of $e^\mp$ can be well detected and reconstructed only when the tracking information from the VELO (or from the TT) to the T stations is available in LHCb, so the requirement of a high quality electron can suppress these backgrounds. In addition, there exist three neutrinos in the final states of the background processes, while the signal processes only comprise a neutrino, the requirement of the invariant mass and line shape of the $\pi^\pm e^\mp$ system can also suppress these backgrounds. In a word, the $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ decays can be searched for at the low background levels.
- (4) In the LHCb experiment, the tracks of $\pi^\pm$ and $e^\mp$ originating from the $K^0(\bar{K}^0) \rightarrow \pi^\mp e^\pm \nu_e$ decays can be well detected and reconstructed when the decay positions of $K^0(\bar{K}^0)$ are located in the vertex detector (VELO) and the Tracker Turicensis (TT). The center of TT is at about 2.5m from the interaction point; this indicates that the flight lengths of $K^0(\bar{K}^0)$ are less than 2.5m if the decay positions of $K^0(\bar{K}^0)$ are located in VELO and TT [67]. When these experimental features are taken into account, t_{up} can be set to a specific value, such as $2/\Gamma_S$.

- (5) However, the flight lengths of $K^0(\bar{K}^0)$ in the LHCb experiment do not match with the specific value of t_{up} in theory; for example, even if the duration time t of $K^0(\bar{K}^0)$ is small in an event, the flight length of $K^0(\bar{K}^0)$ in this event can be longer than the distance from the interaction point to the center of TT when the energy and momentum of $B_c^\pm$ mesons in the laboratory frame are large enough. On the other side, even if the flight length of $K^0(\bar{K}^0)$ in an event is less than $2.5m$, the duration time t of $K^0(\bar{K}^0)$ can be larger than the specific value of t_{up} when the energy and momentum of $B_c^\pm$ mesons in the laboratory frame are small enough in this event. In a word, if a specific value of t_{up} is adopted, the theoretical prediction, which is based on the specific value of t_{up} , cannot match with the LHCb data sample directly.
- (6) There is a method to match the theoretical prediction with the LHCb data sample when a specific value of t_{up} is adopted. First, a sample with a specific value of t_{up} is generated by MC simulations, and then obtain the efficiency of the detecting and event selection. Second, as for every event in the data sample after event selection in the LHCb experiment, the duration time t of $K^0(\bar{K}^0)$ in the laboratory frame can be obtained with the momentum and the flight length of $K^0(\bar{K}^0)$ in the laboratory frame. Third, the duration time t of $K^0(\bar{K}^0)$ in the rest frame of $K^0(\bar{K}^0)$ can be obtained by using the duration time t of $K^0(\bar{K}^0)$ in the laboratory frame and the Lorentz transformations (between the laboratory frame and the rest frame of B_c and between the rest frame of B_c and the rest frame of $K^0(\bar{K}^0)$). Fourth, to obtain the data sample which matches with the theoretical prediction and MC simulations, the requirement that the duration time t of $K^0(\bar{K}^0)$ in the rest frame of $K^0(\bar{K}^0)$ is less than the specific value of t_{up} is applied in the data sample after event selection in the LHCb experiment. Obviously, this method is complicated; moreover, this method cannot be performed perfectly for the $B_c^+ \rightarrow B^+ K^0 + B^+ \bar{K}^0 \rightarrow B^+ \pi^+ e^+ \bar{\nu}_e$ decays.

This is because the final states contain neutrino, and the energy and momentum of $B_c^\pm$ and $K^0(\bar{K}^0)$ mesons in the laboratory frame cannot be determined accurately.

- (7) The events in which the flight length of $K^0(\bar{K}^0)$ is more than $2.5m$ can also be detected by the station (Inner Tracker and Outer Tracker) in the LHCb experiment; the efficiency of the detecting and event selection for these events is very low at LHCb. Meanwhile, when the duration time t of $K^0(\bar{K}^0)$ is small, the $K^0(\bar{K}^0) \rightarrow \pi^\pm e^\mp \nu_e$ decays receive contributions from the K_S^0 decay and the $K_S^0 - K_L^0$ interference; when the duration time t of $K^0(\bar{K}^0)$ is large, the $K^0(\bar{K}^0) \rightarrow \pi^\pm e^\mp \nu_e$ decays receive contributions from the K_L^0 decay and the $K_S^0 - K_L^0$ interference [66,68].
- (8) If we adopt the scenario that t_{up} is infinity, the theoretical prediction is based on the calculations that include the contributions of all of the duration time t of $K^0(\bar{K}^0)$. In experiment, a sample with all of the duration time t of $K^0(\bar{K}^0)$ is generated by MC simulations, and then we obtain the efficiency of the detecting and event selection. Meanwhile, the data sample after event selection also includes the contributions of all of the duration time t of $K^0(\bar{K}^0)$ in the LHCb experiment. Using the data sample after event selection in the LHCb experiment and the efficiency of the detecting and event selection, we can obtain the total event number which includes the contribution of all of the duration time t of $K^0(\bar{K}^0)$ and matches perfectly with the theoretical prediction.

In this paper, we adopt the scenario that t_{up} is infinity in order to match the theoretical prediction with the data sample in experiment.

Similarly, we can derive the time-independent decay widths for the $B_c^- \rightarrow B^- K^0 + B^- \bar{K}^0 \rightarrow B^- \pi^- e^+ \nu_e$ (hereafter for brevity referred to as $B_c^- \rightarrow B^- \pi^- e^+ \nu_e$) and $B_c^\pm \rightarrow B^\pm K^0 + B^\pm \bar{K}^0 \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ (hereafter for brevity referred to as $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e$) decays,

$$\Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e) = \frac{G_F^2 f_K^2 (f_0(m_{K^0}^2))^2}{288\pi m_{B_c^+}^3} f(m_{B_c^+}, m_{B^+}, m_{K^0}) (m_{B_c^+}^2 - m_{B^+}^2)^2 |V_{cs}|^2 |V_{ud}|^2 \cdot \Gamma(K^0 \rightarrow \pi^- e^+ \nu_e) \int_0^\infty |g_+(t)|^2 dt \left[r_{wf}^2 r_g \left| \frac{p}{q} \right|^2 + r_{wf} r_{sf} e^{i(\delta-\phi)} \frac{p}{q} + r_{wf} r_{sf} e^{-i(\delta-\phi)} \frac{p^*}{q^*} + 1 \right], \quad (39)$$

$$\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) = \frac{G_F^2 f_K^2 (f_0(m_{K^0}^2))^2}{288\pi m_{B_c^+}^3} f(m_{B_c^+}, m_{B^+}, m_{K^0}) (m_{B_c^+}^2 - m_{B^+}^2)^2 |V_{cs}|^2 |V_{ud}|^2 \cdot \Gamma(K^0 \rightarrow \pi^- e^+ \nu_e) \int_0^\infty |g_+(t)|^2 dt \left[r_{wf}^2 + r_{wf} r_{sf} e^{i(\delta-\phi)} \frac{p}{q} + r_{wf} r_{sf} e^{-i(\delta-\phi)} \frac{p^*}{q^*} + r_g \left| \frac{p}{q} \right|^2 \right], \quad (40)$$

$$\Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e) = \frac{G_F^2 f_K^2 (f_0(m_{K^0}^2))^2}{288\pi m_{B_c^+}^3} f(m_{B_c^+}, m_{B^+}, m_{K^0}) (m_{B_c^+}^2 - m_{B^+}^2)^2 |V_{cs}|^2 |V_{ud}|^2 \cdot \Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) \int_0^\infty |g_+(t)|^2 dt \left[r_{wf}^2 + r_{wf} r_{sf} e^{i(\delta+\phi)} \frac{q}{p} + r_{wf} r_{sf} e^{-i(\delta+\phi)} \frac{q^*}{p^*} + r_g \left| \frac{q}{p} \right|^2 \right], \quad (41)$$

