

Flavor SU(3) analysis of the charmless semileptonic $B \rightarrow PV\ell^+\nu_\ell$ decays

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All charmless $B \rightarrow PV\ell^+\nu_\ell$ (P denotes the light pseudoscalar meson and V denotes the vector meson) decays have not been experimentally observed to date, but some of them might be measured in the near future. The charmless $B \rightarrow PV\ell^+\nu_\ell$ decays with the axial-vector (A) resonance states, the tensor (T) resonance states, and the excited vector ($V_{E,D}$) resonance states are studied based on flavor SU(3) analysis in this work, where V_E contains $\{\rho(1450), K^*(1410), \omega(1420), \phi(1680)\}$ with quantum numbers $n^{2S+1}L_J = 2^3S_1$ and V_D contains $\{\rho(1700), K^*(1680), \omega(1650), \phi(2170)\}$ with quantum numbers $n^{2S+1}L_J = 1^3D_1$. The hadronic amplitudes of the $B \rightarrow A/T\ell^+\nu_\ell$ decays are related by the nonperturbative parameters, and the branching ratio predictions of $B \rightarrow A/T/V_{E,D}\ell^+\nu_\ell$ are obtained, and then the branching ratios of the $B \rightarrow A(A \rightarrow PV)\ell^+\nu_\ell$, $B \rightarrow T(T \rightarrow PV)\ell^+\nu_\ell$, $B \rightarrow V_E(V_E \rightarrow PV)\ell^+\nu_\ell$, and $B \rightarrow V_D(V_D \rightarrow PV)\ell^+\nu_\ell$ are obtained by the narrow width approximation or further considering the width effects. We find that most branching ratios of the $B \rightarrow A\ell^+\nu_\ell$ decays and many branching ratios of the $B \rightarrow A(A \rightarrow PV)\ell^+\nu_\ell$ decays are on the order of $\mathcal{O}(10^{-4}-10^{-3})$, and all branching ratios with the tensor resonance states and the excited vector resonance states are small. Our results could be tested in the future Belle II and LHCb experiments.

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I. INTRODUCTION

Semileptonic B decays are very interesting in particle physics due to their relatively clean signals. Semileptonic $B \rightarrow PV\ell^+\nu_\ell$ decays can receive contributions from both nonresonant and various resonant contributions. The intermediate resonances in these semileptonic decays provide a clean environment for exploring meson spectroscopy. Although no charmless $B \rightarrow PV\ell^+\nu_\ell$ decay has been measured to date, near future experiments such as Belle II and LHCb are expected to measure these decays.

Since the weak and strong dynamics can be separated in the semileptonic decays, the theoretical description of the semileptonic decays is relatively simple. The hadronic transition form factors contain all the strong dynamics in the initial and final hadrons, making them crucial for testing the theoretical calculations of the involved strong

interactions. However, the calculations of the $B \rightarrow PV$ form factors are more complicated than ones for $B \rightarrow V$ or $B \rightarrow P$ form factors, and reliable calculations are currently lacking. In the absence of reliable calculations, symmetry analysis plays a vital role in understanding semileptonic transitions by providing relations between hadronic form factors or hadronic decay amplitudes. Among these, SU(3) flavor symmetry is one of the symmetries which has attracted a lot of attention. SU(3) flavor symmetry has been widely used to study hadron decays, including b-hadron decays [1–14] and c-hadron decays [13–29].

Four-body semileptonic $D/B \rightarrow PP/PV\ell^+\nu_\ell$ decays have also been explored in previous studies, for instance, in Refs. [30–35]. In this work, we will focus on analyzing four-body semileptonic charmless $B \rightarrow A(A \rightarrow PV)\ell^+\nu_\ell$, $B \rightarrow T(T \rightarrow PV)\ell^+\nu_\ell$, $B \rightarrow V_E(V_E \rightarrow PV)\ell^+\nu_\ell$ and $B \rightarrow V_D(V_D \rightarrow PV)\ell^+\nu_\ell$ decays using the SU(3) flavor symmetry/breaking approach. The analysis begins by obtaining the form factor relations for $B \rightarrow A/T/V_{E,D}\ell^+\nu_\ell$ decays through SU(3) flavor symmetry/breaking. After ignoring the SU(3) flavor breaking, the branching ratios of the $B \rightarrow A/T/V_{E,D}\ell^+\nu_\ell$ decays are predicted. Using these predictions along with previous results for $A/T/V_{E,D} \rightarrow PV$ decays, the branching ratios for $B \rightarrow A(A \rightarrow PV)\ell^+\nu_\ell$,

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$B \rightarrow T(T \rightarrow PV)\ell^+\nu_\ell$, $B \rightarrow V_E(V_E \rightarrow PV)\ell^+\nu_\ell$, and $B \rightarrow V_D(V_D \rightarrow PV)\ell^+\nu_\ell$ decays are obtained by using the narrow width approximation and considering the width effects. Due to limited experimental data on $B \rightarrow M\ell^+\nu_\ell$, $M \rightarrow PV$, and $B \rightarrow M(M \rightarrow PV)\ell^+\nu_\ell$ with $M = A, T, V_E, V_D$, SU(3) flavor breaking effects are ignored. The relevant nonperturbative coefficients and mixing angles are then constrained by using existing experimental data. Finally, the branching ratios that have not yet been measured are predicted by the SU(3) flavor symmetry.

The paper is structured as follows: in Sec. II, three-body semileptonic charmless $B \rightarrow A/T/V_{E,D}\ell^+\nu_\ell$ decays are studied. In Sec. III, four-body semileptonic charmless $B \rightarrow PV\ell^+\nu_\ell$ with the axial-vector resonance states, the tensor resonance states, and the excited vector resonance states are explored. Summary and conclusions are presented in Sec. IV. Additional details are provided in the appendixes.

II. SEMILEPTONIC CHARMLESS $B \rightarrow M\ell^+\nu_\ell$ DECAYS

A. Theoretical frame

The effective Hamiltonian below the electroweak scale can be used to describe the $\bar{b} \rightarrow \bar{u}\ell^+\nu_\ell$ transitions, which can be written as

$$\mathcal{H}_{\text{eff}}(\bar{b} \rightarrow \bar{u}\ell^+\nu_\ell) = \frac{G_F}{\sqrt{2}} V_{ub} \bar{b} \gamma^\mu (1 - \gamma_5) \bar{u} \bar{\nu}_\ell \gamma_\mu (1 - \gamma_5) \ell, \quad (1)$$

where G_F is the Fermi constant, and V_{ub} is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element. The decay amplitude of $B \rightarrow M\ell^+\nu_\ell$ can be expressed as

$$\mathcal{A}(B \rightarrow M\ell^+\nu_\ell) = \frac{G_F}{\sqrt{2}} V_{ub} \sum_{m,n} g_{mn} L_m^{\lambda_\ell \lambda_\nu} H_n^{\lambda_M}, \quad (2)$$

with

$$L_m^{\lambda_\ell \lambda_\nu} = \epsilon_\alpha(m) \bar{\nu}_\ell \gamma_\alpha (1 - \gamma_5) \ell, \quad (3)$$

$$H_n^{\lambda_M} = \epsilon_\beta^*(n) \langle M(p_M, \epsilon^*) | \bar{u} \gamma^\beta (1 - \gamma_5) \bar{b} | B(p_B) \rangle, \quad (4)$$

where $\lambda_M = 0, \pm 1$, $\lambda_\ell = \pm \frac{1}{2}$, and $\lambda_\nu = +\frac{1}{2}$ are the particle helicities, and ϵ and ϵ^* are the polarization vectors of the virtual W -boson and M -meson, respectively. The leptonic parts $L_m^{\lambda_\ell \lambda_\nu}$ use perturbation theory to calculate, while the hadronic parts $H_n^{\lambda_M}$ are encoded in the transition form factors.

The differential branching ratios are [36]

$$\frac{d\mathcal{B}(B \rightarrow M\ell^+\nu_\ell)}{dq^2} = \frac{\tau_B G_F^2 |V_{ub}|^2 \lambda^{1/2} (q^2 - m_\ell^2)^2}{24(2\pi)^3 m_B^3 q^2} \mathcal{H}_{\text{total}} \quad (5)$$

with

$$\mathcal{H}_{\text{total}} \equiv (\mathcal{H}_U + \mathcal{H}_L) \left(1 + \frac{m_\ell^2}{2q^2} \right) + \frac{3m_\ell^2}{2q^2} \mathcal{H}_S, \quad (6)$$

$$\mathcal{H}_U = |H_+^M|^2 + |H_-^M|^2, \quad \mathcal{H}_L = |H_0^M|^2, \quad \mathcal{H}_S = |H_t^M|^2, \quad (7)$$

where $\lambda \equiv \lambda(m_B^2, m_M^2, q^2)$ with $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$, $q^2 \equiv (p_B - p_M)^2$, $m_\ell^2 \leq q^2 \leq (m_B - m_M)^2$, and $H_{\pm,0,t}^M$ are the hadronic amplitudes. For the $B \rightarrow PV\ell^+\nu_\ell$ semileptonic decay, we only consider the axial-vector meson resonant states, the tensor meson resonant states and the excited vector resonance states. The hadronic amplitudes $H_{\pm,0,t}^A$, $H_{\pm,0,t}^T$, and $H_{\pm,0,t}^{V_{E,D}}$ are listed in Appendix A. Then the branching ratios of the $B \rightarrow M\ell^+\nu_\ell$ decays could be obtained by the relevant form factors, which depend on the different methods.

The SU(3) flavor symmetry approach is independent of the detailed dynamics, and it can provide the hadronic amplitude relationships among various decay modes. We will use SU(3) flavor symmetry to analyze relative decays in this work. The SU(3) flavor analysis depends on the SU(3) flavor group, and the relevant meson multiplets of the SU(3) flavor group are listed in Appendix B.

In terms of the SU(3) flavor symmetry, the hadronic amplitudes of the $B \rightarrow M\ell^+\nu_\ell$ decays can be parametrized as

$$\begin{aligned} H(B \rightarrow M\ell^+\nu_\ell) &= c_0^M B^i M_i^j H_j + c_1^M B^a W_a^i M_i^j H_j \\ &+ c_2^M B^i M_i^a W_a^j H_j, \end{aligned} \quad (8)$$

where c_0^M ($c_{1,2}^M$) are the SU(3) flavor symmetry (breaking) nonperturbative coefficients. Please notice that the nonperturbative coefficients for the axial-vector mesons with $J^{PC} = 1^{++}$ and for the axial-vector mesons with $J^{PC} = 1^{+-}$ are different, and we will denote them by $c_{0,1,2}^A$ and $c_{0,1,2}^B$, respectively. The matrix W is related to the SU(3) flavor breaking effects, which mainly come from different masses of u/d and s quarks [37,38]. The hadronic amplitude relations for the $B \rightarrow M\ell^+\nu_\ell$ decays are summarized in Table I. Amplitude relations between different decay processes can be easily obtained from Table I, and the same relationships are also true for the form factors. Since there are not enough experimental data in the $B \rightarrow A/T\ell^+\nu_\ell$ decays, we will only consider the SU(3) flavor symmetry contributions, i.e., $C^M = C'^M = c_0^M$.

B. Numerical results

The theoretical input parameters, such as the lifetimes, the masses, and the experimental data within the 1σ error bar from the Particle Data Group (PDG) [39] will be used in our numerical analysis.

TABLE I. SU(3) amplitudes for $B \rightarrow M\ell^+\nu_\ell$ decays due to the $\bar{b} \rightarrow \bar{u}\ell^+\nu_\ell$ transitions. $C^M = c_0^M + c_1^M + c_2^M$, $C'^M = c_0^M - 2c_1^M + c_2^M$, and $C^M = C'^M$ if ignoring the SU(3) breaking effects.

Decay modes	SU(3) hadronic amplitudes	Decay modes	SU(3) hadronic amplitudes
$B^+ \rightarrow a_1(1260)^0\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}C^A$	$B^+ \rightarrow a_2(1320)^0\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}C^T$
$B^+ \rightarrow f_1(1285)^0\ell^+\nu_\ell$	$\frac{1}{\sqrt{6}}(\sqrt{2}\cos\theta_{3P_1} + \sin\theta_{3P_1})C^A$	$B^+ \rightarrow f_2(1270)\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}\cos\theta_{f_2}C^T$
$B^+ \rightarrow f_1(1420)^0\ell^+\nu_\ell$	$-\frac{1}{\sqrt{6}}(\sqrt{2}\sin\theta_{3P_1} - \cos\theta_{3P_1})C^A$	$B^+ \rightarrow f_2'(1525)\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}\sin\theta_{f_2}C^T$
$B^0 \rightarrow a_1(1260)^-\ell^+\nu_\ell$	C^A	$B^0 \rightarrow a_2(1320)^-\ell^+\nu_\ell$	C^T
$B^+ \rightarrow b_1(1235)^0\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}C^B$	$B_s^0 \rightarrow K_2^*(1430)^-\ell^+\nu_\ell$	C'^T
$B^+ \rightarrow h_1(1170)^0\ell^+\nu_\ell$	$\frac{1}{\sqrt{6}}(\sqrt{2}\cos\theta_{1P_1} + \sin\theta_{1P_1})C^B$		
$B^+ \rightarrow h_1(1415)^0\ell^+\nu_\ell$	$-\frac{1}{\sqrt{6}}(\sqrt{2}\sin\theta_{1P_1} - \cos\theta_{1P_1})C^B$	$B^+ \rightarrow \rho(1450)^0\ell^+\nu_\ell/\rho(1700)^0\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}C^{V_{E/D}}V_{ub}^*$
$B^0 \rightarrow b_1(1235)^-\ell^+\nu_\ell$	C^B	$B^+ \rightarrow \omega(1420)\ell^+\nu_\ell/\omega(1650)\ell^+\nu_\ell$	$\frac{1}{\sqrt{2}}C^{V_{E/D}}V_{ub}^*$
$B_s^0 \rightarrow K_1(1270)^-\ell^+\nu_\ell$	$(\sin\theta_{K_1}C'^A + \cos\theta_{K_1}C'^B)$	$B^0 \rightarrow \rho(1450)^-\ell^+\nu_\ell/\rho(1700)^-\ell^+\nu_\ell$	$C^{V_{E/D}}V_{ub}^*$
$B_s^0 \rightarrow K_1(1400)^-\ell^+\nu_\ell$	$(\cos\theta_{K_1}C'^A - \sin\theta_{K_1}C'^B)$	$B_s^0 \rightarrow K^*(1410)^-\ell^+\nu_\ell/K^*(1680)^-\ell^+\nu_\ell$	$C'^{V_{E/D}}V_{ub}^*$

