

Emergent factorization of the Hilbert space at large N and the black hole information paradox

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We investigate the emergent factorization of Hilbert space in the low-energy description of matrix models, addressing key aspects of the black hole information paradox. We examine the collective description for the low-energy sector of $SU(N)$ matrix model, characterized by a factorized Hilbert space composed of a finite number of boxes and anti-boxes. This factorization leads us to examine the emergence of thermofield dynamics state in the low energy sector from a fine-tuned state. Our investigation of these matrix models elucidates a concrete mechanism for constructing the truncated algebra of accessible observables, thereby facilitating an understanding of black hole complementarity. In the context of the black hole information paradox, we discuss the origin of the island appearing inside the black hole and provide a reinterpretation of the recent proposal—the holography of information.

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I. INTRODUCTION

The study of emergent collective phenomenon in physics has unveiled profound insights into complex systems, where individual components interact to manifest new and often unexpected collective behaviors, including superconductivity, magnetism, and solitons. From the perspective of the AdS/CFT correspondence [1], semiclassical gravity can also be viewed as an emergent collective phenomenon in the large N Conformal Field Theory (CFT) [2–10], emphasizing the emergence of the holographic coordinate from the boundary CFT.

Black holes stand as another quintessential example of complex systems in physics, epitomizing the intricacies of

emergent collective phenomena in gravity. The study of black holes, particularly in the context of the black hole information paradox [11,12], poses one of the most challenging puzzles in theoretical physics. The essence of the paradox lies in the entanglement between Hawking radiation and the inside of the black hole. Consequently, understanding the nature of operators inside black holes, known as *mirror operators* [13,14], is essential for the black hole information paradox.

Recently, black hole microstates have been investigated through cohomology for Bogomol'nyi–Prasad–Sommerfeld (BPS) operators in the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM) [15–18], where the BPS black hole operator, defined by nongraviton cohomologies, has been explicitly identified for small values of $N = 2, 3, 4$. This explicit construction of the black hole operators raises a question regarding the black hole information paradox: What is the microscopic origin of the mirror operators in the framework of constructing the BPS black hole operator? If black hole operators are formulated within a single $\mathcal{N} = 4$ SYM theory in large N limit, the origin of operators inside the black hole could also be traced back to that SYM theory. This aligns with black hole complementarity [13,14,19–25],

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wherein a simple operator inside the black hole can be represented as a complicated operator on the outside.

In this article, we study a simple matrix model to demonstrate that in large N , the mirror operator in the black hole emerges as a collective excitation, analogous to a hole in the band theory. We show how the low-energy description for the matrix model gives rise to an emergent factorization of the Hilbert space, wherein two different branches of the collective fields form two factorized Hilbert spaces in Large N limit—one represents the degrees of freedom outside the black hole while the other corresponds to that of the black hole interior. At finite N , these two branches of collective degrees of freedom are not independent—one branch can be expressed in terms of the other—and the Hilbert space no longer factorizes, thus providing the underlying mechanism for black hole complementarity [14,21].

Based on the emergent factorization of the Hilbert space, we elaborate old and new progresses in the black hole information paradox. We demonstrate that in the low energy sector, the thermofield dynamics (TFD) state, which is the holographic dual state of the AdS black hole and is the entangled state of the factorized Hilbert space at Large N [26], can emerge from a fine-tuned state at finite N , shedding light on the factorization problem of the black hole [8,27–29]. Furthermore, we elucidate the origin of the *island* [30–36] that appears inside the black hole and provide a reinterpretation of the *holography of information* [37–44] from the perspective of emergent factorization.