where $\Gamma(K^0 \rightarrow \pi^- e^+ \nu_e)$ and $\Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e)$ are the decay widths for the $K^0 \rightarrow \pi^- e^+ \nu_e$ and $\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e$ decays, respectively. CPT invariance implies $\Gamma(K^0 \rightarrow \pi^- e^+ \nu_e) = \Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e)$ [61]. Both $\Gamma(K^0 \rightarrow \pi^- e^+ \nu_e)$ and $\Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e)$ can be given by

$$\Gamma(K^0 \rightarrow \pi^- e^+ \nu_e) = \Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) = \Gamma_S \cdot \mathcal{B}(K_S^0 \rightarrow \pi^\pm e^\mp \nu_e) = \Gamma_L \cdot \mathcal{B}(K_L^0 \rightarrow \pi^\pm e^\mp \nu_e), \quad (42)$$

where $\mathcal{B}(K_S^0 \rightarrow \pi^\pm e^\mp \nu_e)$ and $\mathcal{B}(K_L^0 \rightarrow \pi^\pm e^\mp \nu_e)$ are the branching ratios for the $K_S^0 \rightarrow \pi^\pm e^\mp \nu_e$ and $K_L^0 \rightarrow \pi^\pm e^\mp \nu_e$ decays, respectively. Here, we note that $\mathcal{B}(K_S^0 \rightarrow \pi^\pm e^\mp \nu_e)$ is the summation of the branching ratios for the $K_S^0 \rightarrow \pi^+ e^- \bar{\nu}_e$ and $K_S^0 \rightarrow \pi^- e^+ \nu_e$ decays, $\mathcal{B}(K_L^0 \rightarrow \pi^\pm e^\mp \nu_e)$ is the summation of the branching ratios for the $K_L^0 \rightarrow \pi^+ e^- \bar{\nu}_e$ and $K_L^0 \rightarrow \pi^- e^+ \nu_e$ decays. Both $\mathcal{B}(K_S^0 \rightarrow \pi^\pm e^\mp \nu_e)$ and $\mathcal{B}(K_L^0 \rightarrow \pi^\pm e^\mp \nu_e)$ can be obtained from the Particle Data Group [64]. In order to characterize the different effects of CP violation, Eqs. (39) and (41) can also be written as

$$\Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e) = \frac{G_F^2 f_K^2 (f_0(m_{K^0}^2))^2}{288\pi m_{B_c^+}^3} f(m_{B_c^+}, m_{B^+}, m_{K^0}) (m_{B_c^+}^2 - m_{B^+}^2)^2 |V_{cs}|^2 |V_{ud}|^2 \cdot \Gamma(K^0 \rightarrow \pi^- e^+ \nu_e) \int_0^\infty |g_+(t)|^2 dt \left[r_{wf}^2 r_g \left| \frac{p}{q} \right|^2 + r_{wf} r_{sf} e^{i(\delta-\phi)} \frac{q}{p} + r_{wf} r_{sf} e^{-i(\delta-\phi)} \frac{q^*}{p^*} + 1 - r_{wf} r_{sf} e^{i(\delta-\phi)} \left(\frac{q}{p} - \frac{p}{q} \right) - r_{wf} r_{sf} e^{-i(\delta-\phi)} \left(\frac{q^*}{p^*} - \frac{p^*}{q^*} \right) \right], \quad (43)$$

$$\Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e) = \frac{G_F^2 f_K^2 (f_0(m_{K^0}^2))^2}{288\pi m_{B_c^+}^3} f(m_{B_c^+}, m_{B^+}, m_{K^0}) (m_{B_c^+}^2 - m_{B^+}^2)^2 |V_{cs}|^2 |V_{ud}|^2 \cdot \Gamma(\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}_e) \int_0^\infty |g_+(t)|^2 dt \left[r_{wf}^2 + r_{wf} r_{sf} e^{i(\delta+\phi)} \frac{p}{q} + r_{wf} r_{sf} e^{-i(\delta+\phi)} \frac{p^*}{q^*} + r_g \left| \frac{q}{p} \right|^2 - r_{wf} r_{sf} e^{i(\delta+\phi)} \left(\frac{p}{q} - \frac{q}{p} \right) - r_{wf} r_{sf} e^{-i(\delta+\phi)} \left(\frac{p^*}{q^*} - \frac{q^*}{p^*} \right) \right], \quad (44)$$

where the second and third terms in the square brackets of the above equations correspond to the direct CP violation effects. The last two terms in the square brackets of the above equations correspond to the new CP violation effects, which will be discussed in the next section.

Making use of Eqs. (33) and (39)–(41) and multiplying the mean life of the B_c meson, we can obtain the branching ratios for the $B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e$ and $B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e$ decays.

III. THE ASYMMETRIES IN $B_c \rightarrow B\pi^\pm e^\mp \nu_e$ DECAYS

Basing on the partial decay widths for the $B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e$ and $B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e$ decays derived in Sec. II, we can proceed to calculate the asymmetries in these decays.

A. CP asymmetry between the $B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e$ and $B_c^- \rightarrow B^- \pi^- e^+ \nu_e$ decays

In cascade decay chains $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e$, the time-independent CP violation observable is defined as

$$\mathcal{A}_{CP}^{pm} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^+ \rightarrow f_B) - \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^+ \rightarrow f_B) + \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}, \quad (45)$$

where f_B denotes the final state from the decay of the B^+ meson, $\bar{f}_B$ is the CP -conjugate state of f_B , and $\Gamma(B^+ \rightarrow f_B)$ and $\Gamma(B^- \rightarrow \bar{f}_B)$ are the partial decay widths of the $B^+ \rightarrow f_B$ and $B^- \rightarrow \bar{f}_B$ decays, respectively. With Eqs. (A1) and (A2) listed

in Appendix A, we can obtain

$$\begin{aligned}\mathcal{A}_{CP}^{pm} &= \mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) \cdot [1 - \mathcal{A}_{CP}(B \rightarrow f_B) + o(\mathcal{A}_{CP}(B \rightarrow f_B))] \\ &\quad + \mathcal{A}_{CP}(B \rightarrow f_B) \cdot [1 + \mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) + o(\mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e))] \\ &= \mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) \cdot [1 + o(\mathcal{A}_{CP}(B \rightarrow f_B))] \\ &\quad + \mathcal{A}_{CP}(B \rightarrow f_B) \cdot [1 + o(\mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e))],\end{aligned}\quad (46)$$

where $\mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e)$ and $\mathcal{A}_{CP}(B \rightarrow f_B)$ denote the CP asymmetry in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \bar{\nu}_e$ and $B_c^\mp \rightarrow B^\mp \pi^\mp e^\pm \nu_e$ decays and the CP asymmetry in $B^+ \rightarrow f_B$ and $B^- \rightarrow \bar{f}_B$ decays, respectively,

$$\mathcal{A}_{CP}(B \rightarrow f_B) = \frac{\Gamma(B^+ \rightarrow f_B) - \Gamma(B^- \rightarrow \bar{f}_B)}{\Gamma(B^+ \rightarrow f_B) + \Gamma(B^- \rightarrow \bar{f}_B)}, \quad (47)$$

$$\begin{aligned}\mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) \\ = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) - \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) + \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)},\end{aligned}\quad (48)$$

$o(\mathcal{A}_{CP}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e))$ and $o(\mathcal{A}_{CP}(B \rightarrow f_B))$ in Eq. (46) denote the high-order small quantities.

