

Thread embodiment of holographic quantum entanglement

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This paper systematically develops the concept of entanglement threads that characterize the entanglement structure of holographic duality. Behind this framework lies a simple philosophy: holographic quantum entanglement can be visualized using threadlike objects. Inspired by the fact that tensor network models can be deformed into a quantum circuit form with flow-conserving features, we abstract the concept of entanglement threads. These entanglement threads can be understood as a preset ensemble of wires in a holographic quantum circuit, and we propose that they characterize the underlying partially ordered structure of holographic quantum entanglement. Combining the concepts of entanglement threads and kinematic space, a elegant circuit interpretation for the holographic complexity is provided. We also clarify the connection and distinction between entanglement threads and the previously proposed concept of bit threads.

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I. INTRODUCTION

Many phenomena in the holographic principle [1–3], especially the holographic entanglement entropy formula proposed in 2006 (the Ryu-Takayanagi formula) [4–6], reveal a close connection between gravitational spacetime geometry and quantum entanglement. Along the line of thought that spacetime geometry may be constructed from quantum entanglement, an obviously important direction is to fully understand the nature of the entanglement structure that corresponds to classical spacetimes dual to holographic quantum systems in AdS/CFT duality. In 2009, Swingle was the first to propose that tensor networks, developed in the study of quantum many-body systems, could be used to characterize this entanglement structure [7,8]. Tensor networks are widely used in quantum many-body theory as a powerful numerical simulation tool, representing a quantum state as a contraction of many smaller tensors. Each small tensor can be regarded as a local few-body quantum state and can be represented diagrammatically as a vertex from which several legs extend, corresponding to local qudits. With such a diagrammatic representation, the whole quantum state is naturally depicted as a network with geometric structure. The multiscale entanglement renormalization

ansatz (MERA) tensor network [9–11], a tensor network specifically designed to represent the ground states of critical systems (which can be viewed as lattice versions of conformal field theories), exhibits a high-dimensional geometric structure strikingly similar to a spatial slice of AdS space. Moreover, in the MERA tensor network, the minimal cut that divides the network into two parts (i.e., the one that cuts through the minimal number of internal legs) provides an estimate of the entanglement entropy of a subregion. This strongly resembles the RT formula. Therefore, the MERA tensor network is regarded as a concrete realization of the anti-de Sitter/conformal field theory (AdS/CFT) duality. Due to the physical meaning of tensor networks, it is natural that the geometric structure of the MERA tensor network encodes information about quantum entanglement. The approach of using tensor networks to study the entanglement structure in holographic duality has been continuously developed, and many other concrete models of holographic tensor networks have been proposed and improved, see, e.g., [12–38].

Another novel approach to studying the entanglement structure in holographic duality is based on a “thread picture” (as opposed to the “network picture” of tensor networks). This thread-based picture first attracted attention in 2016, when Headrick and collaborators developed the concept of bit threads [39–42]. Consider partitioning a holographic quantum system into two parts, A and $\bar{A}$. A bit thread is a one-dimensional line that starts from region A , traverses the holographic bulk, and connects to region $\bar{A}$. By drawing an analogy with the max-cut/min-flow theorem in network flow theory [43,44], they discovered that if the

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density of bit threads in the bulk is constrained by an upper bound, then the maximum possible number of threads connecting A and $\bar{A}$ is exactly given by the area of the Ryu-Takayanagi surface—a minimal extremal surface. The bit thread bundle vividly characterizes the entanglement entropy between A and $\bar{A}$, as the RT surface area equals the entanglement entropy according to the RT formula. Furthermore, this naturally connects to the concept of entanglement distillation [45–47] in quantum information theory.

In this paper, inspired by the thread picture that naturally emerges in holographic tensor network models, we abstract the notion of entanglement threads. These entanglement threads traverse codimension-one spatial slices of the holographic bulk, with their endpoints anchored on the holographic boundary. We argue that the number of entanglement threads connecting two subregions on the holographic boundary is given by the half-conditional mutual information. Furthermore, we demonstrate that the specific trajectory of each entanglement thread in the bulk is precisely a geodesic. This leads us to view the kinematic space [48,49] as a “circuit board” that organizes qudits, with entanglement threads serving as wires extending from it, utilized to construct quantum circuits for the corresponding holographic states. Therefore, the collection of entanglement threads provides a prestructure for adding basic tensors or quantum gates. In this sense, we propose that entanglement threads encode some underlying partial order information of the entanglement structure in holographic duality. Each entanglement thread characterizes how the qudits it threads are globally entangled. Ultimately, the entanglement relationships are intuitively captured—not only between various subregions on the holographic boundary, but also between different surfaces within the bulk, via entanglement threads that simultaneously pass through them.

The structure of this paper is as follows. In Sec. II, we introduce the background and motivation, presenting the basic setup of the entanglement thread picture along with the inspiration from tensor networks. In Sec. III, we clarify the rules governing the trajectories of entanglement threads in the holographic bulk, helping us confirm that their paths in the bulk slices are exactly geodesics. In Sec. IV, we introduce the mathematical tools of kinematic space, which are particularly well-suited for organizing these entanglement threads. We demonstrate how entanglement threads can be used as wires to construct a quantum circuit representing holographic quantum states, and verify this idea from the perspective of holographic complexity. In Sec. V, we further discuss the physical significance of entanglement threads and their connections and distinctions from the earlier bit thread concept. Section VI is the conclusion. For completeness, a more detailed review of kinematic space is provided in the Appendix.

II. BACKGROUND AND MOTIVATION

A. Representing holographic quantum entanglement as threads

The idea of representing holographic quantum entanglement using threads is, in a sense, simple and straightforward.¹ Since entanglement entropy measures the degree of quantum entanglement, for example, if we consider two complementary systems A and B , and wish to depict a contribution of one unit ($\log 2$) of entanglement to the entanglement entropy, we may schematically draw a line (thread) connecting A and B . Thus, when the entanglement entropy between A and B is $n \log 2$, we draw n threads between A and B . Generalizing this to multiple subsystems $A_1, A_2, A_3, \dots, A_N$ is straightforward; we simply imagine each pair A_i and A_j being connected by a bundle of threads, whose number is denoted by F_{ij} . It is proven in [51] that in order to correctly match the entanglement entropy for any bipartition of the total system (which we assume to be in a pure state), the number of threads F_{ij} connecting regions A_i and A_j is precisely given by one half of the quantum conditional mutual information (qCMI) (as we will review in the following).

In the holographic framework, one observes that in addition to the quantum system itself, there naturally exists a higher-dimensional bulk. This makes the idea of characterizing entanglement via threads even more intuitive. From now on, we shall appropriately refer to these threads as holographic entanglement threads, or simply entanglement threads. We can imagine these threads being carried by—or threading through—codimension-1 spatial slices of the bulk, as originally inspired by the bit threads proposed in [39–42]. In particular, this idea aligns closely with the Ryu-Takayanagi (RT) proposal for holographic entanglement entropy [4–6]. To see this, consider the simplest case of a subregion A in a CFT pure state that is dual to a pure AdS bulk. We may imagine a family of uninterrupted threads, each connecting A and its complement B , passing through the RT surface γ_A , with the number of threads equal to the entanglement entropy between A and B , denoted by $S(A)$ [see Fig. 1(a)]

$$F_{AB} = S_A. \quad (1)$$

By the RT formula [4–6] (setting $4G_N = 1$), the area of γ_A equals S_A , so we further have

¹Motivated by the concept of bit threads [39–42], the thread-kind objects have been used in a series of works (see, e.g., [13,14,50–58]) to attempt to describe the connection between quantum entanglement and spacetime structure. Despite of their different perspectives, these works essentially express a rather natural idea to decompose entanglement entropy in a more refined manner. This paper can be regarded as a unified description of these thread objects, by abstracting the concept of entanglement threads from the general tensor network model.

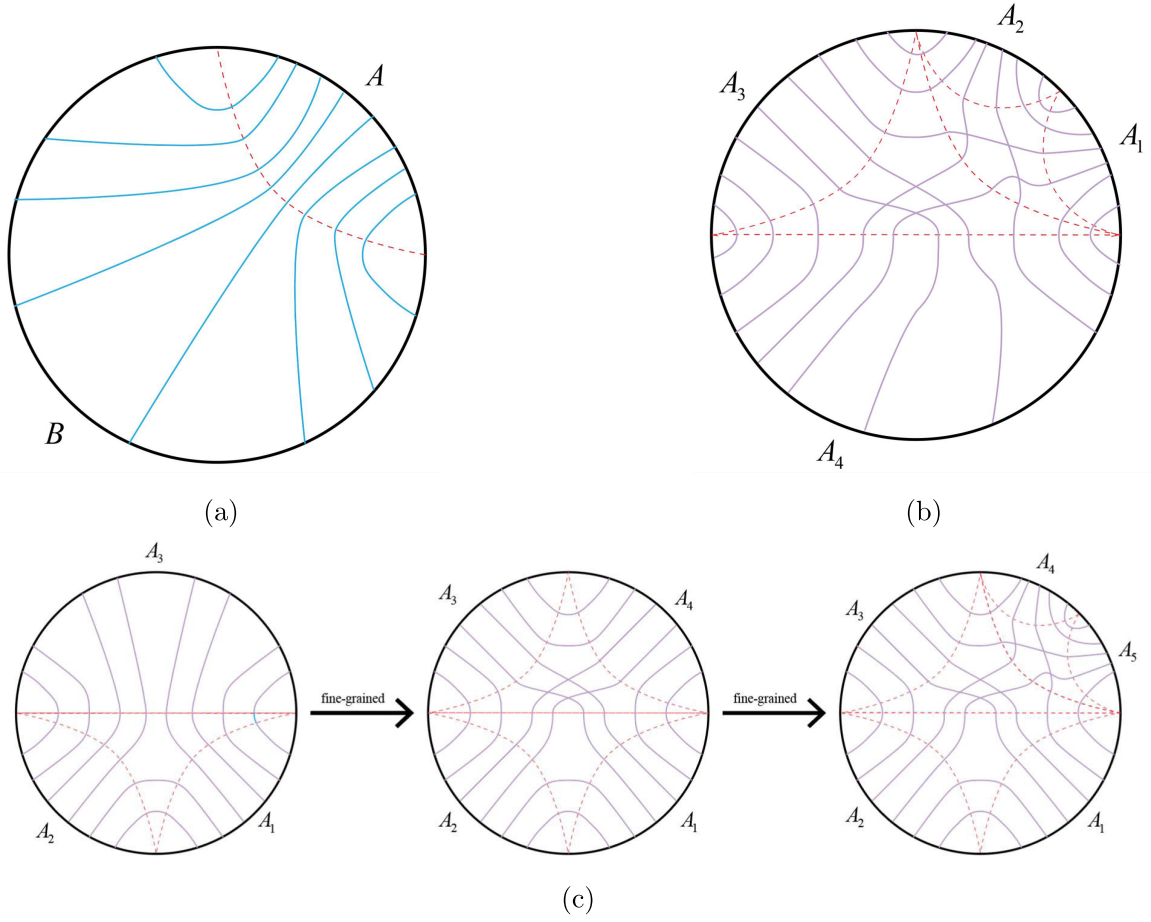


FIG. 1. (a) A holographic “thread” picture characterizing the entanglement entropy between two complementary regions. (b) A more refined thread configuration characterizing a set of entanglement entropies involving more subregions. (c) By iteratively dividing the quantum system, more and more refined thread configurations can be constructed, characterizing the entanglement structure at more and more refined levels. Here the RT surfaces are represented as red dashed lines, while the threads are schematically represented as purple lines. The pictures here are basically taken from another paper [57]. However, please note that the exact trajectories of these entanglement threads have not yet been really specified, which is precisely what this present paper aims to achieve.

$$F_{AB} = \text{Area}(\gamma_A), \quad (2)$$

which means that each unit area of γ_A accommodates exactly one entanglement thread.