Until now, none of the $B \rightarrow A\ell^+\nu_\ell$ decays have been measured, so we cannot determine the $C^{A,B}$ by the experimental data. The form factors of the $B \rightarrow a_1/b_1$ and $B_s \rightarrow K_{1A}/K_{1B}$ transitions are taken from Ref. [40]. For the $B \rightarrow f_1(1285)$, $B \rightarrow f_1(1420)$, $B \rightarrow h_1(1170)$, and $B \rightarrow h_1(1415)$ transitions, two cases are considered. In the “a” case, we use the mixing angles $\theta_{1P_1} \in [5^\circ, 56^\circ]$ and $\theta_{3P_1} \in [56^\circ, 125^\circ]$ defined in Eqs. (B5)–(B6), and the form factors obtained by using the SU(3) flavor symmetry relationships from the $B \rightarrow a_1/b_1$ form factors. In the “b” case, we use the mixing angles $\alpha_{f_1} = (69.7 \pm 8)^\circ$ and $\alpha_{h_1} = (86.7 \pm 6)^\circ$ defined in Eqs. (B7)–(B8) and the form factors of the $B \rightarrow f_{1q,s}/h_{1q,s}$ transitions in Ref. [40]. The predicted branching ratios of the $B \rightarrow A\ell^+\nu_\ell$ decays are listed in Table II. One can see that the errors of some results are a little large, the errors of $\mathcal{B}(B \rightarrow a_1(1260)\ell^+\nu_\ell)$ and $\mathcal{B}(B \rightarrow b_1(1235)\ell^+\nu_\ell)$ mainly come from the form factors and the CKM matrix element V_{ub} , and the errors of other branching ratios also come from the mixing angles θ_{K_1} , θ_{3P_1} , or θ_{1P_1} . For the $\mathcal{B}(B^+ \rightarrow f_1\ell^+\nu_\ell)$ and $\mathcal{B}(B^+ \rightarrow h_1\ell^+\nu_\ell)$, we listed the results in two cases. We find that every branching ratio is obviously different in two cases except $\mathcal{B}(B^+ \rightarrow h_1(1170)\ell^+\nu_\ell)$. For the $\mathcal{B}(B_s^0 \rightarrow K_1^-\ell^+\nu_\ell)$ decays, their branching ratios are affected by the mixing angle θ_{K_1} , the results are obtained by using $\theta_{K_1} \in [52^\circ, 65^\circ]$ [30] and $\theta_{K_1} = (33 \pm 4)^\circ$ [41], which are denoted by $\natural$ and $\sharp$ in Table II, respectively. One can see that the predictions of $\mathcal{B}(B_s^0 \rightarrow K_1(1270)^-\ell^+\nu_\ell)$ in two cases are similar; nevertheless, the predictions of $\mathcal{B}(B_s^0 \rightarrow K_1(1400)^-\ell^+\nu_\ell)$ are slightly different.

As for the $B \rightarrow T\ell^+\nu_\ell$ decays, no decay mode has been measured. The data of $\mathcal{B}(B^+ \rightarrow f_2(1270)\ell^+\nu_\ell)$

may be obtained by $\mathcal{B}(B^+ \rightarrow f_2(1270)^0\ell^+\nu_{\ell'}) = \frac{\mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_{\ell'})^{\text{Exp}}}{\mathcal{B}(f_2(1270)^0 \rightarrow \pi^+\pi^-)^{\text{Exp}}}$ from Ref. [39], as follows:

$$\mathcal{B}(B^+ \rightarrow f_2(1270)^0\ell^+\nu_{\ell'})^{\text{Exp}} = (3.21 \pm 1.65) \times 10^{-5}. \quad (9)$$

The relations of the $B \rightarrow T\ell^+\nu_\ell$ decays in Table I will be used for the form factors $A_1^T(0)$. Other form factors $F_i(0) = V^T(0), A_0^T(0), A_2^T(0)$ can be expressed as $r_i \times A_1^T(0)$, and the values of the ratios $r_i = \frac{F_i(0)}{A_1^T(0)}$ are taken from Ref. [42]. The form factor $A_1^T(0)$ is determined by the experimental data given in Eq. (9), and we obtain $A_1^T(0) = 0.14 \pm 0.06$ (it only include part, such as C^T but not $\frac{1}{\sqrt{2}}\cos\theta_{f_2}$), which is slightly smaller than $0.19_{-0.03}^{+0.04}$ in Ref. [42]. Then the branching ratios

TABLE II. Branching ratios of the $B \rightarrow A\ell^+\nu_\ell$ decays within 1 σ error (in units of 10^{-4}); here, $\ell' = e, \mu$. For $\mathcal{B}(B^+ \rightarrow f_1/h_1\ell^+\nu_\ell)$, ^a denotes the results obtained by using the $\theta_{1P_1}/\theta_{3P_1}$ mixing angles and the form factors obtained by the SU(3) flavor symmetry relationships. ^b denotes ones obtained by using the $\alpha_{f_1}/\alpha_{h_1}$ mixing angles and the $B \rightarrow f_{1q}/h_{1q}$ form factors taken from Ref. [40]. For $\mathcal{B}(B_s^0 \rightarrow K_1^-\ell^+\nu_\ell)$, $\natural$ denotes the results by using $\theta_{K_1} \in [52^\circ, 65^\circ]$ [30], and $\sharp$ denotes ones by using $\theta_{K_1} = (33 \pm 4)^\circ$ [41]. The a, b, $\natural$, $\sharp$ signs are same in Tables V and VI.

Branching ratios	Predictions with $\ell = \ell'$	Predictions with $\ell = \tau$
$\mathcal{B}(B^+ \rightarrow a_1(1260)^0\ell^+\nu_\ell)$	16.33 ± 4.01	6.91 ± 1.87
$\mathcal{B}(B^+ \rightarrow b_1(1235)^0\ell^+\nu_\ell)$	28.16 ± 10.01	13.26 ± 4.42
$\mathcal{B}(B^+ \rightarrow f_1(1285)^0\ell^+\nu_\ell)$	$[8.32 \times 10^{-4}, 15.67]^a$ 13.39 ± 3.99^b	$[3.35 \times 10^{-4}, 6.43]^a$ 5.44 ± 1.69^b
$\mathcal{B}(B^+ \rightarrow f_1(1420)^0\ell^+\nu_\ell)$	9.60 ± 7.99^a 2.24 ± 1.69^b	3.87 ± 3.23^a 0.90 ± 0.67^b
$\mathcal{B}(B^+ \rightarrow h_1(1170)^0\ell^+\nu_\ell)$	30.87 ± 11.65^a 32.82 ± 12.32^b	14.79 ± 4.95^a 15.11 ± 5.97^b
$\mathcal{B}(B^+ \rightarrow h_1(1415)^0\ell^+\nu_\ell)$	$[5.37 \times 10^{-8}, 5.82]^a$ $[2.82 \times 10^{-7}, 0.56]^b$	$[2.36 \times 10^{-8}, 2.45]^a$ $[1.21 \times 10^{-7}, 0.24]^b$
$\mathcal{B}(B^0 \rightarrow a_1(1260)^-\ell^+\nu_\ell)$	30.31 ± 7.39	12.85 ± 3.47
$\mathcal{B}(B^0 \rightarrow b_1(1235)^-\ell^+\nu_\ell)$	52.21 ± 18.57	24.63 ± 8.23
$\mathcal{B}(B_s^0 \rightarrow K_1(1270)^-\ell^+\nu_\ell)$	$72.42 \pm 19.60^\natural$ $74.43 \pm 19.53^\sharp$	$31.24 \pm 9.18^\natural$ $33.19 \pm 9.41^\sharp$
$\mathcal{B}(B_s^0 \rightarrow K_1(1400)^-\ell^+\nu_\ell)$	$5.73 \pm 3.82^\natural$ $3.03 \pm 2.16^\sharp$	$2.89 \pm 1.96^\natural$ $1.17 \pm 0.89^\sharp$

TABLE III. Branching ratios of $B \rightarrow T\ell^+\nu_\ell$ decays within 1σ error (in units of 10^{-5}). † denotes $\mathcal{B}(B^+ \rightarrow f_2(1270)\ell^+\nu_\ell)$ are obtained by $\mathcal{B}(B^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell)_{f_2(1270)}^{\text{Exp.}}/\mathcal{B}(f_2(1270) \rightarrow \pi^+\pi^-)^{\text{Exp}}$ from Ref. [39].

Branching ratios	Our predictions ($\ell = \ell'$)	Our predictions ($\ell = \tau$)
$\mathcal{B}(B^+ \rightarrow a_2(1320)^0\ell^+\nu_\ell)$	2.92 ± 1.51	0.93 ± 0.58
$\mathcal{B}(B^+ \rightarrow f_2(1270)\ell^+\nu_\ell)$	$3.21 \pm 1.65^\dagger$	1.06 ± 0.66
$\mathcal{B}(B^+ \rightarrow f_2'(1525)\ell^+\nu_\ell)$	$(4.56 \pm 3.03) \times 10^{-2}$	$(1.27 \pm 0.91) \times 10^{-2}$
$\mathcal{B}(B^0 \rightarrow a_2(1320)^-\ell^+\nu_\ell)$	5.41 ± 2.79	1.72 ± 1.07
$\mathcal{B}(B_s^0 \rightarrow K_2(1430)^-\ell^+\nu_\ell)$	4.72 ± 2.48	1.50 ± 0.94

of the $B \rightarrow T\ell^+\nu_\ell$ decays can be predicted in terms of the constrained $A_1^T(0)$, and they are listed in Table III.

For the $B \rightarrow V_{E,D}\ell^+\nu_\ell$ decays, there is still no measured mode. The form factors of the $B \rightarrow K^*(1410)$ and $B \rightarrow K^*(1680)$ transitions are taken from Ref. [43], which are calculated by the large energy effective theory, and other form factors of $B \rightarrow V_{E,D}$ are related with them by the SU(3) flavor symmetry given in Table I. Then the branching ratios of the $B \rightarrow V_{E,D}\ell^+\nu_\ell$ decays are obtained and listed in Table IV. In addition, the form factors of $B \rightarrow V_E$ transitions are also calculated by the light-front quark models [44], and they are listed in Table XI of Appendix A. The branching ratios of the $B \rightarrow V_E\ell^+\nu_\ell$ decays are also predicted by using the form factors within the light-front quark models, and they are given in Table IV, too. One can see that branching ratios obtained by the light-front quark models are slightly smaller than ones obtained by the large energy effective theory. In the following analysis, the results from the large energy effective theory will be used.

III. CHARMLESS DECAYS $B \rightarrow PV\ell^+\nu_\ell$ WITH THE RESONANCES

A. $\mathcal{B}(B \rightarrow PV\ell^+\nu_\ell)$ with the narrow width approximation

For the $B \rightarrow PV\ell^+\nu_\ell$ decays with the resonances, if the decay widths of the resonance states are very narrow, the resonant branching ratios can be written as

$$\begin{aligned} \mathcal{B}(B \rightarrow R\ell^+\nu_\ell, R \rightarrow PV) \\ = \mathcal{B}(B \rightarrow R\ell^+\nu_\ell) \times \mathcal{B}(R \rightarrow PV), \end{aligned} \quad (10)$$

where R denotes the resonances, and $R = A/T$ in this work. This relation is also a good approximation for wider resonances [34]. The branching ratios of the two-body nonleptonic decays $\mathcal{B}(A \rightarrow PV)$ and $\mathcal{B}(T \rightarrow PV)$ have been studied in our previous work [30]. Using the expressions of $\mathcal{B}(B \rightarrow A/T\ell^+\nu_\ell)$ given in Sec. II and $\mathcal{B}(A/T \rightarrow PV)$ given in Ref. [30], we may predict the branching ratios of the $B \rightarrow A(A \rightarrow PV)\ell^+\nu_\ell$ and $B \rightarrow T(T \rightarrow PV)\ell^+\nu_\ell$ decays, which are listed in the second and third columns of Tables V and VI, respectively, named as the results in S_1 case.

From the second and third columns of Tables V and VI, one can see that some branching ratios with the axial-vector meson resonances, such as $\mathcal{B}(B^+ \rightarrow \rho^\mp\pi^\pm\ell^+\nu_\ell)_{[a_1(1260)^0, h_1(1170)^0]}$, $\mathcal{B}(B^+ \rightarrow \rho^0\pi^0\ell^+\nu_\ell)_{[h_1(1170)^0]}$, $\mathcal{B}(B^+ \rightarrow \omega\pi^0\ell^+\nu_\ell)_{[b_1(1235)^0]}$, $\mathcal{B}(B^+ \rightarrow \rho^0\eta\ell^+\nu_\ell)_{[b_1(1235)^0]}$, $\mathcal{B}(B^0 \rightarrow \rho^0\pi^-\ell^+\nu_\ell)_{[a_1(1260)^-]}$, $\mathcal{B}(B^0 \rightarrow \rho^-\pi^0\ell^+\nu_\ell)_{[a_1(1260)^-]}$, $\mathcal{B}(B^0 \rightarrow \omega\pi^-\ell^+\nu_\ell)_{[b_1(1235)^-]}$, $\mathcal{B}(B^0 \rightarrow \rho^-\eta\ell^+\nu_\ell)_{[b_1(1235)^-]}$, $\mathcal{B}(B_s^0 \rightarrow K^{*-}\pi^0\ell^+\nu_\ell)_{[K_1(1270)^-]}$, $\mathcal{B}(B_s^0 \rightarrow \bar{K}^{*0}\pi^-\ell^+\nu_\ell)_{[K_1(1270)^-]}$, $\mathcal{B}(B_s^0 \rightarrow \rho^0K^-\ell^+\nu_\ell)_{[K_1(1270)^-]}$, $\mathcal{B}(B_s^0 \rightarrow \rho^-\bar{K}^0\ell^+\nu_\ell)_{[K_1(1270)^-]}$, and $\mathcal{B}(B_s^0 \rightarrow \omega K^-\ell^+\nu_\ell)_{[K_1(1270)^-]}$, are on the order of $\mathcal{O}(10^{-4}-10^{-3})$, and other branching ratios with the axial-vector meson resonances

TABLE IV. Branching ratios of $B \rightarrow V_{E/D}\ell^+\nu_\ell$ decays within 1σ error (in units of 10^{-5}). ^{a} denotes the form factors calculated by the large energy effective theory, and ^{b} denotes the form factors calculated by the light-front quark models.

