

Second-order chiral kinetic theory from the Wigner function approach

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We present a systematic method to derive the chiral kinetic theory from the Wigner equations based on quantum field theory order by order. We take special effort to derive the chiral kinetic theory in seven-dimensional phase space from the eight-dimensional chiral kinetic theory. We give the seven-dimensional chiral kinetic theory up to second order of semiclassical expansion. We find some new second-order contributions compared with other approaches.

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I. INTRODUCTION

In relativistic heavy-ion collisions, it is possible to generate strong magnetic field [1–3] and orbital angular momentum or vorticity [4–11], which can lead to novel chiral effects such as the chiral magnetic effects (CME) [12–14], chiral vorticity effects (CVE) [15–18], and chiral separate effects (CSE) [19,20] and so on. The research on these various chiral effects in relativistic heavy-ion collisions has greatly stimulated the rapid progress on the chiral kinetic theory (CKT) [21–34]. The recent reviews on the chiral transport can be found in Refs. [35–38]. The CKT provides a consistent theoretical framework to merge the chiral anomaly into the kinetic theory by the Berry monopole in momentum space and can be applicable not only in relativistic heavy-ion collisions in high energy nuclear physics but also in condensed matter physics [39–41], astrophysics, and cosmology [42,43].

The CME, CVE, and CSE are all first-order chiral effects in terms of the expansion of spacetime gradient and electromagnetic field, which is equivalent to semiclassical $\hbar$ expansion for Abelian gauge field. The well-established chiral kinetic equation (CKE) is also valid up to first order. There have been already some effort to go beyond first order, studying the second-order chiral effects [44–51] and deriving the second-order chiral kinetic theory [52–54]. In Ref. [52], second-order chiral kinetic equation in seven-dimensional phase space was derived from quantum mechanics by the Hamiltonian for a single chiral fermion, including only second order in electromagnetic field. In Refs. [53,54], second-order chiral kinetic equation in eight-dimensional phase space was derived from quantum field theory by the Wigner function for chiral fermions, including second order in both electromagnetic field and spacetime gradient. In this paper, we will take special effort to derive second-order chiral kinetic equation in seven-dimensional phase space from quantum field theory within Wigner function framework [55–58]. Actually, it is not a trivial task at all to integrate the eight-dimensional CKE to seven-dimensional CKE in Wigner function approach. For brevity, we will restrict ourselves to uniform electromagnetic field.

In Sec. II, we will give a brief review of the Wigner function formalism for chiral system to keep our present work self-contained. In Sec. III, we derive the eight-dimensional chiral kinetic equation up to second order

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of semiclassical $\hbar$ expansion and integrate out timelike component of momentum and obtain the seven-dimensional CKE. Finally, we summarize our results in Sec. IV.

We use the metric convention $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and Levi-Civita tensor convention $\epsilon^{0123} = 1$. We use the natural unit with $\hbar = c = 1$ unless otherwise specified.

II. WIGNER FUNCTION FORMALISM

The Wigner function $W(x, p)$ for Dirac field is 4×4 matrix in spinor space and the matrix element is defined as the ensemble expectation value of the Wigner operator [55–57],

$$W_{\alpha\beta}(x, p) = \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \left\langle \tilde{\psi}_\beta \left(x + \frac{y}{2} \right) U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \times \psi_\alpha \left(x - \frac{y}{2} \right) \right\rangle, \quad (2.1)$$

where U denotes the gauge link along the straight line between $x - y/2$ and $x + y/2$,

$$U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \equiv \text{Exp} \left(-i \int_{x-y/2}^{x+y/2} dz^\mu A_\mu(z) \right). \quad (2.2)$$

We will restrict ourselves to the chiral and collisionless fermionic system under a constant external electromagnetic field $F_{\mu\nu}$ in space and time, i.e. $\partial_x^\lambda F^{\mu\nu} = 0$, hence we have removed the path ordering of the gauge link. It should be noted that the charge carried by fermion has been absorbed into the electromagnetic vector potential A_μ .

The Wigner equation for chiral particle under constant electromagnetic field tensor is given by [57],

$$\gamma_\mu \left(p^\mu + \frac{i}{2} \nabla^\mu \right) W(x, p) = 0, \quad (2.3)$$

where γ^μ 's are Dirac matrices and $\nabla^\mu \equiv \partial_x^\mu - F^\mu{}_\nu \partial_p^\nu$. The subscript or superscript x and p of the derivative denote the gradient for the coordinate and momentum, respectively. We can decompose the Wigner function in terms of 16 independent generators of the Clifford algebra,

$$W = \frac{1}{4} \left[\mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right], \quad (2.4)$$

where we have suppressed all the arguments of the Wigner function for brevity. In the subsequent sections, we will suppress or keep the arguments of the function alternately for convenience. At chiral limit, it is more convenient to define the chiral basis

$$\mathcal{J}_s^\mu \equiv \frac{1}{2} (\mathcal{V}^\mu + s \mathcal{A}^\mu), \quad (2.5)$$

with $s = +1$ or -1 corresponding to right-handed or left-handed component, respectively. Substituting Eq. (2.4) together with Eq. (2.5) into Eq. (2.3), we find that the right-handed or left-handed component will totally decouple from all the other components,

$$\nabla_\mu \mathcal{J}_s^\mu = 0, \quad (2.6)$$

$$p_\mu \mathcal{J}_s^\mu = 0, \quad (2.7)$$

$$p^\mu \mathcal{J}_s^\nu - p^\nu \mathcal{J}_s^\mu = -\frac{s}{2} \hbar \epsilon^{\mu\nu\rho\sigma} \nabla_\rho \mathcal{J}_{s\sigma}, \quad (2.8)$$

where we have recovered the dependence on Planck constant $\hbar$. To simplify the Wigner equations, we can resort to semiclassical expansion in $\hbar$

$$\mathcal{J}_s^\mu = \mathcal{J}_s^{(0)\mu} + \hbar \mathcal{J}_s^{(1)\mu} + \hbar^2 \mathcal{J}_s^{(2)\mu} + \dots \quad (2.9)$$

In the following, we will suppress the subscript “ s ” in the expressions for brevity. Substituting these expansions into the Wigner equations and requiring the equations hold order by order gives rise to zeroth-order equation

$$p_\mu \mathcal{J}^{(0)\mu} = 0, \quad (2.10)$$

$$\nabla_\mu \mathcal{J}^{(0)\mu} = 0, \quad (2.11)$$

$$p^\mu \mathcal{J}^{(0)\nu} - p^\nu \mathcal{J}^{(0)\mu} = 0, \quad (2.12)$$

and the k th order equations with $n \geq 1$

$$p_\mu \mathcal{J}^{(k)\mu} = 0, \quad (2.13)$$

$$\nabla_\mu \mathcal{J}^{(k)\mu} = 0, \quad (2.14)$$

$$p^\mu \mathcal{J}^{(k)\nu} - p^\nu \mathcal{J}^{(k)\mu} = -\frac{s}{2} \epsilon^{\mu\nu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(k-1)}. \quad (2.15)$$

We note that all the components of chiral Wigner function $\mathcal{J}^\mu$ are coupled with each other. In order to disentangle the Wigner equations and find the independent components or equations, we will apply the method proposed in [32] by introducing a timelike constant 4-vector n^μ with normalization $n^2 = 1$. Then we can decompose any 4-vector X^μ as

$$X^\mu = X_n n^\mu + \bar{X}^\mu, \quad (2.16)$$

where $X_n = X \cdot n$ and $\bar{X}^\mu = \Delta^{\mu\nu} X_\nu$ with $\Delta^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$. We can also decompose the antisymmetric tensors $F^{\mu\nu}$ as

$$F^{\mu\nu} = E^\mu n^\nu - E^\nu n^\mu + \epsilon^{\mu\nu\rho\sigma} n_\rho B_\sigma. \quad (2.17)$$

It is easy to verify the following relations

$$E^\mu = F^{\mu\nu} n_\nu, \quad B^\mu = \epsilon^{\mu\nu\rho\sigma} n_\nu F_{\rho\sigma}/2. \quad (2.18)$$

With such decomposition, the derivative operator ∇_μ can be expressed as

$$\nabla^\mu = n^\mu \nabla_n + \bar{\nabla}^\mu, \quad (2.19)$$

$$\nabla_n = n^\mu \nabla_\mu = \partial_n^x + E^\mu \partial_\mu^p, \quad (2.20)$$

$$\bar{\nabla}^\mu = \bar{\partial}_\mu^x - E^\mu \partial_n^p - \bar{\epsilon}^{\mu\nu\rho} B_\rho \partial_\nu^p, \quad (2.21)$$

where we have defined the spacelike antisymmetric Levi-Civita tensor by

$$\bar{\epsilon}^{\mu\rho\nu} = \epsilon^{\mu\sigma\rho\nu} n_\sigma. \quad (2.22)$$

III. THE CKT DERIVED FROM WIGNER FUNCTION

In this section, we display a systematic method to formalize the CKT by disentangling the Wigner equations order by order. In particular, we show how to organize the chiral Wigner function and CKT in seven-dimensional phase space into a unified form. We will present the CKT in seven-dimensional phase space up to second order.