As mentioned earlier, we can go beyond the bipartite case and do better. We can construct increasingly refined thread configurations that allow us to compute the entanglement entropies for not just a single subregion. For instance, as shown in Fig. 1(b), we can further decompose region A into $A = A_1 \cup A_2$, and B into $B = A_3 \cup A_4$. Thus, we can construct a finer thread configuration that enables us to compute the entanglement entropies between six connected regions (A_1, A_2, A_3, A_4 , as well as $A = A_1 \cup A_2$ and $B = A_3 \cup A_4$) and their complements. Specifically, we design the entanglement thread configuration so that the number of threads connecting each of these six regions to their complements matches the corresponding entanglement entropy. Again, using the RT formula, we can describe that on each of the six connected RT surfaces

$\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_{12}, \gamma_{34}$, there is exactly one entanglement thread per unit area. This “thought experiment” can be iterated: we further partition the quantum system M into more adjacent and nonoverlapping “elementary regions” $A_1, A_2, \dots, A_N$,² with

$$A_i \cap A_j = \emptyset, \quad \bigcup_i A_i = M, \quad (3)$$

and obtain a correspondingly finer thread configuration [see Fig. 1(c)]. It is preferable that each elementary region remains much larger than the Planck scale, to carefully ensure the applicability of the RT formula. Let us denote the number of entanglement threads between each pair of regions A_i and A_j as F_{ij} . Following [50,51], we describe this thread configuration using a complete graph, as shown

²We can take N to be very large, expecting to capture more refined entanglement structure.

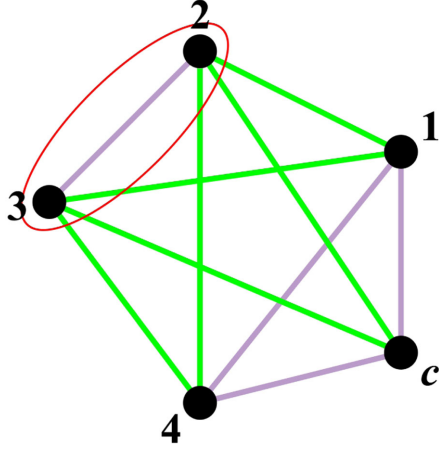


FIG. 2. A simplified network diagram corresponding to the entanglement thread configuration. Each elementary region is simplified by a point and the component flow flux F_{ij} is represented by the line segment connecting two points. Then for example, the entropy of the region in red circle is equal to the sum of the green lines in this figure.

in Fig. 2, where each region A_i is simplified to a boundary vertex and each thread bundle between regions is simplified to an edge connecting two vertices. The value F_{ij} is represented by the weight of the edge. Adjacent elementary regions can be merged into simply connected regions, e.g., $A_{12} = A_1 \cup A_2$, and all such simply connected unions form a set: $\{A_{i(i+1)\dots j} | i \leq j\}$, whose number of elements is $\frac{N(N-1)}{2}$. Each simply connected region $A_{i(i+1)\dots j}$ has an entanglement entropy $S_{i(i+1)\dots j}$ with its complement (with the assumption that M is in a pure state). According to the thread representation of entanglement, we write

$$S_{a(a+1)\dots b} = \sum_{i \in \{a, \dots, b\}, j \notin \{a, \dots, b\}} F_{ij}, \quad (4)$$

This can be intuitively understood as follows: the entanglement entropy between $A = A_{i(i+1)\dots j}$ and its complement $\bar{A}$ comes from the total number of threads connecting elementary regions within A and those in $\bar{A}$. Given the physical meaning of F_{ij} as the number of threads connecting A_i and A_j , the values of F_{ij} are evidently symmetric in i and j , so their total number is also $\frac{N(N-1)}{2}$, matching the number of entropies $S_{i(i+1)\dots j}$. Thus, the system of equations (4) is full rank and has a unique solution [51], given by

$$F_{ij} = \frac{1}{2} I(A_i, A_j | L) = \frac{1}{2} [S(A_i \cup L) + S(A_j \cup L) - S(L) - S(A_i \cup L \cup A_j)], \quad (5)$$

where $L = A_{(i+1)\dots(j-1)}$ denotes the simply connected region between A_i and A_j , encoding the size of their

separation. In other words, the number of threads connecting A_i and A_j is precisely given by the half-conditional mutual information.

At this point, we should note that in using the simplified complete graph and Eq. (4), we have so far assigned these threads only a topological meaning. That is, although we believe the higher-dimensional bulk hosts these threads, we have not specified their actual trajectories in the bulk. In fact, the goal of this paper is precisely to explore:

To what extent is this idea of representing CMI as threads reasonable, and how should we understand these threads, especially their trajectories in the holographic bulk...

It is also worth emphasizing that CMI does not have to be visualized as threadlike objects. This approach is motivated by several reasons. First, it is a naive attempt to depict entanglement between two subsystems by connecting them with threads—the more entangled, the more threads. Second, the idea is inspired by bit threads [39–41], where entanglement entropy can be encoded in the number of threads. Third, in the special context of holography, the extra bulk dimension naturally serves to accommodate the trajectories of these threads.

Finally, as background, we note that conditional mutual information appears naturally in various studies of holographic duality, such as in holographic kinematic space [48,49], the holographic entropy cone [59–62], and holographic partial entanglement entropy [63–66]. The reason is quite understandable: these studies all focus on exploring possible decompositions of holographic entanglement entropy, and the “visual metaphor” we just discussed reveals a very simple underlying essence.

Next, we will soon turn to the topic of tensor networks. After gaining some inspiration there, we will return to the topic of threads.

B. The thread picture in tensor network models

In this section, we briefly shift to the topic of tensor networks, as a review of this will provide us with significant insights into the thread picture. We will realize that in tensor network models of holographic duality, structures resembling the entanglement thread picture described above emerge naturally and ubiquitously. Beyond this phenomenological inspiration, the thread picture within tensor networks also endows entanglement threads with more precise quantum-state interpretations and helps us understand the meaning of their trajectories in the bulk.

We take the example of the “HaPPY tensor network state.” Based on the close connection between holographic states and quantum error-correcting codes [12,67,68] proposed a tensor network composed of perfect tensors, as shown in Fig. 3(a), to describe the quantum states of a

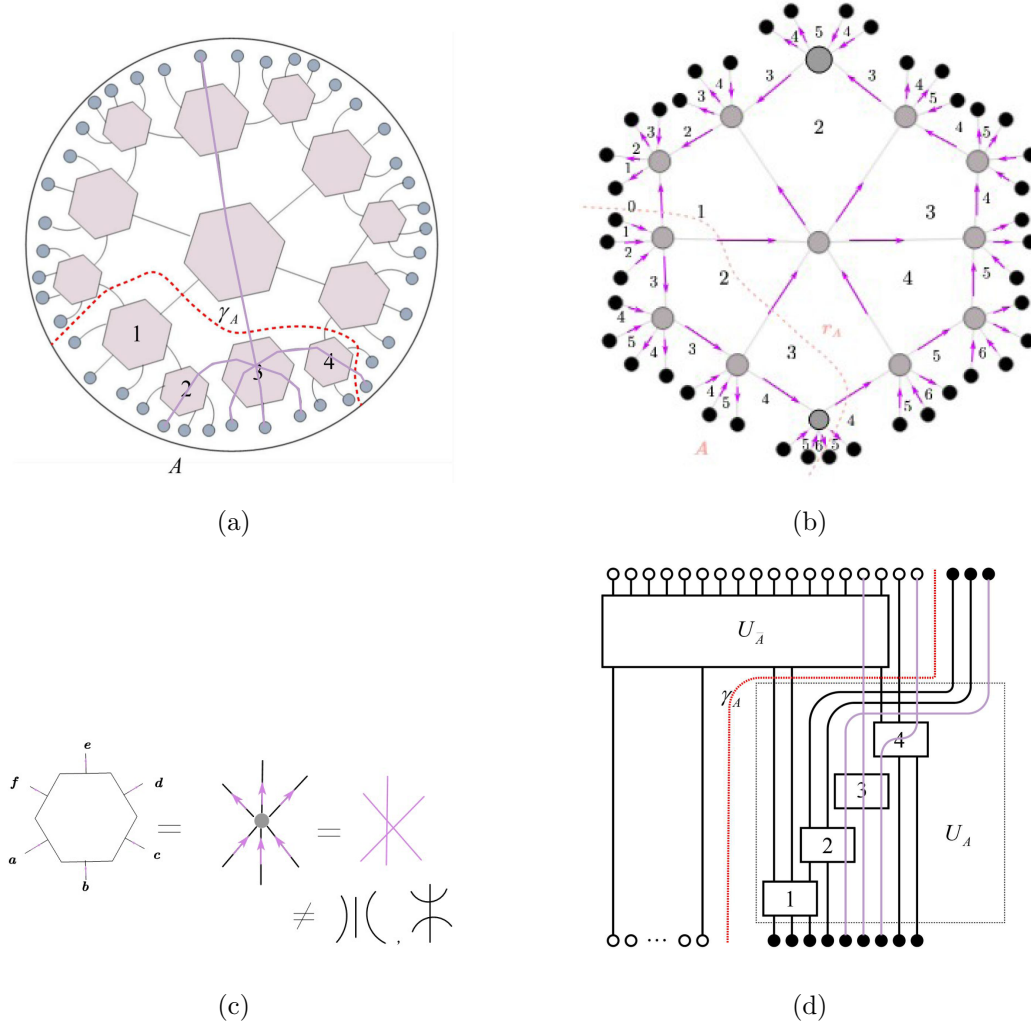


FIG. 3. (a) HaPPY tensor network state and an RT surface (red dashed line) which calculates the entanglement entropy of a subregion A . Furthermore, in Sec. III C, we will obtain the specific trajectory of the entanglement thread (the purple line) therein. (b) Following [12], we assign orientations to the edges (i.e., legs) of the HaPPY network. The global “flow” structure of the entire network naturally generates a picture similar to the thread configuration—by gluing together many small arrows pointing in the same direction to form a single thread. (c) In principle, there is more than one way to connect arrows of the same direction into threadlike objects. However, the rules in Sec. III B fully determine the concrete configuration of entanglement threads. (d) Deforming the tensor network topologically into a quantum circuit. Then an entanglement thread (the purple line) corresponds to a quantum wire therein. In particular, we have demonstrated three entanglement threads as explanatory aids [for both (a) and (d)].

holographic quantum system. Each hexagonal basic unit extends six legs, representing a basic perfect tensor of six qubits T_{abcdef} . A $2s$ -perfect tensor is a pure state of $2s$ qudits such that, for any positive integer $k \leq s$, the mapping from any k qudits to the remaining $2s - k$ qudits is an isometry [69–72].

It is noteworthy that the original paper [12] has already revealed the presence of an evident thread picture in the HaPPY tensor network. More specifically, as pointed out in [12], to calculate the entanglement entropy of a connected region A within the framework of the HaPPY tensor network, one can perform a topological deformation of the tensor network into a quantum circuit. As shown in Fig. 3(d), after deforming the tensor network topologically

into a quantum circuit, the entanglement entropy of A is simply given by the number of wires that pass through the RT surface γ_A . For completeness, we now briefly review the method of transforming the HaPPY tensor network into a quantum circuit. Following [12], we first assign orientations to the edges (i.e., legs) of the HaPPY network. As shown in Fig. 3(b) (where, we represent the hexagonal pattern simply as a circular vertex), this can be achieved by assigning numbers to the nodes of the dual graph.³ The procedure is as follows: first, choose any boundary node

³In graph theory, a node in the dual graph corresponds to a region enclosed by edges in the original network (at the boundary, this is given by the gaps between the external legs).

(i.e., the “gap” between two external legs) as the “starting point” and label it with the number 0. Then, assign a number to each dual node such that the number equals the distance between that node and the starting node. In other words, in the original network, the number assigned to a region is the minimum number of edges that need to be crossed to reach it from region 0. These numbers then determine the orientation of each edge in the HaPPY network, according to the following rule: the number on the right side of an “arrow” (i.e., an directed edge) is always greater than that on the left side. As shown in Fig. 3(b), the global “flow” structure of the entire network naturally generates a picture similar to the thread configuration—the idea is simply to glue together many small arrows pointing in the same direction to form a single thread. As pointed out in [12], this is facilitated by the fact that each perfect tensor has an even number of legs, ensuring that the number of incoming arrows at each vertex in the network equals the number of outgoing arrows. This guarantees “flow conservation,” which makes defining a thread configuration possible. Moreover, the negatively curved tiling pattern of the HaPPY model ensures the uniqueness of edge directionality and the absence of circular flows, which matches the default rule of entanglement threads: each thread connects two boundary points. Using the direction of arrows in the network as a partial order, we can topologically deform the directed HaPPY tensor network graph into a quantum circuit diagram, as shown in Fig. 3(d), where each basic tensor is interpreted as a quantum gate with three inputs and three outputs.