II. MODEL

We begin by the one-dimensional $SU(N)$ matrix model whose Hamiltonian is given by

$$H = \text{tr} \left[U \frac{\partial}{\partial U} U \frac{\partial}{\partial U} \right], \quad (1)$$

where $U(t)$ is the $N \times N$ $SU(N)$ matrix. This Hamiltonian is invariant under the $U(N)$ transformation, $U \rightarrow V^{-1}UV$ where $V \in U(N)$. Hence the $U(N)$ -singlet sector can be expressed in terms of the $U(N)$ -invariant collective field defined by

$$\phi_n \equiv \text{tr}(U^n) \quad (n = 1, 2, \dots). \quad (2)$$

The collective field ϕ_n is also called a plaquette variable of winding number n in the (one-plaquette) Kogut-Susskind lattice gauge theory [45,46]. Note that not all ϕ_n 's are independent at finite N . $\{\phi_1, \phi_2, \dots, \phi_{N-1}\}$ forms the set of independent collective fields, and other collective field, ϕ_m ($m \geq N$), can be expressed in terms of them.

In large N limit, the collective fields ϕ_n ($n \geq 1$) become independent, and the dynamics of the $U(N)$ singlet sector can be described by the collective Hamiltonian of the collective fields [47,48],

$$\begin{aligned} H_{\text{col}}[\phi] = & N \sum_{n=1}^{\infty} n \phi_n \frac{\partial}{\partial \phi_n} + \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} n \phi_m \phi_{n-m} \frac{\partial}{\partial \phi_n} \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nm \phi_{n+m} \frac{\partial}{\partial \phi_n} \frac{\partial}{\partial \phi_m} \\ & - \frac{1}{N} \left[\sum_{n=1}^{\infty} n \phi_n \frac{\partial}{\partial \phi_n} \right]^2. \end{aligned} \quad (3)$$

The cubic interaction terms represent the splitting and joining of the loops, with the coefficients indicating the number of possible ways these processes can occur [48–51]. The Fock space representation [52] can be achieved by substituting ϕ_n with $\sqrt{n}a_n^\dagger$ and $\frac{\partial}{\partial \phi_n}$ with $\frac{1}{\sqrt{n}}a_n$, respectively [48].

It was shown that the collective Hamiltonian (3) is diagonalized by the $SU(N)$ character represented by the Schur polynomials of the collective fields,

$$P(R; \{\phi\}) = \frac{1}{|R|!} \sum_{g \in S_{|R|}} \text{ch}_R(g) \prod_{j=1}^{|R|} (\phi_j)^{\mu(g)_j}, \quad (4)$$

where the energy eigenstate is characterized by the Young tableau R , where $|R|$ denotes the total number of boxes in the Young tableau R . In addition, $\text{ch}_R(g)$ denotes the character of $S_{|R|}$ for the group element $g \in S_{|R|}$ in the irreducible representation associated to the Young tableau R . $\mu(g)_j$ denotes the number of cycles of length j in $g \in S_{|R|}$. The corresponding energy eigenvalue is given by the Casimir of the Young tableau R .

$$E_{\text{SU}(N)}(R) = \sum_{j=1}^n r_j(N + r_j - 2j + 1) - \frac{1}{N} \left(\sum_{j=1}^n r_j \right)^2. \quad (5)$$

Here, r_j denotes the number of boxes in the j th row in the Young tableau R .

The energy eigenvalue (5) of a Young tableau R is identical to that of its conjugate Young tableau, $\bar{R}$, where the conjugate Young tableau $\bar{R}$ is defined by inverting R both upside down and from left to right, and filling it out to a rectangle with N rows [see Fig. 1(a)]. For example the *anti-box* $\bar{\square}$, a column of $N - 1$ boxes, has the same energy as a single box $\square$, which is reminiscent of the particle-hole relationship. Therefore, in the low energy sector, a generic Young tableau consists of a Young tableau λ^L with a finite number of anti-boxes as well as a Young tableau λ^R with a finite number of boxes [see Fig. 1(b)]. The energy eigenvalue corresponding to such a Young tableau denoted by $R = (\lambda^L, \lambda^R)$ can be decomposed into that of λ^L and λ^R with $1/N$ correction,