A. Zeroth-order CKT

We start from the zeroth-order Wigner equations. With the timelike vector n^μ , we can decompose zeroth order equation (2.12) as

$$\bar{p}^\mu \mathcal{J}_n^{(0)} - p_n \bar{\mathcal{J}}^{(0)\mu} = 0, \quad (3.1)$$

$$\bar{p}^\mu \bar{\mathcal{J}}^{(0)\nu} - \bar{p}^\nu \bar{\mathcal{J}}^{(0)\mu} = 0. \quad (3.2)$$

Then the first equation (3.1) gives the relation

$$\bar{\mathcal{J}}^{(0)\mu} = \frac{1}{p_n} \bar{p}^\mu \mathcal{J}_n^{(0)}. \quad (3.3)$$

which implies that only the timelike component $\mathcal{J}_n^{(0)}$ is independent. The Wigner equation automatically expresses other components in terms of $\mathcal{J}_n^{(0)}$. With this relation above, the second equation (3.2) holds automatically. Substituting Eq. (3.3) into Eq. (2.10) gives rise to

$$p^2 \frac{\mathcal{J}_n^{(0)}}{p_n} = 0, \quad (3.4)$$

which implies the on-shell condition with the general solution given by

$$\mathcal{J}_n^{(0)} = p_n J_n^{(0)} \delta(p^2), \quad (3.5)$$

where the distribution function $J_n^{(0)}$ is regular at $p^2 = 0$. It follows that the spacelike component of the chiral Wigner function is given by

$$\bar{\mathcal{J}}^{(0)\mu} = \bar{p}^\mu J_n^{(0)} \delta(p^2). \quad (3.6)$$

During deriving the chiral kinetic equations in seven-dimensional phase space or calculating some physical quantities such as charge currents and energy-momentum tensor, we always encounter some integrations over the Wigner function with a specific integral weight which is a function of p_n . Therefore, we will integrate the Wigner function over p_n with a general weight function $g(p_n)$

$$\int dp_n g(p_n) \mathcal{J}_n^{(0)} = \frac{1}{2} \sum_\lambda g(\lambda |\bar{p}|) \lambda J_n^{(0)}(\lambda), \quad (3.7)$$

$$\int dp_n g(p_n) \bar{\mathcal{J}}^{(0)\mu} = \frac{1}{2} \sum_\lambda g(\lambda |\bar{p}|) \hat{p}^\mu J_n^{(0)}(\lambda), \quad (3.8)$$

where the index λ denotes positive ($\lambda = +1$) or negative ($\lambda = -1$) energy contribution and $J_n^{(0)}(\lambda)$ and $\hat{p}^\mu$ are defined as

$$J_n^{(0)}(\lambda) \equiv J_n^{(0)}|_{p_n=\lambda|\bar{p}|}, \quad \hat{p}^\mu \equiv \frac{\bar{p}^\mu}{|\bar{p}|}. \quad (3.9)$$

With the general expression, we can have specific results

$$\int dp_n \mathcal{J}_n^{(0)\mu} = \frac{1}{2} \sum_\lambda \sqrt{\omega^{(0)}} \dot{x}^{(0)\mu} J_n^{(0)}, \quad (3.10)$$

$$\int dp_n p_n \mathcal{J}_n^{(0)} = \frac{1}{2} \sum_\lambda \sqrt{\omega^{(0)}} \epsilon_n^{(0)} J_n^{(0)}. \quad (3.11)$$

In the expressions above, we have defined some useful variables or functions

$$\sqrt{\omega^{(0)}} = 1, \quad \dot{x}^{(0)\mu} = \lambda n^\mu + \hat{p}^\mu, \quad \epsilon_n^{(0)} = |\bar{p}|, \quad (3.12)$$

where $\sqrt{\omega^{(0)}}$ represents the factor associated with the invariant phase space [23] which is unity at zeroth order, $\dot{x}^{(0)\mu}$ the four-velocity for particles at zeroth order which is just four-velocity for free chiral fermions, and $\epsilon_n^{(0)}$ the energy for particles at zeroth order which is just onshell energy for free chiral fermions. For brevity, we have suppressed above and will suppress below all the arguments associated with λ in $J_n^{(0)}$ or $\dot{x}^{(0)\mu}$.

The chiral kinetic equation at zeroth order could be obtained directly by integrating the zeroth-order Wigner equations (2.11) over the momentum p_n . The integral interval from 0 to $+\infty$ gives the positive-energy contribution which is associated with the particle's distribution

while the integral interval from $-\infty$ to 0 gives the negative-energy contribution which is associated with the antiparticle's distribution. However, it is valuable to integrate Eq. (2.11) together with the arbitrary weight function $g(p_n)$, i.e.

$$\int dp_n g(p_n) \nabla_\mu \mathcal{J}^{(0)\mu} = 0. \quad (3.13)$$

The terms including the derivative with respect to p_n in the operator (2.21) can be integrated with the integral by parts while the derivatives with respect to $\bar{p}_\mu$ can be directly pulled out of the integral

$$E_\mu \int dp_n g'(p_n) \mathcal{J}^{(0)\mu} + \nabla_\mu \int dp_n g(p_n) \mathcal{J}^{(0)\mu} = 0. \quad (3.14)$$

It should be pointed out that we have used the same symbol ∇_μ to represent the corresponding operator where the derivative with respect to p_n has been removed after we integrate over p_n , i.e., in the expression above

$$\nabla^\mu = \partial_\mu^x + E^\mu \bar{\partial}_\mu^p - \bar{\epsilon}^{\mu\rho\nu} B_\rho \bar{\partial}_\nu^p. \quad (3.15)$$

We expect the readers could easily distinguish them from the context. It is remarkable that the final result after integration is given by

$$\sum_\lambda g(\lambda|\bar{p}) \nabla_\mu [\sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} J_n^{(0)}] = 0. \quad (3.16)$$

where the terms including the derivative of $g(\lambda|\bar{p})$ have been canceled. Since this equation holds for the arbitrary function $g(\lambda|\bar{p})$, it is only possible that the equation

$$\dot{x}^{(0)\mu} \nabla_\mu J_n^{(0)} = 0, \quad (3.17)$$

must be satisfied for positive and negative energy contributions separately. According to the correspondence between the positive/negative energy solution and particle/antiparticle or direct calculation from the Wigner function definition (2.1) specific to a free particle, the CKE of the particles with momentum $\bar{p}^\mu$ and helicity s can be obtained by setting $\lambda = +1$, while the CKE of the antiparticles with momentum $\bar{p}^\mu$ and helicity s can be obtained by setting $\lambda = -1$ along with the replacement $\bar{p}^\mu \rightarrow -\bar{p}^\mu$ and $s \rightarrow -s$. Such replacement can be made for the Eqs. (3.7), (3.8), (3.10), and (3.11).

B. First-order CKT

The Wigner tensor equations (2.15) with $k \geq 1$ can be decomposed into two groups of equations according to timelike or spacelike components

$$\bar{p}^\mu \mathcal{J}_n^{(k)} - p_n \tilde{\mathcal{J}}^{(k)\mu} = -\frac{s}{2} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(k-1)}, \quad (3.18)$$

$$\bar{p}^\mu \tilde{\mathcal{J}}^{(k)\nu} - \bar{p}^\nu \tilde{\mathcal{J}}^{(k)\mu} = \frac{s}{2} \bar{\epsilon}^{\mu\nu\rho} (\nabla_n \tilde{\mathcal{J}}_\rho^{(k-1)} - \nabla_\rho \mathcal{J}_n^{(k-1)}). \quad (3.19)$$

Setting $k = 1$, the Eq. (3.18) relates $\tilde{\mathcal{J}}^{(1)\mu}$ to $\mathcal{J}_n^{(1)}$ via

$$\tilde{\mathcal{J}}^{(1)\mu} = \frac{1}{p_n} \bar{p}^\mu \mathcal{J}_n^{(1)} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(0)}, \quad (3.20)$$

which implies that only the timelike component $\mathcal{J}_n^{(1)}$ is independent just similar to the case at zeroth order. It is straightforward to verify that the equation (3.19) is fulfilled automatically once the Eqs. (3.20), (2.10), and (2.11) hold [32]. Substituting the expression (3.20) into Eq. (2.13) with $k = 1$ gives rise to

$$p^2 \frac{\mathcal{J}_n^{(1)}}{p_n} = -\frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} p_\mu \nabla_\rho \mathcal{J}_\sigma^{(0)}. \quad (3.21)$$

The nonvanishing contribution on the right-handed side implies the modification to the on-shell conditions for free chiral particles and the general solution must take the form

$$\begin{aligned} \mathcal{J}_n^{(1)} &= p_n \tilde{J}_n^{(1)} \delta(p^2) - \frac{s}{2p^2} \bar{\epsilon}^{\mu\rho\sigma} p_\mu \nabla_\rho \mathcal{J}_\sigma^{(0)} \\ &= p_n \tilde{J}_n^{(1)} \delta(p^2) + s(B \cdot p) J_n^{(0)} \frac{\delta(p^2)}{p^2}, \end{aligned} \quad (3.22)$$

where we have used the symbol $\tilde{J}_n^{(1)}$ because we will reserve the symbol $J_n^{(1)}$ for further redefinition. Plugging it into Eq. (3.20) leads to

$$\begin{aligned} \tilde{\mathcal{J}}^{(1)\mu} &= \bar{p}^\mu \left[\tilde{J}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ &\quad + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho [p_\sigma J_n^{(0)} \delta(p^2)]. \end{aligned} \quad (3.23)$$

Following the similar approach used in zeroth order, we integrate $\mathcal{J}_n^{(1)}$ over p_n with arbitrary weight function $g(p_n)$

$$\begin{aligned} &\int dp_n g(p_n) \mathcal{J}_n^{(1)} \\ &= \frac{1}{2} \sum_\lambda g \left[\lambda \tilde{J}_n^{(1)} + \frac{s(B \cdot \bar{p})}{2|\bar{p}|^2} \lambda J_n^{(0)'} - \frac{s(B \cdot \bar{p})}{2|\bar{p}|^3} J_n^{(0)} \right] \\ &\quad + \frac{1}{2} \sum_\lambda g' \frac{s(B \cdot \bar{p})}{2|\bar{p}|^2} \lambda J_n^{(0)}. \end{aligned} \quad (3.24)$$

We emphasize again that we have suppressed all the arguments associated with λ in the functions $g, \tilde{J}_n^{(1)}$,

$J_n^{(0)}$, and $J_n^{(0)}$ as well as $J_n^{(0)'}$ and g' . The prime on the functions $J_n^{(0)}$ and g denotes derivative as usual

$$J_n^{(0)'} \equiv \frac{\partial J_n^{(0)}}{\partial p_n} \Big|_{p_n=\lambda|\vec{p}|}, \quad g' \equiv \frac{dg(p_n)}{dp_n} \Big|_{p_n=\lambda|\vec{p}|}. \quad (3.25)$$

From Eqs. (3.24) and (3.23), we obtain

$$\begin{aligned} & \int dp_n g(p_n) \tilde{\mathcal{J}}^{(1)\mu} \\ &= \frac{1}{2} \sum_{\lambda} \left\{ g \hat{p}^{\mu} \left[\tilde{J}_n^{(1)} + \frac{s(B \cdot \vec{p})}{2|\vec{p}|^2} J_n^{(0)'} - \frac{s(B \cdot \vec{p})}{2|\vec{p}|^3} \lambda J_n^{(0)} \right] \right. \\ & \quad + \left(g' - \frac{\lambda g}{|\vec{p}|} \right) \hat{p}^{\mu} \frac{s(B \cdot \vec{p})}{2|\vec{p}|^2} J_n^{(0)} \\ & \quad + s \left(\frac{\lambda g'}{2|\vec{p}|^2} - \frac{g}{2|\vec{p}|^3} \right) \bar{\epsilon}^{\mu\rho\sigma} E_{\rho} \vec{p}_{\sigma} J_n^{(0)} \\ & \quad \left. + s \sum_{\lambda} \bar{\epsilon}^{\mu\rho\sigma} \bar{\nabla}_{\rho} \left(\frac{\lambda g}{2|\vec{p}|^2} \vec{p}_{\sigma} J_n^{(0)} \right) \right\}. \quad (3.26) \end{aligned}$$