Based on the above observations and insights, there are several comments as follows. First, as seen in Fig. 3(b), the partial ordering defined on the HaPPY tensor network naturally gives rise to the concept of “local flow lines” or “arrows,” which strongly suggests that one can glue segments of local flow lines into global entanglement threads. However, at this stage, [12] does not specify the concrete rules for such gluing. From the diagram, it seems that there is significant freedom in gluing the local arrows into global entanglement threads. In the quantum circuit representation in Fig. 3(d), this degree of freedom is reflected in the fact that the number of wires are conserved in the quantum circuit. Since one can always insert auxiliary two-qudit SWAP gates to redefine the “apparent wires,”⁴ the diagrammatic representation of a quantum circuit is not unique. In Sec. VB, we will propose an interesting way to reconcile the nonuniqueness of such circuit drawings with the definite trajectories of entanglement threads, by comparing the concept of entanglement threads with the earlier concept of bit threads [39–41].

⁴By apparent wires, we refer to the visually parallel straight lines in the standard quantum circuit diagrams, representing the evolution of individual qudits.

The issue of how to glue local “arrows” into global entanglement threads is in fact equivalent to the question of determining the trajectories of entanglement threads through the bulk. This is the central topic we will address in detail in the following sections. What is important for now is that, in the context of tensor networks, we have already obtained a concrete way to define the trajectories of entanglement threads: the trajectory of a thread can be defined as the ordered sequence of basic tensors (or graphically, “hexagonal tiles”) in the bulk that it passes through. Moreover, since each leg extending from a basic tile represents a qudit, each entanglement thread can actually be considered as a collection of qudits. We shall adopt and further develop these insights in the following sections.

III. TRAJECTORIES OF HOLOGRAPHIC ENTANGLEMENT THREADS IN THE BULK

A. The “no-return” rule of entanglement threads on RT surfaces

As mentioned in the previous section, according to our understanding, each entanglement thread should be constructed by gluing together a series of adjacent arrows. However, at least in the case of HaPPY tensor network, the partial ordering information characterized by the collection of all local directed arrows is not sufficient to fully determine the specific trajectory of each entanglement thread. The question of how local arrows are glued into global entanglement threads is essentially equivalent to the question of how entanglement threads traverse the bulk.

To come straight to the point, in this section, we will further demonstrate that the trajectories of these entanglement threads are precisely geodesics in the holographic bulk. This conclusion only requires us to pay attention to the following phenomenological fact: according to the thread scheme described in Sec. II A, an entanglement thread cannot pass through a given simply connected RT surface more than once. Figure 4(a) illustrates some examples of disallowed entanglement thread configurations. The reason lies in the constraint from the holographic principle. For convenience, we adopt the unit convention $4G_N = 1$. Under this convention, the entanglement entropy between a simply connected region A and its complement $\bar{A}$ is exactly equal to the area of the RT surface γ_A . Since we require that the number of entanglement threads connecting A and $\bar{A}$ be precisely equal to this entropy, it follows that each unit area on the RT surface γ_A can accommodate exactly one entanglement thread, and each thread is allowed to occupy that unit area only once. Similarly, an entanglement thread with both endpoints in A , or both in $\bar{A}$, is not allowed to pass through γ_A , because the area of γ_A is only sufficient to accommodate “travelers” whose starting and ending points lie in A and $\bar{A}$, respectively.

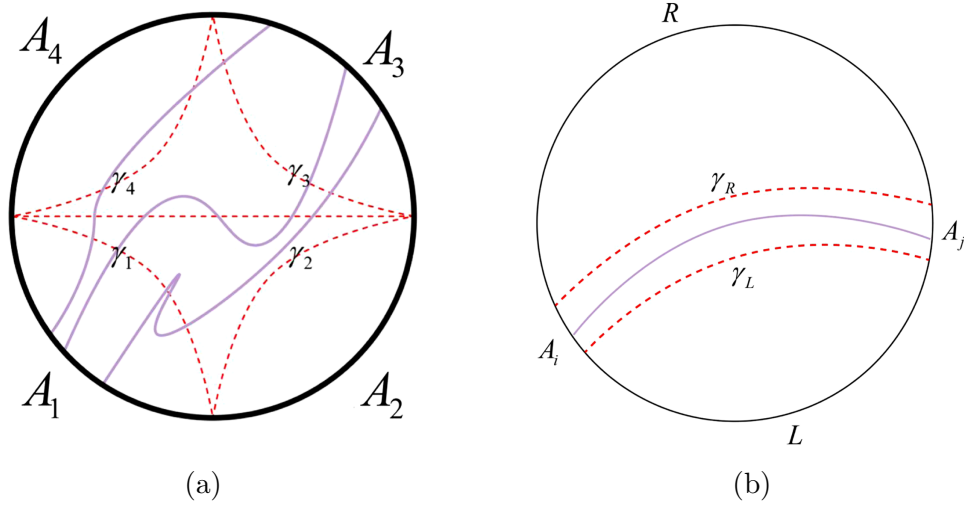


FIG. 4. (a) Some counterexamples that violate the thread trajectory rules. (b) By continuously shrinking the sizes of A_i and A_j , in the limit, ζ_{ij} must coincide with the geodesics.

According to this “no-return” rule of entanglement threads on RT surfaces, it is not hard to conjecture that the trajectories of entanglement threads in the bulk should coincide with geodesics. To intuitively understand this conjecture requires only a few lines of argument. As shown in Fig. 4(b), let us suppose that the sizes of two elementary regions A_i and A_j are made very small. Denote the regions between them as L and R , respectively. Then, according to the no-return rule, the entanglement thread ζ_{ij} connecting A_i and A_j will be confined within a narrow channel bounded by two RT surfaces γ_L and γ_R , because ζ_{ij} is not allowed to pass through either γ_L or γ_R more than once. Therefore, as we continuously shrink the sizes of A_i and A_j , in the limiting case, the trajectory ζ_{ij} must coincide with the geodesic segment defined between γ_L and γ_R .

B. The entanglement meaning of thread trajectories

Although we have already argued from a phenomenological perspective that entanglement threads follow certain trajectories, it is still necessary to discuss their precise meaning at the level of quantum information theory, so as to provide a more convincing foundation for this concept. In this subsection, with the intuition of tensor network picture in mind, we present an analysis from the quantum state perspective concerning the trajectories of entanglement threads in the bulk. Note that our following discussion does not rely on assuming the AdS bulk to be exactly modeled by a HaPPY tensor network; it can be fully applied to more abstract and general classes of tensor networks. This is one of the advantages of the entanglement thread approach over specific tensor network constructions.

The basic idea is to locally decompose the entire bulk. Without loss of generality, we consider (as shown in Fig. 5) a decomposition of the bulk slice W into four parts W_1, W_2, W_3, W_4 by two intersecting RT surfaces γ_{12} and γ_{23} . This

can also be viewed as a four-part division of a holographic tensor network, as long as we conventionally define RT surfaces as minimal cuts of internal bonds within the tensor network. Let us, for instance, focus on the small region W_1 adjacent to A_1 , whose boundary is labeled as $\partial W_1 = A_1 \cup \sigma_1 \cup \sigma_3$. Treating the bulk slice as a tensor network,⁵ we are led to two insights. First, the boundary $A_1 \cup \sigma_1 \cup \sigma_3$ can be regarded as corresponding to a pure state, which in principle can be considered as constructed from contracting the basic tensors within the subnetwork W_1 . This idea is referred to as the surface-state correspondence in [73,74]. Second, one can imagine that the local arrows in W_1 are sufficient to be glued into a local entanglement thread configuration V_{W_1} , which characterizes the entanglement structure of the sub-bulk region W_1 and its boundary dual state—just as when we consider the full bulk region W , all the local arrows can be glued into a global thread configuration V_W . In particular, V_{W_1} should be a local portion of the full thread configuration V_W , since they share the same set of local arrow “material” in region W_1 .

After abstracting away from specific tensor network models and returning to the pure AdS setting, the above argument motivates the following proposal: as illustrated in Fig. 5, when the entire holographic boundary is partitioned into four regions,⁶ we should at least be able to construct a compatible thread configuration, such that: not only should

⁵Regardless of whether the HaPPY network is a precise model, one can always imagine a sufficiently good holographic tensor network to make the discussion satisfactory. We note that recent work [28–30] has made a remarkable step toward constructing such concrete tensor networks.

⁶Of course, in our framework, one can expect more refined thread configurations when the boundary is partitioned into more regions.

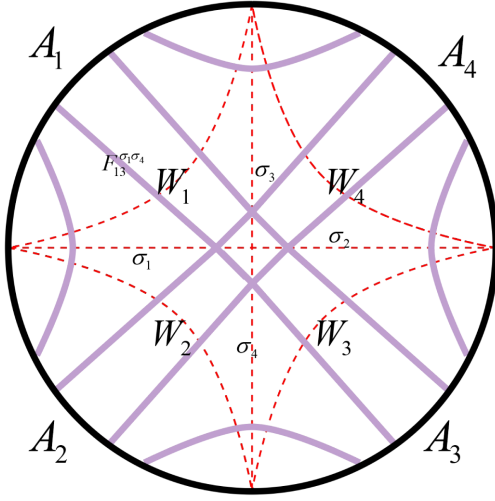


FIG. 5. A decomposition of the bulk slice W into four parts W_1, W_2, W_3, W_4 by two intersecting RT surfaces $\gamma_{12} = \sigma_3 \cup \sigma_4$ and $\gamma_{23} = \sigma_1 \cup \sigma_2$. Further considering the trajectory of the entanglement threads in each fragment, now a total of 8 independent thread bundles (each bundle is represented by a thick purple line) are required, and they are marked using the generalized RT surfaces (bulk extremal surfaces) they pass through.

it reproduce, via the prescription in Sec. II A, the holographic entanglement entropies for regions A_1, A_2, A_3, A_4 and their unions related to the full bulk slice W , but also, for each W_i , it should reproduce the holographic entanglement entropies of the related boundary regions (e.g., for W_1 : A_1, σ_1, σ_3 , and their unions), using the same prescription.

Now we proceed with the analysis. From a global perspective, following the formula in Sec. II A, to characterize the entanglement structure among the four elementary regions A_1, A_2, A_3, A_4 , we only need six independent bundles: $\{F_{12}, F_{13}, F_{14}, F_{23}, F_{24}, F_{34}\}$, which are sufficient to capture the set of entropies $\{S_1, S_2, S_3, S_4, S_{12}, S_{23}\}$. Furthermore, in the metaphor of tensor networks discussed in Sec. II B, solving for these six values of F_{ij} amounts to determining the number of entanglement threads connecting each pair (A_i, A_j) , assuming that the local arrows have been glued into a global configuration. However, this does not yet specify the exact paths these threads take through the bulk network. Now, by decomposing W into four regions W_1, W_2, W_3, W_4 , and applying the entanglement thread formula to each subregion, we gain further information about the trajectories of the threads. For example, consider W_1 and its three boundary regions A_1, σ_1, σ_3 . Constructing V_{W_1} requires, in principle, three thread bundles with fluxes denoted by $\{F_{A_1\sigma_1}, F_{A_1\sigma_3}, F_{\sigma_1\sigma_3}\}$. However, since V_{W_1} is just a fragment of V_W , we expect the entanglement entropies of A_1, σ_1, σ_3 to be fully captured using only the threads already present in V_W . This in turn constrains the possible trajectories of the threads in V_W . To see this, observe that, by the no-return rule for entanglement threads, a bundle

going from A_1 to A_3 has two possible routes: via $\sigma_3 \rightarrow \sigma_2$ or via $\sigma_1 \rightarrow \sigma_4$. Likewise, the bundle from A_2 to A_4 has two possible paths. We can write

$$\begin{aligned} F_{13} &= F_{13}^{\sigma_1\sigma_4} + F_{13}^{\sigma_3\sigma_2}, \\ F_{24} &= F_{24}^{\sigma_1\sigma_3} + F_{24}^{\sigma_4\sigma_2}. \end{aligned} \quad (6)$$