Branching ratios	Our predictions ($\ell = \ell'$)	Our predictions ($\ell = \tau$)
$\mathcal{B}(B^+ \rightarrow \rho(1450)^0\ell^+\nu_\ell)$	$10.36 \pm 1.52^a, 8.14 \pm 1.15^b$	$4.43 \pm 0.71^a, 3.18 \pm 0.48^b$
$\mathcal{B}(B^+ \rightarrow \omega(1420)\ell^+\nu_\ell)$	$11.34 \pm 2.20^a, 9.00 \pm 1.77^b$	$5.02 \pm 1.14^a, 3.64 \pm 0.82^b$
$\mathcal{B}(B^0 \rightarrow \rho(1450)^-\ell^+\nu_\ell)$	$19.26 \pm 2.87^a, 15.14 \pm 2.15^b$	$8.23 \pm 1.34^a, 5.91 \pm 0.91^b$
$\mathcal{B}(B_s^0 \rightarrow K^*(1410)^-\ell^+\nu_\ell)$	$24.14 \pm 3.17^a, 23.69 \pm 3.19^b$	$11.21 \pm 1.55^a, 10.16 \pm 1.44^b$
$\mathcal{B}(B^+ \rightarrow \rho(1700)^0\ell^+\nu_\ell)$	5.25 ± 0.73^a	1.87 ± 0.29^a
$\mathcal{B}(B^+ \rightarrow \omega(1650)\ell^+\nu_\ell)$	5.66 ± 0.85^a	2.10 ± 0.35^a
$\mathcal{B}(B^0 \rightarrow \rho(1700)^-\ell^+\nu_\ell)$	9.77 ± 1.35^a	3.47 ± 0.53^a
$\mathcal{B}(B_s^0 \rightarrow K^*(1680)^-\ell^+\nu_\ell)$	11.35 ± 1.55^a	4.33 ± 0.65^a

TABLE V. SU(3) branching ratios of $B \rightarrow PV\ell'^+\nu_{\ell'}$ decays within 1σ error (in units of 10^{-4}). $\mathcal{B}_{[A/T]}$ denote the branching ratio with the axial-vector/tensor meson resonant contributions. In the S_1 case, the results are obtained by using the narrow width approximation in Eq. (10). In the S_2 case, the width effects are considered as given in Eq. (11). The definitions of $\mathcal{B}_{A/T}$ and $S_{1,2}$ cases are same in Table VI.

Decay modes	$\mathcal{B}_{[A]}$ in S_1 case	$\mathcal{B}_{[T]}$ in S_1 case	$\mathcal{B}_{[A]}$ in S_2 case	$\mathcal{B}_{[T]}$ in S_2 case
$B^+ \rightarrow \rho^-\pi^+\ell'^+\nu_{\ell'}$ or $B^+ \rightarrow \rho^+\pi^-\ell'^+\nu_{\ell'}$	$6.58 \pm 2.71_{[a_1(1260)^0]}$ 6.93 ± 3.85^a $7.53 \pm 3.10^b_{[h_1(1170)^0]}$ 0.64 ± 0.64^a $0.045 \pm 0.045^b_{[h_1(1415)^0]}$	$0.10 \pm 0.06_{[a_2(1320)^0]}$	$3.68 \pm 1.42_{[a_1(1260)^0]}$ 4.27 ± 2.45^a $4.59 \pm 1.62^b_{[h_1(1170)^0]}$ 0.54 ± 0.54^a $0.029 \pm 0.029^b_{[h_1(1415)^0]}$	$0.090 \pm 0.048_{[a_2(1320)^0]}$
$B^+ \rightarrow \rho^0\pi^0\ell'^+\nu_{\ell'}$	6.96 ± 3.87^a $7.57 \pm 3.12^b_{[h_1(1170)^0]}$ 0.64 ± 0.64^a $0.045 \pm 0.045^b_{[h_1(1415)^0]}$		4.35 ± 2.50^a $4.66 \pm 1.69^b_{[h_1(1170)^0]}$ 0.54 ± 0.54^a $0.029 \pm 0.029^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \omega\eta\ell'^+\nu_{\ell'}$	0.66 ± 0.35^a $0.68 \pm 0.26^b_{[h_1(1170)^0]}$ 0.150 ± 0.150^a $0.012 \pm 0.012^b_{[h_1(1415)^0]}$		0.24 ± 0.12^a $0.25 \pm 0.09^b_{[h_1(1170)^0]}$ 0.10 ± 0.10^a $0.0067 \pm 0.0067^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \phi\eta\ell'^+\nu_{\ell'}$	0.022 ± 0.022^a $0.018 \pm 0.018^b_{[h_1(1170)^0]}$ 0.0225 ± 0.0225^a $0.0032 \pm 0.0032^b_{[h_1(1415)^0]}$		0.0053 ± 0.0053^a $0.0055 \pm 0.0055^b_{[h_1(1170)^0]}$ 0.0050 ± 0.0050^a $0.00061 \pm 0.00061^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \omega\pi^0\ell'^+\nu_{\ell'}$	$18.50 \pm 7.32_{[b_1(1235)^0]}$		$15.59 \pm 5.70_{[b_1(1235)^0]}$	
$B^+ \rightarrow \rho^0\eta\ell'^+\nu_{\ell'}$	$1.44 \pm 0.61_{[b_1(1235)^0]}$		$0.41 \pm 0.17_{[b_1(1235)^0]}$	
$B^+ \rightarrow K^{*+}K^-\ell'^+\nu_{\ell'}$ or $B^+ \rightarrow K^{*-}K^+\ell'^+\nu_{\ell'}$	$0.32 \pm 0.14_{[a_1(1260)^0]}$ $0.12 \pm 0.05_{[b_1(1235)^0]}$ 0.045 ± 0.045^a $0.08 \pm 0.04^b_{[f_1(1285)^0]}$ 2.20 ± 2.20^a $0.47 \pm 0.47^b_{[f_1(1420)^0]}$ 0.27 ± 0.20^a $0.29 \pm 0.21^b_{[h_1(1170)^0]}$ 0.127 ± 0.127^a $0.037 \pm 0.037^b_{[h_1(1415)^0]}$	$0.00016 \pm 0.00009_{[a_2(1320)^0]}$ $0.00025 \pm 0.00015_{[f_2(1270)^0]}$ $0.00028 \pm 0.00020_{[f_2'(1525)^0]}$	$0.15 \pm 0.06_{[a_1(1260)^0]}$ $0.059 \pm 0.021_{[b_1(1235)^0]}$ $0^{a,b}_{[f_1(1285)^0]}$ 1.34 ± 1.34^a $0.27 \pm 0.27^b_{[f_1(1420)^0]}$ 0.081 ± 0.057^a $0.094 \pm 0.062^b_{[h_1(1170)^0]}$ 0.096 ± 0.096^a $0.024 \pm 0.024^b_{[h_1(1415)^0]}$	$0.000079 \pm 0.000046_{[a_2(1320)^0]}$ $0.000095 \pm 0.000058_{[f_2(1270)^0]}$ $0.00022 \pm 0.00015_{[f_2'(1525)^0]}$
$B^+ \rightarrow K^{*0}\bar{K}^0\ell'^+\nu_{\ell'}$ or $B^+ \rightarrow \bar{K}^{*0}K^0\ell'^+\nu_{\ell'}$	$0.31 \pm 0.13_{[a_1(1260)^0]}$ $0.11 \pm 0.04_{[b_1(1235)^0]}$ 0.012 ± 0.012^a $0.02 \pm 0.01^b_{[f_1(1285)^0]}$ 1.96 ± 1.96^a $0.42 \pm 0.42^b_{[f_1(1420)^0]}$ 0.26 ± 0.19^a $0.28 \pm 0.20^b_{[h_1(1170)^0]}$ 0.109 ± 0.109^a $0.032 \pm 0.032^b_{[h_1(1415)^0]}$	$0.00013 \pm 0.00007_{[a_2(1320)^0]}$ $0.00022 \pm 0.00013_{[f_2(1270)^0]}$ $0.00025 \pm 0.00018_{[f_2'(1525)^0]}$	$0.14 \pm 0.06_{[a_1(1260)^0]}$ $0.051 \pm 0.018_{[b_1(1235)^0]}$ $0^{a,b}_{[f_1(1285)^0]}$ 1.19 ± 1.19^a $0.24 \pm 0.24^b_{[f_1(1420)^0]}$ 0.075 ± 0.053^a $0.089 \pm 0.059^b_{[h_1(1170)^0]}$ 0.085 ± 0.085^a $0.021 \pm 0.021^b_{[h_1(1415)^0]}$	$0.000064 \pm 0.000037_{[a_2(1320)^0]}$ $0.000085 \pm 0.000052_{[f_2(1270)^0]}$ $0.00019 \pm 0.00014_{[f_2'(1525)^0]}$
$B^0 \rightarrow \rho^0\pi^-\ell'^+\nu_{\ell'}$	$12.19 \pm 5.02_{[a_1(1260)^-]}$	$0.19 \pm 0.10_{[a_2(1320)^-]}$	$6.84 \pm 2.63_{[a_1(1260)^-]}$	$0.17 \pm 0.09_{[a_2(1320)^-]}$
$B^0 \rightarrow \rho^-\pi^0\ell'^+\nu_{\ell'}$	$12.25 \pm 5.06_{[a_1(1260)^-]}$	$0.19 \pm 0.10_{[a_2(1320)^-]}$	$6.89 \pm 2.64_{[a_1(1260)^-]}$	$0.17 \pm 0.09_{[a_2(1320)^-]}$
$B^0 \rightarrow \omega\pi^-\ell'^+\nu_{\ell'}$	$34.16 \pm 13.48_{[b_1(1235)^-]}$		$6.37 \pm 2.26_{[b_1(1235)^-]}$	
$B^0 \rightarrow \rho^-\eta\ell'^+\nu_{\ell'}$	$2.69 \pm 1.14_{[b_1(1235)^-]}$		$0.52 \pm 0.22_{[b_1(1235)^-]}$	
$B^0 \rightarrow K^{*0}K^-\ell'^+\nu_{\ell'}$	$1.17 \pm 0.50_{[a_1(1260)^-]}$ $0.42 \pm 0.17_{[b_1(1235)^-]}$	$0.00053 \pm 0.00030_{[a_2(1320)^-]}$	$0.54 \pm 0.22_{[a_1(1260)^-]}$ $0.21 \pm 0.08_{[b_1(1235)^-]}$	$0.00027 \pm 0.00015_{[a_2(1320)^-]}$
$B^0 \rightarrow K^{*-}K^0\ell'^+\nu_{\ell'}$	$1.18 \pm 0.51_{[a_1(1260)^-]}$ $0.43 \pm 0.17_{[b_1(1235)^-]}$	$0.00056 \pm 0.00032_{[a_2(1320)^-]}$	$0.54 \pm 0.22_{[a_1(1260)^-]}$ $0.21 \pm 0.09_{[b_1(1235)^-]}$	$0.00026 \pm 0.00015_{[a_2(1320)^-]}$
$B_s^0 \rightarrow K^{*-}\pi^0\ell'^+\nu_{\ell'}$	$5.39 \pm 3.23^\natural$ $5.60 \pm 3.40^\natural_{[K_1(1270)^-]}$ $1.73 \pm 1.19^\natural$ $0.95 \pm 0.69^\natural_{[K_1(1400)^-]}$	$0.040 \pm 0.022_{[K_2^*(1430)^-]}$	$4.48 \pm 2.50^\natural$ $4.55 \pm 2.50^\natural_{[K_1(1270)^-]}$ $1.20 \pm 0.83^\natural$ $0.82 \pm 0.58^\natural_{[K_1(1400)^-]}$	$0.035 \pm 0.020_{[K_2^*(1430)^-]}$
$B_s^0 \rightarrow \bar{K}^{*0}\pi^-\ell'^+\nu_{\ell'}$	$10.58 \pm 6.34^\natural$ $10.97 \pm 6.66^\natural_{[K_1(1270)^-]}$ $3.42 \pm 2.35^\natural$ $1.89 \pm 1.36^\natural_{[K_1(1400)^-]}$	$0.076 \pm 0.043_{[K_2^*(1430)^-]}$	$8.76 \pm 4.87^\natural$ $8.89 \pm 4.89^\natural_{[K_1(1270)^-]}$ $2.36 \pm 1.64^\natural$ $1.61 \pm 1.15^\natural_{[K_1(1400)^-]}$	$0.067 \pm 0.038_{[K_2^*(1430)^-]}$
$B_s^0 \rightarrow \rho^0K^-\ell'^+\nu_{\ell'}$	$12.39 \pm 3.49^\natural$ $12.64 \pm 3.38^\natural_{[K_1(1270)^-]}$ $0.066 \pm 0.066^\natural$ $0.022 \pm 0.022^\natural_{[K_1(1400)^-]}$	$0.014 \pm 0.008_{[K_2^*(1430)^-]}$	$5.73 \pm 1.47^\natural$ $5.83 \pm 1.33^\natural_{[K_1(1270)^-]}$ $0.037 \pm 0.037^\natural$ $0.016 \pm 0.016^\natural_{[K_1(1400)^-]}$	$0.015 \pm 0.009_{[K_2^*(1430)^-]}$

(Table continued)

TABLE V. (Continued)

Decay modes	$\mathcal{B}_{[A]}$ in S_1 case	$\mathcal{B}_{[T]}$ in S_1 case	$\mathcal{B}_{[A]}$ in S_2 case	$\mathcal{B}_{[T]}$ in S_2 case
$B_s^0 \rightarrow \rho^- \bar{K}^0 \ell'^+ \nu_{\ell'}$	$23.97 \pm 6.63^{\text{a}}$ $24.51 \pm 6.56^{\text{a}} [K_1(1270)^-]$ $0.13 \pm 0.13^{\text{a}}$ $0.044 \pm 0.044^{\text{a}} [K_1(1400)^-]$	$0.027 \pm 0.016 [K_2^*(1430)^-]$	$10.62 \pm 2.73^{\text{a}}$ $10.88 \pm 2.49^{\text{a}} [K_1(1270)^-]$ $0.072 \pm 0.072^{\text{a}}$ $0.032 \pm 0.032^{\text{a}} [K_1(1400)^-]$	$0.028 \pm 0.017 [K_2^*(1430)^-]$
$B_s^0 \rightarrow \omega K^- \ell'^+ \nu_{\ell'}$	$6.88 \pm 1.92^{\text{a}}$ $7.01 \pm 1.87^{\text{a}} [K_1(1270)^-]$ $0.064 \pm 0.064^{\text{a}}$ $0.021 \pm 0.021^{\text{a}} [K_1(1400)^-]$	$0.015 \pm 0.011 [K_2^*(1430)^-]$	$4.94 \pm 1.30^{\text{a}}$ $5.08 \pm 1.18^{\text{a}} [K_1(1270)^-]$ $0.035 \pm 0.035^{\text{a}}$ $0.016 \pm 0.016^{\text{a}} [K_1(1400)^-]$	$0.016 \pm 0.011 [K_2^*(1430)^-]$
$B_s^0 \rightarrow K^{*-} \eta \ell'^+ \nu_{\ell'}$	$0.37 \pm 0.19^{\text{a}}$ $0.38 \pm 0.19^{\text{a}} [K_1(1270)^-]$ $0.15 \pm 0.10^{\text{a}}$ $0.11 \pm 0.10^{\text{a}} [K_1(1400)^-]$	$0.0028 \pm 0.0017 [K_2^*(1430)^-]$	$0.15 \pm 0.08^{\text{a}}$ $0.14 \pm 0.08^{\text{a}} [K_1(1270)^-]$ $0.078 \pm 0.052^{\text{a}}$ $0.080 \pm 0.068^{\text{a}} [K_1(1400)^-]$	$0.0014 \pm 0.0009 [K_2^*(1430)^-]$
$B_s^0 \rightarrow \phi K^- \ell'^+ \nu_{\ell'}$	$0.22 \pm 0.15^{\text{a}}$ $0.12 \pm 0.09^{\text{a}} [K_1(1400)^-]$	$0.00019 \pm 0.00012 [K_2^*(1430)^-]$	$0.11 \pm 0.08^{\text{a}}$ $0.082 \pm 0.059^{\text{a}} [K_1(1400)^-]$	$0.00012 \pm 0.00007 [K_2^*(1430)^-]$

and all branching ratios with the tensor meson resonances are small.