Additional benefit by introducing an arbitrary weight function $g(p_n)$ is that we can easily recognize that some functions always appear as a whole. As we all know, after integration over p_n , there is no relation between $J_n^{(0)'}$ and $J_n^{(0)}$ any more. However, from the expression (3.24), we note that $J_n^{(0)'}$ and $\tilde{J}_n^{(1)}$ always appear together as a whole, we can redefine a new distribution function by

$$J_n^{(1)} \equiv \tilde{J}_n^{(1)} + \frac{s(B \cdot \vec{p})}{2|\vec{p}|^2} J_n^{(0)'}, \quad (3.27)$$

This redefinition has a very clear physical meaning: the second term is the first-order correction for $J_n^{(0)}$ due to the energy shift $s(B \cdot \vec{p})/2|\vec{p}|^2$ generated from the interaction between magnetic field and spin. With this new distribution function and specific cases of $g(p_n) = 1$ and $g(p_n) = p_n$, we can have

$$\int dp_n \mathcal{J}_n^{(1)} = \frac{1}{2} \sum_{\lambda} \lambda (\sqrt{\omega}^{(0)} J_n^{(1)} + \sqrt{\omega}^{(1)} J_n^{(0)}), \quad (3.28)$$

$$\begin{aligned} \int dp_n p_n \mathcal{J}_n^{(1)} &= \frac{1}{2} \sum_{\lambda} (\sqrt{\omega}^{(0)} \epsilon^{(0)} J_n^{(1)} + \sqrt{\omega}^{(0)} \epsilon^{(1)} \tilde{J}_n^{(0)} \\ & \quad + \sqrt{\omega}^{(1)} \epsilon^{(0)} J_n^{(0)}) \end{aligned} \quad (3.29)$$

where

$$\sqrt{\omega}^{(1)} = -\frac{s\lambda(B \cdot \vec{p})}{2|\vec{p}|^3}, \quad \epsilon^{(1)} = \frac{s\lambda(B \cdot \vec{p})}{2|\vec{p}|^2} \quad (3.30)$$

denotes the first-order modification to the invariant phase space and the particle's energy [23], respectively.

The chiral Wigner function after integration over p_n can be written as

$$\begin{aligned} \int dp_n \mathcal{J}^{(1)\mu} &= \frac{1}{2} \sum_{\lambda} [\sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} J_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} J_n^{(0)} \\ & \quad + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} J_n^{(0)}] \end{aligned} \quad (3.31)$$

where first-order modification to the four-velocity $\dot{x}^{(0)\mu}$ is given by

$$\begin{aligned} \dot{x}^{(1)\mu} &= -\frac{s\lambda(B \cdot \vec{p})}{2|\vec{p}|^3} \hat{p}^{\mu} - \frac{s}{2|\vec{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_{\rho} \vec{p}_{\sigma} \\ & \quad + \frac{s\lambda}{2} \bar{\epsilon}^{\mu\rho\sigma} \bar{\nabla}_{\rho} \frac{\vec{p}_{\sigma}}{|\vec{p}|^2}. \end{aligned} \quad (3.32)$$

It needs to be emphasized that we should regard $\dot{x}^{(1)\mu}$ as a operator instead of a ordinary function. The operator $\bar{\nabla}_{\rho}$ in the last term of the Eq. (3.32) acts on not only $\vec{p}_{\sigma}/|\vec{p}|^2$ but also the distribution function $J_n^{(0)}$. The first-order CKE in seven-dimensional phase space can be derived from Eq. (2.14) with $k = 1$ by following the similar procedure in zeroth order from Eq. (3.13) to Eq. (3.16). It turns out that all the terms including the derivative of g have been still canceled at first order just like the case at zeroth-order and only an overall factor g remains. It follows that the following equation must hold for both positive and negative energy parts separately

$$\begin{aligned} 0 &= \nabla_{\mu} [\sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} J_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} J_n^{(0)} \\ & \quad + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} J_n^{(0)}]. \end{aligned} \quad (3.33)$$

We can move the operator ∇_{μ} adjacent to the distribution function

$$\begin{aligned} 0 &= \sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} \nabla_{\mu} J_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} \nabla_{\mu} J_n^{(0)} \\ & \quad + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} \nabla_{\mu} J_n^{(0)} + [\nabla_{\mu}, \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu}] J_n^{(0)} \\ & \quad + [\nabla_{\mu}, \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu}] J_n^{(0)}. \end{aligned} \quad (3.34)$$

By utilizing the zeroth-order CKE (3.17) for the second term on the right-handed side of the equation above, we have

$$\begin{aligned} 0 &= \dot{x}^{(0)\mu} \nabla_{\mu} J_n^{(1)} + \dot{x}^{(1)\mu} \nabla_{\mu} J_n^{(0)} + \dot{x}^{(0)\mu} [\nabla_{\mu}, \sqrt{\omega}^{(1)}] J_n^{(0)} \\ & \quad + [\nabla_{\mu}, \dot{x}^{(1)\mu}] J_n^{(0)}. \end{aligned} \quad (3.35)$$

Substituting the specific results for $\sqrt{\omega}^{(1)}$ in Eq. (3.30) and $\dot{x}^{(1)\mu}$ in Eq. (3.32) into the equation above and dealing with some vector analysis, we obtain

$$0 = \dot{x}^{(0)\mu} \nabla_\mu J_n^{(1)} - \frac{sB \cdot \bar{p}}{2|\bar{p}|^3} \hat{p}^\mu \bar{\nabla}_\mu J_n^{(0)} - \frac{\lambda s}{2|\bar{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma \bar{\nabla}_\mu J_n^{(0)}. \quad (3.36)$$

Remember that the operator ∇_μ includes only derivatives on $\bar{p}^\mu$ after integration over p_n and the CKE for the particles with momentum $\bar{p}^\mu$ and helicity s is obtained by setting $\lambda = +1$, while the CKE for the antiparticles with momentum $\bar{p}^\mu$ and helicity s by setting $\lambda = -1$ along with the replacement $\bar{p}^\mu \rightarrow -\bar{p}^\mu$ and $s \rightarrow -s$.

C. Second-order CKT

Now we come to our major task of deriving second-order CKT in seven-dimensional phase space. From the second-order Wigner equation (3.18) by setting $k = 2$, we can express $\tilde{\mathcal{J}}^{(2)\mu}$ in terms of $\mathcal{J}_n^{(2)}$ via

$$\tilde{\mathcal{J}}^{(2)\mu} = \frac{1}{p_n} \bar{p}^\mu \mathcal{J}_n^{(2)} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(1)}. \quad (3.37)$$

Similar to the case at zeroth and first order, we can verify that the Eq. (3.19) holds automatically given the Eqs. (3.37) and (3.20) together with the specific case $k = 1$ for (2.13) and (2.14). Actually such remarkable procedure can be generalized to any higher order and has been verified systematically in Ref. [32]. Substituting the expression (3.20) into Eq. (2.13) with $k = 2$ gives rise to

$$p^2 \frac{\mathcal{J}_n^{(2)}}{p_n} = -\frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} p_\mu \nabla_\rho \mathcal{J}_\sigma^{(1)}, \quad (3.38)$$

which implies the correction to on-shell condition of the second-order Wigner function $\mathcal{J}_n^{(2)}$. Inserting the expressions (3.22) and (3.23) into the right-hand side of the equation above, the general expression for $\mathcal{J}_{sn}^{(2)}$ must take the form

$$\begin{aligned} \mathcal{J}_n^{(2)} &= p_n \tilde{\mathcal{J}}_n^{(2)} \delta(p^2) - \frac{s}{2p^2} \bar{\epsilon}^{\mu\rho\sigma} p_\mu \nabla_\rho \mathcal{J}_\sigma^{(1)} \\ &= p_n \tilde{\mathcal{J}}_n^{(2)} \delta(p^2) + \frac{sB \cdot p}{p^2} \left[\tilde{\mathcal{J}}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ &\quad - \frac{s}{2p^2} \bar{\epsilon}^{\mu\rho\sigma} p_\mu \nabla_\rho \left\{ \frac{s}{2p_n} \bar{\epsilon}_{\sigma\alpha\beta} \nabla^\alpha [p^\beta J_n^{(0)} \delta(p^2)] \right\} \\ &= p_n \tilde{\mathcal{J}}_n^{(2)} \delta(p^2) + \frac{sB \cdot p}{p^2} \left[\tilde{\mathcal{J}}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ &\quad + \frac{s}{2p^2} (\bar{p}_\mu \bar{\nabla}_\rho - \bar{p}_\rho \bar{\nabla}_\mu) \left\{ \frac{s}{2p_n} \nabla^\mu [p^\rho J_n^{(0)} \delta(p^2)] \right\}, \end{aligned} \quad (3.39)$$

where we have used the identity

$$\bar{\epsilon}^{\mu\nu\alpha} \bar{\epsilon}_{\alpha\rho\sigma} = -\Delta^\mu{}_\rho \Delta^\nu{}_\sigma + \Delta^\nu{}_\rho \Delta^\mu{}_\sigma. \quad (3.40)$$

It is much more complicated and difficult to carry out the integral over p_n for $\mathcal{J}_n^{(2)}$ together with a weight function $g(p_n)$. The more details can be found in the Appendix. Here we only present the final result

$$\begin{aligned} \int dp_n g(p_n) \mathcal{J}_n^{(2)} &= \frac{1}{2} \sum_\lambda \left\{ \lambda g(J_n^{(2)} + \sqrt{\omega}^{(1)} J_n^{(1)} + \sqrt{\omega}^{(2)} J_{sn}^{(0)}) + \frac{1}{2} \lambda g' \frac{sB \cdot \bar{p}}{|\bar{p}|^2} J_n^{(1)} \right. \\ &\quad + \frac{1}{24} \left(\frac{g'''}{|\bar{p}|^3} - \frac{3\lambda g''}{|\bar{p}|^4} + \frac{3g'}{|\bar{p}|^5} \right) [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)} + \frac{1}{24} \left(\frac{g'''}{|\bar{p}|^3} - \frac{3\lambda g''}{|\bar{p}|^4} + \frac{3g'}{|\bar{p}|^5} \right) [(E \cdot \bar{p})^2 - \bar{p}^2 E^2] J_n^{(0)} \\ &\quad + \frac{1}{12} \left(\frac{\lambda g'''}{|\bar{p}|^2} - \frac{3g''}{|\bar{p}|^3} + \frac{6\lambda g'}{|\bar{p}|^4} \right) \bar{\epsilon}^{\mu\rho\sigma} E_\mu B_\rho \bar{p}_\sigma J_n^{(0)} + \frac{1}{8} \left(\frac{g''}{|\bar{p}|^3} - \frac{2\lambda g'}{|\bar{p}|^4} \right) [(E \cdot \bar{p}) \bar{p}^\mu - \bar{p}^2 E^\mu + \lambda \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma] \bar{\nabla}_\mu J_n^{(0)} \\ &\quad \left. + \frac{1}{8} \frac{g'}{|\bar{p}|^3} (\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}) \bar{\nabla}^\mu \bar{\nabla}^\nu J_n^{(0)} \right\}, \end{aligned} \quad (3.41)$$

where $\sqrt{\omega}^{(2)}$ can be regarded as the second-order correction to the invariant phase space

$$\begin{aligned} \sqrt{\omega}^{(2)} &= -\frac{1}{2|\bar{p}|^5} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \\ &\quad - \frac{\lambda}{4} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^4}. \end{aligned} \quad (3.42)$$