The CP violation effects in the nonleptonic decays of the $B^\pm$ can reach the value larger than 0.1 [64], so they cannot be neglected. Fortunately, we only investigate the semileptonic decays of the $B^\pm$ mesons in this paper; the direct CP violations in these decays do not occur within the Standard Model [69]. However, $\mathcal{A}_{CP}(B \rightarrow f_B)$ should also receive the contributions from the CP violations in charm meson decays for the semileptonic decays of the $B^\pm$ to charmed meson; for example, the CP violation in $B^\pm \rightarrow \bar{D}^0 l^\pm \nu_l \rightarrow f_{\bar{D}} l^\pm \nu_l$ decay chains is given by

$$\begin{aligned}\mathcal{A}_{CP}(B \rightarrow f_B) &= \frac{\Gamma(B^+ \rightarrow \bar{D}^0 l^+ \nu_l) - \Gamma(B^- \rightarrow D^0 l^- \bar{\nu}_l)}{\Gamma(B^+ \rightarrow \bar{D}^0 l^+ \nu_l) + \Gamma(B^- \rightarrow D^0 l^- \bar{\nu}_l)} \\ &\quad + \frac{\Gamma(\bar{D}^0 \rightarrow f_{\bar{D}}) - \Gamma(D^0 \rightarrow \bar{f}_D)}{\Gamma(\bar{D}^0 \rightarrow f_{\bar{D}}) + \Gamma(D^0 \rightarrow \bar{f}_D)},\end{aligned}\quad (49)$$

where $(\Gamma(\bar{D}^0 \rightarrow f_{\bar{D}}) - \Gamma(D^0 \rightarrow \bar{f}_D))/(\Gamma(\bar{D}^0 \rightarrow f_{\bar{D}}) + \Gamma(D^0 \rightarrow \bar{f}_D))$ is the CP asymmetry in $\bar{D}^0 \rightarrow f_{\bar{D}}$ and $D^0 \rightarrow \bar{f}_D$ decays, $f_{\bar{D}}$ denotes the final state from the decay of the $\bar{D}^0$ meson, and $\bar{f}_D$ is the CP -conjugate state of $f_{\bar{D}}$. In this paper, the $\bar{D}^0 \rightarrow K^+ e^- \bar{\nu}_e$, $\bar{D}^0 \rightarrow K^+ \mu^- \bar{\nu}_\mu$, $\bar{D}^0 \rightarrow K^+ \pi^-$, $\bar{D}^0 \rightarrow K^+ \pi^- \pi^0$, $\bar{D}^0 \rightarrow K^+ \pi^- \pi^- \pi^+$, $\bar{D}^0 \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0$, and $\bar{D}^0 \rightarrow K^+ \pi^- \pi^0 \pi^0$ decays are involved. For the semileptonic decays of the D^0 meson, there is no CP violation in the Standard Model. For the nonleptonic D^0 decays listed above, the CP violation has not been discovered in experiment. With the measurement of the CP asymmetry in $D^0 \rightarrow \pi^+ \pi^-$ decay, the branching ratio for the $D^0 \rightarrow \pi^+ \pi^-$ decay, and the large branching ratio for the nonleptonic D^0 decays listed above, the orders of the CP violations in the nonleptonic D^0 decays listed above can be estimated to be $\mathcal{O}(10^{-5})$ or smaller, which can be neglected safely [64,70–72]. In summary, the CP asymmetry in the semileptonic decays of the $B^\pm$ is

$$\mathcal{A}_{CP}(B \rightarrow f_B) = \frac{\Gamma(B^+ \rightarrow f_B) - \Gamma(B^- \rightarrow \bar{f}_B)}{\Gamma(B^+ \rightarrow f_B) + \Gamma(B^- \rightarrow \bar{f}_B)} \approx 0, \quad (50)$$

combining Eqs. (46)–(48) and (50), we can obtain

$$\mathcal{A}_{CP}^{pm} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) - \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) + \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}. \quad (51)$$

By combining Eqs. (20), (33), (34), (37), (43), and (51), we can derive

$$\mathcal{A}_{CP}^{pm} = \mathcal{A}_{CP,\text{mix}}^{pm} + \mathcal{A}_{CP,\text{dir}}^{pm} + \mathcal{A}_{CP,\text{int}}^{pm}, \quad (52)$$

$$\mathcal{A}_{CP,\text{mix}}^{pm} = \frac{r_{wf}^2 \cdot r_g \cdot (|\frac{q}{p}|^2 - |\frac{p}{q}|^2)}{r_{wf}^2 \cdot r_g \cdot (|\frac{q}{p}|^2 + |\frac{p}{q}|^2) + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi} (\frac{q^*}{p^*} e^{-i\delta} + \frac{p}{q} e^{i\delta})) + 2}, \quad (53)$$

$$\mathcal{A}_{CP,\text{dir}}^{pm} = \frac{-4r_{wf} \cdot r_{sf} \cdot |\frac{q}{p}| \cdot \sin \phi \cdot \sin \delta_S^{pm}}{r_{wf}^2 \cdot r_g \cdot (|\frac{q}{p}|^2 + |\frac{p}{q}|^2) + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi} (\frac{q^*}{p^*} e^{-i\delta} + \frac{p}{q} e^{i\delta})) + 2}, \quad (54)$$

$$\mathcal{A}_{CP,\text{int}}^{pm} = \frac{2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i(\phi-\delta)} (\frac{q}{p} - \frac{p}{q}))}{r_{wf}^2 \cdot r_g \cdot (|\frac{q}{p}|^2 + |\frac{p}{q}|^2) + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi} (\frac{q^*}{p^*} e^{-i\delta} + \frac{p}{q} e^{i\delta})) + 2}, \quad (55)$$

where $\mathcal{A}_{CP,\text{mix}}^{pm}$ denotes the indirect CP violation in kaon mixing [22,44], which is induced by the difference between the time evolutions of an initially ($t = 0$) pure K^0 and $\bar{K}^0$ as shown in Eqs. (16) and (17). $\mathcal{A}_{CP,\text{dir}}^{pm}$ denotes the direct CP violation originating from the interference between the Cabibbo-favored and doubly Cabibbo-suppressed amplitudes. The parameter δ_S^{pm} in Eq. (54) denotes the strong phase difference of the direct CP asymmetry $\mathcal{A}_{CP,\text{dir}}^{pm}$, which is given by

$$\delta_S^{pm} = \arg\left(\frac{q}{p}\right) + \delta = -\arctan\left(\frac{2\text{Im}(\epsilon)}{1 - |\epsilon|^2}\right) - \arctan\left(\frac{2\Delta m}{\Gamma_S - \Gamma_L} \cdot \frac{\Gamma_S \Gamma_L}{\Gamma^2 + (\Delta m)^2}\right). \quad (56)$$

From the above equation, we can see that the strong phase difference of the direct CP asymmetry $\mathcal{A}_{CP,\text{dir}}^{pm}$ is associated with the phases of q/p and $\int_0^\infty g_-(t)(g_+(t))^* dt$, i.e., the strong phase difference of the direct CP asymmetry arises from $K^0 - \bar{K}^0$ mixing parameters, and thus prevent pollution from strong dynamics. $\mathcal{A}_{CP,\text{int}}^{pm}$ represents the new CP violation effect (hereafter for brevity referenced as the CP violation in the interference), which originates from the interference between the amplitude of the $B_c^- \rightarrow B^- K^0 \rightarrow B^- \pi^- e^+ \nu_e$ decay with the difference between the mixing effect of $K^0 \rightarrow \bar{K}^0$ and that of $\bar{K}^0 \rightarrow K^0$, which can be derived from Eqs. (16), (17), (33), and (43). Combining Eqs. (20) and (55), we can obtain that

$$\mathcal{A}_{CP,\text{int}}^{pm} = \frac{-8r_{wf} \cdot r_{sf} \cdot (\text{Re}\epsilon \cdot \cos(\phi - \delta) + \text{Im}\epsilon \cdot \sin(\phi - \delta))}{r_{wf}^2 \cdot r_g \cdot (|\frac{q}{p}|^2 + |\frac{p}{q}|^2) + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi}(\frac{q^*}{p^*}e^{-i\delta} + \frac{p}{q}e^{i\delta})) + 2}. \quad (57)$$

Obviously, $\mathcal{A}_{CP,\text{int}}^{pm}$ involves the parameter r_{wf} and the weak phase difference ϕ , which relate to the CKM matrix elements.