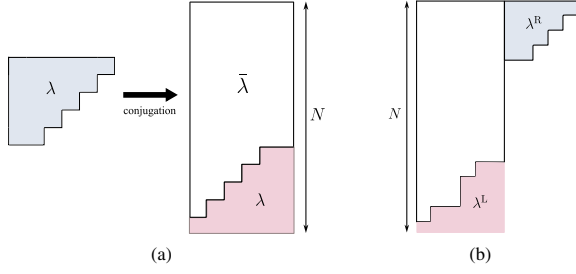


FIG. 1. (a) For a Young tableau λ , the conjugate Young tableau $\bar{\lambda}$ is defined by flipping λ both vertically and horizontally. (b) A generic Young tableau in the low energy sector of $SU(N)$ matrix model. It consists of $\mathcal{O}(N^0)$ number of boxes (blue) and anti-boxes (red), respectively.

$$E_{SU(N)}(R) = E_{SU(N)}(\lambda^L) + E_{SU(N)}(\lambda^R) + \frac{2|\lambda^L||\lambda^R|}{N}. \quad (6)$$

The details are found in the Appendix A. Hence, the low energy sector includes Young tableaux with $\mathcal{O}(N)$ numbers of boxes related to the λ^L .

In order to describe the low energy sector of the $SU(N)$ matrix model (1), the Hilbert space for the collective Hamiltonian (3) needs to be expanded. The collective Hamiltonian in Eq. (3) is formulated based on the collective field ϕ_n which is considered to be independent in the large N limit. However, this framework cannot accommodate Young tableaux with a number of boxes of order $\mathcal{O}(N)$ due to the breakdown of the independence of the collective fields.

In the low energy sector, we define the two branches of the collective field as follows:

$$\phi_n \equiv \text{tr}(U^n), \quad \bar{\phi}_n = \text{tr}(U^{-n}). \quad (7)$$

While ϕ_n and $\bar{\phi}_m$ are not independent at finite N , they become independent in the large N limit. In this low energy description, ϕ_n and $\bar{\phi}_n$ are unrestricted because we consider a finite number of boxes and anti-boxes in the large N limit. In terms of ϕ_n and $\bar{\phi}_n$, the collective Hamiltonian, which reproduces the low energy spectrum (6), is found to be

$$H[\phi, \bar{\phi}] = H_{\text{col}}[\phi] + H_{\text{col}}[\bar{\phi}] + \frac{2}{N} \sum_{n=1}^{\infty} n \phi_n \frac{\partial}{\partial \phi_n} \sum_{m=1}^{\infty} m \bar{\phi}_m \frac{\partial}{\partial \bar{\phi}_m}, \quad (8)$$

where $H_{\text{col}}[\phi]$ is defined in Eq. (3), and similar for $H_{\text{col}}[\bar{\phi}]$.

The Hilbert space of the low energy sector is factorized into two Fock spaces, $\mathcal{H}_\phi \otimes \mathcal{H}_{\bar{\phi}}$, which is constructed by two branches of the collective fields, $\{\phi_n\}$ and $\{\bar{\phi}_n\}$, respectively. This factorization of the Hilbert space is an emergent phenomenon in the low energy sector because $\mathcal{H}_{\bar{\phi}}$ is not independent of $\mathcal{H}_\phi$ at finite N (See Fig. 2). For example, using the constraint in the Schur polynomial of

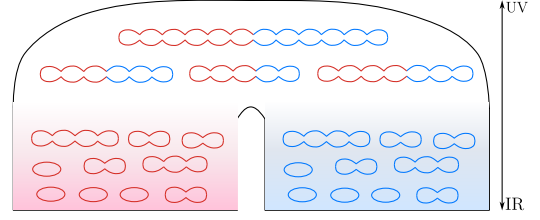


FIG. 2. Emergent factorization of Hilbert space in the low energy sector. In the low energy description, the loops ϕ_n (blue) and antiloops $\bar{\phi}_n$ (red) are independent. But, in the UV regime, they are no longer independent due to the constraints of the matrix model.

$SU(N)$ matrix, the state corresponding to a single anti-box can be expressed in terms of that of the $N - 1$ boxes at finite N . Hence the collective description at finite N should take into account this relation (see Ref. [53]).