Just like $\dot{x}^{(1)\mu}$, $\sqrt{\omega}^{(2)}$ is also identified as an operator because the operators $\bar{\nabla}$ in the second term on the right-hand side of the equal sign above actually act on all the terms following it including the distribution function $J_n^{(0)}$. Besides, we have defined a new distribution function $\mathcal{J}_n^{(2)}$ via

$$J_n^{(2)} = \tilde{J}_n^{(2)} + \frac{s\mathbf{B} \cdot \bar{\mathbf{p}}}{2|\bar{\mathbf{p}}|^2} \tilde{J}_n^{(1)'} + \frac{1}{2} \left(\frac{s\mathbf{B} \cdot \bar{\mathbf{p}}}{2|\bar{\mathbf{p}}|^2} \right)^2 J_n^{(0)''} \quad (3.43)$$

$$\begin{aligned} & + \frac{1}{24} (\mathbf{B} \cdot \bar{\mathbf{p}})^2 \left(\frac{\lambda}{|\bar{\mathbf{p}}|^3} J_n^{(0)'''} - \frac{9\lambda}{|\bar{\mathbf{p}}|^5} J_n^{(0)'} \right) + \frac{1}{24} B^2 \left(\frac{\lambda}{|\bar{\mathbf{p}}|} J_n^{(0)'''} - \frac{3}{|\bar{\mathbf{p}}|^2} J_n^{(0)''} - \frac{3\lambda}{|\bar{\mathbf{p}}|^3} J_n^{(0)'} \right) \\ & + \frac{1}{24} [(E \cdot \bar{\mathbf{p}})^2 - E^2 \bar{p}^2] \left(\frac{\lambda}{|\bar{\mathbf{p}}|^3} J_n^{(0)'''} - \frac{3}{|\bar{\mathbf{p}}|^4} J_n^{(0)''} + \frac{3\lambda}{|\bar{\mathbf{p}}|^5} J_n^{(0)'} \right) \\ & + \frac{1}{12} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \left(\frac{1}{|\bar{\mathbf{p}}|^2} J_n^{(0)'''} - \frac{3\lambda}{|\bar{\mathbf{p}}|^3} J_n^{(0)''} + \frac{3}{|\bar{\mathbf{p}}|^4} J_n^{(0)'} \right) \\ & - \frac{1}{8} [(E \cdot \bar{\mathbf{p}}) \bar{p}^\mu - \bar{p}^2 E^\mu + \lambda |\bar{\mathbf{p}}| \bar{\epsilon}^{\mu\nu\lambda} B_\nu \bar{p}_\lambda] \left(\frac{\lambda}{|\bar{\mathbf{p}}|^3} \bar{\nabla}_\mu J_n^{(0)''} - \frac{2}{|\bar{\mathbf{p}}|^4} \bar{\nabla}_\mu J_n^{(0)'} \right) \\ & + \frac{\lambda}{8|\bar{\mathbf{p}}|^3} (\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}) \bar{\nabla}^\mu \bar{\nabla}^\nu J_n^{(0)'}, \end{aligned} \quad (3.44)$$

where the second term on the right-handed side of the first line above can be identified as the first-order correction for $J_n^{(1)}$ while the third term as the second-order correction for $J_n^{(1)}$ due to the energy shift $s(\mathbf{B} \cdot \bar{\mathbf{p}})/2|\bar{\mathbf{p}}|^2$. We can identify the terms associated with $J_n^{(0)'}$ as the second-order correction for the energy shift. However, all other terms associated with $J_n^{(0)'''}$, $J_n^{(0)''}$, or operator $\bar{\nabla}^\mu$ have no explicit intuitive physical meaning except that the term of last line originate from the uncertainty principle. These terms have not been obtained in previous work.

For the specific cases $g(p_n) = 1$ and $g(p_n) = p_n$, we have

$$\begin{aligned} & \int dp_n \mathcal{J}_n \\ & = \frac{1}{2} \lambda [\sqrt{\omega^{(0)}} \tilde{J}_n^{(2)} + \sqrt{\omega^{(1)}} \tilde{J}_n^{(1)} + \sqrt{\omega^{(2)}} J_n^{(0)}], \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} & \int dp_n p_n \mathcal{J}_n^{(2)} \\ & = \frac{1}{2} [\sqrt{\omega^{(0)}} \epsilon^{(0)} \tilde{J}_n^{(2)} + \sqrt{\omega^{(1)}} \epsilon^{(0)} \tilde{J}_n^{(1)} + \sqrt{\omega^{(0)}} \epsilon^{(1)} \tilde{J}_n^{(1)} \\ & + \sqrt{\omega^{(1)}} \epsilon^{(1)} J_n^{(0)} + \sqrt{\omega^{(0)}} \epsilon^{(2)} J_n^{(0)} + \sqrt{\omega^{(2)}} \epsilon^{(0)} J_n^{(0)}], \end{aligned} \quad (3.46)$$

where $\epsilon^{(2)}$ can be interpreted as second-order correction to the energy for free chiral fermions

$$\begin{aligned} \epsilon^{(2)} & = \frac{1}{8|\bar{\mathbf{p}}|^5} [(B \cdot \bar{\mathbf{p}})^2 + B^2 \bar{p}^2 + (E \cdot \bar{\mathbf{p}})^2 - \bar{p}^2 E^2] \\ & + \frac{\lambda}{4|\bar{\mathbf{p}}|^4} \bar{\epsilon}^{\mu\rho\sigma} E_\mu B_\rho \bar{p}_\sigma \\ & - \frac{\lambda}{8} \bar{\nabla}_\mu \frac{(E \cdot \bar{\mathbf{p}}) \bar{p}^\mu - \bar{p}^2 E^\mu - \lambda |\bar{\mathbf{p}}| \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma}{|\bar{\mathbf{p}}|^4} \\ & + \frac{1}{8} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{\mathbf{p}}|^3}. \end{aligned} \quad (3.47)$$

Just like $\sqrt{\omega^{(2)}}$ and $\dot{x}^{(1)\mu}$, the terms with $\bar{\nabla}$ denote operators which act on all the terms following it. It should be noted that the operator term in Eq. (3.42), which also appears in Eq. (3.47), contributes when the system is inhomogeneous in phase space. Such inhomogeneity can arise from many-body effects, but it can also originate from a single particle in the form of a wave packet. Such term is associated with the uncertainty principle and can modify the on-shell condition, as discussed in Ref. [57].

Now let us calculate the integral of the chiral Wigner function $\mathcal{J}^{(2)\mu}$

$$\begin{aligned} \int dp_n \mathcal{J}^{(2)\mu} & = \int dp_n \left[\frac{1}{p_n} p^\mu \mathcal{J}_n^{(2)} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \tilde{\mathcal{J}}_\sigma^{(1)} \right] \\ & = n^\mu \int dp_n \mathcal{J}_n^{(2)} + \bar{p}^\mu \int dp_n \frac{1}{p_n} \mathcal{J}_n^{(2)} \\ & - \frac{s}{2} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \int dp_n \frac{1}{p_n^2} \tilde{\mathcal{J}}_\sigma^{(1)} \\ & + \frac{s}{2} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \int dp_n \frac{1}{p_n} \tilde{\mathcal{J}}_\sigma^{(1)}. \end{aligned} \quad (3.48)$$

The first term after the last equal sign can be obtained by Eq. (3.45), the second term by setting $g(p_n) = 1/p_n$ in Eq. (3.41), and the third and fourth terms by setting $g(p_n) = 1/p_n^2$ and $g(p_n) = 1/p_n$ in Eq. (3.26), respectively. The final result can be organized into

$$\begin{aligned} \int dp_n \mathcal{J}^{(2)\mu} & = \frac{1}{2} [\sqrt{\omega^{(0)}} \dot{x}^{(0)\mu} J_n^{(2)} + \sqrt{\omega^{(1)}} \dot{x}^{(0)\mu} J_n^{(1)} \\ & + \sqrt{\omega^{(0)}} \dot{x}^{(1)\mu} J_n^{(1)} + \sqrt{\omega^{(1)}} \dot{x}^{(1)\mu} J_n^{(0)} \\ & + \sqrt{\omega^{(0)}} \dot{x}^{(2)\mu} J_n^{(0)} + \sqrt{\omega^{(2)}} \dot{x}^{(0)\mu} J_n^{(0)}], \end{aligned} \quad (3.49)$$

where $\dot{x}^{(2)\mu}$ denotes second-order correction to the four-velocity $\dot{x}^{(0)\mu}$ and actually represents an operator

$$\begin{aligned}
\dot{x}^{(2)\mu} \equiv & \frac{3|\bar{p}|(E \cdot p)E^\mu - 5|\bar{p}|(B \cdot p)B^\mu + \lambda(B \cdot p)\bar{\epsilon}^{\mu\alpha\beta}E_\alpha\bar{p}_\beta}{8|\bar{p}|^6} - \frac{11(B \cdot \bar{p})^2 + B^2\bar{p}^2 + 5(E \cdot \bar{p})^2 - 11\bar{p}^2E^2 + 4\lambda|\bar{p}|\bar{\epsilon}^{\nu\alpha\beta}E_\nu B_\alpha\bar{p}_\beta}{16|\bar{p}|^6}\hat{p}^\mu \\
& - \bar{\nabla}_\alpha\bar{\nabla}_\beta \frac{\bar{p}^\alpha\bar{p}^\beta\bar{p}^\mu - \bar{p}^2\bar{p}^\mu\Delta^{\alpha\beta} + 2\bar{p}^2\bar{p}^\mu\Delta^{\alpha\beta} - 2\bar{p}^2\bar{p}^\alpha\Delta^{\mu\beta}}{16|\bar{p}|^5} + \lambda E_\alpha\bar{\nabla}_\beta \frac{2\bar{p}^\alpha\bar{p}^\beta\bar{p}^\mu + \bar{p}^2\bar{p}^\mu\Delta^{\alpha\beta} - \bar{p}^2\bar{p}^\alpha\Delta^{\mu\beta} - 2\bar{p}^2\bar{p}^\beta\Delta^{\mu\alpha}}{8|\bar{p}|^6} \\
& + B_\alpha\bar{\nabla}_\beta \frac{\bar{\epsilon}^{\beta\alpha\nu}\bar{p}_\nu\bar{p}^\mu + 2\bar{\epsilon}^{\mu\beta\nu}\bar{p}_\nu\bar{p}^\alpha}{8|\bar{p}|^5}
\end{aligned} \tag{3.50}$$