B. CP asymmetry between the $B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e$ and $B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e$ decays

In cascade decay chains $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$, the time-independent CP violation observable is defined as

$$\mathcal{A}_{CP}^{mp} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) - \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) + \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}. \quad (58)$$

Using the same method as that for deriving the expression of $\mathcal{A}_{CP}^{pm}$ and Eqs. (A1) and (A2), we can derive

$$\mathcal{A}_{CP}^{mp} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) - \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e)}{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) + \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e)}. \quad (59)$$

Using Eqs. (20), (40), (44), and (59), we can obtain

$$\mathcal{A}_{CP}^{mp} = \mathcal{A}_{CP,\text{mix}}^{mp} + \mathcal{A}_{CP,\text{dir}}^{mp} + \mathcal{A}_{CP,\text{int}}^{mp}, \quad (60)$$

$$\mathcal{A}_{CP,\text{mix}}^{mp} = \frac{r_g \cdot (|\frac{p}{q}|^2 - |\frac{q}{p}|^2)}{2r_{wf}^2 + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi}(\frac{p}{q}e^{i\delta} + \frac{q^*}{p^*}e^{-i\delta})) + r_g \cdot (|\frac{p}{q}|^2 + |\frac{q}{p}|^2)}, \quad (61)$$

$$\mathcal{A}_{CP,\text{dir}}^{mp} = \frac{4r_{wf} \cdot r_{sf} \cdot |\frac{p}{q}| \cdot \sin \phi \cdot \sin \delta_S^{mp}}{2r_{wf}^2 + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi}(\frac{p}{q}e^{i\delta} + \frac{q^*}{p^*}e^{-i\delta})) + r_g \cdot (|\frac{p}{q}|^2 + |\frac{q}{p}|^2)}, \quad (62)$$

$$\mathcal{A}_{CP,\text{int}}^{mp} = \frac{2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{i(\phi+\delta)}(\frac{p}{q} - \frac{q}{p}))}{2r_{wf}^2 + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi}(\frac{p}{q}e^{i\delta} + \frac{q^*}{p^*}e^{-i\delta})) + r_g \cdot (|\frac{p}{q}|^2 + |\frac{q}{p}|^2)}, \quad (63)$$

where $\mathcal{A}_{CP,\text{mix}}^{mp}$, $\mathcal{A}_{CP,\text{dir}}^{mp}$, and $\mathcal{A}_{CP,\text{int}}^{mp}$ denote the indirect CP violation in kaon mixing, the direct CP violation, and the CP violation in the interference, respectively. The parameter δ_S^{mp} in Eq. (62) denotes the strong phase difference in the direct CP violation $\mathcal{A}_{CP,\text{dir}}^{mp}$, which is given by

$$\delta_S^{mp} = \arg\left(\frac{p}{q}\right) + \delta = \arctan\left(\frac{2\text{Im}(\epsilon)}{1 - |\epsilon|^2}\right) - \arctan\left(\frac{2\Delta m}{\Gamma_S - \Gamma_L} \cdot \frac{\Gamma_S \Gamma_L}{\Gamma^2 + (\Delta m)^2}\right). \quad (64)$$

As can be seen in the above equation, the strong phase difference of the direct CP asymmetry arises from kaon mixing parameters, and is thus free of strong pollution. Using Eqs. (20) and (63), we can rewrite the CP violation in the interference $\mathcal{A}_{CP,\text{int}}^{mp}$ as follows:

$$\mathcal{A}_{CP,\text{int}}^{mp} = \frac{8r_{wf} \cdot r_{sf} \cdot (\text{Re}\epsilon \cdot \cos(\phi + \delta) - \text{Im}\epsilon \cdot \sin(\phi + \delta))}{2r_{wf}^2 + 2r_{wf} \cdot r_{sf} \cdot \text{Re}(e^{-i\phi}(\frac{p}{q}e^{i\delta} + \frac{q^*}{p^*}e^{-i\delta})) + r_g \cdot (|\frac{p}{q}|^2 + |\frac{q}{p}|^2)}. \quad (65)$$

Similarly, $\mathcal{A}_{CP,\text{int}}^{mp}$ involves the parameter r_{wf} and the weak phase difference ϕ , which are determined by the CKM matrix elements.

C. The asymmetries between the $B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e$ decays and their partially conjugated decays $B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e$

Beside the CP asymmetries in $B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e$ and their conjugated decays, we can also define another two asymmetry observables, which are given by

$$\mathcal{A}_{CP}^{pp} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^+ \rightarrow f_B) - \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^+ \rightarrow f_B) + \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}, \quad (66)$$

and

$$\mathcal{A}_{CP}^{mm} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) - \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) + \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B)}. \quad (67)$$

Using Eqs. (B1) and (B2) and employing the same methods in Sec. III A, we can simplify Eqs. (66) and (67) as

$$\mathcal{A}_{CP}^{pp} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) - \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e)}{\Gamma(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) + \Gamma(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e)}, \quad (68)$$

$$\mathcal{A}_{CP}^{mm} = \frac{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) - \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}{\Gamma(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) + \Gamma(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}. \quad (69)$$

By substituting Eqs. (33), (39), (40), and (41) into Eqs. (68) and (69), we can obtain

$$\mathcal{A}_{CP}^{pp} = \frac{|p|^2 - |q|^2 \cdot r_g}{|p|^2 + |q|^2 \cdot r_g} = \frac{|p|^2 - |q|^2 + |q|^2 \cdot (1 - r_g)}{|p|^2 + |q|^2 \cdot r_g}, \quad (70)$$

$$\mathcal{A}_{CP}^{mm} = -\frac{|q|^2 - |p|^2 \cdot r_g}{|q|^2 + |p|^2 \cdot r_g} = \frac{|p|^2 - |q|^2 - |p|^2 \cdot (1 - r_g)}{|q|^2 + |p|^2 \cdot r_g}. \quad (71)$$

From the above equations, we can see that the dominant contributions to the asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$ arise from $K^0 - \bar{K}^0$ mixing, and the reasons are as follows: in the asymmetry observable $\mathcal{A}_{CP}^{pp}$, the involved decays are not conjugated with each other, but have the same final states ($\pi^+ e^- \bar{\nu}_e$) of the neutral kaon, so not only the weak phase difference but also the strong phase difference between the Cabibbo-favored and doubly Cabibbo-suppressed amplitudes of the decay process $B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e$ are opposite to that of the decay process $B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e$, which can be seen in Eqs. (29) and (32). As a result, the interference between the Cabibbo-favored and doubly

Cabibbo-suppressed amplitudes of the decay process $B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e$ is the same as that of the $B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e$ decay, as shown in Eqs. (33) and (41), so there is no direct CP violation. Meanwhile, because the decays in $\mathcal{A}_{CP}^{pp}$ have the same final states ($\pi^+ e^- \bar{\nu}_e$) of the neutral kaon, only the mixing effect of $K^0 \rightarrow \bar{K}^0$ is involved, so $\mathcal{A}_{CP}^{pp}$ have no CP violation in the interference, which originates from the interference between the amplitude of the $B_c^- \rightarrow B^- K^0 (\bar{K}^0) \rightarrow B^- \pi^- e^+ \nu_e (\pi^+ e^- \bar{\nu}_e)$ decay with the difference between the mixing effect of $K^0 \rightarrow \bar{K}^0$ and that of $\bar{K}^0 \rightarrow K^0$. The situation in the asymmetry observable $\mathcal{A}_{CP}^{mm}$ is the same as that in the asymmetry observable $\mathcal{A}_{CP}^{pp}$.