Here, the superscripts indicate the intermediate surfaces traversed by the threads. On the other hand, according to the no-return rule, the threads from A_1 to A_2 must all pass through σ_1 , and cannot pass through σ_2 , or else they would cross the RT surface γ_{12} at least twice. Applying similar reasoning, we now have eight independent thread bundles in total, that is $\{F_{12}^{\sigma_1}, F_{23}^{\sigma_4}, F_{34}^{\sigma_2}, F_{14}^{\sigma_3}, F_{13}^{\sigma_1\sigma_4}, F_{13}^{\sigma_3\sigma_2}, F_{24}^{\sigma_1\sigma_3}, F_{24}^{\sigma_4\sigma_2}\}$. We can now solve for the flux values of these eight bundles. This is because, in addition to the six entanglement entropies of the boundary regions of W (which correspond to the areas of RT surfaces $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_{12}, \gamma_{23}$), we now also have entanglement entropies of the boundary subregions of W_1, W_2, W_3, W_4 as further constraints. For example, for W_1 , the entanglement entropy of σ_1 (whose RT surface is itself, hence the entropy equals its area) provides a constraint

$$\text{Area}(\sigma_1) = F_{12}^{\sigma_1} + F_{13}^{\sigma_1\sigma_4} + F_{24}^{\sigma_1\sigma_3}. \quad (7)$$

Note that since the areas of all six simply connected RT surfaces have already been used, once the area of σ_1 is determined, the area of σ_2 is also determined, since their sum is exactly the area of σ_3 . We are then left with only one last constraint, which can be chosen as, for instance, the entropy of σ_3 (as a boundary subregion of W_4)

$$\text{Area}(\sigma_3) = F_{14}^{\sigma_3} + F_{13}^{\sigma_3\sigma_2} + F_{24}^{\sigma_1\sigma_3}. \quad (8)$$

Finally, we obtain eight effective constraints to solve for the eight flux values.⁷ In particular, we can calculate the flux values of the splitting flow from F_{13} and F_{24} explicitly as

$$\begin{aligned} F_{13}^{\sigma_1\sigma_4} &= \frac{1}{2}(\text{Area}(\sigma_1) + \text{Area}(\sigma_4) - \text{Area}(\gamma_{23})), \\ F_{13}^{\sigma_3\sigma_2} &= \frac{1}{2}(\text{Area}(\sigma_2) + \text{Area}(\sigma_3) - \text{Area}(\gamma_{14})), \\ F_{24}^{\sigma_1\sigma_3} &= \frac{1}{2}(\text{Area}(\sigma_1) + \text{Area}(\sigma_3) - \text{Area}(\gamma_{12})), \\ F_{24}^{\sigma_4\sigma_2} &= \frac{1}{2}(\text{Area}(\sigma_2) + \text{Area}(\sigma_4) - \text{Area}(\gamma_{13})). \end{aligned} \quad (9)$$

We find that each formula in (9) corresponds exactly to the conditional mutual information between two boundary

⁷Here we omit the other six equations corresponding to the RT surfaces $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_{12}, \gamma_{23}$, since they are already self-evident according to Sec. II A.

regions of a sub-bulk. For instance, $F_{24}^{\sigma_1\sigma_3}$ is precisely the conditional mutual information between σ_1 and σ_3 within W_1 . This shows that, within W_1 , the entanglement threads connecting σ_1 and σ_3 are entirely derived from the portion of the global thread bundle connecting A_2 and A_4 that passes through those surfaces—just as we would expect.

We have come full circle to present the following argument to the reader: Not only can we, in the sense of Sec. II A, discuss the flux values of entanglement threads between elementary boundary regions (which rely only on the topological structure of threads, and can be simply encoded in a complete graph), but we can also discuss how threads traverse the bulk slice—and show that their precise trajectories can, in principle, be determined from more detailed entanglement information within the bulk. In Sec. III A, we boldly conjectured that the trajectories of entanglement threads are geodesics. In Sec. IV, we will use the language of kinematic space [48,49] to verify that assuming threads follow geodesics indeed leads to the result in (9).

C. Trajectories of entanglement threads in the HaPPY tensor network

As an example of the close connection between the entanglement thread approach and the tensor network approach, let us now, based on the analysis of Sec. III B, proceed to rigorously determine the trajectories of entanglement threads in the HaPPY tensor network model, extending the results of [12]. The insight from Sec. III B is to consider a subregion of the bulk—or in the tensor network context, a subnetwork. Now let us take the subnetwork to be a single six-legged tensor T itself. Applying the labeling and orientation rule from Sec. II B—where numbers are assigned to the dual graph—we find that the six legs of T must be oriented such that its three incoming legs are adjacent to each other. Accordingly, we can assign the six indices a, b, c, d, e, f (physically representing six qudits) to the six legs in a unique way, starting from the first incoming leg and proceeding counterclockwise. Again, we observe that, in principle, there is more than one way to connect arrows of the same direction into threadlike objects. However, if we treat the six qudits as six elementary regions and consider all simply connected regions such as $a \cup f$, $a \cup f \cup e$, etc., then—similar to the argument in Sec. III B—we can compute that, within this subtensor network, we have

$$F_{ad} = F_{be} = F_{cf} = \log d, \quad (10)$$

where d is the dimension of each qudit, and all other fluxes are zero. Let $d = 2^\chi$, then Eq. (10) implies that, in the HaPPY tensor network, for each subnetwork represented by a single tensor, the entanglement thread distribution is that: there are χ threads connecting a to d , χ threads connecting b to e and χ threads connecting c to f

[see Fig. 3(c)]. Only in this way can the flux of threads connecting each simply connected region and its complement correctly match the entanglement entropy. For the HaPPY tensor network, this simple analysis fully determines the concrete configuration of entanglement threads, as shown in Fig. 3(a). Note that since this tensor network is merely a discrete depiction of a spatial slice of pure AdS, its entanglement thread configuration inevitably loses some finer details. Nevertheless, we observe that each entanglement thread in the figure indeed appears as a geodesic, consistent with our argument in Sec. III A.

Recall that, as reviewed in Sec. II B, the HaPPY tensor network can be topologically deformed into a conventional quantum circuit diagram. We can now also mark the entanglement thread trajectories in the quantum circuit diagram. Figure 3(d) shows examples of a few entanglement threads and their corresponding paths. It is evident that, since we arbitrarily chose a particular representation of the apparent wire set in the quantum circuit, these entanglement thread trajectories are not simple straight lines. In principle, one can choose a canonical representation of the quantum circuit such that each apparent wire precisely corresponds to a single entanglement thread. In the next section, we will use tools from kinematic space to explicitly construct such a canonical representation.

IV. ENTANGLEMENT THREADS, KINEMATIC SPACE, AND QUANTUM CIRCUITS

A. Characterizing entanglement threads through kinematic space

We have proposed in Sec. III that the traveling trajectory of an entanglement thread in the bulk is exactly a geodesic. This naturally relates to the concept of kinematic space [48,49], which is defined as the set of all boundary-anchored geodesics in the holographic bulk. Indeed, we should have expected this, because in the formulation of kinematic space, conditional mutual information plays a central role in the metric of the space due to its elegant mathematical structure. However, we arrived here from a different direction—namely, from the perspective of quantum entanglement. We place a review of the kinematic space in the Appendix. Here, to verify that the trajectories of entanglement threads are indeed geodesics, we can adopt the mathematical techniques used to characterize geodesics in kinematic space to compute the fluxes in the example of Fig. 5.

As a demonstration, let us compute the flux $F_{13}^{\sigma_1\sigma_4}$, which denotes the number of threads starting from A_1 , passing through the surfaces σ_1 and σ_4 , and reaching A_3 . Since we now assume that the trajectories of these entanglement threads precisely coincide with geodesics, it becomes relatively straightforward to identify the set of geodesics just enough to satisfy the trajectory constraints, which, in

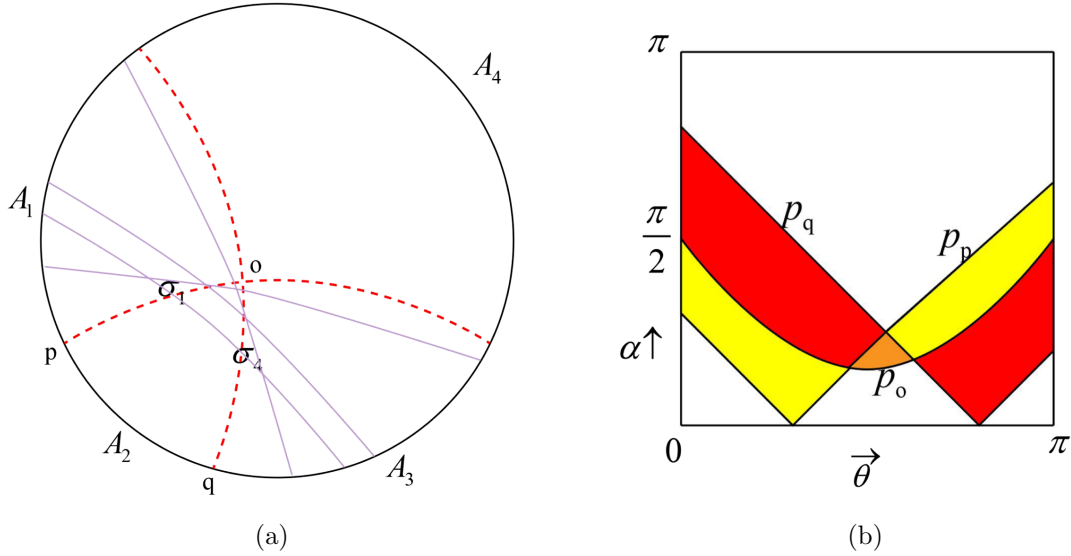


FIG. 6. (a) Some geodesics accounting for $F_{13}^{\sigma_1\sigma_4}$ are shown. (b) The set of geodesics contributing to $F_{13}^{\sigma_1\sigma_4}$ corresponds to the orange region, which is the intersection of the red and yellow regions.

kinematic space, form particular curves made up of collections of points. These borderline curves in kinematic space correspond precisely to the so-called point-curves associated with the three points involved in the problem, namely o , p , and q , as illustrated in Fig. 6. In Fig. 6(b), the red region is bounded by the point-curves corresponding to o and q , representing those geodesics in the original space that intersect the segment oq . The yellow region is bounded by the point-curves corresponding to o and p , representing geodesics intersecting the segment op in the original AdS space. Clearly, the geodesics that contribute to $F_{13}^{\sigma_1\sigma_4}$ are those lying in the intersection of the red and yellow regions—that is, the orange region shown in the figure. The area of the orange region directly gives the value of $F_{13}^{\sigma_1\sigma_4}$. Using the notation from the Appendix, we can compute

$$F_{13}^{\sigma_1\sigma_4} = \frac{1}{2} \left(\frac{1}{4} \int_{p_o \Delta p_q} \tilde{\omega} + \frac{1}{4} \int_{p_o \Delta p_p} \tilde{\omega} - \frac{1}{4} \int_{p_p \Delta p_q} \tilde{\omega} \right). \quad (11)$$

From Eq. (A14), we can identify that the last term inside the parentheses is precisely $S_2 = \frac{\text{Area}(\gamma_2)}{4G_N}$, and we can also recognize that the first two terms are exactly the same as the corresponding terms in Eq. (9). Thus, (11) is completely consistent with (9).

We have therefore verified that the mathematics of kinematic space can indeed be correctly applied to study the fluxes of entanglement threads. This provides us with a systematic mathematical tool for analyzing the entanglement structure in holographic duality using holographic entanglement threads. The simple example shown here is not sufficient to fully demonstrate the power of the mathematical structure of kinematic space. However, for more general problems—such as more complex divisions

of boundary regions, or more general holographic scenarios like wormholes—the simple method of using linear combinations of the areas of a few RT surfaces in the bulk will no longer suffice. In such cases, the mathematical language of integral geometry will offer powerful tools. We leave these computational challenges for future exploration.