Following the similar procedure at zeroth or first order, second-order CKE is given by

$$0 = \nabla_\mu [\sqrt{\omega}^{(0)}\dot{x}^{(0)\mu}J_n^{(2)} + \sqrt{\omega}^{(1)}\dot{x}^{(0)\mu}J_n^{(1)} + \sqrt{\omega}^{(0)}\dot{x}^{(1)\mu}J_n^{(1)} + \sqrt{\omega}^{(1)}\dot{x}^{(1)\mu}J_n^{(0)} + \sqrt{\omega}^{(0)}\dot{x}^{(2)\mu}J_n^{(0)} + \sqrt{\omega}^{(2)}\dot{x}^{(0)\mu}J_n^{(0)}] \tag{3.51}$$

Moving the operator ∇_μ all through $\sqrt{\omega}$ and $\dot{x}^\mu$ to J_n gives rise to

$$\begin{aligned}
0 = & \sqrt{\omega}^{(0)}\dot{x}^{(0)\mu}\nabla_\mu J_n^{(2)} + \sqrt{\omega}^{(1)}\dot{x}^{(0)\mu}\nabla_\mu J_n^{(1)} + \sqrt{\omega}^{(0)}\dot{x}^{(1)\mu}\nabla_\mu J_n^{(1)} + \sqrt{\omega}^{(1)}\dot{x}^{(1)\mu}\nabla_\mu J_n^{(0)} + \sqrt{\omega}^{(0)}\dot{x}^{(2)\mu}\nabla_\mu J_n^{(0)} + \sqrt{\omega}^{(2)}\dot{x}^{(0)\mu}\nabla_\mu J_n^{(0)} \\
& + [\nabla_\mu, \sqrt{\omega}^{(0)}\dot{x}^{(0)\mu}]J_n^{(2)} + [\nabla_\mu, \sqrt{\omega}^{(1)}\dot{x}^{(0)\mu}]J_n^{(1)} + [\nabla_\mu, \sqrt{\omega}^{(0)}\dot{x}^{(1)\mu}]J_n^{(1)} + [\nabla_\mu, \sqrt{\omega}^{(1)}\dot{x}^{(1)\mu}]J_n^{(0)} + [\nabla_\mu, \sqrt{\omega}^{(0)}\dot{x}^{(2)\mu}]J_n^{(0)} \\
& + [\nabla_\mu, \sqrt{\omega}^{(2)}\dot{x}^{(0)\mu}]J_n^{(0)}.
\end{aligned} \tag{3.52}$$

Using the zeroth-order and first-order CKE (3.17) and (3.35), we can simplify the second-order CKE to

$$\begin{aligned}
0 = & \dot{x}^{(0)\mu}\nabla_\mu J_n^{(2)} + \dot{x}^{(1)\mu}\nabla_\mu J_n^{(1)} + \dot{x}^{(2)\mu}\nabla_\mu J_n^{(0)} + [\nabla_\mu, \dot{x}^{(0)\mu}]J_n^{(2)} + [\nabla_\mu, \sqrt{\omega}^{(1)}\dot{x}^{(0)\mu}]J_n^{(1)} + [\nabla_\mu, \dot{x}^{(1)\mu}]J_n^{(1)} \\
& + [\nabla_\mu, \sqrt{\omega}^{(1)}\dot{x}^{(1)\mu}]J_n^{(0)} + [\nabla_\mu, \dot{x}^{(2)\mu}]J_n^{(0)} + [\nabla_\mu, \sqrt{\omega}^{(2)}\dot{x}^{(0)\mu}]J_n^{(0)} - \sqrt{\omega}^{(1)}\dot{x}^{(0)\mu}[\nabla_\mu, \sqrt{\omega}^{(1)}]J_n^{(0)} \\
& - \sqrt{\omega}^{(1)}[\nabla_\mu, \dot{x}^{(1)\mu}]J_n^{(0)}.
\end{aligned} \tag{3.53}$$

Substituting the specific expressions for $\dot{x}^{(1)\mu}$, $\dot{x}^{(2)\mu}$, $\sqrt{\omega}^{(1)}$ and $\sqrt{\omega}^{(2)}$ into the formal result above, we obtain the final second-order CKE

$$\begin{aligned}
0 = & \dot{x}^{(0)\mu}\nabla_\mu J_n^{(2)} - \frac{sB \cdot \bar{p}}{2|\bar{p}|^3}\hat{p}^\mu\bar{\nabla}_\mu J_n^{(1)} - \frac{\lambda s}{2|\bar{p}|^3}\bar{\epsilon}^{\mu\rho\sigma}E_\rho\bar{p}_\sigma\bar{\nabla}_\mu J_n^{(1)} + \frac{3(B \cdot \bar{p})^2 - \bar{p}^2B^2 - 5(E \cdot \bar{p})^2 - \bar{p}^2E^2 - 4\lambda|\bar{p}|\bar{\epsilon}^{\nu\rho\sigma}\bar{p}_\nu E_\rho B_\sigma}{8|\bar{p}|^6}\lambda\hat{p}^\mu\bar{\nabla}_\mu J_n^{(0)} \\
& + \frac{(B \cdot \bar{p})}{4|\bar{p}|^6}\bar{\epsilon}^{\mu\rho\sigma}E_\rho\bar{p}_\sigma\bar{\nabla}_\mu J_n^{(0)} - \frac{3(E \cdot \bar{p})}{4|\bar{p}|^5}\lambda E^\mu\bar{\nabla}_\mu J_n^{(0)} - \frac{\bar{p}_\rho\bar{p}_\sigma - \bar{p}^2\Delta_{\rho\sigma}}{8|\bar{p}|^4}\lambda\hat{p}^\mu\bar{\nabla}_\mu\bar{\nabla}^\rho\bar{\nabla}^\sigma J_n^{(0)} + \frac{\bar{\epsilon}^{\nu\rho\sigma}\bar{p}_\rho B_\sigma}{4|\bar{p}|^4}\lambda\hat{p}^\mu\bar{\nabla}_\mu\bar{\nabla}_\nu J_n^{(0)} \\
& + \frac{2(E \cdot \bar{p})(\bar{p}^2\Delta^{\mu\nu} - \bar{p}^\mu\bar{p}^\nu) + \bar{p}^2[\bar{p}^\mu E^\nu - (E \cdot \bar{p})\Delta^{\mu\nu}]}{4|\bar{p}|^6}\bar{\nabla}_\mu\bar{\nabla}_\nu J_n^{(0)}.
\end{aligned} \tag{3.54}$$

D. The summed CKT

In previous three subsections, we provide a systematic method to derive the CKT in seven-dimensional phase space from the original Wigner equations in eight-dimensional phase space from zeroth to second order. This procedure is very consistent and can be generalized to any higher order systematically. At the end of this section, we will recombine these perturbative result into a unified form. To do that, we can define full distribution function in seven-dimensional phase space

$$J_n = J_n^{(0)} + \hbar J_n^{(1)} + \hbar^2 J_n^{(2)} + \dots \tag{3.55}$$

Then in the light of the expressions (3.10), (3.31), and (3.48), we can write the integration of the full $\mathcal{J}^\mu$ as

$$\int dp_n \mathcal{J}^\mu = \frac{1}{2}\sqrt{\omega}\dot{x}^\mu J_n. \tag{3.56}$$

The full CKE in seven-dimensional phase space will be given by

$$0 = \nabla_\mu (\sqrt{\omega}\dot{x}^\mu J_n), \tag{3.57}$$

or in another form

$$0 = \dot{x}^\mu \nabla_\mu J_n + \sqrt{\omega}^{-1} [\nabla_\mu, \sqrt{\omega} \dot{x}^\mu] J_n. \quad (3.58)$$

When we expand this equations order by order, we will reproduce the results given in Eqs. (3.17), (3.35), and

(3.53). Hence the Eq. (3.62) can be identified as the full equation of CKE in seven-dimensional phase space. Up to second order, we have

$$\sqrt{\omega} = 1 - \frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^3} - \frac{1}{2|\bar{p}|^5} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu - \frac{\lambda}{4} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^4}, \quad (3.59)$$

$$\begin{aligned} \varepsilon = |\bar{p}| + \frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^2} + \frac{1}{8|\bar{p}|^5} [(B \cdot \bar{p})^2 + B^2 \bar{p}^2 + (E \cdot \bar{p})^2 - \bar{p}^2 E^2] + \frac{\lambda}{4|\bar{p}|^4} \bar{\epsilon}^{\mu\rho\sigma} E_\mu B_\rho \bar{p}_\sigma \\ - \frac{\lambda}{8} \bar{\nabla}_\mu \frac{(E \cdot p) \bar{p}^\mu - \bar{p}^2 E^\mu - \lambda |\bar{p}| \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma}{|\bar{p}|^4} + \frac{1}{8} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^3}, \end{aligned} \quad (3.60)$$

$$\begin{aligned} \dot{x}^\mu = \lambda n^\mu + \hat{p}^\mu - \frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^3} \hat{p}^\mu - \frac{s}{2|\bar{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma + \frac{s\lambda}{2} \bar{\epsilon}^{\mu\rho\sigma} \bar{\nabla}_\rho \frac{\bar{p}_\sigma}{|\bar{p}|^2} + \frac{3|\bar{p}|(E \cdot p) E^\mu - 5|\bar{p}|(B \cdot p) B^\mu + \lambda(B \cdot p) \bar{\epsilon}^{\mu\alpha\beta} E_\alpha \bar{p}_\beta}{8|\bar{p}|^6} \\ - \frac{11(B \cdot \bar{p})^2 + B^2 \bar{p}^2 + 5(E \cdot \bar{p})^2 - 11\bar{p}^2 E^2 + 4\lambda |\bar{p}| \bar{\epsilon}^{\nu\alpha\beta} E_\nu B_\alpha \bar{p}_\beta}{16|\bar{p}|^6} \hat{p}^\mu - \bar{\nabla}_\alpha \bar{\nabla}_\beta \frac{\bar{p}^\alpha \bar{p}^\beta \bar{p}^\mu - \bar{p}^2 \bar{p}^\mu \Delta^{\alpha\beta} + 2\bar{p}^2 \bar{p}^\mu \Delta^{\alpha\beta} - 2\bar{p}^2 \bar{p}^\alpha \Delta^{\mu\beta}}{16|\bar{p}|^5} \\ + \lambda E_\alpha \bar{\nabla}_\beta \frac{2\bar{p}^\alpha \bar{p}^\beta \bar{p}^\mu + \bar{p}^2 \bar{p}^\mu \Delta^{\alpha\beta} - \bar{p}^2 \bar{p}^\alpha \Delta^{\mu\beta} - 2\bar{p}^2 \bar{p}^\beta \Delta^{\mu\alpha}}{8|\bar{p}|^6} + B_\alpha \bar{\nabla}_\beta \frac{\bar{\epsilon}^{\beta\alpha\nu} \bar{p}_\nu \bar{p}^\mu + 2\bar{\epsilon}^{\mu\beta\nu} \bar{p}_\nu \bar{p}^\alpha}{8|\bar{p}|^5}. \end{aligned} \quad (3.61)$$