IV. NUMERICAL CALCULATION

A. Input parameters

Using the theoretical expressions for the branching ratios and the asymmetry observables derived in Secs. II and III, we are able to calculate the observables numerically. First, we collect the input parameters used in this work as below [62,64]:

$$\begin{aligned} m_{B_c^+} &= 6.274 \text{ GeV}, & m_{B^+} &= 5.279 \text{ GeV}, \\ \Delta m &= (3.484 \pm 0.006) \times 10^{-15} \text{ GeV}, \\ G_F &= 1.166 \times 10^{-5} \text{ GeV}^{-2}, \\ \Gamma_S &= (7.351 \pm 0.003) \times 10^{-15} \text{ GeV}, \\ \Gamma_L &= (1.287 \pm 0.005) \times 10^{-17} \text{ GeV}, \\ m_{D^*} &= 2.007 \text{ GeV}, \\ m_{D_0^*} &= (2.343 \pm 0.010) \text{ GeV}, \end{aligned} \quad (72)$$

$$\begin{aligned}\tau_{B_c} &= (0.510 \pm 0.009) \times 10^{-12} \text{ s}, & f_K &= (0.154 \pm 0.002) \text{ GeV}, \\ \text{Re}(\epsilon) &= (1.66 \pm 0.02) \times 10^{-3}, & \text{Im}(\epsilon) &= (1.57 \pm 0.02) \times 10^{-3}, \\ m_{K^0} &= 0.498 \text{ GeV}, & \mathcal{B}(K_S^0 \rightarrow \pi^\pm e^\mp \nu_e) &= (7.14 \pm 0.06) \times 10^{-4}.\end{aligned}$$

In order to see physics more transparently, we use the Wolfenstein parametrization of the CKM matrix elements [64,73–75],

$$\begin{aligned}V_{ud} &= \sqrt{1 - \lambda^2} \sqrt{1 - A^2 \lambda^6 (\rho^2 + \eta^2)}, & V_{cd} &= -\lambda \sqrt{1 - A^2 \lambda^4} - A^2 \lambda^5 \sqrt{1 - \lambda^2} (\rho + i\eta), \\ V_{us} &= \lambda \sqrt{1 - A^2 \lambda^6 (\rho^2 + \eta^2)}, & V_{cs} &= \sqrt{1 - \lambda^2} \sqrt{1 - A^2 \lambda^4} - A^2 \lambda^6 (\rho + i\eta),\end{aligned}\quad (73)$$

with λ, A, ρ , and η are the real parameters. The latest results fitted by the UTfit collaboration are presented as follows [76]:

$$\lambda = 0.2251 \pm 0.0008, \quad A = 0.828 \pm 0.010, \quad \rho = 0.164 \pm 0.009, \quad \eta = 0.355 \pm 0.009. \quad (74)$$

B. The numerical results of the branching ratios

By substituting the values of the parameters listed in Sec. IV A into Eqs. (33), (39), (40), and (41), we can obtain the numerical values of the branching ratios for the $B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e$ and $B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e$ decays, which are shown as follows:

$$\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) = \frac{\mathcal{B}(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}{2} = (5.17_{+0.53}^{-0.52}) \times 10^{-4}, \quad (75)$$

$$\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) = \frac{\mathcal{B}(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e)}{2} = (5.13_{+0.52}^{-0.51}) \times 10^{-4}, \quad (76)$$

$$\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^+ e^- \bar{\nu}_e) = \frac{\mathcal{B}(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e)}{2} = (5.13_{+0.52}^{-0.51}) \times 10^{-4}, \quad (77)$$

$$\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^- e^+ \nu_e) = \frac{\mathcal{B}(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^- e^+ \nu_e)}{2} = (5.17_{+0.53}^{-0.52}) \times 10^{-4}. \quad (78)$$

By combining the above branching ratios with the branching ratios for the consequent decays of the $B^\pm$ mesons given by the Particle Data Group, we can calculate the results of the branching ratios $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e)$ [hereafter for brevity collectively referred to as $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi e \nu_e \rightarrow f_{B^\pm} \pi e \nu_e)$], which are equal to $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) \cdot \mathcal{B}(B^\pm \rightarrow f_{B^\pm})$ and listed in Table I. We only consider the decay channels with the branching ratios are larger than 5.0×10^{-7} and the semileptonic decays of $B^\pm$ mesons, which are hopefully to be marginally observed by the current experiments.

According to these numerical results of $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi e \nu_e \rightarrow f_{B^\pm} \pi e \nu_e)$ and the following formula [77–79],

$$(\epsilon_f N)_B = \frac{9}{\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi e \nu_e \rightarrow f_{B^\pm} \pi e \nu_e)}, \quad (79)$$

we can estimate how many $B_c^\pm$ events times efficiency are needed to observe these decays with three standard deviations (3σ). In Eq. (79), ϵ_f is the detecting efficiency of the final states, and N is the number of the $B_c^\pm$ meson needed to observe the decays with three standard deviations (3σ). With the numerical results of the branching ratios in Table I, we can obtain the numerical results of $(\epsilon_f N)_B$ for the $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e$ channels, which are also listed in Table I. Similarly, we can calculate the branching ratios and the numerical results of $(\epsilon_f N)_B$ for the $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$, $B_c^\pm \rightarrow B^\pm \pi^+ e^- \bar{\nu}_e \rightarrow f_{B^\pm} \pi^+ e^- \bar{\nu}_e$, and $B_c^\pm \rightarrow B^\pm \pi^- e^+ \nu_e \rightarrow f_{B^\pm} \pi^- e^+ \nu_e$ decays, which are listed in Tables II–IV, respectively. Here, we note that the branching ratios for the $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$, $B_c^\pm \rightarrow B^\pm \pi^+ e^- \bar{\nu}_e \rightarrow f_{B^\pm} \pi^+ e^- \bar{\nu}_e$, and $B_c^\pm \rightarrow B^\pm \pi^- e^+ \nu_e \rightarrow f_{B^\pm} \pi^- e^+ \nu_e$ decays [hereafter for brevity

TABLE IV. The total branching fractions and the numerical results of $(\epsilon_f N)_B$ for the $B_c^\pm \rightarrow B^\pm \pi^- e^+ \nu_e \rightarrow f_{B^\pm} \pi^- e^+ \nu_e$ decays.

The $B^+ \rightarrow f_{B^+}$ decays	The branching ratio	$(\epsilon_f N)_B$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 l^+ \nu_l$	$(1.64_{-0.19}^{+0.18}) \times 10^{-6}$	$(4.91 \sim 6.16) \times 10^6$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ l^+ \nu_l$	$(9.39_{-1.00}^{+0.98}) \times 10^{-7}$	$(0.87 \sim 1.07) \times 10^7$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l$	$(1.01 \pm 0.11) \times 10^{-6}$	$(8.02 \sim 9.96) \times 10^6$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l$	$(6.56_{-0.73}^{+0.71}) \times 10^{-7}$	$(1.23 \sim 1.54) \times 10^7$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \mu^- \bar{\nu}_\mu \pi^0 l^+ \nu_l$	$(6.31_{-0.70}^{+0.69}) \times 10^{-7}$	$(1.28 \sim 1.60) \times 10^7$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 l^+ \nu_l$	$(7.30_{-0.81}^{+0.79}) \times 10^{-7}$	$(1.11 \sim 1.38) \times 10^7$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l$	$(2.66 \pm 0.31) \times 10^{-6}$	$(3.02 \sim 3.82) \times 10^6$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0 l^+ \nu_l$	$(1.52 \pm 0.17) \times 10^{-6}$	$(5.33 \sim 6.65) \times 10^6$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 \pi^0 l^+ \nu_l$	$(1.64_{-0.19}^{+0.18}) \times 10^{-6}$	$(4.93 \sim 6.18) \times 10^6$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0 \pi^0 l^+ \nu_l$	$(7.95_{-1.15}^{+1.14}) \times 10^{-7}$	$(0.99 \sim 1.32) \times 10^7$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \gamma l^+ \nu_l$	$(1.45 \pm 0.17) \times 10^{-6}$	$(5.53 \sim 7.02) \times 10^6$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \gamma l^+ \nu_l$	$(8.29_{-0.94}^{+0.93}) \times 10^{-7}$	$(0.97 \sim 1.22) \times 10^7$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 \gamma l^+ \nu_l$	$(8.94_{-1.03}^{+1.01}) \times 10^{-7}$	$(0.90 \sim 1.14) \times 10^7$

collectively referred to as $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi e \nu_e \rightarrow f_{B^\pm} \pi e \nu_e)$ are equal to $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$, $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \bar{\nu}_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$, and $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^- e^+ \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$, respectively.