B. $\mathcal{B}(B \rightarrow PV \ell^+ \nu_{\ell})$ with the width effects

Some decay widths of the resonance states, such as ones of $h_1(1170)$, $h_1(1415)$, $b_1(1235)$, $a_1(1260)$, $K_1(1270)$, $K_1(1400)$, $f_2(1270)$, $f_2'(1270)$, $a_2(1320)$, and $K_2(1400)$, are not very narrow. We will consider the width effects of the resonance states. After considering the width effects of the resonance states, the decay branching ratios of $B \rightarrow PV \ell^+ \nu_{\ell}$ are [45,46]

$$\mathcal{B}(B \rightarrow PV \ell^+ \nu_{\ell})_R = \frac{1}{\pi} \int_{(m_R - n\Gamma_R)^2}^{(m_R + n\Gamma_R)^2} dt_R \int_{m_{\ell}^2}^{(m_B - \sqrt{t_R})^2} dq^2 \times \frac{\sqrt{t_R} d\mathcal{B}(B \rightarrow R \ell^+ \nu_{\ell}, t_R) / dq^2 \mathcal{B}(R \rightarrow PV, t_R) \Gamma_R}{(t_R - m_R^2)^2 + m_R^2 \Gamma_R^2}, \quad (11)$$

where $d\mathcal{B}(B \rightarrow R \ell^+ \nu_{\ell}, t_R) / dq^2$ and $\mathcal{B}(R \rightarrow PV, t_R)$ are obtained from $d\mathcal{B}(B \rightarrow R \ell^+ \nu_{\ell}) / dq^2$ and $\mathcal{B}(R \rightarrow PV)$ by replacing $m_R \rightarrow \sqrt{t_R}$. The form factors of the $B \rightarrow R \ell^+ \nu_{\ell}$ decays and the coefficients of the R-V-P vertexes follow the SU(3) flavor symmetry relations. And then the predicted branching ratios are obtained, which are listed in the last two columns of Tables V and VI and named as ones in the S_2 case. Comparing with the results with the narrow width approximation, one can see that some results considered the resonant width effects are obviously smaller. For $\mathcal{B}(B^+ \rightarrow K^{*\pm} K^{\mp} \ell^+ \nu_{\ell})_{[f_1(1285)^0]}$, $\mathcal{B}(B^+ \rightarrow K^{*0} \bar{K}^0 \ell^+ \nu_{\ell})_{[f_1(1285)^0]}$, and $\mathcal{B}(B^+ \rightarrow \bar{K}^{*0} K^0 \ell^+ \nu_{\ell})_{[f_1(1285)^0]}$, they are not allowed in the S_2 case, since the decay width of $f_1(1285)^0$ is small. Note that for $f_1(1285)^0 \rightarrow K^{*+} K^-$, $K^{*-} K^+$, $K^{*0} \bar{K}^0$, $\bar{K}^{*0} K^0$ decays, their phase spaces are very narrow, and their branching ratios have been obtained by considering the decay widths of both $f_1(1285)^0$ and K^* mesons in Ref. [30]. However, we only consider the decay width of $f_1(1285)^0$ meson in the S_2 case, and this is the reason

that the predictions are different between the S_1 and S_2 cases.

C. $\mathcal{B}(B \rightarrow PV \ell^+ \nu_{\ell})$ with the excited vector resonance states

In addition, the excited vector meson contributions to the branching ratios of four-body semileptonic decays will be calculated by the narrow width approximation in this subsection. Since $\mathcal{B}(B \rightarrow V_{E,D} \ell^+ \nu_{\ell})$ have been listed in Table IV, now we just need to calculate $\mathcal{B}(V_{E,D} \rightarrow PV)$ to get $\mathcal{B}(B \rightarrow PV \ell^+ \nu_{\ell})$ with the excited vector resonance states.

The branching ratios of the $V_{E,D} \rightarrow PV$ strong decays can be written as [47]

$$\mathcal{B}(V' \rightarrow PV) = \frac{\tau_{V'} p_c^3}{12\pi m_{V'}^2} g_{V' \rightarrow PV}^2, \quad (12)$$

where $V' = V_E$ or V_D , and $p_c \equiv \frac{\sqrt{\lambda(m_{V'}^2, m_P^2, m_V^2)}}{2m_{V'}}$ and $g_{V' \rightarrow PV}$ are the strong coupling constants. $g_{V' \rightarrow PV}$ can be parametrized by the SU(3) flavor symmetry

$$g_{V' \rightarrow PV} = g_{V'} V'^i P_i^k V_k^j, \quad (13)$$

where $g_{V'}$ are the corresponding nonperturbative parameters. The strong coupling constants of these decays are listed in Table VII.

For the $V_D \rightarrow PV$ strong decays, only $K^*(1680) \rightarrow \rho K, K^* \pi$ decays have been measured [39], as follows:

$$\begin{aligned} \mathcal{B}(K^*(1680) \rightarrow \rho K)^{\text{Exp}} &= (31.4_{-2.1}^{+5.0})\%, \\ \mathcal{B}(K^*(1680) \rightarrow K^* \pi)^{\text{Exp}} &= (29.9_{-5.0}^{+2.2})\%, \end{aligned} \quad (14)$$

and they can be used to constrain the nonperturbative parameter g_{V_D} . We obtain $g_{V_D} = 6.48 \pm 1.59$ from $\mathcal{B}(K^*(1680) \rightarrow K^* \pi)^{\text{Exp}}$ and $g_{V_D} = 5.33 \pm 1.33$ from $\mathcal{B}(K^*(1680) \rightarrow \rho K)^{\text{Exp}}$. The communication between them, $g_{V_D} \in [4.88, 6.66]$, will be used to obtain the

TABLE VI. SU(3) branching ratios of $B^+ \rightarrow PV\tau^+\nu_\tau$ decays within 1σ error (in units of 10^{-4}).

Decay modes	$\mathcal{B}_{[A]}$ in S_1 case	$\mathcal{B}_{[T]}$ in S_1 case	$\mathcal{B}_{[A]}$ in S_2 case	$\mathcal{B}_{[T]}$ in S_2 case
$B^+ \rightarrow \rho^- \pi^+ \tau^+ \nu_\tau$	$2.79 \pm 1.19_{[a_1(1260)^0]}$	$0.033 \pm 0.021_{[a_2(1320)^0]}$	$1.41 \pm 0.56_{[a_1(1260)^0]}$	$0.028 \pm 0.018_{[a_2(1320)^0]}$
or	3.22 ± 1.67^a $3.56 \pm 1.30^b_{[h_1(1170)^0]}$		1.98 ± 1.07^a $2.08 \pm 0.67^b_{[h_1(1170)^0]}$	
$B^+ \rightarrow \rho^+ \pi^- \tau^+ \nu_\tau$	0.27 ± 0.27^a $0.019 \pm 0.019^b_{[h_1(1415)^0]}$		0.23 ± 0.23^a $0.013 \pm 0.013^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \rho^0 \pi^0 \tau^+ \nu_\tau$	3.24 ± 1.67^a $3.58 \pm 1.31^b_{[h_1(1170)^0]}$ 0.27 ± 0.27^a $0.019 \pm 0.019^b_{[h_1(1415)^0]}$		2.00 ± 1.09^a $2.08 \pm 0.67^b_{[h_1(1170)^0]}$ 0.23 ± 0.23^a $0.013 \pm 0.013^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \omega \eta \tau^+ \nu_\tau$	0.30 ± 0.15^a $0.32 \pm 0.12^b_{[h_1(1170)^0]}$ 0.066 ± 0.066^a $0.0050 \pm 0.0050^b_{[h_1(1415)^0]}$		0.094 ± 0.049^a $0.10 \pm 0.04^b_{[h_1(1170)^0]}$ 0.042 ± 0.042^a $0.0029 \pm 0.0029^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \phi \eta \tau^+ \nu_\tau$	0.010 ± 0.010^a $0.0088 \pm 0.0088^b_{[h_1(1170)^0]}$ 0.010 ± 0.010^a $0.0014 \pm 0.0014^b_{[h_1(1415)^0]}$		0.0019 ± 0.0019^a $0.0019 \pm 0.0019^b_{[h_1(1170)^0]}$ 0.0019 ± 0.0019^a $0.00025 \pm 0.00025^b_{[h_1(1415)^0]}$	
$B^+ \rightarrow \omega \pi^0 \tau^+ \nu_\tau$	$8.38 \pm 3.00_{[b_1(1235)^0]}$		$7.18 \pm 2.31_{[b_1(1235)^0]}$	
$B^+ \rightarrow \rho^0 \eta \tau^+ \nu_\tau$	$0.67 \pm 0.26_{[b_1(1235)^0]}$		$0.18 \pm 0.07_{[b_1(1235)^0]}$	
$B^+ \rightarrow K^{*+} K^- \tau^+ \nu_\tau$	$0.13 \pm 0.05_{[a_1(1260)^0]}$	$0.000053 \pm 0.000035_{[a_2(1320)^0]}$	$0.045 \pm 0.019_{[a_1(1260)^0]}$	$0.000022 \pm 0.000014_{[a_2(1320)^0]}$
or	$0.057 \pm 0.022_{[b_1(1235)^0]}$	$0.000080 \pm 0.000055_{[f_2(1270)^0]}$	$0.025 \pm 0.009_{[b_1(1235)^0]}$	$0.000024 \pm 0.000016_{[f_2(1270)^0]}$
$B^+ \rightarrow K^{*-} K^+ \tau^+ \nu_\tau$	0.019 ± 0.018^a $0.032 \pm 0.018^b_{[f_1(1285)^0]}$ 0.89 ± 0.89^a $0.19 \pm 0.19^b_{[f_1(1420)^0]}$ 0.13 ± 0.09^a $0.14 \pm 0.10^b_{[h_1(1170)^0]}$ 0.056 ± 0.056^a $0.016 \pm 0.016^b_{[h_1(1415)^0]}$	$0.000079 \pm 0.000060_{[f_2'(1525)^0]}$	$0_{[f_1(1285)^0]}$ 0.49 ± 0.49^a $0.10 \pm 0.10^b_{[f_1(1420)^0]}$ 0.031 ± 0.022^a $0.037 \pm 0.025^b_{[h_1(1170)^0]}$ 0.040 ± 0.040^a $0.0099 \pm 0.0099^b_{[h_1(1415)^0]}$	$0.000059 \pm 0.000044_{[f_2'(1525)^0]}$
$B^+ \rightarrow K^{*0} \bar{K}^0 \tau^+ \nu_\tau$	$0.13 \pm 0.05_{[a_1(1260)^0]}$	$0.000042 \pm 0.000028_{[a_2(1320)^0]}$	$0.043 \pm 0.017_{[a_1(1260)^0]}$	$0.000018 \pm 0.000011_{[a_2(1320)^0]}$
or	$0.050 \pm 0.020_{[b_1(1235)^0]}$	$0.000071 \pm 0.000049_{[f_2(1270)^0]}$	$0.022 \pm 0.008_{[b_1(1235)^0]}$	$0.000022 \pm 0.000014_{[f_2(1270)^0]}$
$B^+ \rightarrow \bar{K}^{*0} K^0 \tau^+ \nu_\tau$	0.0047 ± 0.0048^a $0.0075 \pm 0.0044^b_{[f_1(1285)^0]}$ 0.80 ± 0.80^a $0.17 \pm 0.17^b_{[f_1(1420)^0]}$ 0.12 ± 0.09^a $0.13 \pm 0.10^b_{[h_1(1170)^0]}$ 0.047 ± 0.047^a $0.013 \pm 0.013^b_{[h_1(1415)^0]}$	$0.000071 \pm 0.000053_{[f_2'(1525)^0]}$	$0_{[f_1(1285)^0]}$ 0.43 ± 0.43^a $0.09 \pm 0.09^b_{[f_1(1420)^0]}$ 0.029 ± 0.021^a $0.035 \pm 0.023^b_{[h_1(1170)^0]}$ 0.035 ± 0.035^a $0.0088 \pm 0.0088^b_{[h_1(1415)^0]}$	$0.000053 \pm 0.000039_{[f_2'(1525)^0]}$
$B^0 \rightarrow \rho^0 \pi^- \tau^+ \nu_\tau$	$5.17 \pm 2.21_{[a_1(1260)^-]}$	$0.062 \pm 0.039_{[a_2(1320)^-]}$	$2.64 \pm 1.06_{[a_1(1260)^-]}$	$0.052 \pm 0.033_{[a_2(1320)^-]}$
$B^0 \rightarrow \rho^- \pi^0 \tau^+ \nu_\tau$	$5.20 \pm 2.22_{[a_1(1260)^-]}$	$0.062 \pm 0.040_{[a_2(1320)^-]}$	$2.65 \pm 1.04_{[a_1(1260)^-]}$	$0.053 \pm 0.033_{[a_2(1320)^-]}$
$B^0 \rightarrow \omega \pi^- \tau^+ \nu_\tau$	$15.48 \pm 5.53_{[b_1(1235)^-]}$		$2.88 \pm 0.92_{[b_1(1235)^-]}$	
$B^0 \rightarrow \rho^- \eta \tau^+ \nu_\tau$	$1.25 \pm 0.50_{[b_1(1235)^-]}$		$0.23 \pm 0.09_{[b_1(1235)^-]}$	
$B^0 \rightarrow K^{*0} K^- \tau^+ \nu_\tau$	$0.49 \pm 0.20_{[a_1(1260)^-]}$ $0.20 \pm 0.08_{[b_1(1235)^-]}$	$0.00017 \pm 0.00011_{[a_2(1320)^-]}$	$0.16 \pm 0.07_{[a_1(1260)^-]}$ $0.088 \pm 0.036_{[b_1(1235)^-]}$	$0.000073 \pm 0.000047_{[a_2(1320)^-]}$
$B^0 \rightarrow K^{*-} K^0 \tau^+ \nu_\tau$	$0.49 \pm 0.20_{[a_1(1260)^-]}$ $0.20 \pm 0.08_{[b_1(1235)^-]}$	$0.00018 \pm 0.00012_{[a_2(1320)^-]}$	$0.16 \pm 0.07_{[a_1(1260)^-]}$ $0.087 \pm 0.036_{[b_1(1235)^-]}$	$0.000073 \pm 0.000047_{[a_2(1320)^-]}$
$B_s^0 \rightarrow K^{*-} \pi^0 \tau^+ \nu_\tau$	$2.37 \pm 1.45^\sharp$ $2.53 \pm 1.57^\sharp_{[K_1(1270)^-]}$ $0.86 \pm 0.60^\sharp$ $0.36 \pm 0.28^\sharp_{[K_1(1400)^-]}$	$0.013 \pm 0.008_{[K_2^*(1430)^-]}$	$1.93 \pm 1.07^\sharp$ $2.01 \pm 1.13^\sharp_{[K_1(1270)^-]}$ $0.57 \pm 0.39^\sharp$ $0.31 \pm 0.23^\sharp_{[K_1(1400)^-]}$	$0.011 \pm 0.007_{[K_2^*(1430)^-]}$
$B_s^0 \rightarrow \bar{K}^{*0} \pi^- \tau^+ \nu_\tau$	$4.66 \pm 2.85^\sharp$ $4.95 \pm 3.09^\sharp_{[K_1(1270)^-]}$ $1.71 \pm 1.19^\sharp$ $0.71 \pm 0.55^\sharp_{[K_1(1400)^-]}$	$0.024 \pm 0.016_{[K_2^*(1430)^-]}$	$3.78 \pm 2.09^\sharp$ $3.92 \pm 2.19^\sharp_{[K_1(1270)^-]}$ $1.12 \pm 0.77^\sharp$ $0.60 \pm 0.45^\sharp_{[K_1(1400)^-]}$	$0.021 \pm 0.014_{[K_2^*(1430)^-]}$