As we mentioned in previous sections, these quantities should be regarded as an operator instead of usual functions because the operator ∇ also acts on the distribution function. The specific expression for the CKE up to the second order is given by

$$\begin{aligned} 0 = \dot{x}^{(0)\mu} \nabla_\mu J_n - \frac{sB \cdot \bar{p}}{2|\bar{p}|^3} \hat{p}^\mu \bar{\nabla}_\mu J_n - \frac{\lambda s}{2|\bar{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma \bar{\nabla}_\mu J_n + \frac{3(B \cdot \bar{p})^2 - \bar{p}^2 B^2 - 5(E \cdot \bar{p})^2 - \bar{p}^2 E^2 - 4\lambda |\bar{p}| \bar{\epsilon}^{\nu\rho\sigma} \bar{p}_\nu E_\rho B_\sigma}{8|\bar{p}|^6} \lambda \hat{p}^\mu \bar{\nabla}_\mu J_n \\ + \frac{(B \cdot \bar{p})}{4|\bar{p}|^6} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma \bar{\nabla}_\mu J_n - \frac{3(E \cdot \bar{p})}{4|\bar{p}|^5} \lambda E^\mu \bar{\nabla}_\mu J_n - \frac{\bar{p}_\rho \bar{p}_\sigma - \bar{p}^2 \Delta_{\rho\sigma}}{8|\bar{p}|^4} \lambda \hat{p}^\mu \bar{\nabla}_\mu \bar{\nabla}^\rho \bar{\nabla}^\sigma J_n + \frac{\bar{\epsilon}^{\nu\rho\sigma} \bar{p}_\rho B_\sigma}{4|\bar{p}|^4} \lambda \hat{p}^\mu \bar{\nabla}_\mu \bar{\nabla}_\nu J_n \\ + \frac{2(E \cdot \bar{p})(\bar{p}^2 \Delta^{\mu\nu} - \bar{p}^\mu \bar{p}^\nu) + \bar{p}^2 [\bar{p}^\mu E^\nu - (E \cdot \bar{p}) \Delta^{\mu\nu}]}{4|\bar{p}|^6} \bar{\nabla}_\mu \bar{\nabla}_\nu J_n. \end{aligned} \quad (3.62)$$

We have organized the CKT in a different form from the conventional one given in [21–25]. This form is more convenient to be derived from the Wigner function formalism and can be generalized to any higher order. If we drop all the second-order terms in the CKE above (3.62), only the first three terms on the right-hand side of the first equation are retained. The remained first-order CKE can be verified to be equivalent to the usual form [21–25] up to first-order approximation, but in a more simple and elegant way. All the terms associated with ∇ in Eqs. (3.59), (3.60), and (3.61) and the terms including two or three operators ∇ in (3.62) are new and have not been obtained in Ref. [52]. It should be useful to display the results for $\sqrt{\omega}$ and ε given in Ref. [52] with our symbol conventions

$$\begin{aligned} \sqrt{\omega} = 1 - \frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^3} + \frac{1}{4|\bar{p}|^6} [2(B \cdot p)^2 - \bar{p}^2 B^2] \\ - \frac{\lambda}{2|\bar{p}|^5} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu, \end{aligned} \quad (3.63)$$

$$\begin{aligned} \varepsilon = |\bar{p}| + \frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^2} + \frac{1}{16|\bar{p}|^5} [-(B \cdot \bar{p})^2 + 2B^2 \bar{p}^2] \\ + \frac{\lambda}{4|\bar{p}|^4} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu. \end{aligned} \quad (3.64)$$

We note that the zeroth- and first-order expressions are the same as our results, corresponding to the first and second terms on the right-handed sides of both equations, respectively. The terms associated with $\bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu$ are also the same. For $\sqrt{\omega}$, the third term on the right-handed side of the Eq. (3.63) is absent in our result (3.59). Such difference can be eliminated by redefining second-order distribution via the replacement in (3.59)

$$J_{sn}^{(2)} \rightarrow J_{sn}^{(2)} - \frac{1}{4|\bar{p}|^6} [2(B \cdot p)^2 - \bar{p}^2 B^2] J_{sn}^{(0)}. \quad (3.65)$$

Such replacement does not change the result (3.60). We note that the term associated with B^2 is the same between

(3.60) and (3.64) while the term associated with $(B \cdot p)^2$ is different. Another difference is that our result (3.60) includes the terms E^2 and $(E \cdot p)^2$ but they are absent in (3.64). How these terms associated with E^2 and $(E \cdot p)^2$ arise can be seen more clearly when we rewrite Eq. (3.47) as the following form

$$\begin{aligned} \varepsilon^{(2)} = & \frac{1}{8|\bar{p}|^5} [(\tilde{B} \cdot \bar{p})^2 + \tilde{B}^2 \bar{p}^2] + \frac{1}{8} \bar{\nabla}_\mu \frac{\tilde{\varepsilon}^{\mu\rho\sigma} \tilde{B}_\rho \bar{p}_\sigma}{|\bar{p}|^3} \\ & + \frac{1}{8} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^3}. \end{aligned} \quad (3.66)$$

where we have introduced the effective magnetic field $\tilde{B}^\mu$

$$\tilde{B}^\mu \equiv B^\mu + \frac{\lambda}{|\bar{p}|} \tilde{\varepsilon}^{\alpha\mu\beta} E_\alpha \bar{p}_\beta \quad (3.67)$$

This expression shows that both the forms of $\tilde{\varepsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu$, E^2 and $(E \cdot \bar{p})^2$ come from the effective magnetic field $\tilde{B}^\mu$. We interpret all these differences as well as the addition terms with ∇ in our result as the quantum effects from quantum field theory, which might not be able to be captured from semiclassical approach in quantum mechanics. The difference between our result and the one in [53,54] is obvious. Our result here has already been written in seven-dimensional phase space and the Dirac delta functions or their derivatives have been integrated out. The results (3.59), (3.60), (3.61), and (3.62) can be used for numerical calculation directly and constitutes the main result of this paper.

IV. SUMMARY

Within the Wigner function formalism, we have derived the CKE in seven-dimensional phase space up to the second order in semiclassical expansion $\hbar$. We find that second-

order CKT includes various quantum effects which cannot be captured by the conventional semiclassical approach, such as the higher-derivative terms in coordinate space or momentum space. While this derived equation can be the starting point for numerical evaluation on the transport process of nonlinear chiral effects associated with space-time derivative and electromagnetic fields, this paper also provides a systematic and consistent approach to derive the CKE in seven-dimensional phase space from the original covariant Wigner equations in eight-dimensional phase space order by order. In the present work, we restrict ourselves to the constant electromagnetic field in flat space-time. The generalization from constant to varying electromagnetic field, from massless to massive fermions, from Abelian to non-Abelian gauge field, or from flat to curved space-time is straightforward and will be investigated in the future.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: SOME CALCULATION DETAILS AT SECOND ORDER

In this Appendix, we will give some details on calculating the integral in Eq. (3.41). Let us start from Eq. (3.39) by moving the factor $1/p^2$ in the last term to be adjacent to $\delta(p^2)$ and moving $\bar{p}_\mu$ and $\bar{p}_\rho$ to the left of the operator ∇_ρ and ∇_μ , respectively. After such manipulation, we obtain

$$\begin{aligned} \mathcal{J}_n^{(2)} = & p_n \tilde{\mathcal{J}}_n^{(2)} \delta(p^2) + \frac{sB \cdot p}{p^2} \left[\tilde{\mathcal{J}}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] + \frac{\tilde{\varepsilon}^{\rho\mu\nu} p_\rho E_\mu B_\nu}{2p_n^2} J_n^{(0)} \frac{\delta(p^2)}{p^2} \\ & - \frac{1}{2} \tilde{\varepsilon}_{\rho\mu\nu} B^\nu \left(\nabla^\mu - \frac{2}{p^2} F^{\mu\sigma} p_\sigma \right) \left[\frac{\bar{p}^\rho}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] + \frac{1}{4} \left(\bar{\nabla}_\rho - \frac{2}{p^2} F_{\rho\sigma} p^\sigma \right) \left[\tilde{\varepsilon}^{\rho\mu\nu} B_\nu \frac{p_\mu}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ & - \frac{1}{4} \left(\bar{\nabla}_\rho - \frac{2}{p^2} F_{\rho\sigma} p^\sigma \right) \left[\frac{\bar{p}_\mu}{p_n^2} (E^\mu \bar{p}^\rho - E^\rho \bar{p}^\mu) J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] + \frac{1}{4} \bar{\nabla}^\mu \bar{\nabla}^\nu \left[\frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ & - \frac{1}{2} \left(F^{\nu\rho} \nabla^\mu \frac{p_\rho}{p^2} + F^{\mu\rho} \nabla^\nu \frac{p_\rho}{p^2} \right) \left[\frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ & - \frac{1}{2} F^{\mu\rho} F^{\nu\sigma} \left(\frac{g_{\rho\sigma}}{p^2} - 4 \frac{p_\rho p_\sigma}{p^4} \right) \left[\frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right]. \end{aligned} \quad (A1)$$

where the operator ∇ acts on all the terms following it. Using the relations

$$\delta'''(p^2) = -\frac{6\delta(p^2)}{p^6}, \quad \delta''(p^2) = \frac{2\delta(p^2)}{p^4}, \quad \delta'(p^2) = -\frac{\delta(p^2)}{p^2}, \quad (\text{A2})$$