B. Kinematic space, quantum circuits, and complexity

Identifying each entanglement thread's trajectory as a geodesic in the bulk is a nontrivial assertion. Although we have seen various roles that geodesics play in different threadlike pictures in the literature—such as bit threads, string condensations [52], and PEE threads [53,54]—in our framework, this conclusion is reached based on an assumption that carries a direct and explicit meaning in quantum information theory, rather than an indirect one. In the following, we will demonstrate that, using the language of kinematic space, we can construct a canonical quantum circuit in which each apparent wire precisely corresponds to an entanglement thread in the bulk—that is, a geodesic. In other words, each geodesic in the original tensor network is “straightened” into a wire in this canonical quantum circuit, while each tensor is bisected and mapped to a quantum gate. Specifically, when two geodesics intersect in the original tensor network, they are entangled through a basic tensor (or a simple combination of a few basic tensors).⁸ Conversely, in the resulting quantum circuit, whenever two wires are coupled through a quantum gate

⁸Taking the HaPPY tensor network as an example, as shown in Fig. 3, two intersecting entanglement thread bundles are coupled within a single 3-to-3 quantum gate. However, it can be shown that such a 3-to-3 perfect tensor gate can be decomposed into a composition of three two-qudit quantum gates [75] (see Fig. 12).

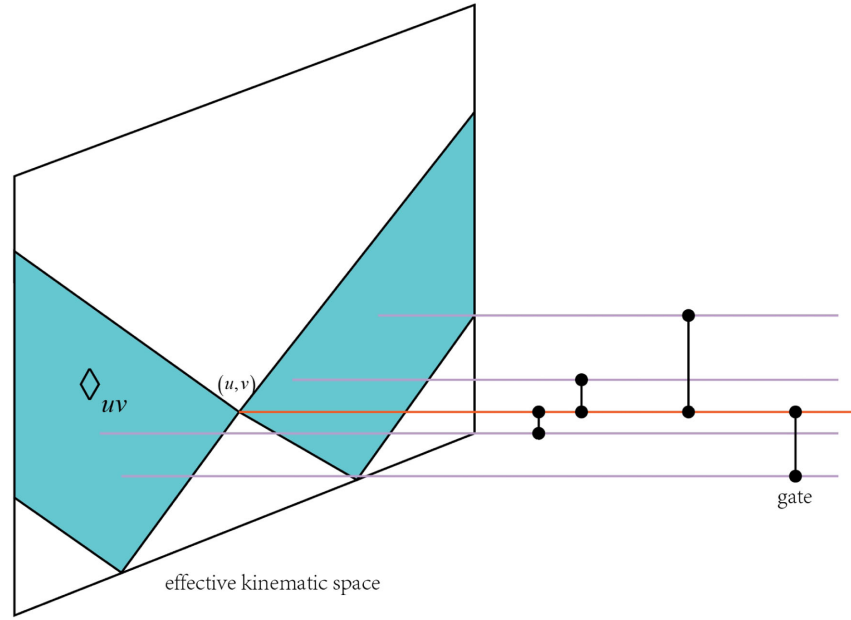


FIG. 7. Regarding kinematic space as an input board, a qubit (u, v) will be coupled through a quantum gate with each wire emanating from the $\diamond_{uv}$ region.

(or a combination of a few basic quantum gates), it means that the corresponding geodesics in the original tensor network intersect.

Now we turn to Fig. 7. Since each point in kinematic space K corresponds precisely to a geodesic, kinematic space provides a convenient way to arrange the relative positions of wires after the topological deformation of the tensor network into a quantum circuit. In other words, kinematic space can be viewed as a “input board” for the quantum circuit, where each point (u, v) corresponds to one of the input qudits.⁹ These input qudits extend into wires, with different wires entangled through quantum gates. Compared to Fig. 3(d), which is a relatively arbitrary deformation of the original tensor network, it can be seen that the wires in the current quantum circuit are perfectly arranged as a series of parallel straight lines, conforming to the usual convention in quantum circuit literature. This indicates that the kinematic space is a very elegant way of organizing the entanglement structure. The special arrangement of qubits on the kinematic space has the advantage of expediently describing how the wires are coupled through quantum gates. As shown in Fig. 7, consider the qubit (u, v) , which in the original space N represents a geodesic

ζ_{uv} connecting two anchor points u and v on the boundary. Which wires will couple with this qubit? We have already argued that it should be the set of other geodesics that intersect ζ_{uv} , which corresponds precisely to the diamond-shaped region $\diamond_{uv}$ in kinematic space K . In a word, regarding kinematic space as an input board, a qubit (u, v) will be coupled through a quantum gate with each wire emanating from the $\diamond_{uv}$ region. It turns out that this conclusion leads us to the concept of holographic complexity.

Let us briefly review the concept of complexity. As the name suggests, in the context of quantum circuits, the complexity of a state is the minimum number of gates required to transform a reference state into the target state. The complexity of an operator is the minimum number of gates required to simulate that operator using a quantum circuit. For quantum theories defined in finite-dimensional Hilbert spaces, this definition is straightforward. However, defining the complexity of a field theory state is notoriously challenging. Nevertheless, in the context of holographic duality, the complexity of a field theory has been proposed to have a bulk dual. In the first wave of proposals for holographic duals of field theory complexity, [76–79] considered a system composed of two copies of CFT initially entangled through a thermofield double state. Then the complexity of the system evolving in time is argued to be holographic dual to either the volume of the Einstein-Rosen bridge of the dual two-sided black hole or the action of a Wheeler-DeWitt patch. The first proposal, which relates complexity to volume, is known as the “complexity = volume” (CV) proposal [76,77]. Based on complexity = volume, [80,81] also proposed the

⁹We can use a discrete picture to describe this precisely: first, regularize kinematic space by dividing it into small diamonds with volume equal to 1. Thus, each small diamond corresponds to one thread. On the other hand, recall that the RT formula applies to regions far larger than the Planck scale ℓ_p , therefore each diamond $[\diamond_{A_i, A_j} L]$, corresponding to regions far larger than ℓ_p , can be viewed as a fundamental cell that contains considerable number of threads therein.

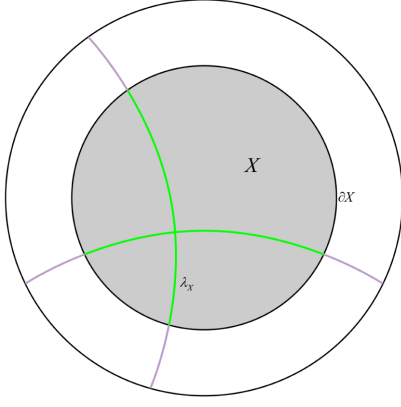


FIG. 8. The volume of a bulk region X (marked in gray) can be given by an integral over the length of all geodesic chords (marked in green).

subregion complexity, which states that for a boundary subregion A , the volume of the bulk region enclosed by its corresponding RT surface gives the complexity of the reduced density matrix for A . Specifically, when A is taken to be the entire boundary system, the original CV proposal is recovered.

Returning to our context, a useful formula is the expression of CV complexity in the language of kinematic space [82,83]. As shown in Fig. 8, the formula states that the volume of a bulk region X can be given by an integral over the length of all geodesic chords. Here, the chord length of a geodesic refers to the length of the segment of the geodesic that lies within X (the green line in the figure). More explicitly

$$\frac{\text{vol}(X)}{4G_N} = \frac{1}{2\pi} \int_{G_X} \lambda_X \omega, \quad (12)$$

where G_X is the set of all geodesics $\zeta \in K$ that intersect X , and λ_X is the length of the chord $\zeta \cap X$.¹⁰ Applying (12) to the entire holographic bulk region X gives an expression for global CV complexity, while applying it to a subregion's entanglement wedge $W(A)$ yields an expression for subregion CV complexity.

The quantum circuit interpretation of kinematic space explicitly explain the meaning of CV complexity as expressed in (12). Let us begin by considering the case where X is taken as the entire holographic bulk region. In this case, (12) indicates that CV complexity should be given by the total length of all geodesics in the bulk. In other words, G_X is the set of all geodesics, and λ_X is the full length of each geodesic. In our explanation in Fig. 7, the right-hand side of (12) essentially counts the number of quantum gates in the quantum circuit deformed from the

tensor network. Because for each geodesic (u, v) acting as a wire, the number of quantum gates it couples with is exactly given by the number of wires in the region $\diamond_{uv}$, which also corresponds to the length of the geodesic ς_{uv} . As we sum over each wire, the total number of quantum gates in the circuit naturally emerges as the sum of the gates that each wire couples with. Thus (12) explains the complexity.

We can also understand subregion complexity. As shown in Fig. 9(a), as an example, we consider a subregion W_r within the bulk, which is defined as the bulk region enclosed by a cutoff surface M_r . Let us focus on a geodesic (marked in red) that passes through the M_r surface. In the quantum circuit perspective, this corresponds to a wire extending from a specific point on the “kinematic space input board” (see Fig. 7). Now consider two other geodesics, one marked in blue that intersects the red geodesic and also passes through the region W_r , and another marked in cyan that does not lie within the W_r region (maybe or may not intersect with the red thread). The corresponding points of these two geodesics on the input board are also present in Fig. 9(b). Recall that in the sense of surface/state duality, M_r , as a closed surface, corresponds to a pure state Ψ_r . This leads to the complexity meaning of Eq. (12): it gives the number of quantum gates needed to construct the state Ψ_r . The idea is that we need the quantum gates inside W_r to construct the state Ψ_r , while the quantum gates outside W_r are not needed.¹¹ According to our interpretation, the intersection of the blue and red threads in the figure counts as a quantum gate for constructing Ψ_r , while the possible intersection of the gray and red threads does not contribute to constructing Ψ_r . We can rephrase this from the input board perspective as shown in Fig. 9(b): let ∂W_r (in this case, i.e., M_r) correspond to a region $\Omega_{\partial W_r}$ in kinematic space (in this case, simple a rectangle). For a selected wire ς_{uv} (the red thread), only the quantum gates coupling with the wires located within the region $\Omega_{\partial W_r} \cap \diamond_{uv}$ (i.e., the set of blue wires) are needed to construct the quantum state $\Psi_{\partial W_r}$ (i.e., Ψ_r in this case). Alternatively, we can equivalently imagine constructing $\Psi_{\partial W_r}$ by removing the unnecessary quantum gates (i.e., those corresponding to intersections of the cyan threads with the red threads) from the original quantum circuit (Fig. 7). Note that this argument applies to a more general subregion X in the bulk, such as the entanglement wedge $W(A)$ of a boundary subregion A , where ∂W_r is the union of A and its RT surface.

Interestingly, in a sense, our work provides a more explicit quantum information theory interpretation for the kinematic space, that is, we can understand the kinematic

¹⁰Equation (12) is a standard result in integral geometry, see Ch. 17 of [84] for example.

¹¹Here, we adopt an assumption from surface-state duality implicitly: a point in the interior of W_r with no volume corresponds to a state with no internal entanglement, suitable as the initial state for the quantum circuit.

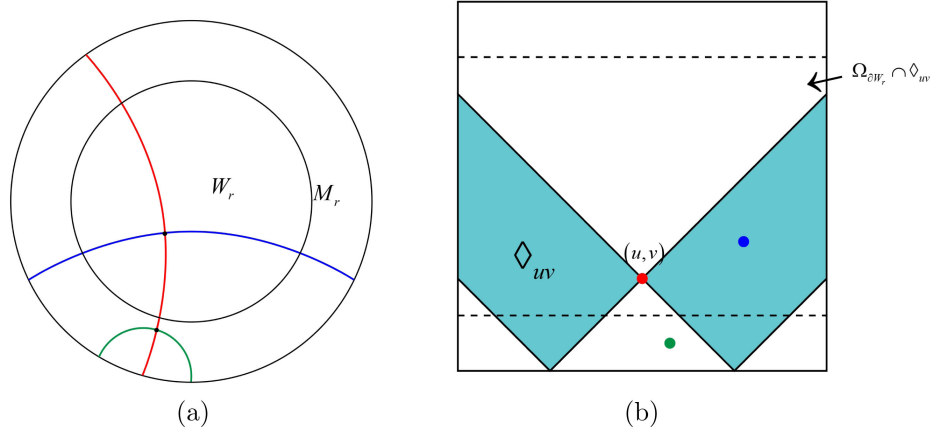


FIG. 9. (a) Focusing on a geodesic (marked in red) that passes through the M_r surface, there exist geodesics that intersect it and also pass through the region W_r (blue lines), and geodesics do not lie within the W_r region (cyan lines). (b) The kinematic picture corresponding to the left figure.

space as a “circuit input board” that depicts the entanglement structure of holographic spacetime.