(Table continued)

TABLE VI. (Continued)

Decay modes	$\mathcal{B}_{[A]}$ in S_1 case	$\mathcal{B}_{[T]}$ in S_1 case	$\mathcal{B}_{[A]}$ in S_2 case	$\mathcal{B}_{[T]}$ in S_2 case
$B_s^- \rightarrow \rho^0 K^- \tau^+ \nu_\tau$	$5.34 \pm 1.63^{\pm 1}$ $5.62 \pm 1.62^{\pm 2} [K_1(1270)^-]$ $0.031 \pm 0.031^{\pm 1}$ $0.0085 \pm 0.0085^{\pm 2} [K_1(1400)^-]$	$0.0044 \pm 0.0029 [K_2^*(1430)^-]$	$2.32 \pm 0.64^{\pm 1}$ $2.45 \pm 0.61^{\pm 2} [K_1(1270)^-]$ $0.016 \pm 0.016^{\pm 1}$ $0.0064 \pm 0.0064^{\pm 2} [K_1(1400)^-]$	$0.0046 \pm 0.0030 [K_2^*(1430)^-]$
$B_s^0 \rightarrow \rho^- \bar{K}^0 \tau^+ \nu_\tau$	$10.33 \pm 3.11^{\pm 1}$ $10.92 \pm 3.16^{\pm 2} [K_1(1270)^-]$ $0.061 \pm 0.061^{\pm 1}$ $0.017 \pm 0.017^{\pm 2} [K_1(1400)^-]$	$0.0086 \pm 0.0057 [K_2^*(1430)^-]$	$4.29 \pm 1.17^{\pm 1}$ $4.51 \pm 1.12^{\pm 2} [K_1(1270)^-]$ $0.031 \pm 0.031^{\pm 1}$ $0.013 \pm 0.013^{\pm 2} [K_1(1400)^-]$	$0.0087 \pm 0.0058 [K_2^*(1430)^-]$
$B_s^0 \rightarrow \omega K^- \tau^+ \nu_\tau$	$2.97 \pm 0.89^{\pm 1}$ $3.11 \pm 0.89^{\pm 2} [K_1(1270)^-]$ $0.030 \pm 0.030^{\pm 1}$ $0.0082 \pm 0.0082^{\pm 2} [K_1(1400)^-]$	$0.0049 \pm 0.0036 [K_2^*(1430)^-]$	$1.99 \pm 0.54^{\pm 1}$ $2.10 \pm 0.53^{\pm 2} [K_1(1270)^-]$ $0.015 \pm 0.015^{\pm 1}$ $0.0062 \pm 0.0062^{\pm 2} [K_1(1400)^-]$	$0.0051 \pm 0.0038 [K_2^*(1430)^-]$
$B_s^0 \rightarrow K^{*-} \eta \tau^+ \nu_\tau$	$0.17 \pm 0.09^{\pm 1}$ $0.17 \pm 0.09^{\pm 2} [K_1(1270)^-]$ $0.079 \pm 0.052^{\pm 1}$ $0.043 \pm 0.036^{\pm 2} [K_1(1400)^-]$	$0.00089 \pm 0.00060 [K_2^*(1430)^-]$	$0.056 \pm 0.033^{\pm 1}$ $0.057 \pm 0.033^{\pm 2} [K_1(1270)^-]$ $0.034 \pm 0.022^{\pm 1}$ $0.028 \pm 0.024^{\pm 2} [K_1(1400)^-]$	$0.00041 \pm 0.00027 [K_2^*(1430)^-]$
$B_s^0 \rightarrow \phi K^- \tau^+ \nu_\tau$	$0.11 \pm 0.08^{\pm 1}$ $0.048 \pm 0.037^{\pm 2} [K_1(1400)^-]$	$0.000059 \pm 0.000041 [K_2^*(1430)^-]$	$0.044 \pm 0.03^{\pm 1}$ $0.027 \pm 0.020^{\pm 2} [K_1(1400)^-]$	$0.000032 \pm 0.000021 [K_2^*(1430)^-]$

branching ratios and decay widths of the $V_D \rightarrow PV$ strong decays. After considering the bounds of $\mathcal{B}(K^*(1680) \rightarrow PV) \leq 63.2\%$, $\mathcal{B}(\rho(1700) \rightarrow PV) \leq 100\%$, and $\mathcal{B}(\omega(1650) \rightarrow PV) \leq 130\%$, the predictions are obtained and listed in Table VIII. Please note that $\mathcal{B}(\omega(1650) \rightarrow PV)$, which is defined as $\mathcal{B}(\omega(1650) \rightarrow \rho\pi) + \mathcal{B}(\omega(1650) \rightarrow K^*K) + \mathcal{B}(\omega(1650) \rightarrow \omega\eta)$, is always

larger than 1, so we have to extend that $\mathcal{B}(\omega(1650) \rightarrow PV) \leq 130\%$, which means that all branching ratio predictions of every $\omega(1650) \rightarrow PV$ modes should be smaller than our predictions. Moreover, we set $\mathcal{B}(K^*(1680) \rightarrow PV) \leq 63.2\%$, since $\mathcal{B}(K^*(1680) \rightarrow K\pi) = (38.7 \pm 2.5)\%$ and $\mathcal{B}(K^*(1680) \rightarrow K\eta) = (1.4_{-0.8}^{+1.0})\%$ are deducted [39].

TABLE VII. Strong coupling constants of the $V' \rightarrow PV$ decays by the SU(3) flavor symmetry.

Decay modes	Strong couplings $g_{V' \rightarrow PV}$	Decay modes	Strong couplings $g_{V' \rightarrow PV}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow \rho^0 K^+$	$\frac{1}{\sqrt{2}} g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow \rho^- K^+$	$g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow \omega K^+$	$\frac{1}{\sqrt{2}} g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow \rho^0 K^0$	$\frac{1}{\sqrt{2}} g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow \rho^+ \bar{K}^0$	$g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow \omega K^0$	$\frac{1}{\sqrt{2}} g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow \phi K^+$	$g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow \phi K^0$	$g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow K^{*+} \pi^0$	$\frac{1}{\sqrt{2}} g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow K^{*+} \pi^-$	$g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow \bar{K}^{*0} \pi^+$	$g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow K^{*0} \pi^0$	$-\frac{1}{\sqrt{2}} g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow K^{*+} \eta$	$(\frac{\cos \theta_P}{\sqrt{6}} - \frac{\sin \theta_P}{\sqrt{3}}) g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow K^{*0} \eta$	$(\frac{\cos \theta_P}{\sqrt{6}} - \frac{\sin \theta_P}{\sqrt{3}}) g_{V'}$
$K^*(1410)^+/K^*(1680)^+ \rightarrow K^{*+} \eta'$	$(\frac{\sin \theta_P}{\sqrt{6}} + \frac{\cos \theta_P}{\sqrt{3}}) g_{V'}$	$K^*(1410)^0/K^*(1680)^0 \rightarrow K^{*0} \eta'$	$(\frac{\sin \theta_P}{\sqrt{6}} + \frac{\cos \theta_P}{\sqrt{3}}) g_{V'}$
$\rho(1450)^0/\rho(1700)^0 \rightarrow \rho^0 \eta$	$(\frac{2\cos \theta_P}{\sqrt{6}} - \frac{2\sin \theta_P}{\sqrt{3}}) g_{V'}$	$\rho(1450)^+/ \rho(1700)^+ \rightarrow \rho^+ \eta$	$(\frac{2\cos \theta_P}{\sqrt{6}} - \frac{2\sin \theta_P}{\sqrt{3}}) g_{V'}$
$\rho(1450)^0/\rho(1700)^0 \rightarrow \rho^0 \eta'$	$(\frac{2\sin \theta_P}{\sqrt{6}} + \frac{2\cos \theta_P}{\sqrt{3}}) g_{V'}$	$\rho(1450)^+/ \rho(1700)^+ \rightarrow \rho^+ \eta'$	$(\frac{2\sin \theta_P}{\sqrt{6}} + \frac{2\cos \theta_P}{\sqrt{3}}) g_{V'}$
$\rho(1450)^0/\rho(1700)^0 \rightarrow \omega \pi^0$	$\sqrt{2} g_{V'}$	$\rho(1450)^+/ \rho(1700)^+ \rightarrow \omega \pi^+$	$\sqrt{2} g_{V'}$
$\rho(1450)^0/\rho(1700)^0 \rightarrow K^{*+} K^-$	$\frac{1}{\sqrt{2}} g_{V'}$	$\rho(1450)^+/ \rho(1700)^+ \rightarrow K^{*+} \bar{K}^0$	$g_{V'}$
$\rho(1450)^0/\rho(1700)^0 \rightarrow K^{*-} K^+$	$\frac{1}{\sqrt{2}} g_{V'}$	$\rho(1450)^+/ \rho(1700)^+ \rightarrow \bar{K}^{*0} K^+$	$g_{V'}$
$\rho(1450)^0/\rho(1700)^0 \rightarrow \bar{K}^{*0} K^0$	$-\frac{1}{\sqrt{2}} g_{V'}$		
$\rho(1450)^0/\rho(1700)^0 \rightarrow K^{*0} \bar{K}^0$	$-\frac{1}{\sqrt{2}} g_{V'}$		
$\omega(1420)/\omega(1650) \rightarrow \rho^0 \eta$	$(\frac{2\cos \theta_P}{\sqrt{6}} - \frac{2\sin \theta_P}{\sqrt{3}}) g_{V'}$	$\omega(1420)/\omega(1650) \rightarrow K^{*+} K^-$	$\frac{1}{\sqrt{2}} g_{V'}$
$\omega(1420)/\omega(1650) \rightarrow \rho^0 \eta'$	$(\frac{2\sin \theta_P}{\sqrt{6}} + \frac{2\cos \theta_P}{\sqrt{3}}) g_{V'}$	$\omega(1420)/\omega(1650) \rightarrow K^{*-} K^+$	$\frac{1}{\sqrt{2}} g_{V'}$
$\omega(1420)/\omega(1650) \rightarrow \rho^0 \pi^0$	$\sqrt{2} g_{V'}$	$\omega(1420)/\omega(1650) \rightarrow \bar{K}^{*0} K^0$	$\frac{1}{\sqrt{2}} g_{V'}$
$\omega(1420)/\omega(1650) \rightarrow \rho^+ \pi^-$	$\sqrt{2} g_{V'}$	$\omega(1420)/\omega(1650) \rightarrow K^{*0} \bar{K}^0$	$\frac{1}{\sqrt{2}} g_{V'}$
$\omega(1420)/\omega(1650) \rightarrow \rho^- \pi^+$	$\sqrt{2} g_{V'}$		

TABLE VIII. Branching ratios and decay widths of the $V_D \rightarrow PV$ strong decays by the SU(3) flavor symmetry.