we can decompose $\mathcal{J}_n^{(2)}$ into seven parts

$$\mathcal{J}_n^{(2)} = \sum_{i=1}^7 \mathcal{J}_{n,i}^{(2)} \quad (\text{A3})$$

where

$$\begin{aligned} \mathcal{J}_{n,1}^{(2)} &= p_n \tilde{J}_n^{(2)} \delta(p^2) + \frac{sB \cdot p}{p^2} \left[\tilde{J}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} J_n^{(0)} \frac{\delta(p^2)}{p^2} \right] \\ \mathcal{J}_{n,2}^{(2)} &= -\frac{1}{2} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \frac{1}{p_n^2} J_n^{(0)} \delta'(p^2) \\ \mathcal{J}_{n,3}^{(2)} &= \frac{3}{4} \bar{\epsilon}_{\rho\mu\nu} B^\nu [\bar{\nabla}^\mu \delta'(p^2) + (\bar{\epsilon}^{\mu\sigma\lambda} B_\sigma \bar{p}_\lambda + E^\mu p_n) \delta''(p^2)] \frac{\bar{p}^\rho}{p_n} J_n^{(0)} \\ \mathcal{J}_{n,4}^{(2)} &= \frac{1}{4} [\bar{\nabla}_\rho \delta'(p^2) + (\bar{\epsilon}_{\rho\sigma\nu} B^\sigma \bar{p}^\nu + E_\rho p_n) \delta''(p^2)] \frac{\bar{p}_\mu}{p_n^2} (E^\mu \bar{p}^\rho - E^\rho \bar{p}^\mu) J_n^{(0)} \\ \mathcal{J}_{n,5}^{(2)} &= -\frac{1}{2} (\epsilon^{\mu\sigma\lambda} B_\sigma \nabla^\nu \bar{p}_\lambda + E^\mu \nabla^\nu p_n) \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \delta''(p^2) \\ \mathcal{J}_{n,6}^{(2)} &= -\frac{1}{4} \Delta_\tau^\mu F^{\tau\rho} \Delta_\kappa^\nu F^{\kappa\sigma} \left[g_{\rho\sigma} \delta''(p^2) + \frac{4}{3} p_\rho p_\sigma \delta'''(p^2) \right] \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \\ \mathcal{J}_{n,7}^{(2)} &= -\frac{1}{4} \bar{\nabla}^\mu \bar{\nabla}^\nu \left[\frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \delta'(p^2) \right] \end{aligned} \quad (\text{A4})$$

Then we can finish integrating over p_n by using the identities

$$\delta(p^2) = \frac{1}{2|\bar{p}|} \sum_\lambda \delta(p_n - \lambda|\bar{p}|) \quad (\text{A5})$$

$$\delta'(p^2) = \frac{1}{4p_n|\bar{p}|} \sum_\lambda \delta'(p_n - \lambda|\bar{p}|) \quad (\text{A6})$$

$$\delta''(p^2) = \frac{1}{8p_n|\bar{p}|^2} \sum_\lambda \left[\lambda \delta''(p_n - \lambda|\bar{p}|) + \frac{1}{|\bar{p}|} \delta'(p_n - \lambda|\bar{p}|) \right] \quad (\text{A7})$$

$$\delta'''(p^2) = \frac{1}{16p_n|\bar{p}|^3} \sum_\lambda \left[\delta'''(p_n - \lambda|\bar{p}|) + \frac{3\lambda}{|\bar{p}|} \delta''(p_n - \lambda|\bar{p}|) + \frac{3}{|\bar{p}|^2} \delta'(p_n - \lambda|\bar{p}|) \right] \quad (\text{A8})$$

The results after integration can be decomposed into seven parts

$$J_{g,n}^{(2)} \equiv \int dp_n g(p_n) \mathcal{J}_n^{(2)} = \sum_{i=1}^7 J_{g,i}^{(2)}, \quad \text{with} \quad J_{g,i}^{(2)} \equiv \int dp_n g(p_n) \mathcal{J}_{n,i}^{(2)} \quad (\text{A9})$$

where

$$\begin{aligned} J_{g,1}^{(2)} &= \int dp_n g(p_n) \left\{ p_n \tilde{J}_n^{(2)} \delta(p^2) + sB \cdot p \left[-\tilde{J}_n^{(1)} \delta'(p^2) + \frac{sB \cdot p}{2p_n} J_n^{(0)} \delta''(p^2) \right] \right\} \\ &= \sum_\lambda \left\{ \frac{g}{2} \left[\lambda \tilde{J}_n^{(2)} + \frac{sB \cdot \bar{p}}{2|\bar{p}|^2} \lambda \tilde{J}_n^{(1)'} - \frac{sB \cdot \bar{p}}{2|\bar{p}|^3} \left(\tilde{J}_n^{(1)} + \frac{sB \cdot \bar{p}}{2|\bar{p}|^2} J_n^{(0)'} \right) \right] + \frac{g}{16|\bar{p}|^2} (B \cdot \bar{p})^2 \left(\lambda J_n^{(0)''} - \frac{3}{|\bar{p}|} J_n^{(0)'} + \frac{8\lambda}{|\bar{p}|^2} J_n^{(0)} \right) \right. \\ &\quad \left. + \frac{g'}{4|\bar{p}|^2} (sB \cdot \bar{p}) \left(\lambda \tilde{J}_n^{(1)} + \frac{sB \cdot \bar{p}}{2|\bar{p}|^2} \lambda J_n^{(0)'} - \frac{5}{4} \frac{sB \cdot \bar{p}}{|\bar{p}|^3} J_n^{(0)} \right) + \frac{1}{16} g'' \frac{(B \cdot \bar{p})^2}{|\bar{p}|^4} \lambda J_n^{(0)} \right\} \end{aligned} \quad (\text{A10})$$

$$\begin{aligned}
J_{g,2}^{(2)} &= -\frac{1}{2} \int dp_n g(p_n) \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \frac{1}{\bar{p}_n^2} J_n^{(0)} \delta'(p^2) \\
&= \sum_\lambda \left\{ \frac{g}{8|\bar{p}|^4} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \left(\lambda J_n^{(0)'} - \frac{3}{|\bar{p}|} J_n^{(0)} \right) + \frac{g'}{8|\bar{p}|^4} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \lambda J_n^{(0)} \right\}
\end{aligned} \tag{A11}$$

$$\begin{aligned}
J_{g,3}^{(2)} &= \frac{3}{4} \int dp_n g(p_n) \bar{\epsilon}_{\rho\mu\nu} B^\nu [\bar{\nabla}^\mu \delta'(p^2) + (\bar{\epsilon}^{\mu\sigma\lambda} B_\sigma \bar{p}_\lambda + E^\mu p_n) \delta''(p^2)] \frac{\bar{p}^\rho}{p_n} J_n^{(0)} \\
&= \sum_\lambda \left\{ -\frac{3g}{16} \bar{\epsilon}_{\rho\mu\nu} B^\nu \bar{\nabla}^\mu \frac{\bar{p}^\rho}{|\bar{p}|^3} \left(J_n^{(0)'} - \frac{2\lambda}{|\bar{p}|} J_n^{(0)} \right) + \frac{3g}{32|\bar{p}|^4} [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] \left(\lambda J_n^{(0)''} - \frac{5}{|\bar{p}|} J_n^{(0)'} + \frac{8\lambda}{|\bar{p}|^2} J_n^{(0)} \right) \right. \\
&\quad + \frac{3g}{32|\bar{p}|^3} \bar{\epsilon}_{\rho\mu\nu} \bar{p}^\rho E^\mu B^\nu \left(J_n^{(0)''} - \frac{3\lambda}{|\bar{p}|} J_n^{(0)'} + \frac{3}{|\bar{p}|^2} J_n^{(0)} \right) - \frac{3g'}{16|\bar{p}|^3} \bar{\epsilon}_{\rho\mu\nu} \bar{p}^\rho B^\nu \bar{\nabla}^\mu J_n^{(0)} + \frac{g'}{8|\bar{p}|^5} [3(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)} \\
&\quad \left. - \frac{3}{32|\bar{p}|^4} \left(\frac{1}{|\bar{p}|} g' + \lambda g'' \right) [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)} + \frac{3}{32|\bar{p}|^3} \left(\frac{\lambda}{|\bar{p}|} g' - g'' \right) \bar{\epsilon}_{\rho\mu\nu} \bar{p}^\rho E^\mu B^\nu J_n^{(0)} \right\}
\end{aligned} \tag{A12}$$

$$\begin{aligned}
J_{g,4}^{(2)} &= \frac{1}{4} \int dp_n g(p_n) [\bar{\nabla}_\rho \delta'(p^2) + (\bar{\epsilon}_{\rho\sigma\lambda} B^\sigma \bar{p}^\lambda + E_\rho p_n) \delta''(p^2)] \frac{\bar{p}_\mu}{p_n^2} (E^\mu \bar{p}^\rho - E^\rho \bar{p}^\mu) J_n^{(0)} \\
&= \sum_\lambda \left\{ -\frac{g}{16} \bar{\nabla}_\rho \frac{(E \cdot \bar{p}) \bar{p}^\rho - E^\rho \bar{p}^2}{|\bar{p}|^4} \left(\lambda J_n^{(0)'} - \frac{3}{|\bar{p}|} J_n^{(0)} \right) + \frac{g}{32|\bar{p}|^4} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] \left(\lambda J_n^{(0)''} - \frac{5}{|\bar{p}|} J_n^{(0)'} + \frac{8\lambda}{|\bar{p}|^2} J_n^{(0)} \right) \right. \\
&\quad + \frac{g}{32|\bar{p}|^3} \bar{\epsilon}_{\rho\mu\nu} \bar{p}^\rho E^\mu B^\nu \left(J_n^{(0)''} - \frac{7\lambda}{|\bar{p}|} J_n^{(0)'} + \frac{15}{|\bar{p}|^2} J_n^{(0)} \right) - \frac{\lambda g'}{16|\bar{p}|^4} [(E \cdot \bar{p}) \bar{p}^\rho - E^\rho \bar{p}^2] \bar{\nabla}_\rho J_n^{(0)} \\
&\quad \left. + \frac{1}{32|\bar{p}|^4} \left(\frac{1}{|\bar{p}|} g' - \lambda g'' \right) [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] J_n^{(0)} + \frac{1}{32|\bar{p}|^3} \left(\frac{\lambda}{|\bar{p}|} g' - g'' \right) \bar{\epsilon}_{\rho\mu\nu} \bar{p}^\rho E^\mu B^\nu J_n^{(0)} \right\}
\end{aligned} \tag{A13}$$