V. DISCUSSION OF ENTANGLEMENT THREADS

In this section we further discuss the significance carried by entanglement threads. The main point is that the entire configuration of threads can be regarded as a global skeleton: it encodes a global pattern of “which degrees of freedom can be strung together,” rather than the local details of any particular patch of the tensor network. We will also discuss the relation between entanglement threads and the well-known bit threads [39–42], as well as possible schemes for generalizing our construction to higher dimensions.

A. A global perspective on holographic entanglement structure

In the previous sections we have seen that holographic entanglement can be encoded by a collection of entanglement threads, whose trajectories coincide with bulk geodesics. Making further use of kinematic space [48,49], we constructed a canonical quantum–circuit picture: each apparent wire in the circuit can be put in one-to-one correspondence with a thread, while the elementary tensors in the holographic tensor network are mapped to quantum gates acting on these wires.

A more abstract way to phrase this is the following. Instead of postulating a continuous bulk metric from the outset, we may first assume that there exists some holographic tensor-network/quantum-circuit realization. Its local structure (tensors, gate set) may be unknown or nonunique, but the connectivity of the wires is stable. In this language, the “trajectory of a thread” can be understood as the partial order defined by the sequence of quantum gates that a given wire passes through in the circuit. When we smoothly deform the tensor network, or even change

our interpretation of the bulk metric, this partial order remains invariant. In this sense it is closer to the ontic structure of entanglement than to any particular geometric background.

To formalize this intuition, we may temporarily “switch off” all nontrivial quantum gates and keep only the wires themselves. Equivalently, we consider an idealized thread–skeleton state. For each thread ζ we describe all lattice degrees of freedom pierced by ζ by the simplest Greenberger-Horne-Zeilinger-type superposition

$$|\zeta\rangle = \frac{1}{\sqrt{2}}(|0_1 0_2 \cdots 0_n\rangle + |1_1 1_2 \cdots 1_n\rangle), \quad (13)$$

where 1 and n label the two endpoints of the thread on the boundary, and the remaining indices label the bulk lattice sites (for instance, the internal legs of each multipartite tensor in a HaPPY network) that the thread passes through inside the tensor network. The whole configuration of threads naturally corresponds to a “skeleton state”

$$|\Psi_{\text{thread}}\rangle = \bigotimes_{\text{all } \zeta} |\zeta\rangle, \quad (14)$$

namely the tensor product of the states carried by all threads. This state is not intended to reproduce the full holographic CFT state. Instead, it keeps precisely the piece of structure that we care about: which sites are simultaneously pierced by a given thread. After this skeleton has been fixed, one can “fill in” a finite set of local quantum gates on top of it to obtain a concrete holographic tensor-network model. The differences between different models are then reflected in the choice of local gate sets and in the fine-grained entanglement they generate, rather than in the skeleton of threads itself.

Even at such a coarse-grained level, however, $|\Psi_{\text{thread}}\rangle$ already contains rather rich information. Intuitively, each

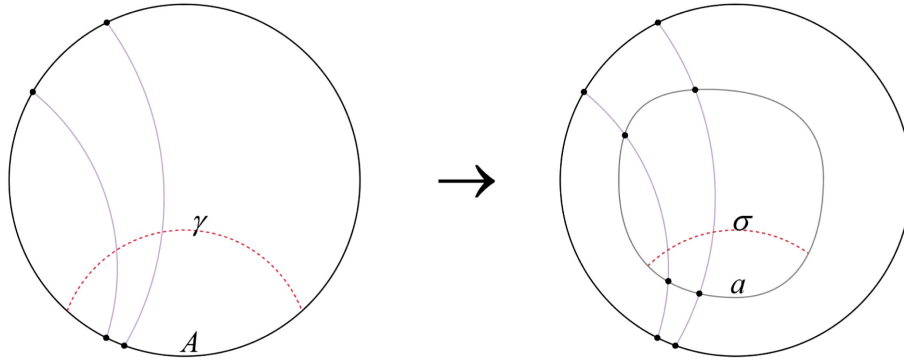


FIG. 10. The RT formula is satisfied in an evident way by interpreting a threadlike object as a Bell state associated with its two endpoints. To account for the generalized RT formula for a more general surface in the bulk, it is naturally to assign an analogous state interpretation to the two intersection points between the surface and the thread. This implies a simple correspondence (13) for each thread to realize the generalized RT formula (15) universally.

small segment of a thread can be viewed as carrying a Bell-pair of entanglement between its two endpoints. For any minimal bulk surface σ , the von Neumann entropy of the reduced density matrix obtained by restricting to σ is simply

$$S(\sigma) = \# \text{ of threads crossing } \sigma. \quad (15)$$

By construction of the thread configuration, this number is equal to $\text{Area}(\sigma)/(4G_N)$. In this way the generalized RT formula [73,74] are naturally explained, in direct analogy with many earlier works that interpret RT formula using threadlike objects (see, e.g., [39,58,85]), where each thread is treated as a Bell pair between its endpoint qudits. For a visual illustration of this logical analogy, see Fig. 10.¹²

In summary, the entanglement threads we introduced provide a skeleton that records “which degrees of freedom are wired together,” even though we have not yet asked the question “what can actually be done using these wires?” Interestingly, we find that this latter question naturally leads us to the notion of “quantum information flow” [86,87] developed in categorical quantum mechanics [88–90], although these terms may sound unfamiliar to most people working on holography. There, entanglement is not treated as a table of coefficients in some particular basis, but as a structural resource that permits certain abstract “flow channels” to exist.

To give the reader an intuitive picture, Fig. 11 uses the simplest quantum teleportation protocol [91] as an example and deliberately avoids the abstract categorical-quantum-mechanics notation in order to illustrate the “quantum

information flow” picture [86,87]. In the right-hand panel the teleportation process is represented as what Coecke calls an entanglement network [86,87],¹³ he then defines a quantum information flow, drawn as a solid green line (to be distinguished from the dashed line denoting classical information flow, i.e., classical communication), which captures the ability—rooted in quantum entanglement—to transmit a qubit from Alice to Bob.

In a forthcoming paper [94] we will show that, for holographic systems, entanglement threads effectively provide a skeleton of potential flux lines, which can be used, under suitable operational rules, to analyze the quantum information flow between any chosen pair of boundary subregions. In other words, in the holographic setting a given entanglement thread configuration plays the role of a preinstalled wiring diagram. Once the threads on some bulk slice have been laid down, we can ask more operational questions, for example: Given two boundary subregions A_i and A_j , if we are only allowed to perform LOCC on the boundary and act on certain restricted bulk degrees of freedom, how many mutually independent logical qudits can be transmitted in parallel from A_i to A_j ? Such questions no longer focus on a static entropy value; instead, they probe how large a quantum information flow capacity the thread skeleton can support under a prescribed class of protocols. In our follow-up work, along the lines of Coecke’s quantum information flow program [86,87], we will give a more precise definition of this capacity and show that, under appropriate operational constraints, the maximal information flow between two boundary regions is in fact governed by certain familiar

¹²In fact, the idea of promoting the information-theoretic meaning of a thread from a Bell pair between two qudits to the multiqubit state (13) was already proposed, in a slightly different form, in our previous works [50,55,56], where it was referred to as “thread-state correspondence.” We refer interested readers to [50,55,56] for further details.

¹³In fact this can be formalized within the categorical-quantum-mechanics framework as a dagger-compact symmetric monoidal category [88–90,92], or more intuitively expressed using the diagrammatic ZX-calculus [90,93] developed later; here we deliberately avoid that formalism and will expand on it in a future paper [94]—interested readers can consult the literature [90].

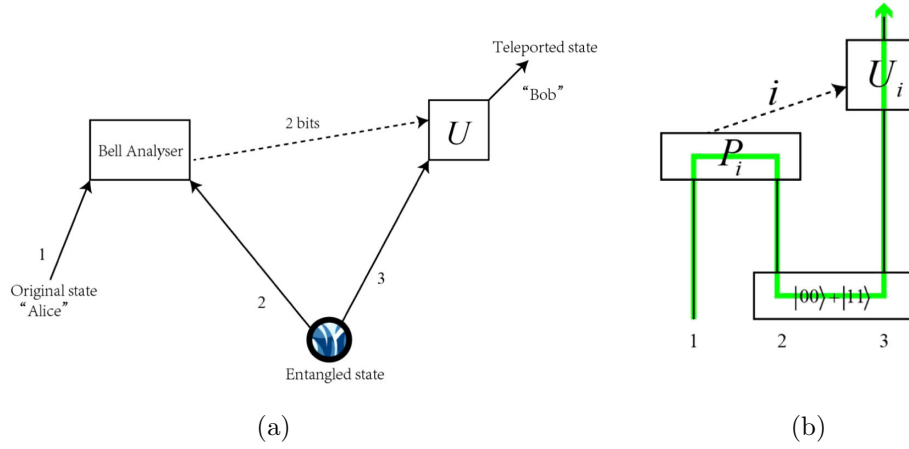


FIG. 11. (a) Schematical picture of quantum teleportation [91]. The box “U” denotes the unitary transformation Bob should perform in order to retrieve the original state. (b) The corresponding “entanglement network” [86,87], where the green thick line represents the quantum information flow, while the dotted line represents the classical information flow.

geometric quantities. For reasons of focus, we will not elaborate on the technical definition of this Coecke-style flow or on the proofs in the present section. Here we only stress the following point: it is precisely because entanglement threads inherently encode such a global skeleton that they can capture some more fundamental aspects of holographic entanglement structure, without relying on the microscopic details of any specific tensor network realization.

B. Comparison between entanglement threads and bit threads

As mentioned in the introduction, the thread-based perspective on entanglement in holographic duality first attracted attention in 2016 with the concept of *bit threads* developed by Headrick *et al.* Bit threads are unoriented bulk curves that end on the boundary and subject to the rule that the thread density is less than $\frac{1}{4G_N}$ everywhere [39–41].¹⁴ In particular, this density bound implies that the number of threads passing through a minimal surface $\gamma(A)$, which separates a boundary subregion A from its complement, cannot exceed its area times $\frac{1}{4G_N}$. Therefore, the flux of threads connecting A and its complement, $\text{Flux}(A)$, does not exceed this value

$$\text{Flux}(A) \leq \frac{1}{4G_N} \text{Area}(\gamma(A)). \quad (16)$$

Using terminology from network flow theory, we say that if inequality (16) is saturated, then a thread configuration *locks the region* A . In fact, this bound is tight. For any A , there always exists a locking thread configuration such that

¹⁴When one takes the Hodge dual of bit threads one gets calibrated geometries, which mathematicians (geometers) use to identify minimal surfaces. This is a viewpoint adopted in [95].

$$\text{Flux}_{\text{locking}}(A) = \frac{1}{4G_N} \text{Area}(\gamma(A)). \quad (17)$$

This theorem is known as max flow-min cut theorem (see [41] and references therein), that is, the maximal flux of bit threads (over all possible bit thread configurations) through a boundary subregion A is equal to the area of the bulk minimal surface $\gamma(A)$ homologous to A . Therefore, the famous RT formula which relates the entanglement entropy of a boundary subregion A and the area of the bulk minimal extremal surface $\gamma(A)$ homologous to A

$$S(A) = \frac{\text{Area}(\gamma(A))}{4G_N}, \quad (18)$$

can be expressed in another way, that is, the entropy of a boundary subregion A is proportional to the flux of the locking thread configuration passing through A

$$S(A) = \text{Flux}_{\text{locking}}(A). \quad (19)$$

When the bit threads are required to be locally parallel, one can use the language of *flow* to describe the behavior of bit threads conveniently in mathematics, that is, using a vector field $\vec{v}$ to describe the bit threads, just as using the magnetic field $\vec{B}$ to describe the magnetic field lines. The difference is that for the latter we regard the magnetic field itself as the more fundamental concept, while for the former we consider the threads to be more fundamental. The constraints on the bit threads can then be expressed as the requirements for the flow $\vec{v}$ as follows,

$$\nabla \cdot \vec{v} = 0, \quad \rho(\vec{v}) \equiv |\vec{v}| \leq \frac{1}{4G_N}. \quad (20)$$

For situations involving more than one pair of boundary subregions, the concept of *threadbundles* is also useful.