$V_D \rightarrow PV$ decays	Branching ratios (%)	Decay width (MeV)	$V_D \rightarrow PV$ decays	Branching ratios (%)	Decay width (MeV)
$K^*(1680)^+ \rightarrow \rho^0 K^+$	5.90 ± 1.43	21.61 ± 2.63	$K^*(1680)^0 \rightarrow \rho^- K^+$	11.80 ± 2.86	43.25 ± 5.27
$K^*(1680)^+ \rightarrow \omega K^+$	5.76 ± 1.40	21.11 ± 2.59	$K^*(1680)^0 \rightarrow \rho^0 K^0$	5.84 ± 1.42	21.39 ± 2.61
$K^*(1680)^+ \rightarrow \rho^+ \bar{K}^0$	11.68 ± 2.83	42.81 ± 5.23	$K^*(1680)^0 \rightarrow \omega K^0$	5.70 ± 1.39	20.90 ± 2.57
$K^*(1680)^+ \rightarrow \phi K^+$	3.72 ± 1.13	13.39 ± 2.45	$K^*(1680)^0 \rightarrow \phi K^0$	3.63 ± 1.11	13.07 ± 2.42
$K^*(1680)^+ \rightarrow K^{*+} \pi^0$	7.55 ± 1.82	27.44 ± 3.03	$K^*(1680)^0 \rightarrow K^{*+} \pi^-$	15.06 ± 3.63	54.70 ± 6.04
$K^*(1680)^+ \rightarrow \bar{K}^{*0} \pi^+$	14.90 ± 3.59	54.15 ± 6.01	$K^*(1680)^0 \rightarrow K^{*0} \pi^0$	7.48 ± 1.80	27.16 ± 3.01
$K^*(1680)^+ \rightarrow K^{*+} \eta$	1.87 ± 0.72	6.69 ± 1.83	$K^*(1680)^0 \rightarrow K^{*0} \eta$	1.84 ± 0.71	6.54 ± 1.79
$K^*(1680)^+ \rightarrow K^{*+} \eta'$	...	...	$K^*(1680)^0 \rightarrow K^{*0} \eta'$	...	...
$K^*(1680) \rightarrow K^* \pi$	22.50 ± 5.41	81.74 ± 9.03	$K^*(1680) \rightarrow K^* \eta$	1.85 ± 0.71	6.62 ± 1.81
$K^*(1680) \rightarrow \rho K$	17.61 ± 4.27	64.53 ± 7.87	$K^*(1680) \rightarrow \omega K$	5.73 ± 1.40	21.00 ± 2.58
$K^*(1680) \rightarrow \phi K$	3.67 ± 1.12	13.23 ± 2.43			
$\rho(1700)^0 \rightarrow \rho^0 \eta$	14.99 ± 5.41	44.08 ± 10.83	$\rho(1700)^+ \rightarrow \rho^+ \eta$	14.99 ± 5.41	44.11 ± 10.83
$\rho(1700)^0 \rightarrow \rho^0 \eta'$	0.020 ± 0.020	0.048 ± 0.048	$\rho(1700)^+ \rightarrow \rho^+ \eta'$	0.020 ± 0.020	0.050 ± 0.050
$\rho(1700)^0 \rightarrow \omega \pi^0$	48.50 ± 12.03	141.91 ± 15.14	$\rho(1700)^+ \rightarrow \omega \pi^+$	48.37 ± 12.00	141.56 ± 15.12
$\rho(1700)^0 \rightarrow K^{*+} K^-$	4.77 ± 1.31	14.09 ± 2.05	$\rho(1700)^+ \rightarrow K^{*+} \bar{K}^0$	9.42 ± 2.59	27.78 ± 4.06
$\rho(1700)^0 \rightarrow K^{*-} K^+$	4.77 ± 1.31	14.08 ± 2.05	$\rho(1700)^+ \rightarrow \bar{K}^{*0} K^+$	9.39 ± 2.59	27.69 ± 4.06
$\rho(1700)^0 \rightarrow \bar{K}^{*0} K^0$	4.63 ± 1.28	13.65 ± 2.01	$\rho(1700) \rightarrow \rho \eta$	14.99 ± 5.41	44.09 ± 10.83
$\rho(1700)^0 \rightarrow K^{*0} \bar{K}^0$	4.63 ± 1.28	13.65 ± 2.01	$\rho(1700) \rightarrow \rho \eta'$	0.020 ± 0.020	0.049 ± 0.049
$\rho(1700) \rightarrow \omega \pi$	48.43 ± 12.02	141.74 ± 15.13	$\rho(1700) \rightarrow K^* K$	18.81 ± 5.19	55.48 ± 8.11
$\omega(1650) \rightarrow \omega \eta$	10.13 ± 2.51	34.84 ± 9.12	$\omega(1650) \rightarrow K^{*+} K^-$	3.11 ± 0.50	10.80 ± 1.73
$\omega(1650) \rightarrow \omega \eta'$	...	...	$\omega(1650) \rightarrow K^{*-} K^+$	3.11 ± 0.50	10.80 ± 1.73
$\omega(1650) \rightarrow \rho^0 \pi^0$	36.43 ± 2.98	126.71 ± 10.30	$\omega(1650) \rightarrow \bar{K}^{*0} K^0$	3.00 ± 0.49	10.41 ± 1.72
$\omega(1650) \rightarrow \rho^+ \pi^-$	36.32 ± 2.97	126.33 ± 10.27	$\omega(1650) \rightarrow K^{*0} \bar{K}^0$	3.00 ± 0.49	10.41 ± 1.72
$\omega(1650) \rightarrow \rho^- \pi^+$	36.32 ± 2.97	126.33 ± 10.27	$\omega(1650) \rightarrow K^* K$	12.21 ± 1.97	42.42 ± 6.89
$\omega(1650) \rightarrow \rho \pi$	108.97 ± 8.90	378.99 ± 30.82			

TABLE IX. Branching ratios and decay widths of the $V_E \rightarrow PV$ strong decays by the SU(3) flavor symmetry.

$V_E \rightarrow PV$ decays	Branching ratios (%)	Decay width (MeV)	Decay modes	Branching ratios (%)	Decay width (MeV)
$K^*(1410)^+ \rightarrow \rho^0 K^+$	3.47 ± 2.53	7.58 ± 5.25	$K^*(1410)^0 \rightarrow \rho^- K^+$	6.94 ± 5.06	15.16 ± 10.51
$K^*(1410)^+ \rightarrow \omega K^+$	3.22 ± 2.36	7.03 ± 4.89	$K^*(1410)^0 \rightarrow \rho^0 K^0$	3.35 ± 2.45	7.32 ± 5.08
$K^*(1410)^+ \rightarrow \rho^+ \bar{K}^0$	6.70 ± 4.90	14.64 ± 10.17	$K^*(1410)^0 \rightarrow \omega K^0$	3.10 ± 2.27	6.77 ± 4.73
$K^*(1410)^+ \rightarrow \phi K^+$	...	...	$K^*(1410)^0 \rightarrow \phi K^0$	...	...
$K^*(1410)^+ \rightarrow K^{*+} \pi^0$	8.06 ± 5.61	17.71 ± 11.55	$K^*(1410)^0 \rightarrow K^{*+} \pi^-$	16.00 ± 11.14	35.14 ± 22.93
$K^*(1410)^+ \rightarrow \bar{K}^{*0} \pi^+$	15.69 ± 10.94	34.47 ± 22.51	$K^*(1410)^0 \rightarrow K^{*0} \pi^0$	7.91 ± 5.51	17.37 ± 11.34
$K^*(1410)^+ \rightarrow K^{*+} \eta$	...	...	$K^*(1410)^0 \rightarrow K^{*0} \eta$	...	...
$K^*(1410)^+ \rightarrow K^{*+} \eta'$	...	...	$K^*(1410)^0 \rightarrow K^{*0} \eta'$	...	...
$K^*(1410) \rightarrow K^* \pi$	23.83 ± 16.60	52.35 ± 34.16	$K^*(1410) \rightarrow \omega K$	3.16 ± 2.31	6.90 ± 4.81
$K^*(1410) \rightarrow \rho K$	10.23 ± 7.47	22.35 ± 15.50			
$\rho(1450)^0 \rightarrow \rho^0 \eta$	5.05 ± 4.07	19.56 ± 15.22	$\rho(1450)^+ \rightarrow \rho^+ \eta$	5.07 ± 4.09	19.55 ± 15.21
$\rho(1450)^0 \rightarrow \rho^0 \eta'$	...	...	$\rho(1450)^+ \rightarrow \rho^+ \eta'$	...	...
$\rho(1450)^0 \rightarrow \omega \pi^0$	32.93 ± 22.86	125.28 ± 80.47	$\rho(1450)^+ \rightarrow \omega \pi^+$	32.78 ± 22.77	124.72 ± 80.13
$\rho(1450)^0 \rightarrow K^{*+} K^-$	0.94 ± 0.80	3.43 ± 2.78	$\rho(1450)^+ \rightarrow K^{*+} \bar{K}^0$	1.78 ± 1.52	6.48 ± 5.30
$\rho(1450)^0 \rightarrow K^{*-} K^+$	0.94 ± 0.80	3.43 ± 2.78	$\rho(1450)^+ \rightarrow \bar{K}^{*0} K^+$	1.78 ± 1.52	6.47 ± 5.30
$\rho(1450)^0 \rightarrow \bar{K}^{*0} K^0$	0.84 ± 0.72	3.05 ± 2.52	$\rho(1450) \rightarrow \rho \eta$	5.06 ± 4.08	19.56 ± 15.21
$\rho(1450)^0 \rightarrow K^{*0} \bar{K}^0$	0.84 ± 0.72	3.05 ± 2.52	$\rho(1450) \rightarrow K^* K$	3.55 ± 3.03	12.95 ± 10.60
$\rho(1450) \rightarrow \omega \pi$	32.85 ± 22.82	125.00 ± 80.30			
$\omega(1420) \rightarrow \omega \eta$	2.17 ± 2.06	9.70 ± 9.35	$\omega(1420) \rightarrow K^{*+} K^-$	0.37 ± 0.37	1.65 ± 1.65
$\omega(1420) \rightarrow \rho^0 \eta'$	...	...	$\omega(1420) \rightarrow K^{*-} K^+$	0.37 ± 0.37	1.65 ± 1.65
$\omega(1420) \rightarrow \rho^0 \pi^0$	20.54 ± 12.84	95.67 ± 60.95	$\omega(1420) \rightarrow \bar{K}^{*0} K^0$	0.32 ± 0.32	1.42 ± 1.42
$\omega(1420) \rightarrow \rho^+ \pi^-$	20.42 ± 12.77	95.14 ± 60.57	$\omega(1420) \rightarrow K^{*0} \bar{K}^0$	0.32 ± 0.32	1.42 ± 1.42
$\omega(1420) \rightarrow \rho^- \pi^+$	20.42 ± 12.77	95.14 ± 60.57	$\omega(1420) \rightarrow K^* K$	1.37 ± 1.37	6.14 ± 6.14
$\omega(1420) \rightarrow \rho \pi$	61.27 ± 38.31	285.42 ± 181.72			

From Table VIII, one can see that the errors of some predictions are large, and the errors mainly come from the nonperturbative parameter g_{V_D} and the decay widths such as $\Gamma_{K^*(1680)} = 320 \pm 110$ MeV, $\Gamma_{\rho(1700)} = 250 \pm 100$ MeV, and $\Gamma_{\omega(1650)} = 315 \pm 35$ MeV [39]. Our prediction of $\mathcal{B}(K^*(1680) \rightarrow K^*\pi)$ is consistent with its experimental data within 1σ error, nevertheless, our prediction of $\mathcal{B}(K^*(1680) \rightarrow \rho K)$ is smaller than its experimental data, since we only use the experimental data to obtain $g_{V_D} \in [4.88, 6.66]$, but do not use the data given in Eq. (14) to further constrain the predictions. The reason is that within the 1σ error of the theoretical inputs and the experimental data, it is difficult to satisfy $\mathcal{B}(K^*(1680) \rightarrow \rho K)^{\text{Exp}}$ and $\mathcal{B}(K^*(1680) \rightarrow K^*\pi)^{\text{Exp}}$ at the same time. If we extend the experimental errors to 2σ , the experimental data of $\mathcal{B}(K^*(1680) \rightarrow \rho K)$ and $\mathcal{B}(K^*(1680) \rightarrow K^*\pi)$ can be satisfied at the same time, and the errors of the $\mathcal{B}(K^*(1680) \rightarrow PV)$ predictions are very small, for example:

$$\begin{aligned}\mathcal{B}(K^*(1680) \rightarrow K^*\pi) &= (34.12 \pm 0.18)\%, \\ \mathcal{B}(K^*(1680) \rightarrow \rho K) &= (27.34 \pm 0.14)\%, \\ \mathcal{B}(K^*(1680) \rightarrow \omega K) &= (8.90 \pm 0.05)\%, \\ \mathcal{B}(K^*(1680) \rightarrow K^*\eta) &= (2.85 \pm 0.44)\%, \\ \mathcal{B}(K^*(1680) \rightarrow \phi K) &= (5.89 \pm 0.18)\%.\end{aligned}\quad (15)$$

Nevertheless, other predictions including $\Gamma(K^*(1680) \rightarrow PV)$ still contain large errors. The branching ratio predictions in Table VIII will be used to obtain $\mathcal{B}(B \rightarrow PV\ell^+\nu_\ell)_{[V_D]}$ later.

Strong decays of the excited vector mesons have been studied by using effective relativistic quantum field theoretical model based on flavor symmetry [48], in which the charges of the final state particle are not distinguished. For convenient comparison with them, our decay width predictions are also listed in Table VIII. Our prediction of $\Gamma(K^*(1680) \rightarrow K^*\eta)$ is 1 order of magnitude larger than the one in Ref. [48]. Our predictions of other decay widths of the $V_D \rightarrow PV$ decays are on the same order of magnitude as the ones in Ref. [48] with smaller errors.

As for the $V_E \rightarrow PV$ decays, no modes have been measured at present, and there are only two limits [39], as follows:

$$\begin{aligned}\mathcal{B}(K^*(1410) \rightarrow K^*\pi) &> 40\%, \\ \mathcal{B}(K^*(1410) \rightarrow \rho K) &< 7\%, \quad \text{at the } 95\% \text{ CL},\end{aligned}\quad (16)$$

and they cannot be satisfied at same time within the SU(3) flavor symmetry. Therefore, $g_{V_E} = \frac{18.4 \pm 3.8}{16.5 \pm 3.5} g_{V_D}$ from Ref. [48] with $g_{V_D} \in [4.88, 6.66]$ will be used to obtain the predictions of the $V_E \rightarrow PV$ decays. After further

TABLE X. Excited vector resonance state $V_{E/D}$ contributions to the branching ratios of the $B \rightarrow PV\ell^+\nu_\ell$ decays within 1σ error (in units of 10^{-4}).