$$\begin{aligned}
J_{g,5}^{(2)} &= -\frac{1}{2} \int dp_n g(p_n) (\bar{\epsilon}^{\mu\sigma\lambda} B_\sigma \bar{\nabla}^\nu \bar{p}_\lambda + E^\mu \bar{\nabla}^\nu p_n) \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \delta''(p^2) \\
&= \sum_\lambda \left\{ -\frac{g}{16} \bar{\epsilon}^{\mu\sigma\lambda} B_\sigma \bar{\nabla}_\mu \frac{\bar{p}_\lambda}{|\bar{p}|^2} \left(\lambda J_n^{(0)''} - \frac{5}{|\bar{p}|} J_n^{(0)'} + \frac{8\lambda}{|\bar{p}|^2} J_n^{(0)} \right) - \frac{g}{16} E^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^3} \left(J_n^{(0)''} - \frac{3\lambda}{|\bar{p}|} J_n^{(0)'} + \frac{3}{|\bar{p}|^2} J_n^{(0)} \right) \right. \\
&\quad + \frac{1}{16|\bar{p}|^2} \left(\frac{5}{|\bar{p}|} g' - \lambda g'' \right) \bar{\epsilon}^{\mu\sigma\lambda} B_\sigma \bar{p}_\lambda \bar{\nabla}_\mu J_n^{(0)} + \frac{5g'}{16|\bar{p}|^5} [B^2 \bar{p}^2 - 3(B \cdot \bar{p})^2] J_n^{(0)} + \frac{\lambda g''}{8|\bar{p}|^4} (B \cdot \bar{p})^2 J_n^{(0)} \\
&\quad + \frac{1}{16|\bar{p}|^3} \left(\frac{3\lambda}{|\bar{p}|} g' - g'' \right) [(E \cdot \bar{p}) \bar{p}_\nu - \bar{p}^2 E_\nu] \bar{\nabla}^\nu J_n^{(0)} - \frac{1}{16|\bar{p}|^3} \left(\frac{8}{|\bar{p}|^2} g' - \frac{5\lambda}{|\bar{p}|} g'' + g''' \right) [(B \cdot p)^2 - B^2 \bar{p}^2] J_n^{(0)} \\
&\quad - \frac{1}{16|\bar{p}|^2} \left(\frac{14\lambda}{|\bar{p}|^2} g' - \frac{8}{|\bar{p}|} g'' + 2\lambda g''' \right) \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu J_n^{(0)} - \frac{1}{16|\bar{p}|^3} \left(\frac{3}{|\bar{p}|^2} g' - \frac{3\lambda}{|\bar{p}|} g'' + g''' \right) [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] J_n^{(0)} \\
&\quad - \frac{\lambda g'}{8|\bar{p}|^2} \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma \bar{\nabla}_\mu J_n^{(0)'} + \frac{\lambda g'}{4|\bar{p}|^4} (B \cdot \bar{p})^2 J_n^{(0)'} - \frac{g'}{8|\bar{p}|^3} [(E \cdot \bar{p}) \bar{p}_\nu - \bar{p}^2 E_\nu] \bar{\nabla}^\nu J_n^{(0)'} \\
&\quad - \frac{1}{16|\bar{p}|^3} [(B \cdot p)^2 - B^2 \bar{p}^2] \left(g' J_n^{(0)''} - \frac{5\lambda g'}{|\bar{p}|} J_n^{(0)'} + 2g'' J_n^{(0)'} \right) - \frac{1}{16|\bar{p}|^2} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \left(2\lambda g' J_n^{(0)''} - \frac{8g'}{|\bar{p}|} J_n^{(0)'} + 4\lambda g'' J_n^{(0)'} \right) \\
&\quad \left. - \frac{1}{16|\bar{p}|^3} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] \left(g' J_n^{(0)''} - \frac{3\lambda g'}{|\bar{p}|} J_n^{(0)'} + 2g'' J_n^{(0)'} \right) \right\}
\end{aligned} \tag{A14}$$

$$\begin{aligned}
J_{g,6}^{(2)} = & -\frac{1}{4} \int dp_n g(p_n) \Delta_\lambda^\mu F^{\lambda\rho} \Delta_\kappa^\nu F^{\kappa\sigma} \left[g_{\rho\sigma} \delta''(p^2) + \frac{4}{3} p_\rho p_\sigma \delta'''(p^2) \right] \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \\
= & \sum_\lambda \left\{ -\frac{g}{32|\bar{p}|^4} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2 + (B \cdot \bar{p})^2 + B^2 \bar{p}^2] \left(\lambda J_n^{(0)''} - \frac{5}{|\bar{p}|} J_n^{(0)'} + \frac{8\lambda}{|\bar{p}|^2} J_n^{(0)} \right) \right. \\
& + \frac{g}{48|\bar{p}|^3} [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] \left(J_n^{(0)'''} - \frac{9\lambda}{|\bar{p}|} J_n^{(0)''} + \frac{33}{|\bar{p}|^2} J_n^{(0)'} - \frac{48\lambda}{|\bar{p}|^6} J_n^{(0)} \right) \\
& + \frac{g}{24|\bar{p}|^2} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \left(\lambda J_n^{(0)'''} - \frac{6}{|\bar{p}|} J_n^{(0)''} + \frac{15\lambda}{|\bar{p}|^2} J_n^{(0)'} - \frac{15}{|\bar{p}|^3} J_n^{(0)} \right) \\
& + \frac{g}{48|\bar{p}|^3} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] \left(J_n^{(0)'''} - \frac{3\lambda}{|\bar{p}|} J_n^{(0)''} + \frac{3}{|\bar{p}|^2} J_n^{(0)'} \right) \\
& + \frac{1}{32|\bar{p}|^4} \left(\frac{5}{|\bar{p}|} g' - \lambda g'' \right) [(E \cdot \bar{p})^2 - E^2 \bar{p}^2 + (B \cdot \bar{p})^2 + B^2 \bar{p}^2] J_n^{(0)} \\
& + \frac{1}{48|\bar{p}|^3} \left(\frac{33}{|\bar{p}|^2} g' - \frac{9\lambda}{|\bar{p}|} g'' + g''' \right) [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)} + \frac{1}{48|\bar{p}|^3} \left(\frac{3}{|\bar{p}|^2} g' - \frac{3\lambda}{|\bar{p}|} g'' + g''' \right) [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] J_n^{(0)} \\
& + \frac{1}{24|\bar{p}|^2} \left(\frac{15\lambda}{|\bar{p}|^2} g' - \frac{6}{|\bar{p}|} g'' + \lambda g''' \right) \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu J_n^{(0)} - \frac{\lambda g'}{16|\bar{p}|^4} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2 + (B \cdot \bar{p})^2 + B^2 \bar{p}^2] J_n^{(0)'} \\
& + \frac{1}{16|\bar{p}|^3} [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] \left(g' J_n^{(0)''} - \frac{6\lambda}{|\bar{p}|} g' J_n^{(0)'} + g'' J_n^{(0)'} \right) + \frac{1}{16|\bar{p}|^3} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] \left(g' J_n^{(0)''} - \frac{2\lambda}{|\bar{p}|} g' J_n^{(0)'} + g'' J_n^{(0)'} \right) \\
& \left. + \frac{1}{8|\bar{p}|^2} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \left(\lambda g' J_n^{(0)''} - \frac{4}{|\bar{p}|} g' J_n^{(0)'} + \lambda g'' J_n^{(0)'} \right) \right\} \quad (A15)
\end{aligned}$$

$$\begin{aligned}
J_{g,7}^{(2)} = & -\frac{1}{4} \int dp_n g(p_n) \bar{\nabla}^\mu \bar{\nabla}^\nu \left[\frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{p_n} J_n^{(0)} \delta'(p^2) \right] \\
= & \sum_\lambda \left\{ \frac{g}{16} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^3} \left(J_n^{(0)'} - \frac{2\lambda}{|\bar{p}|} J_n^{(0)} \right) + \frac{g'}{16} (\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}) \frac{1}{|\bar{p}|^3} \bar{\nabla}^\mu \bar{\nabla}^\nu J_n^{(0)} \right. \\
& - \frac{1}{8|\bar{p}|^2} \left(\frac{2}{|\bar{p}|} g' - \lambda g'' \right) \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma \bar{\nabla}_\mu J_n^{(0)} - \frac{1}{8|\bar{p}|^3} \left(\frac{2\lambda}{|\bar{p}|} g' - g'' \right) [(E \cdot \bar{p}) \bar{p}_\nu - \bar{p}^2 E_\nu] \bar{\nabla}^\nu J_n^{(0)} \\
& + \frac{g'}{4|\bar{p}|^5} [2(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)} - \frac{\lambda g''}{8|\bar{p}|^4} (B \cdot \bar{p})^2 J_n^{(0)} - \frac{1}{16|\bar{p}|^3} \left(\frac{2\lambda}{|\bar{p}|} g'' - g''' \right) [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)} \\
& - \frac{1}{16|\bar{p}|^3} \left(\frac{2\lambda}{|\bar{p}|} g'' - g''' \right) [(E \cdot \bar{p})^2 - \bar{p}^2 E^2] J_n^{(0)} + \frac{1}{8|\bar{p}|^2} \left(\frac{2\lambda}{|\bar{p}|^2} g' - \frac{2}{|\bar{p}|} g'' + \lambda g''' \right) \bar{\epsilon}^{\mu\rho\sigma} E_\mu B_\rho \bar{p}_\sigma J_n^{(0)} \\
& + \frac{\lambda g'}{8|\bar{p}|^2} \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma \bar{\nabla}_\mu J_n^{(0)'} + \frac{g'}{8|\bar{p}|^3} E^\nu (\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}) \bar{\nabla}^\mu J_n^{(0)'} - \frac{\lambda g'}{8|\bar{p}|^4} (B \cdot \bar{p})^2 J_n^{(0)'} + \frac{g''}{16|\bar{p}|^3} [(B \cdot \bar{p})^2 - B^2 \bar{p}^2] J_n^{(0)'} \\
& \left. + \frac{g''}{16|\bar{p}|^3} [(E \cdot \bar{p})^2 - \bar{p}^2 E^2] J_n^{(0)'} + \frac{\lambda g''}{8|\bar{p}|^2} \bar{\epsilon}^{\mu\rho\sigma} E_\mu B_\rho \bar{p}_\sigma J_n^{(0)'} \right\} \quad (A16)
\end{aligned}$$

After putting all the $J_{g,i}^{(2)}$ together, we arrive at Eq. (3.41). We find that the terms associated with the mixed derivative terms of g and $J_n^{(0)}$ including the last five lines of $J_{g,5}^{(2)}$ in Eq. (A14), the last four lines of $J_{g,6}^{(2)}$ in Eq. (A15), and the last three lines of $J_{g,7}^{(2)}$ in Eq. (A16) cancel each other. Such cancellation makes the final physical quantities only depend on the distribution function $J_n^{(2)}$, $J_n^{(1)}$, and $J_n^{(0)}$.

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