C. The numerical results of the CP asymmetries

Now, we proceed to calculate the numerical results of the CP asymmetries in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays. By substituting the values of the parameters in Sec. IV A into Eqs. (52)–(55) and (60)–(63), we can obtain the numerical results of the CP asymmetries in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays, which are shown in Table V. From these numerical values, we can obtain the following points:

- (1) Using the numerical results of the parameters in Sec. IV A: $\Gamma_S/\Gamma_L \approx 571$, $\Delta m/\Gamma_L \approx 271$, $\Gamma/\Gamma_L \approx 286$, and Eqs. (35)–(37), we can obtain

$$\int_0^\infty (g_-(t))^* g_+(t) dt \approx 1.94 \times 10^{16} + I \cdot 6.78 \times 10^{13}, \quad (80)$$

$$\begin{aligned} \int_0^\infty |g_-(t)|^2 dt &\approx 1.94 \times 10^{16}, \\ \int_0^\infty |g_+(t)|^2 dt &\approx 1.95 \times 10^{16}, \end{aligned} \quad (81)$$

so $r_g \approx 0.99$, $r_{sf} \approx 0.99$, and $\delta \approx -3.50 \times 10^{-3}$ rad, which can be derived from Eq. (34).

- (2) Using the numerical results of the CKM matrix elements in Sec. IV A and Eq. (34), we can obtain

$$r_{wf} \approx 0.053, \quad \phi \approx -6.42 \times 10^{-4} \text{ rad}. \quad (82)$$

- (3) From Eqs. (56) and (64), we can see that the strong phase differences δ_S^{pm} and δ_S^{mp} suffer from the $\text{Im}(\epsilon)$ and δ suppressions; their numerical values are very small,

$$\begin{aligned} \delta_S^{pm} &\approx -6.64 \times 10^{-3} \text{ rad}, \\ \delta_S^{mp} &\approx -3.55 \times 10^{-4} \text{ rad}. \end{aligned} \quad (83)$$

- (4) As can be seen from Eqs. (54) and (62), the direct CP asymmetry $\mathcal{A}_{CP,\text{dir}}^{pm}$ ($\mathcal{A}_{CP,\text{dir}}^{mp}$) suffers from the r_{wf} , $\sin \phi$ and $\sin \delta_S^{pm}$ ($\sin \delta_S^{mp}$) suppressions, thus its numerical value is of the order of 10^{-7} (10^{-8}), which can be neglected safely.
- (5) As shown in Eqs. (53) and (57), the new CP violation effect, i.e., the CP violation in the interference, $\mathcal{A}_{CP,\text{int}}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\mp \nu_e$ decays contains the term proportional to $\cos(\phi - \delta)$, so it merely suffers from the r_{wf} and $\text{Re}\epsilon$ suppression, while the indirect CP violation in kaon mixing $\mathcal{A}_{CP,\text{mix}}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\mp \nu_e$ decays suffers from both r_{wf}^2 and $|q/p|^2 - |p/q|^2$ suppressions. The new CP violation effect $\mathcal{A}_{CP,\text{int}}^{pm}$ is dominant in the CP asymmetry $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\mp \nu_e$ decays; the indirect CP violation in kaon mixing $\mathcal{A}_{CP,\text{mix}}^{pm}$ only accounts for about 5% of the CP asymmetry $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\mp \nu_e$ decays.
- (6) As for the CP asymmetry $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays, the indirect CP violation in kaon mixing $\mathcal{A}_{CP,\text{mix}}^{mp}$ merely suffers from the $|p/q|^2 - |q/p|^2$ suppression, while the new CP violation $\mathcal{A}_{CP,\text{int}}^{mp}$ suffers from the r_{wf} and $\text{Re}\epsilon$ suppression, so the indirect CP violation in kaon

mixing $\mathcal{A}_{CP,\text{mix}}^{mp}$ is dominant in the CP asymmetry $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays; the new CP violation effect $\mathcal{A}_{CP,\text{int}}^{mp}$ only accounts for about 5% of the CP asymmetry $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays. All of these conclusions can be obtained from Eqs. (61) and (65).

In a word, the new CP violation effect $\mathcal{A}_{CP,\text{int}}^{pm}$ dominates the CP asymmetry $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e$ decays, and the indirect CP violation in kaon mixing $\mathcal{A}_{CP,\text{mix}}^{mp}$ dominates the CP asymmetry $\mathcal{A}_{CP}^{mp}$ in

$B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays, so the CP asymmetry $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e$ decays provides an ideal place to study the new CP violation effect.

According to the numerical results of the CP asymmetries in Table V, we can estimate how many B_c events times efficiency are needed to establish the CP asymmetries to three standard deviations (3σ). When the CP violation $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e$ decays is observed at the three standard deviations (3σ) level, the number of B_c events times efficiency needed reads

$$(\epsilon_f N)_{CP}^{pm} = \frac{9}{[\mathcal{B}(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e \rightarrow f_{B^+} \pi^+ e^- \bar{\nu}_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^- e^+ \nu_e \rightarrow f_{B^-} \pi^- e^+ \nu_e)] \cdot |\mathcal{A}_{CP}^{pm}|} \approx \frac{9}{2 \cdot \mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+}) \cdot |\mathcal{A}_{CP}^{pm}|}. \quad (84)$$

By substituting the numerical results of $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$ in Table I and that of $\mathcal{A}_{CP}^{pm}$ in Table V into Eq. (84), we can obtain the numerical results of $(\epsilon_f N)_{CP}^{pm}$, which are listed in Table VI.

Similarly, the numbers of B_c events times efficiency needed to observe the CP violation $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays at a significance of 3σ reads

$$(\epsilon_f N)_{CP}^{mp} = \frac{9}{[\mathcal{B}(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e \rightarrow f_{B^+} \pi^- e^+ \nu_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e \rightarrow f_{B^-} \pi^+ e^- \bar{\nu}_e)] \cdot |\mathcal{A}_{CP}^{mp}|} \approx \frac{9}{2 \cdot \mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+}) \cdot |\mathcal{A}_{CP}^{mp}|}. \quad (85)$$

TABLE V. The numerical results of the CP asymmetries in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm} \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm} \pi^\mp e^\pm \nu_e$ decays.

$\mathcal{A}_{CP}^{pm} = (-3.36 \pm 0.05) \times 10^{-4}$	$\mathcal{A}_{CP}^{mp} = (6.30 \pm 0.08) \times 10^{-3}$
$\mathcal{A}_{CP,\text{mix}}^{pm} = (-1.69 \pm 0.03) \times 10^{-5}$	$\mathcal{A}_{CP,\text{mix}}^{mp} = (5.98 \pm 0.07) \times 10^{-3}$
$\mathcal{A}_{CP,\text{dir}}^{pm} = (-4.06 \pm 0.17) \times 10^{-7}$	$\mathcal{A}_{CP,\text{dir}}^{mp} = (2.20 \pm 0.28) \times 10^{-8}$
$\mathcal{A}_{CP,\text{int}}^{pm} = (-3.18 \pm 0.04) \times 10^{-4}$	$\mathcal{A}_{CP,\text{int}}^{mp} = (3.21 \pm 0.04) \times 10^{-4}$

TABLE VI. The numerical results of $(\epsilon_f N)_{CP}^{pm}$ and $(\epsilon_f N)_{CP}^{mp}$.