Decay modes	$\mathcal{B}_{[V_E]}$ with $\ell = \ell'$	$\mathcal{B}_{[V_D]}$ with $\ell = \ell'$	$\mathcal{B}_{[V_E]}$ with $\ell = \tau$	$\mathcal{B}_{[V_D]}$ with $\ell = \tau$
$B^+ \rightarrow \rho^-\pi^+\ell^+\nu_\ell$ or $B^+ \rightarrow \rho^+\pi^-\ell^+\nu_\ell$ or $B^+ \rightarrow \rho^0\pi^0\ell^+\nu_\ell$	$0.26 \pm 0.18_{[\omega(1420)]}$	$0.22 \pm 0.03_{[\omega(1650)]}$	$0.12 \pm 0.08_{[\omega(1420)]}$	$0.080 \pm 0.014_{[\omega(1650)]}$
$B^+ \rightarrow \omega\eta\ell^+\nu_\ell$	$0.023 \pm 0.022_{[\omega(1420)]}$	$0.057 \pm 0.015_{[\omega(1650)]}$	$0.0098 \pm 0.0092_{[\omega(1420)]}$	$0.021 \pm 0.005_{[\omega(1650)]}$
$B^+ \rightarrow \omega\pi^0\ell^+\nu_\ell$	$0.34 \pm 0.24_{[\rho(1450)]}$	$0.26 \pm 0.09_{[\rho(1700)]}$	$0.15 \pm 0.10_{[\rho(1450)]}$	$0.094 \pm 0.031_{[\rho(1700)]}$
$B^+ \rightarrow \rho^0\eta\ell^+\nu_\ell$	$0.052 \pm 0.042_{[\rho(1450)]}$	$0.081 \pm 0.032_{[\rho(1700)]}$	$0.022 \pm 0.018_{[\rho(1450)]}$	$0.029 \pm 0.011_{[\rho(1700)]}$
$B^+ \rightarrow \rho^0\eta'\ell^+\nu_\ell$	...	$0.00010 \pm 0.00010_{[\rho(1700)]}$	...	$0.000036 \pm 0.000036_{[\rho(1700)]}$
$B^+ \rightarrow K^{*+}K^-\ell^+\nu_\ell$	$0.0097 \pm 0.0081_{[\rho(1450)]}$	$0.025 \pm 0.008_{[\rho(1700)]}$	$0.0041 \pm 0.0034_{[\rho(1450)]}$	$0.0090 \pm 0.0029_{[\rho(1700)]}$
or $B^+ \rightarrow K^{*-}K^+\ell^+\nu_\ell$	$0.0039 \pm 0.0039_{[\omega(1420)]}$	$0.018 \pm 0.003_{[\omega(1650)]}$	$0.0017 \pm 0.0017_{[\omega(1420)]}$	$0.0065 \pm 0.0011_{[\omega(1650)]}$
$B^+ \rightarrow K^{*0}\bar{K}^0\ell^+\nu_\ell$	$0.0086 \pm 0.0074_{[\rho(1450)]}$	$0.024 \pm 0.008_{[\rho(1700)]}$	$0.0036 \pm 0.0031_{[\rho(1450)]}$	$0.0087 \pm 0.0028_{[\rho(1700)]}$
or $B^+ \rightarrow \bar{K}^{*0}K^0\ell^+\nu_\ell$	$0.0034 \pm 0.0034_{[\omega(1420)]}$	$0.017 \pm 0.003_{[\omega(1650)]}$	$0.0014 \pm 0.0014_{[\omega(1420)]}$	$0.0063 \pm 0.0010_{[\omega(1650)]}$
$B^0 \rightarrow \omega\pi^-\ell^+\nu_\ell$	$0.63 \pm 0.44_{[\rho(1450)]}$	$0.48 \pm 0.16_{[\rho(1700)]}$	$0.27 \pm 0.19_{[\rho(1450)]}$	$0.17 \pm 0.06_{[\rho(1700)]}$
$B^0 \rightarrow \rho^-\eta\ell^+\nu_\ell$	$0.096 \pm 0.078_{[\rho(1450)]}$	$0.15 \pm 0.06_{[\rho(1700)]}$	$0.040 \pm 0.033_{[\rho(1450)]}$	$0.054 \pm 0.021_{[\rho(1700)]}$
$B^0 \rightarrow \rho^-\eta'\ell^+\nu_\ell$	...	$0.00020 \pm 0.00020_{[\rho(1700)]}$	...	$0.000070 \pm 0.000070_{[\rho(1700)]}$
$B^0 \rightarrow K^{*0}K^-\ell^+\nu_\ell$	$0.034 \pm 0.029_{[\rho(1450)]}$	$0.091 \pm 0.030_{[\rho(1700)]}$	$0.014 \pm 0.012_{[\rho(1450)]}$	$0.033 \pm 0.011_{[\rho(1700)]}$
$B^0 \rightarrow K^{*-}K^0\ell^+\nu_\ell$	$0.034 \pm 0.029_{[\rho(1450)]}$	$0.092 \pm 0.030_{[\rho(1700)]}$	$0.014 \pm 0.012_{[\rho(1450)]}$	$0.033 \pm 0.011_{[\rho(1700)]}$
$B_s^0 \rightarrow K^{*-}\pi^0\ell^+\nu_\ell$	$0.19 \pm 0.13_{[K^*(1410)]}$	$0.088 \pm 0.027_{[K^*(1680)]}$	$0.090 \pm 0.062_{[K^*(1410)]}$	$0.034 \pm 0.011_{[K^*(1680)]}$
$B_s^0 \rightarrow \bar{K}^{*0}\pi^-\ell^+\nu_\ell$	$0.38 \pm 0.26_{[K^*(1410)]}$	$0.17 \pm 0.05_{[K^*(1680)]}$	$0.18 \pm 0.12_{[K^*(1410)]}$	$0.067 \pm 0.021_{[K^*(1680)]}$
$B_s^0 \rightarrow \rho^0K^-\ell^+\nu_\ell$	$0.082 \pm 0.058_{[K^*(1410)]}$	$0.069 \pm 0.021_{[K^*(1680)]}$	$0.038 \pm 0.027_{[K^*(1410)]}$	$0.026 \pm 0.008_{[K^*(1680)]}$
$B_s^0 \rightarrow \rho^-\bar{K}^0\ell^+\nu_\ell$	$0.16 \pm 0.11_{[K^*(1410)]}$	$0.14 \pm 0.04_{[K^*(1680)]}$	$0.073 \pm 0.052_{[K^*(1410)]}$	$0.052 \pm 0.016_{[K^*(1680)]}$
$B_s^0 \rightarrow \omega K^-\ell^+\nu_\ell$	$0.076 \pm 0.054_{[K^*(1410)]}$	$0.067 \pm 0.020_{[K^*(1680)]}$	$0.035 \pm 0.025_{[K^*(1410)]}$	$0.026 \pm 0.008_{[K^*(1680)]}$
$B_s^0 \rightarrow K^{*-}\eta\ell^+\nu_\ell$	...	$0.022 \pm 0.009_{[K^*(1680)]}$	...	$0.0082 \pm 0.0033_{[K^*(1680)]}$
$B_s^0 \rightarrow \phi K^-\ell^+\nu_\ell$	...	$0.043 \pm 0.014_{[K^*(1680)]}$	...	$0.016 \pm 0.005_{[K^*(1680)]}$

TABLE XI. Form factors of $B \rightarrow V_E$ transitions obtained by the light-front quark models [44], $F(q^2)$ can be similarly obtained as the ones in Ref. [42], and $F^{B \rightarrow \omega(1420)}$ are the same as $F^{B \rightarrow \rho(1450)}$.

Form factors	$F(0)$	b	Form factors	$F(0)$	b
$V^{B \rightarrow \rho(1450)}$	0.33	-1.07	$V^{B_s \rightarrow K^*(1410)}$	0.38	-2.19
$A_0^{B \rightarrow \rho(1450)}$	0.28	-1.25	$A_0^{B_s \rightarrow K^*(1410)}$	0.30	-2.56
$A_1^{B \rightarrow \rho(1450)}$	0.21	5.43	$A_1^{B_s \rightarrow K^*(1410)}$	0.24	3.18
$A_2^{B \rightarrow \rho(1450)}$	0.16	3.63	$A_2^{B_s \rightarrow K^*(1410)}$	0.20	1.02

considering $\mathcal{B}(K^*(1410) \rightarrow PV) \leq 94.7\%$, $\mathcal{B}(\rho(1450) \rightarrow PV) \leq 100\%$, and $\mathcal{B}(\omega(1420) \rightarrow PV) \leq 100\%$, the branching ratio predictions and the decay width predictions of the $V_E \rightarrow PV$ decays are obtained and listed in Table IX. Our predictions of the decay widths are consistent with ones in Ref. [48]; nevertheless, our errors of the predictions are quite large, due to the nonperturbative parameter $g_{V_E} = 5.78 \pm 2.07$ and the decay widths of the excited vector mesons such as $\Gamma_{K^*(1410)} = 232 \pm 21$ MeV, $\Gamma_{\rho(1450)} = 400 \pm 60$ MeV, and $\Gamma_{\omega(1650)} = 290 \pm 190$ MeV.

Now, in terms of the narrow width approximation given in Eq. (10), $\mathcal{B}(B \rightarrow V_{E,D} \ell^+ \nu_\ell)$ given in Table IV, and $\mathcal{B}(V_{E,D} \rightarrow PV)$ given in Tables VIII and IX, the branching ratios of the $B \rightarrow V_{E,D}(V_{E,D} \rightarrow PV) \ell^+ \nu_\ell$ are obtained, and are listed in Table X. Comparing with the contributions of the axial-vector resonance states and the tensor resonance states given in Tables V and VI, one can see that only the $\rho(1700)$ resonance state contributes to the $B^+ \rightarrow \rho^0 \eta' \ell^+ \nu_\ell$ and $B^0 \rightarrow \rho^- \eta' \ell^+ \nu_\ell$ decays, the branching ratios with some excited vector resonance state contributions and ones with the tensor resonance state contributions are on the same order of magnitude; other excited vector resonance state contributions are at least 1 order larger than the tensor ones. In all $B \rightarrow PV \ell^+ \nu_\ell$ decays except $B \rightarrow \rho \eta' \ell^+ \nu_\ell$, the dominant contributions still come from the axial-vector resonance states.

Before we conclude, two comments of the interference effects and the SU(3) flavor breaking effects are given.

Please note that the total branching ratios of the $B \rightarrow PV \ell^+ \nu_\ell$ are not given in this work, since there are the interference terms between different resonant contributions and between nonresonant and resonant contributions, which we cannot determine by the SU(3) flavor symmetry now. As discussed in Refs. [33,49], the interference effects between different partial waves vanish upon angular integration in the branching ratios. Nevertheless, there still are the interference effects between different resonances with the same partial wave. We take $B^+ \rightarrow \rho^- \pi^+ \ell^+ \nu_\ell$ as an example, both $a_1(1260)^0$ and $h_1(1170)^0$ resonances give large contributions to $\mathcal{B}(B^+ \rightarrow \rho^- \pi^+ \ell^+ \nu_\ell)$, hence their interference terms might be large. Likewise, the

interference effects might also be important for the $B^+ \rightarrow \phi \eta \ell^+ \nu_\ell$, $B \rightarrow K^* \bar{K} \ell^+ \nu_\ell$, $B_s \rightarrow K^* \pi \ell^+ \nu_\ell$, and $B_s^0 \rightarrow K^{*-} \eta \ell^+ \nu_\ell$ decays, in which the two or more kinds of contributions are important.

Although the interference effects have not been studied in this work, the total branching ratios of the $B \rightarrow PV \ell^+ \nu_\ell$ decays can be estimated based our results. $\mathcal{B}(B^+ \rightarrow \rho^\mp \pi^\pm \ell^+ \nu_\ell / \rho^0 \pi^0 \ell^+ \nu_\ell / \omega \pi^0 \ell^+ \nu_\ell / K^{*\mp} K^\pm \ell^+ \nu_\ell)$, $\mathcal{B}(B^0 \rightarrow \omega \pi^- \ell^+ \nu_\ell)$, and $\mathcal{B}(B_s^0 \rightarrow K^{*-} \pi^0 \ell^+ \nu_\ell / \bar{K}^{*0} \pi^- \ell^+ \nu_\ell / \rho^- \bar{K}^0 \ell^+ \nu_\ell)$ might reach the order of 10^{-3} . $\mathcal{B}(B^+ \rightarrow \rho^0 \pi^0 \tau^+ \nu_\tau / \omega \eta \ell^+ \nu_\ell / K^{*\mp} K^\pm \tau^+ \nu_\tau / K^{*0} \bar{K}^0 \ell^+ \nu_\ell / \bar{K}^{*0} K^0 \ell^+ \nu_\ell)$, $\mathcal{B}(B^0 \rightarrow \rho^0 \pi^- \ell^+ \nu_\ell / \rho^- \pi^0 \ell^+ \nu_\ell / \omega \pi^- \tau^+ \nu_\tau / \rho^- \eta \ell^+ \nu_\ell / K^{*-} K^- \ell^+ \nu_\ell / K^{*0} K^0 \ell^+ \nu_\ell)$, and $\mathcal{B}(B_s^0 \rightarrow \rho^0 K^- \ell^+ \nu_\ell / \omega K^- \ell^+ \nu_\ell / K^{*-} \pi^0 \tau^+ \nu_\tau / \rho^- \bar{K}^0 \tau^+ \nu_\tau)$ might reach the order of 10^{-4} . $\mathcal{B}(B^+ \rightarrow \rho^0 \eta \ell^+ \nu_\ell / \omega \eta \tau^+ \nu_\tau)$, $\mathcal{B}(B^0 \rightarrow \rho^- \eta \tau^+ \nu_\tau / K^{*0} K^- \tau^+ \nu_\tau / K^{*-} K^0 \tau^+ \nu_\tau)$, and $\mathcal{B}(B_s^0 \rightarrow K^{*-} \eta \ell^+ \nu_\ell / \phi K^- \ell^+ \nu_\ell)$ might reach the order of 10^{-5} . $\mathcal{B}(B^+ \rightarrow \phi \eta \ell^+ \nu_\ell)$ and $\mathcal{B}(B_s^0 \rightarrow \phi K^- \tau^+ \nu_\tau)$ might reach the order of 10^{-6} .

In addition, the SU(3) flavor breaking corrections due to a nonzero strange quark mass are not considered in this work. There are very little experimental data in the $B \rightarrow A/T/V_{E,D} \ell^+ \nu_\ell$ and $B \rightarrow PV \ell^+ \nu_\ell$ decays, therefore, we cannot judge how large are the possible SU(3) breaking effects. The SU(3) breaking effects are usually of the order of $\sim 20\%$ in some amplitudes, which would conservatively expect an error of 40% in some branching ratios [37,38,50–53].

IV. SUMMARY

The four-body semileptonic $B \rightarrow PV \ell^+ \nu_\ell$ decays are more difficult to calculate than the three-body semileptonic $B \rightarrow M \ell^+ \nu_\ell$ decays. SU(3) flavor symmetry can provide an opportunity to relate different decay modes and can give some valuable information about the decays. The charmless $B \rightarrow PV \ell^+ \nu_\ell$ decays with the axial-vector resonance states, the tensor resonance states, and the excited vector resonance states have been studied by the SU(3) flavor analysis in this work. Our main results can be summarized as follows.

The hadronic amplitudes of the $B \rightarrow A/T/V_{E,D} \ell^+ \nu_\ell$ decays have been related by the nonperturbative parameters, and the branching ratio predictions of $B \rightarrow A/T/V_{E,D} \ell^+ \nu_\ell$ have been obtained. Most branching ratios of the $B \rightarrow A \ell^+ \nu_\ell$ decays are on the order of $\mathcal{O}(10^{-4} - 10^{-3})$, all branching ratios of the $B \rightarrow V_{E,D} \ell^+ \nu_\ell$ decays are on the order of $\mathcal{O}(10^{-5} - 10^{-4})$, most branching ratios of the $B \rightarrow T \ell^+ \nu_\ell$ decays are on the order of $\mathcal{O}(10^{-5})$, and they are reached in current experiments.