On the other hand, in the low energy sector, the constraints in the matrix model that dictate the relation between boxes and anti-boxes is absent, because we consider a finite number of boxes and anti-boxes. As a result, in the low energy sector the anti-boxes emerge as an independent degree of freedom, or in other words, a quasiparticle or a collective excitation of boxes similar to a hole in the band theory.

The energy eigenstate corresponding to a Young tableau $R = (\lambda^L, \lambda^R)$ is also factorized into two Schur polynomials of ϕ 's and $\bar{\phi}$'s, respectively,

$$\Psi_{(\lambda^L, \lambda^R)} = P(\lambda^L, \{\bar{\phi}\}) P(\lambda^R, \{\phi\}). \quad (9)$$

Here, one should not take it simply as a product of two Schur polynomials. Since the Hilbert space in the low energy sector is factorized, it is a tensor product of two Schur polynomials,

$$\Psi_{(\lambda^L, \lambda^R)} = (P(\lambda^L, \{\bar{\phi}\}), P(\lambda^R, \{\phi\})). \quad (10)$$

Therefore, one should not apply the fusion rule identity between the two Schur polynomials, $P(\lambda^L, \{\bar{\phi}\})$ and $P(\lambda^R, \{\phi\})$, as it is based on the constraint of the matrix absent in the low energy description.

We also study the collective Hamiltonian of the $U(N)$ matrix model for the semiclassical description of “particle-hole” fluctuations around a background Young tableau in Appendix B.

III. EMERGENT TFD STATE

The TFD state [54,55], which is holographically dual to eternal Anti-de Sitter spacetime (AdS) black hole [26] in the context of AdS/CFT correspondence, can emerge in the low energy sector because of the emergent factorization of the Hilbert space. In $SU(N)$ case with a finite N , we consider a fine-tuned state in the form of

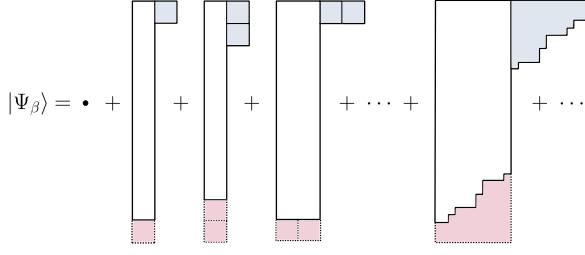


FIG. 3. Fine-tuned state at finite N , which becomes a TFD state in the low energy sector.

$$\Psi_\beta = \sum_{R=(\lambda,\lambda), |\lambda| < N/2} e^{-\frac{\beta}{2} E_{\text{SU}(N)}(R)} P(R, \{\phi\}), \quad (11)$$

where the Young tableau R comprises two Young tableaux of identical shape λ , one of boxes and the other of anti-boxes (see Fig. 3). Hence these Young tableaux $R = (\lambda, \lambda)$ are self-conjugate, i.e., $\bar{R} = R$. We limit the number of boxes in the Young tableau λ to ensure that the Young tableau of boxes does not overlap with the Young tableau of anti-boxes. Such a large Young tableau will be irrelevant in the low energy description.

As the Hilbert space is factorized in the low energy sector, the state Ψ_β becomes the TFD state with the inverse temperature β ,

$$\Psi_\beta = \sum_{\lambda} e^{-\frac{\beta}{2} (E_{\text{SU}(N)}(\lambda) + \frac{|\lambda|^2}{N})} P(\lambda, \{\bar{\phi}\}) P(\lambda, \{\phi\}). \quad (12)$$

In the strict large N limit, it turns into the TFD state for the free loop gas, which is analogous to the *generalized free field* in the holography,

$$\Psi_\beta = \sum_{n=0}^{\infty} e^{-\frac{\beta N n}{2}} \sum_{\nu \vdash n} \prod_{j=1}^n \frac{\phi_j^{\nu_j} \bar{\phi}_j^{\nu_j}}{j^{\nu_j} \nu_j!}. \quad (13)$$

In the context of a black hole, the collective field, $\bar{\phi}_n$, can be viewed as the mirror operator of ϕ_n , arising from the smoothness condition of black hole horizon [56].