The threads in each thread bundle are required to connect only a specified pair of boundary subregions, while still satisfy the constraints of bit threads. Specifically, one can use a set of vector fields $\vec{v}_{ij}$ to represent each thread bundle connecting the A_i region and A_j region respectively. The set V of vector fields $\vec{v}_{ij}$ is referred to as a *multiflow*, and each $\vec{v}_{ij}$ is called a *componentflow*, satisfying (Note that in the present paper we will define $\vec{v}_{ij}$ only with $i < j$ for convenience, which is slightly different from (but equivalent) convention adopted in [40], where the fields $\vec{v}_{ij}$ were also defined for $i \geq j$, but with the constraint $\vec{v}_{ji} = -\vec{v}_{ij}$)

$$\begin{aligned} \nabla \cdot \vec{v}_{ij} &= 0, \\ \rho(V) &\leq \frac{1}{4G_N}, \\ \hat{n} \cdot \vec{v}_{ij}|_{A_k} &= 0, \quad (\text{for } k \neq i, j). \end{aligned} \quad (21)$$

It is worth noting that, since in the situation of multiflows, the threads are not necessarily locally parallel, there are various natural ways the density can be defined, and therefore bounded. In particular, [96] discusses how different definitions of thread density can affect the ability to lock a set of boundary regions.

As we have seen in the main text, many properties of our entanglement threads align with the intuition behind bit threads. First, we assume that entanglement threads traverse the bulk continuously without divergence, matching the divergence-free condition of bit threads. Second, we argued that entanglement threads cannot “turn back” on RT surfaces, while similarly in an optimal locking thread configuration bit threads do not cross any minimal surface more than once—otherwise they would occupy positions needed by other threads connecting two complementary regions, preventing maximal flux. Most importantly, both frameworks assign entanglement entropy meaning via thread flux.

However, on the other hand, there are clear differences between the two types of thread configurations. First, bit threads are subject to a density bound, and this bound makes the existence of an optimal bit thread configuration for a given set of regions a nontrivial problem. In simple terms, due to the overly strict constraints on the thread density, it has created an excessive obstacle to the existence of the bit thread configuration that can lock arbitrary number of subregions simultaneously (see the detailed discussion in [96]). Second, the density bound, combined with the max-flow min-cut theorem allow more than one thread configurations achieving the same maximal entropy. In other words, the optimal (or locking) bit-thread configuration can be nonunique. By contrast, our entanglement thread configuration is unique—since we have asserted that each thread’s trajectory is precisely a bulk geodesic. This is the key difference between the two. It is worthwhile adding two remarks that suggest a deeper connection may exist

despite these differences. First, in seeking concrete constructions of bit threads, [97] discovered that geodesics in the bulk naturally serve as the trajectories of bit threads (though, by definition, the bit thread configuration is not unique). Second, [66] were the first to point out that the bit thread picture is closely related to the concepts of partial entanglement entropy [63] and conditional mutual information, while in our framework, the entanglement thread flux is also given by half the conditional mutual information.

These observations imply a deeper relationship between the two thread-based constructions. One possible explanation is that the bit thread configuration captures the distillable portion of entanglement, whereas the entanglement threads define a scaffold that encodes the full entanglement information—on top of which quantum gates may be added to actualize the holographic state. Within our framework, we can attempt to understand the relationship between entanglement thread and bit thread configurations as follows: an entanglement thread configuration is defined by reorganizing the tensor network that encodes the entanglement of a holographic slice into a quantum circuit, whose wires represent entanglement threads. In particular, as in Fig. 7, when the circuit wires are arranged following the kinematic space organization, each apparent wire corresponds exactly to an entanglement thread following a null-like geodesic trajectory across the slice.

The key point is that in a quantum circuit, due to the conservation of the number of qubits, we have a certain degree of freedom to define the concept of apparent wires (a set of apparently parallel lines in a quantum circuit). A schematic diagram illustrating this freedom is shown in Fig. 12. From this point of view, each optimal bit thread configuration corresponds to a choice of apparent wire sets for the quantum circuit, while the canonical one is Fig. 7, wherein the entanglement threads in the quantum circuit are perfectly arranged as a series of parallel straight lines, that is to say, each apparent wire corresponds precisely to a geodesic in the holographic geometry. Thus different locking bit-thread configurations may be understood as different choices of apparent-wire conventions. In any case, more subtle relations between entanglement threads and bit threads may be a fruitful topic for further exploration.

C. Higher-dimensional generalization

In this work we have focused on the $\text{AdS}_3/\text{CFT}_2$ setting, where the constant-time slice of the bulk is a two-dimensional hyperbolic space and the kinematic space can naturally be identified with the space of bulk geodesics, which is in turn isomorphic to a two-dimensional de Sitter space [48]. Extending our ideas to higher-dimensional AdS/CFT dualities is certainly desirable and a rich toolkit already exists in the literature, but several structural subtleties arise. As emphasized in [98,99], there is no unique notion of “kinematic space” in higher dimensions: one

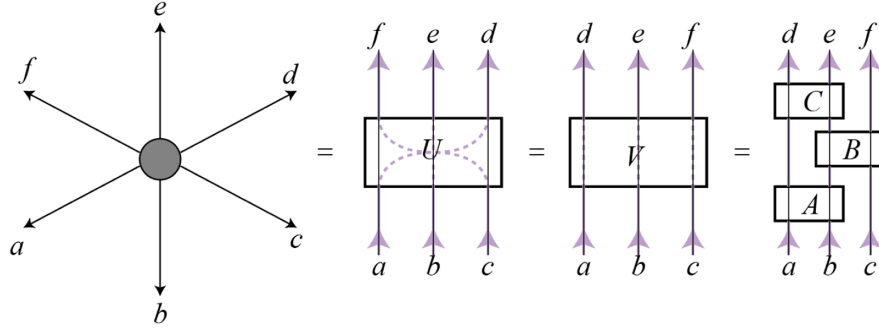


FIG. 12. Due to the conservation of the number of qubits, we have a certain degree of freedom to define the concept of “apparent wires”(the apparently parallel black lines). Therefore, the configuration of the entanglement threads (purple lines) remains unchanged, while the relatively different locking configurations of bit thread (black lines) may be understood as different choices of apparent-wire conventions. Note that for $d \geq 5$, a six-rank perfect tensor can be decomposed into a combination of three two-qudit quantum gates [75].

common choice is a geodesic kinematic space, whose points represent bulk geodesics, while another natural choice is a hypersurface-based kinematic space, whose points represent codimension-one extremal hypersurfaces (for example the RT hemisurfaces anchored on boundary balls). In the vacuum, the latter is typically isomorphic to a d -dimensional de Sitter space and underlies the “ball-kinematic-space” approach [100].

On the other hand, the construction of Lin *et al.* [54] shows that even in arbitrary spacetime dimension one can, in a precise sense, retain threadlike objects as the fundamental kinematic degrees of freedom. By assigning to each boundary-anchored bulk geodesic a weight determined by a two-point partial entanglement entropy (PEE), they obtain a natural measure on the space of geodesics and show that the area of any codimension-two bulk surface can be written as a Crofton-type integral, that is, as the weighted intersection number of that surface with the ensemble of “PEE threads.” This may be viewed as another nontrivial higher-dimensional generalization of the AdS_3 kinematic-space picture that still keeps geodesics at center stage. The most delicate point, however, concerns the elegant result we obtained in the AdS_3 case: the volume of a bulk time slice can be expressed as an integral over kinematic space of the chord lengths of geodesics within the region, providing a geometric underpinning for tensor-network or quantum-circuit interpretations of holographic complexity [82,83]. At a structural level, this formula crucially exploits a low-dimensional accident: in a two-dimensional hyperbolic space, geodesics are codimension-one hypersurfaces, and hence serve simultaneously as natural Crofton probes for both areas (i.e., lengths) and volumes. A closer inspection suggests that, in higher-dimensional AdS backgrounds, simply promoting the thread-density expressions *à la* [54] to higher dimensions is not sufficient to guarantee the existence of an equally simple volume formula that depends only on geodesic chord lengths. From the viewpoint of integral geometry, it is natural in higher dimensions to expect bulk-volume-type Crofton formulas to be built

from codimension-one totally geodesic hypersurfaces (e.g., geodesic hyperplanes or RT hypersurfaces), while one-dimensional geodesics are the canonical Crofton probes for the areas of codimension-one surfaces.

For these reasons, a complete higher-dimensional generalization of our volume formula and of its interpretation in terms of holographic complexity, while retaining the advantages of the thread-based kinematic description, will require a more systematic analysis of higher-dimensional Crofton formulas and of the metric structure on the corresponding kinematic spaces. We leave a thorough investigation of this problem to future work.

VI. CONCLUSIONS AND OUTLOOK

This paper has depicted a thread picture using entanglement threads to characterize entanglement structures in holographic duality. This framework is based on two important insights. The first is a simple idea or philosophy: “represent (holographic) quantum entanglement with connecting lines.” The second is that the entanglement thread picture naturally and universally emerges in tensor network models of holographic duality. For instance, the partial order structure associated with the HaPPY tensor network naturally gives rise to local directed arrows. By gluing together these local arrows, one obtains global entanglement threads. By abstracting these concepts from specific tensor network models, we can imagine entanglement threads traversing through spatial slices of the holographic bulk, with their endpoints anchored on the holographic boundary. This is very similar to the earlier concept of bit threads. However, there exist important distinctions between the two.

We argued that the specific trajectories of entanglement threads in the bulk are exact geodesics. This naturally connects to the concept of kinematic space, which is defined as the collection of all bulk geodesics anchored to boundary points. As a result, kinematic space not only provides mathematical tools for computing the flux of entanglement threads, but also offers a natural way to

organize qudits, enabling the construction of a canonical quantum circuit in which each apparent wire precisely corresponds to an entanglement thread. In other words, by interpreting entanglement threads as a set of predefined wires, one can generate holographic quantum states simply by continuously inserting basic quantum gates. We have also tested this idea using the concept of holographic complexity. Combining the concepts of entanglement threads and kinematic space, a elegant circuit interpretation for the holographic complexity is provided. In a certain sense, our work provides a clearer quantum information-theoretic meaning for kinematic space, namely, we can view kinematic space as a “circuit board” that encodes the entanglement structure of spacetime.

As emphasized in Sec. V, the concept of entanglement thread trajectory does not require a preexisting spacetime background with a metric; rather, it can essentially be defined as a partial order specifying the sequence in which it passes through a set of basic quantum gates. During the deformation of the original tensor network, although the metric concept of space is eliminated, this partial order is preserved. We can also assign to each entanglement thread a thread-state correspondence interpretation, which expresses the global entanglement among the qudits it traverses. Ultimately, through the quantum state of the entire thread configuration, not only the entanglement among subregions on the holographic boundary, but also the entanglement relations between different surfaces on the bulk slice are intuitively characterized by the threads that simultaneously pass through them.

The idea of entanglement threads offers a framework to explore many interesting questions. Apart from what we specifically mentioned in Sec. V, namely considering its relationship with the quantum information flow within the framework of categorical quantum mechanics, its relationship with bit threads, and its higher-dimensional generalization, we can also study, for instance, more types of holographic geometries, such as wormholes [101]¹⁵ and the dynamical aspects of perturbed AdS spaces. Furthermore, what we studied here is a “thread game” for spacelike geodesics. An interesting idea is to consider a similar thread game for timelike geodesics. We conjecture this may be related to pseudoentropy [103] (since it is a kind of timelike entanglement entropy [104]) or complexity [76–79] (inspired by the Lorentzian bit threads proposed in [105–107]). In [108,109] (see also [110]), we have tried to use the thread picture to understand holographic BCFT and the island phenomenon, and found interesting connections between the island and perfect entanglement. It is also worthwhile to further understand those results from the

perspective of quantum circuits and kinematic space developed in this paper.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: REVIEW OF KINEMATIC SPACE

1. Kinematic space and the Crofton form

The concept of kinematic space [48,49] originates from the application of the Crofton formula in integral geometry [84] and its extension in holographic geometry. The Crofton formula roughly states that the length of a curve is measured by the number of geodesics intersecting it. In the context of holographic duality, consider a static slice of a holographic geometry. Without loss of generality, we take an equal-time slice N of AdS_3 space, with the metric given by

$$ds^2 = l_{\text{AdS}}^2(d\rho^2 + \sinh^2 \rho d\phi^2). \quad (\text{A1})$$

Each geodesic can be parametrized as:

$$\cos(\alpha) = \tanh(\rho) \cos(\phi - \theta), \quad \alpha \in (0, \pi), \quad \theta \in S^1, \quad (\text{A2})$$

as shown in Fig. 13, here α intuitively corresponds to half the opening angle of the geodesic, while θ labels the center of the boundary interval corresponding to the geodesic on the boundary $M = \partial N = S^1$. Each complete oriented geodesic is uniquely specified by a pair (θ, α) . Note that reversing the direction of the geodesic (θ, α) corresponds to $(\theta + \pi, \pi - \alpha)$.