In terms of the predictions of the $B \rightarrow A/T/V_{E,D} \ell^+ \nu_\ell$ and $V_{E,D} \rightarrow PV$ decays in this work and the $A/T \rightarrow PV$ decays in Ref. [30], the branching ratios of the $B \rightarrow A(A \rightarrow PV) \ell^+ \nu_\ell$, $B \rightarrow T(T \rightarrow PV) \ell^+ \nu_\ell$, and $B \rightarrow V_{E,D}(V_{E,D} \rightarrow PV) \ell^+ \nu_\ell$ decays have been obtained

by the narrow width approximation or further considering the width effects. We have found that $B^+ \rightarrow \phi \eta \ell^+ \nu_\ell$ only receive the contributions of the axial-vector meson resonances; $B^+ \rightarrow \rho^0 \eta' \ell^+ \nu_\ell$ and $B^0 \rightarrow \rho^- \eta' \ell^+ \nu_\ell$ only receive the contributions of the excited vector resonances; $B^+ \rightarrow \rho^0 \pi^0 \ell^+ \nu_\ell$, $B^+ \rightarrow \omega \eta \ell^+ \nu_\ell$, $B^+ \rightarrow \omega \pi^0 \ell^+ \nu_\ell$, $B^+ \rightarrow \rho^0 \eta \ell^+ \nu_\ell$, $B^0 \rightarrow \omega \pi^- \ell^+ \nu_\ell$, and $B^0 \rightarrow \rho^- \eta \ell^+ \nu_\ell$ receive the contributions of both axial-vector and excited vector meson resonances; and other decays receive the contributions of all three kinds of resonance contributions. Some predicted branching ratios with the axial-vector meson resonances are on the order of $\mathcal{O}(10^{-4}-10^{-3})$, and other branching ratios with the axial-vector meson resonances and all branching ratios with the tensor and excited vector meson resonances are small. After considering the resonant width effects of the axial-vector and tensor mesons, some branching ratios are obviously smaller than ones obtained by the narrow width approximation.

SU(3) flavor symmetry is a powerful tool in analyzing decays. The branching ratios of the $B \rightarrow A/T/V_{E,D} \ell^+ \nu_\ell$ and $B \rightarrow A/T/V_{E,D} (A/T/V_{E,D} \rightarrow PV) \ell^+ \nu_\ell$ decays have been calculated with the SU(3) flavor symmetry. According to our rough predictions, some decays could be tested in the future experiments such as Belle II and LHCb. They are useful for testing and understanding the nonperturbative effects in the semileptonic decays.

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX A: HADRONIC AMPLITUDES AND FORM FACTS

The hadronic amplitudes can be written as [54–56]

$$H_\pm^A = (m_{B_q} - m_A) V_1^A(q^2) \mp \frac{2m_{B_q} |\vec{p}_A|}{(m_{B_q} - m_A)} A(q^2), \quad (\text{A1})$$

$$H_0^A = \frac{1}{2m_A \sqrt{q^2}} \left[(m_{B_q}^2 - m_A^2 - q^2)(m_{B_q} - m_A) V_1^A(q^2) - \frac{4m_{B_q}^2 |\vec{p}_A|^2}{m_{B_q} - m_A} V_2^A(q^2) \right], \quad (\text{A2})$$

$$H_t^A = \frac{2m_{B_q} |\vec{p}_A|}{\sqrt{q^2}} V_0^A(q^2), \quad (\text{A3})$$

for the $B \rightarrow A \ell^+ \nu_\ell$ decays,

$$H_\pm^T = \frac{2|\vec{p}_T|}{2\sqrt{2}m_T} \left[(m_{B_q} + m_T) A_1^T(q^2) \mp \frac{2m_{B_q} |\vec{p}_T|}{(m_{B_q} + m_T)} V^T(q^2) \right], \quad (\text{A4})$$

$$H_0^T = \frac{|\vec{p}_T|}{\sqrt{2}m_T} \frac{1}{2m_T \sqrt{q^2}} \left[(m_{B_q}^2 - m_T^2 - q^2)(m_{B_q} + m_T) A_1^T(q^2) - \frac{4m_{B_q}^2 |\vec{p}_T|^2}{m_{B_q} + m_T} A_2^T(q^2) \right], \quad (\text{A5})$$

$$H_t^T = \frac{|\vec{p}_T|}{\sqrt{2}m_T} \frac{2m_{B_q} |\vec{p}_T|}{\sqrt{q^2}} A_0^T(q^2), \quad (\text{A6})$$

for the $B \rightarrow T \ell^+ \nu_\ell$ decays,

$$H_\pm^{V'} = (m_{B_q} + m_{V'}) A_1^{V'}(q^2) \mp \frac{2m_{B_q} |\vec{p}_{V'}|}{(m_{B_q} + m_{V'})} V(q^2), \quad (\text{A7})$$

$$H_0^{V'} = \frac{1}{2m_{V'} \sqrt{q^2}} \left[(m_{B_q}^2 - m_{V'}^2 - q^2)(m_{B_q} + m_{V'}) A_1^{V'}(q^2) - \frac{4m_{B_q}^2 |\vec{p}_{V'}|^2}{m_{B_q} + m_{V'}} A_2^{V'}(q^2) \right], \quad (\text{A8})$$

$$H_t^{V'} = \frac{2m_{B_q} |\vec{p}_{V'}|}{\sqrt{q^2}} A_0^{V'}(q^2), \quad (\text{A9})$$

for $B \rightarrow V' \ell^+ \nu_\ell$ decays with $V' = V_E$ or V_D , where $|\vec{p}_M| \equiv \sqrt{\lambda}/2m_{B_q}$. The form factors $A(q^2)$ and $V_{0,1,2,3}^A(q^2)$ ($V^T(q^2)$ and $A_{0,1,2}^T(q^2)$) are given by [42,57,58]

$$\begin{aligned} \langle A(p, \epsilon^*) | \bar{u}_k \gamma_\mu (1 - \gamma_5) \bar{b} | B(p_B) \rangle &= \frac{2iA(q^2)}{m_B - m_A} \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\nu} p_B^\alpha p^\beta \\ &\quad - \left[\epsilon_\mu^* (m_B - m_A) V_1^A(q^2) - (p_B + p)_\mu (\epsilon^* \cdot p_B) \frac{V_2^A(q^2)}{m_B - m_A} \right] \\ &\quad + q_\mu (\epsilon^* \cdot p_B) \frac{2m_A}{q^2} [V_3^A(q^2) - V_0^A(q^2)], \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} \langle T(p, \epsilon^*) | \bar{u}_k \gamma_\mu (1 - \gamma_5) b | B(p_B) \rangle &= \frac{2iV^T(q^2)}{m_B + m_T} \epsilon_{\mu\nu\alpha\beta} e^{*\nu} p_B^\alpha p^\beta \\ &\quad + 2m_T \frac{e^* \cdot q}{q^2} q_\mu A_0^T(q^2) + (m_B + m_T) \left(e_\mu^* - \frac{e^* \cdot q}{q^2} q_\mu \right) A_1^T(q^2) \\ &\quad - \frac{e^* \cdot q}{m_B + m_T} \left((p_B + p)_\mu - \frac{m_B^2 - m_T^2}{q^2} q_\mu \right) A_2^T(q^2), \end{aligned} \quad (\text{A11})$$

$$\begin{aligned} \langle V'(p, \epsilon^*) | \bar{u}_k \gamma_\mu (1 - \gamma_5) b | B(p_B) \rangle &= \frac{2V^{V'}(q^2)}{m_B + m_{V'}} \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\nu} p_B^\alpha p^\beta \\ &\quad - i \left[\epsilon_\mu^* (m_B + m_{V'}) A_1^{V'}(q^2) - (p_B + p)_\mu (\epsilon^* \cdot p_B) \frac{A_2^{V'}(q^2)}{m_B + m_{V'}} \right] \\ &\quad + i q_\mu (\epsilon^* \cdot p_B) \frac{2m_{V'}}{q^2} [A_3^{V'}(q^2) - A_0^{V'}(q^2)], \end{aligned} \quad (\text{A12})$$

with $e^{*\nu} \equiv \frac{\epsilon^{*\mu\nu} p_{B\mu}}{m_B}$.

APPENDIX B: MESON MULTIPLY

Bottom pseudoscalar triplet B_i is composed of an anti-b quark and a light u/d/s quark, as follows:

$$B_i = (B^+(\bar{b}u), B^0(\bar{b}d), B_s^0(\bar{b}s)). \quad (\text{B1})$$

The octet and the singlet components of axial-vector mesons are presented in the matrix of A and B [59], as follows:

$$A = \begin{pmatrix} \frac{a_1^0}{\sqrt{2}} + \frac{f_1}{\sqrt{3}} + \frac{f_8}{\sqrt{6}} & a_1^+ & K_{1A}^+ \\ a_1^- & -\frac{a_1^0}{\sqrt{2}} + \frac{f_1}{\sqrt{3}} + \frac{f_8}{\sqrt{6}} & K_{1A}^0 \\ K_{1A}^- & \bar{K}_{1A}^0 & \frac{f_1}{\sqrt{3}} - \frac{2f_8}{\sqrt{6}} \end{pmatrix}, \quad (\text{B2})$$

$$B = \begin{pmatrix} \frac{b_1^0}{\sqrt{2}} + \frac{h_1}{\sqrt{3}} + \frac{h_8}{\sqrt{6}} & b_1^+ & K_{1B}^+ \\ b_1^- & -\frac{b_1^0}{\sqrt{2}} + \frac{h_1}{\sqrt{3}} + \frac{h_8}{\sqrt{6}} & K_{1B}^0 \\ K_{1B}^- & \bar{K}_{1B}^0 & \frac{h_1}{\sqrt{3}} - \frac{2h_8}{\sqrt{6}} \end{pmatrix}, \quad (\text{B3})$$

where A denotes the axial-vector mesons with the spin-parity quantum number of $J^{PC} = 1^{++}$, and B corresponds to the mesons with $J^{PC} = 1^{+-}$.

Mesons $K_1(1270)$ and $K_1(1400)$ can be regarded as the mixtures of K_{1A} and K_{1B} with the mixing angle θ_{K_1} , as follows:

$$\begin{pmatrix} K_1(1270) \\ K_1(1400) \end{pmatrix} = \begin{pmatrix} \sin \theta_{K_1} & \cos \theta_{K_1} \\ \cos \theta_{K_1} & -\sin \theta_{K_1} \end{pmatrix} \begin{pmatrix} K_{1A} \\ K_{1B} \end{pmatrix}. \quad (\text{B4})$$

The value range of θ_{K_1} has been investigated in our previous work; $\theta_{K_1} \in [52^\circ, 65^\circ]$ within 1σ error [30] will be used in this work.

Mesons $f_1(1285)$, $f_1(1420)$ can be expressed by the mixing angle θ_{3P_1} while $h_1(1170)$, $h_1(1415)$ are mixed by θ_{1P_1} , as follows:

$$\begin{pmatrix} f_1(1285) \\ f_1(1420) \end{pmatrix} = \begin{pmatrix} \cos \theta_{3P_1} & \sin \theta_{3P_1} \\ -\sin \theta_{3P_1} & \cos \theta_{3P_1} \end{pmatrix} \begin{pmatrix} f_1 \\ f_8 \end{pmatrix}, \quad (\text{B5})$$

$$\begin{pmatrix} h_1(1170) \\ h_1(1415) \end{pmatrix} = \begin{pmatrix} \cos \theta_{1P_1} & \sin \theta_{1P_1} \\ -\sin \theta_{1P_1} & \cos \theta_{1P_1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_8 \end{pmatrix}, \quad (\text{B6})$$

where $\theta_{1P_1} \in [5^\circ, 56^\circ]$, $\theta_{3P_1} \in [56^\circ, 125^\circ]$ within 1σ error [30] will be used in our analysis. There are different definitions of the mixing angles for f_1 and h_1 [41], as follows:

$$A_L = \sin \alpha_A A_q + \cos \alpha_A A_s, \quad (\text{B7})$$

$$A_H = \cos \alpha_A A_q - \sin \alpha_A A_s, \quad (\text{B8})$$

where A_L (A_H) is the light (heavier) axial vector, A_q and A_s denotes the corresponding components $(u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ in the wave functions. $\alpha_{f_1} = (69.7 \pm 8)^\circ$ and $\alpha_{h_1} = (86.7 \pm 6)^\circ$ are also analyzed in this work [41].

Light pseudoscalar mesons and vector mesons are represented as [60]

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} + \frac{\eta_1}{\sqrt{3}} \end{pmatrix}, \quad (\text{B9})$$

$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}. \quad (\text{B10})$$

The excited vector mesons V_E and V_D are similar to V given in Eq. (B10). The pseudoscalar mesons η and η' are the mixtures of $\eta_1 = \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}$ and $\eta_8 = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}$ with a mixing angle denoted as θ_P [61], as follows:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos \theta_P & -\sin \theta_P \\ \sin \theta_P & \cos \theta_P \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix}. \quad (\text{B11})$$

In our numerical analysis, we will utilize the mixing angle range of $\theta_P = [-20^\circ, -10^\circ]$ from the PDG [39].

The exchange of the lowest multiplet of p-wave tensor mesons with $J^{PC} = 2^{++}$ takes the form [62,63]

$$T = \begin{pmatrix} \frac{a_2^0}{\sqrt{2}} + \frac{f_2^q}{\sqrt{2}} & a_2^+ & K_2^{*+} \\ a_2^- & -\frac{a_2^0}{\sqrt{2}} + \frac{f_2^q}{\sqrt{2}} & K_2^{*0} \\ K_2^{*-} & \bar{K}_2^{*0} & f_2^s \end{pmatrix}. \quad (\text{B12})$$

In this paper, the isovector mesons $a_2(1320)$, isodoublet states $K_2^*(1430)$ and two isosinglet mesons $f_2(1270)$ and $f_2'(1525)$ are investigated; $f_2(1270)$ and $f_2'(1525)$ are mixed by f_2^q and f_2^s with the mixing angle θ_{f_2} , as follows:

$$\begin{pmatrix} f_2(1270) \\ f_2'(1525) \end{pmatrix} = \begin{pmatrix} \cos \theta_{f_2} & \sin \theta_{f_2} \\ \sin \theta_{f_2} & -\cos \theta_{f_2} \end{pmatrix} \begin{pmatrix} f_2^q \\ f_2^s \end{pmatrix}, \quad (\text{B13})$$

where $\theta_{f_2} \in [8^\circ, 10^\circ]$ [64] will be used in our calculation.

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