The $B^+ \rightarrow f_{B^+}$ decays	$(\epsilon_f N)_{CP}^{pm}$	$(\epsilon_f N)_{CP}^{mp}$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 l^+ \nu_l$	$(7.33 \sim 9.21) \times 10^9$	$(3.92 \sim 4.92) \times 10^8$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ l^+ \nu_l$	$(1.29 \sim 1.60) \times 10^{10}$	$(6.91 \sim 8.56) \times 10^8$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l$	$(1.20 \sim 1.49) \times 10^{10}$	$(6.40 \sim 7.95) \times 10^8$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l$	$(1.84 \sim 2.30) \times 10^{10}$	$(0.99 \sim 1.23) \times 10^9$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \mu^- \bar{\nu}_\mu \pi^0 l^+ \nu_l$	$(1.92 \sim 2.39) \times 10^{10}$	$(1.02 \sim 1.28) \times 10^9$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 l^+ \nu_l$	$(1.66 \sim 2.07) \times 10^{10}$	$(0.89 \sim 1.11) \times 10^9$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l$	$(4.51 \sim 5.72) \times 10^9$	$(2.41 \sim 3.06) \times 10^8$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0 l^+ \nu_l$	$(7.94 \sim 9.94) \times 10^9$	$(4.25 \sim 5.31) \times 10^8$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 \pi^0 l^+ \nu_l$	$(7.36 \sim 9.24) \times 10^9$	$(3.94 \sim 4.94) \times 10^8$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0 \pi^0 l^+ \nu_l$	$(1.47 \sim 1.97) \times 10^{10}$	$(0.79 \sim 1.05) \times 10^9$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \gamma l^+ \nu_l$	$(0.82 \sim 1.05) \times 10^{10}$	$(4.41 \sim 5.61) \times 10^8$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \gamma l^+ \nu_l$	$(1.45 \sim 1.83) \times 10^{10}$	$(7.77 \sim 9.76) \times 10^8$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 \gamma l^+ \nu_l$	$(1.35 \sim 1.70) \times 10^{10}$	$(7.20 \sim 9.07) \times 10^8$

Using the numerical results of $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$ in Table II, the numerical results of $\mathcal{A}_{CP}^{mp}$ in Table V, and Eq. (85), we can obtain the numerical results of $(\epsilon_f N)_{CP}^{mp}$, which are also listed in Table VI.

From the numerical results of $(\epsilon_f N)_{CP}^{pm}$ and $(\epsilon_f N)_{CP}^{mp}$ in Table VI, we can see that the numbers of the $B_c^\pm$ events times efficiency, which are needed to observe the CP asymmetries $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$ at the three standard deviations (3σ) level, are of the order of 10^8 – 10^{10} , so they are possible to be marginally observed at the LHC experiment and the HL-LHC experiment.

D. The numerical results of the asymmetry observables

$$\mathcal{A}_{CP}^{pp} \text{ and } \mathcal{A}_{CP}^{mm}$$

Now, we move on to calculate the numerical results of the asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$. By substituting the values of the parameters in Eqs. (72) and (74) into (70) and (71), we have

$$\begin{aligned} (\epsilon_f N)_{CP}^{pp} &= \frac{9}{[\mathcal{B}(B_c^+ \rightarrow B^+ \pi^+ e^- \bar{\nu}_e \rightarrow f_{B^+} \pi^+ e^- \bar{\nu}_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^- e^+ \nu_e \rightarrow f_{B^-} \pi^- e^+ \nu_e)] \cdot |\mathcal{A}_{CP}^{pp}|} \\ &\approx \frac{9}{2 \cdot \mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+}) \cdot |\mathcal{A}_{CP}^{pp}|}, \end{aligned} \quad (88)$$

and

$$\begin{aligned} (\epsilon_f N)_{CP}^{mm} &= \frac{9}{[\mathcal{B}(B_c^+ \rightarrow B^+ \pi^- e^+ \nu_e \rightarrow f_{B^+} \pi^- e^+ \nu_e) + \mathcal{B}(B_c^- \rightarrow B^- \pi^+ e^- \bar{\nu}_e \rightarrow f_{B^-} \pi^+ e^- \bar{\nu}_e)] \cdot |\mathcal{A}_{CP}^{mm}|} \\ &\approx \frac{9}{2 \cdot \mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+}) \cdot |\mathcal{A}_{CP}^{mm}|}. \end{aligned} \quad (89)$$

By substituting the numerical results of the asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$ in Eqs. (86) and (87), the numerical results of $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$ in Table III, and that of $\mathcal{B}(B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e) \cdot \mathcal{B}(B^+ \rightarrow f_{B^+})$ in Table IV into Eqs. (88) and (89), we can obtain the numerical results of $(\epsilon_f N)_{CP}^{pp}$ and $(\epsilon_f N)_{CP}^{mm}$, which are listed in Table VII. Obviously, the

TABLE VII. The numerical results of $(\epsilon_f N)_{CP}^{pp}$ and $(\epsilon_f N)_{CP}^{mm}$.

The $B^+ \rightarrow f_{B^+}$ decays	$(\epsilon_f N)_{CP}^{pp}$	$(\epsilon_f N)_{CP}^{mm}$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 l^+ \nu_l$	$(3.93 \sim 4.93) \times 10^8$	$(0.72 \sim 1.01) \times 10^{10}$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ l^+ \nu_l$	$(6.92 \sim 8.57) \times 10^8$	$(1.27 \sim 1.76) \times 10^{10}$
$B^+ \rightarrow \bar{D}^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l$	$(6.41 \sim 7.96) \times 10^8$	$(1.18 \sim 1.64) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ e^- \bar{\nu}_e \pi^0 l^+ \nu_l$	$(0.99 \sim 1.23) \times 10^9$	$(1.82 \sim 2.53) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \mu^- \bar{\nu}_\mu \pi^0 l^+ \nu_l$	$(1.03 \sim 1.28) \times 10^9$	$(1.89 \sim 2.63) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 l^+ \nu_l$	$(0.89 \sim 1.11) \times 10^9$	$(1.64 \sim 2.27) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 l^+ \nu_l$	$(2.42 \sim 3.06) \times 10^8$	$(4.47 \sim 6.26) \times 10^9$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0 l^+ \nu_l$	$(4.26 \sim 5.32) \times 10^8$	$(0.79 \sim 1.09) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 \pi^0 l^+ \nu_l$	$(3.95 \sim 4.95) \times 10^8$	$(0.73 \sim 1.01) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \pi^0 l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \pi^0 \pi^0 l^+ \nu_l$	$(0.79 \sim 1.06) \times 10^9$	$(1.47 \sim 2.15) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \gamma l^+ \nu_l$	$(4.42 \sim 5.62) \times 10^8$	$(0.82 \sim 1.15) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^- \pi^+ \gamma l^+ \nu_l$	$(7.79 \sim 9.77) \times 10^8$	$(1.44 \sim 2.00) \times 10^{10}$
$B^+ \rightarrow \bar{D}^*(2007)^0 l^+ \nu_l \rightarrow \bar{D}^0 \gamma l^+ \nu_l \rightarrow K^+ \pi^- \pi^0 \pi^0 \gamma l^+ \nu_l$	$(7.22 \sim 9.09) \times 10^8$	$(1.33 \sim 1.86) \times 10^{10}$

numbers of the $B_c^\pm$ events times efficiency, which are needed to observe the asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$ at three standard deviations (3σ) level, are of the order of 10^8 – 10^{10} , so they are hopefully to be marginally observed at the LHC experiment and the HL-LHC experiment.

V. CONCLUSIONS

In this work, we derive the branching ratios for the $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays. Combining the numerical results of the branching ratios for these decays with the branching ratios for the consequent decays of the $B^\pm$ mesons given by the Particle Data Group, we obtain the total branching ratios of the decay chains $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm \pi^\pm e^\mp \nu_e}$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm \pi^\mp e^\pm \nu_e}$. We find that some of the $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm \pi^\pm e^\mp \nu_e}$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm \pi^\mp e^\pm \nu_e}$ decay chains have large branching ratios, whose maximum value can reach 2.66×10^{-6} . We estimate the numbers of the $B_c^\pm$ events times efficiency needed to observe the $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm \pi^\pm e^\mp \nu_e}$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm \pi^\mp e^\pm \nu_e}$ decays at the three standard deviations (3σ) level and find that they are of the order of 10^6 – 10^7 , so these decays may be measurable at the LHC experiment.