As illustrated in Fig. 13, by mapping each geodesic's label (θ, α) to a point (θ, α) in a new space K , the counting of geodesics is translated into volume measurement in the new space. This new space K , dual to the original space N , is called the kinematic space. More specifically, it was proposed that the length of a curve e in N is given by:

$$\text{Length}(e) = \frac{1}{4} \int_K \omega(\theta, \alpha) n_e(\theta, \alpha), \quad (\text{A3})$$

where $n_e(\theta, \alpha)$ is the number of intersections between the geodesic (θ, α) and the curve e , and the integration measure ω is physically interpreted as the “geodesic density function.” It is the volume form in kinematic space K , also known as the Crofton form. Specifically, for a static slice of AdS_3 , the Crofton form is given by

¹⁵By the time this manuscript was finalized, our related work on the entanglement-thread structure of the planar BTZ black hole had also been published [102].

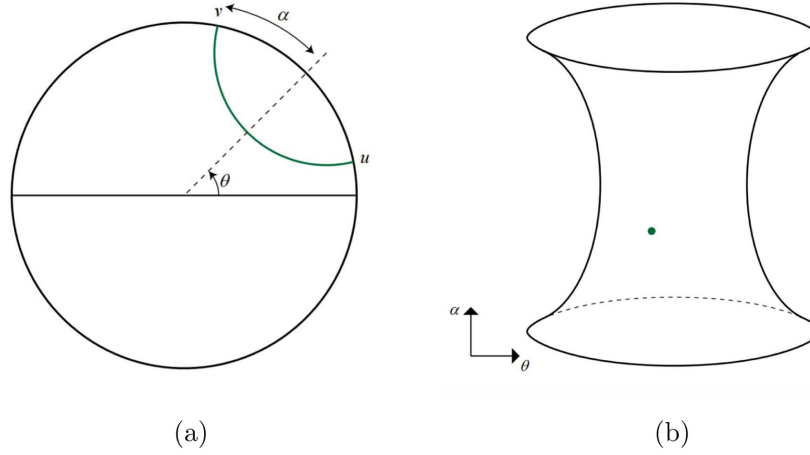


FIG. 13. (a) α is the half-opening angle of the geodesic, and θ marks the center point of the corresponding boundary subregion. (b) Each point in kinematic space K represents a geodesic in the original space N .

$$\omega(\theta, \alpha) = -\frac{1}{\sin^2 \alpha} d\alpha \wedge d\theta. \quad (\text{A4})$$

In the literature, another set of parameters is also commonly used

$$u = \theta - \alpha, \quad v = \theta + \alpha. \quad (\text{A5})$$

Under this change of variables, Eq. (A4) becomes

$$\omega(u, v) = \frac{1}{2 \sin^2(\frac{v-u}{2})} du \wedge dv. \quad (\text{A6})$$

In other words, the kinematic space K is a manifold with the metric

$$ds^2 = \frac{1}{2 \sin^2(\frac{v-u}{2})} dudv, \quad (\text{A7})$$

which is, in fact, a dS_2 (two-dimensional de Sitter) space.

For more general static slices N of holographic geometries, the Crofton measure ω can be derived as follows: first obtain the length $\lambda(u, v)$ of a geodesic in N parametrized by (u, v) (note that u, v are exactly the boundary spatial coordinates of the two end points of the geodesic on the boundary S^1), then

$$\omega = \frac{\partial^2 \lambda(u, v)}{\partial u \partial v} du \wedge dv. \quad (\text{A8})$$

2. Point-curves and conditional mutual information

Let us illustrate how the language of kinematic space can be used to characterize the geometric information in the original space N , via an example of computing the length of a geodesic in N . A point a in N can be mapped to K in the following way: find all geodesics in N that pass through the

point a . Each such geodesic corresponds to a point in K , so the set of all such geodesics forms a trajectory in kinematic space K , denoted p_a , which is called a point-curve. In particular, for pure AdS_3 (and also BTZ black holes), it can be shown that the point-curves are spacelike geodesics in K [111]. In the pure AdS_3 case, the point (ρ, ϕ) in the coordinate system (A1) corresponds to a point-curve in K under coordinates (θ, α) given explicitly by [48,111]

$$\alpha(\theta) = \cos^{-1}(\tanh \rho \cdot \cos(\theta - \phi)). \quad (\text{A9})$$

Now, using the mathematical language of kinematic space, computing the length of the geodesic e_{ab} between two bulk points a and b in the original space N becomes straightforward: we respectively draw the point curves p_a and p_b corresponding to a and b in kinematic space K , which enclose a region denoted by $p_a \Delta p_b$. Next, we can apply formula (A3). First, we specify the integration region as $p_a \Delta p_b$; second, noting that since e_{ab} is itself a geodesic, other geodesics can cross it at most once, the formula (A3) simplifies to

$$\text{Length}(e_{ab}) = \frac{1}{4} \int_{p_a \Delta p_b} \omega. \quad (\text{A10})$$

In particular, let us consider the case when both a and b lie on the boundary $\partial N = M$. Note that since $\partial K = \partial N = M$, the corresponding point curve of a boundary point a in N in kinematic space will exactly be the “light cone” emanating from the same boundary point a in K , as shown in Fig. 14. Now a very interesting result emerges: according to formula (A10), the length of the geodesic between two boundary points a and b in the original space N is given by the volume of the blue region in K space shown in Fig. 14(b). Let the coordinates of points a and b on the boundary be u and v , and denote the geodesic length between them as $\lambda(u, v)$. Let the blue region be denoted by

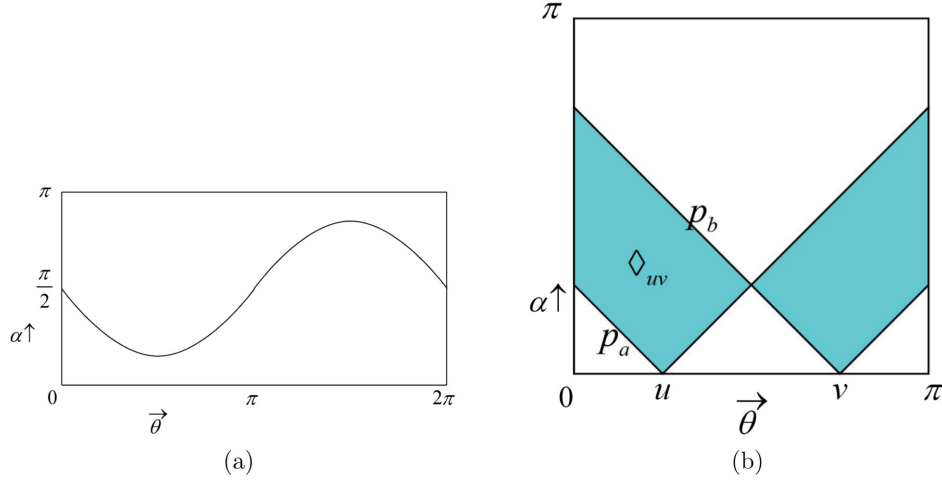


FIG. 14. (a) A point-curve in K corresponding to a point in N . (b) Computing the geodesic distance between two boundary points in N . Note that half of K space already contains all the geodesic information in N space.

$\diamond_{uv}$, then we obtain the formula for computing geodesic lengths using the language of kinematic space

$$\lambda(u, v) = \frac{1}{4} \int_{\diamond_{uv}} \omega. \quad (\text{A11})$$

However, from the RT formula, we know that the left-hand side of (A11) precisely gives the entropy of the segment $R = [u, v]$, namely $S(R) = S(u, v)$, that is

$$S(u, v) = \frac{\lambda(u, v)}{4G_N}. \quad (\text{A12})$$

Thus, a close connection arises between the kinematic space and the concept of entanglement entropy. By renormalizing the Crofton form, define

$$\tilde{\omega} = \frac{\omega}{4G_N}, \quad (\text{A13})$$

then we obtain

$$S(u, v) = \frac{1}{4} \int_{\diamond_{uv}} \tilde{\omega}, \quad (\text{A14})$$

that is, the entanglement entropy of a (connected) sub-region R in the boundary CFT can exactly be represented as the volume of a corresponding region $\diamond_{uv}$ in the kinematic space. The region $\diamond_{uv}$, called the causal diamond, is defined as the diamond-shaped region in kinematic space bounded by the light rays emanating from the two end points u and v of R (and its complement $\bar{R}$).

The half-conditional mutual information also has an intuitive graphical representation in kinematic space. As shown in the Fig. 15, consider two fundamental regions A_i and A_j of the boundary system, with the intermediate region denoted by L . Then, the half-conditional mutual

information $\frac{1}{2}I(A_i, A_j|L)$ [see (5)] is exactly given by the volume of a diamond region $\diamond_{A_i, A_j|L}$ in kinematic space. It is defined as the region enclosed by intersecting light rays emanating from the end points of A_i and A_j at the boundary of kinematic space. It can be verified that

$$\frac{1}{2}I(A_i, A_j|L) = \frac{1}{4} \int_{\diamond_{A_i, A_j|L}} \tilde{\omega}. \quad (\text{A15})$$

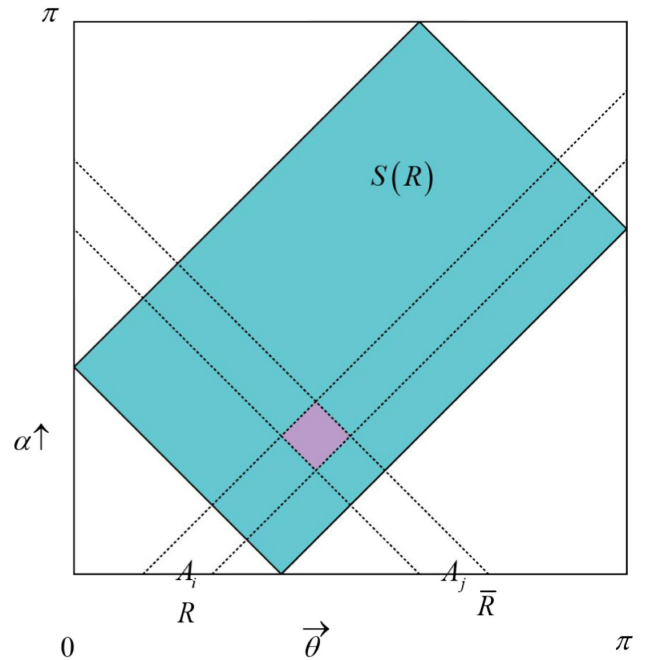


FIG. 15. The half-qCMI has a direct and intuitive geometric interpretation in kinematic space. $\frac{1}{2}I(A_i, A_j|L)$ corresponds to the volume of purple diamond region. The coordinate θ has been chosen such that one of the endpoints of R is exactly the origin.

In our setup, we usually take the fundamental regions A_i and A_j to be very small, so this formula can also be directly seen from (A8).

In the context of our study of holographic entanglement threads, since we identify each geodesic as an entanglement thread, this intuitive correspondence of half-conditional mutual information is consistent with our Eq. (4). As shown in the figure, suppose the fundamental regions A_i and A_j respectively belong to a bipartition of the boundary system, with $A_i \subseteq R, A_j \subseteq \bar{R}$. The figure also shows the diamond

region $\diamond_R$ used to compute the entanglement entropy between R and $\bar{R}$. Obviously, by decomposing R into a series of A_i and $\bar{R}$ into a series of A_j , the volume of $\diamond_R$ can exactly be partitioned into the sum of the volumes of the corresponding $\diamond_{A_i, A_j|L}$. This precisely expresses that the number of threads connecting R and $\bar{R}$ can be decomposed into the contributions of thread bundles connecting each pair of elementary regions A_i and A_j , with a flux value of $\frac{1}{2}I(A_i, A_j|L)$. A preliminary discussion can be seen in [55].

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