We investigate the CP asymmetry observables $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays, which consist of three parts: the indirect CP violations in $K^0 - \bar{K}^0$ mixing $\mathcal{A}_{CP,\text{mix}}^{pm}$ and $\mathcal{A}_{CP,\text{mix}}^{mp}$, the direct CP violations in B_c decay $\mathcal{A}_{CP,\text{dir}}^{pm}$ and $\mathcal{A}_{CP,\text{dir}}^{mp}$, and the new CP violation effects $\mathcal{A}_{CP,\text{int}}^{pm}$ and $\mathcal{A}_{CP,\text{int}}^{mp}$, which originate from the interference between the amplitude of the $B_c^- \rightarrow B^- K^0 (\bar{K}^0) \rightarrow B^- \pi^- e^+ \nu_e (\pi^+ e^- \bar{\nu}_e)$ decay with the difference between the mixing effect of $K^0 \rightarrow \bar{K}^0$ and that of $\bar{K}^0 \rightarrow K^0$. The strong phase differences of the direct CP asymmetries $\mathcal{A}_{CP,\text{dir}}^{pm}$ and $\mathcal{A}_{CP,\text{dir}}^{mp}$ arise from $K^0 - \bar{K}^0$ mixing parameters, and thus prevent pollution from strong dynamics.

We present the numerical results of the CP asymmetry observables $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays; we find that $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ are of the order of $\mathcal{O}(10^{-4})$ and $\mathcal{O}(10^{-3})$, respectively. Meanwhile, the direct CP asymmetries $\mathcal{A}_{CP,\text{dir}}^{pm}$ and $\mathcal{A}_{CP,\text{dir}}^{mp}$ have a very small value and can be neglected safely. For the CP asymmetry observable $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ decays, the new CP violation effect $\mathcal{A}_{CP,\text{int}}^{pm}$ is dominant, and the indirect CP violation in kaon mixing $\mathcal{A}_{CP,\text{mix}}^{pm}$ only accounts for about 5% of the CP asymmetry observable $\mathcal{A}_{CP}^{pm}$. However, for the CP asymmetry observable $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow$

$B^\pm \pi^\mp e^\pm \nu_e$ decays, the indirect CP violation in kaon mixing $\mathcal{A}_{CP,\text{mix}}^{mp}$ is dominant, and the new CP violation effect $\mathcal{A}_{CP,\text{int}}^{mp}$ only accounts for about 5% of the CP asymmetry observable $\mathcal{A}_{CP}^{mp}$. So the CP asymmetry observable $\mathcal{A}_{CP}^{pm}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ decays provides an ideal place to study the new CP violation effect.

Besides the CP asymmetry observables $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e$ decays, we also define another two asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$ in $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e$ decays and their partially conjugated decays $B_c^\mp \rightarrow B^\mp \pi^\mp e^\mp \nu_e$. We derive the expressions for the asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$, which are dominated by $K^0 - \bar{K}^0$ mixing. We calculate the numerical results of the asymmetry observables $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$ and find that $\mathcal{A}_{CP}^{pp}$ is of the order of $\mathcal{O}(10^{-3})$, while $\mathcal{A}_{CP}^{mm}$ is of the order of $\mathcal{O}(10^{-4})$.

Together with the branching ratios for the $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \nu_e \rightarrow f_{B^\pm \pi^\pm e^\mp \nu_e}$, $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm \pi^\mp e^\pm \nu_e}$, $B_c^\pm \rightarrow B^\pm \pi^\pm e^\mp \bar{\nu}_e \rightarrow f_{B^\pm \pi^\pm e^\mp \bar{\nu}_e}$, and $B_c^\pm \rightarrow B^\pm \pi^\mp e^\pm \nu_e \rightarrow f_{B^\pm \pi^\mp e^\pm \nu_e}$ decay chains and the observables $\mathcal{A}_{CP}^{pm}$, $\mathcal{A}_{CP}^{mp}$, $\mathcal{A}_{CP}^{pp}$, and $\mathcal{A}_{CP}^{mm}$, we calculate the numbers of the $B_c^\pm$ events times efficiency needed to establish the observables $\mathcal{A}_{CP}^{pm}$, $\mathcal{A}_{CP}^{mp}$, $\mathcal{A}_{CP}^{pp}$, and $\mathcal{A}_{CP}^{mm}$ to three standard deviations in the decay processes with large branching ratios. The numbers of the $B_c^\pm$ events times efficiency needed to observe the asymmetry observables $\mathcal{A}_{CP}^{pm}$, $\mathcal{A}_{CP}^{mp}$, $\mathcal{A}_{CP}^{pp}$, and $\mathcal{A}_{CP}^{mm}$ at a significance of 3σ in the above-mentioned decays are of the order of 10^8 – 10^{10} , so these observables are hopefully to be marginally observed at the LHC experiment and the HL-LHC experiment.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: THE RELATIONS USED IN $\mathcal{A}_{CP}^{pm}$ AND $\mathcal{A}_{CP}^{mp}$ CALCULATIONS

Below we present the relations used in evaluating the expression of $\mathcal{A}_{CP}^{pm}$ and $\mathcal{A}_{CP}^{mp}$ in Secs. III A and III B:

$$\begin{aligned} & \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) - \Gamma(B_c^- \rightarrow B^- \pi^\mp e^\pm \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\ &= \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot (\Gamma(B^+ \rightarrow f_B) - \Gamma(B^- \rightarrow \bar{f}_B)) \\ &+ (\Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) - \Gamma(B_c^- \rightarrow B^- \pi^\mp e^\pm \nu_e)) \cdot \Gamma(B^- \rightarrow \bar{f}_B), \end{aligned} \quad (\text{A1})$$

and

$$\begin{aligned}
& \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) + \Gamma(B_c^- \rightarrow B^- \pi^\mp e^\pm \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&= \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot (\Gamma(B^+ \rightarrow f_B) + \Gamma(B^- \rightarrow \bar{f}_B)) \\
&\quad - (\Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) - \Gamma(B_c^- \rightarrow B^- \pi^\mp e^\pm \nu_e)) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&= (\Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) + \Gamma(B_c^- \rightarrow B^- \pi^\mp e^\pm \nu_e)) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&\quad + \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot (\Gamma(B^+ \rightarrow f_B) - \Gamma(B^- \rightarrow \bar{f}_B)).
\end{aligned} \tag{A2}$$

APPENDIX B: THE RELATIONS USED IN $\mathcal{A}_{CP}^{pp}$ AND $\mathcal{A}_{CP}^{mm}$ CALCULATIONS

In Sec. III C, the following equations are employed to calculate the expression of $\mathcal{A}_{CP}^{pp}$ and $\mathcal{A}_{CP}^{mm}$:

$$\begin{aligned}
& \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) - \Gamma(B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&= \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot (\Gamma(B^+ \rightarrow f_B) - \Gamma(B^- \rightarrow \bar{f}_B)) \\
&\quad + (\Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) - \Gamma(B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e)) \cdot \Gamma(B^- \rightarrow \bar{f}_B),
\end{aligned} \tag{B1}$$

and

$$\begin{aligned}
& \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot \Gamma(B^+ \rightarrow f_B) + \Gamma(B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&= \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot (\Gamma(B^+ \rightarrow f_B) + \Gamma(B^- \rightarrow \bar{f}_B)) \\
&\quad - (\Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) - \Gamma(B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e)) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&= (\Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) + \Gamma(B_c^- \rightarrow B^- \pi^\pm e^\mp \nu_e)) \cdot \Gamma(B^- \rightarrow \bar{f}_B) \\
&\quad + \Gamma(B_c^+ \rightarrow B^+ \pi^\pm e^\mp \nu_e) \cdot (\Gamma(B^+ \rightarrow f_B) - \Gamma(B^- \rightarrow \bar{f}_B)).
\end{aligned} \tag{B2}$$

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