The construction of the mirror operator is state dependent in the low energy description while it is state independent at finite N regime. The constraints in the matrix model give rise to the operator relation between the collective field ϕ_n and its mirror operator $\bar{\phi}_n$, which is independent of states,

$$P(\lambda, \bar{\phi}) = P(\bar{\lambda}, \phi). \quad (14)$$

On the other hand, due to the absence of these constraints in the low energy description, the construction of the mirror operator becomes state dependent,

$$\bar{\phi}_n \Psi_\beta = e^{\frac{\beta n}{2}} n \frac{\partial}{\partial \phi_n} \Psi_\beta. \quad (15)$$

Here we considered the TFD state (13) in the generalized free field limit. We also find that the state-dependent construction of the mirror operator with respect to the TFD state (12) beyond the generalized free field limit can be approximated to be

$$P(\nu, \{\bar{\phi}_n\}) \Psi_\beta \approx e^{\frac{\beta}{2} E_{\text{SU}(N)}(\nu)} P\left(\nu, \left\{n \frac{\partial}{\partial \phi_n}\right\}\right) \Psi_\beta. \quad (16)$$

The details are given in the Appendix E.

IV. BLACK HOLE COMPLEMENTARITY

In the low energy description for the matrix model, accessible observables are limited because one cannot probe an observable associated with the large number of boxes. Therefore, the set of accessible observables in the singlet sector, which are polynomials of ϕ 's with a finite degree, should have a cutoff in the number of associated boxes. Due to the cutoff, the set of accessible observables denoted by $\mathcal{A}_\phi^{\text{trunc}}$ does not form an algebra. This truncation by cutoff can also be observed in Quantum Field Theory (QFT) on the curved background where a product of exponentially large number of matters would, in principle, produce a black hole operator, but we do not include the black hole operator in the set of accessible observables in the semiclassical calculations. This *truncated algebra* $\mathcal{A}_\phi^{\text{trunc}}$ has been explored in Refs. [13,14,21–25] to reinterpret the black hole complementarity [19,20] from the operator perspective.

The construction of the mirror operators $\bar{\phi}_n$ in the previous section elucidates a mechanism for the revised concept of black hole complementarity asserting that the degree of freedom in black hole interior is not independent of that of the outside [13,14,21–25]. At finite N , the mirror operator $\bar{\phi}_n$ is a polynomial of ϕ 's associated with large Young tableaux. In the low energy sector, such large Young tableaux are beyond the truncated algebra $\mathcal{A}_\phi^{\text{trunc}}$, allowing us to consider ϕ_n and its mirror operator $\bar{\phi}_m$ as independent degrees of freedom. As we begin to probe large Young tableaux (e.g., higher-point correlations or the Hilbert space of the Hawking radiations after Page time), the constraints of the matrix model come into play. This leads to the breakdown of the semiclassical description.

The bulk locality is also an emergent phenomenon in the semiclassical description. $\bar{\phi}$ is viewed as an emergent quasiparticle that becomes independent in the low energy description, ensuring the exact bulk locality in this regime,

$$\left[\bar{\phi}_n, \frac{\partial}{\partial \phi_m} \right] = 0. \quad (17)$$

However, at finite N , the mirror operator $\bar{\phi}_n$ no longer maintains independence from ϕ 's, resulting in the breakdown of the bulk locality,

$$\left[\bar{\phi}_1, \frac{\partial}{\partial \phi_m} \right] = \left[P(\bar{\square}, \phi), \frac{\partial}{\partial \phi_m} \right] \neq 0. \quad (18)$$

This bulk nonlocality, manifesting as an island in the black hole, will be explored in the next section.

V. PAGE CURVE AND ISLAND CONJECTURE

The emergent factorization in the matrix model can provide insight into the black hole information paradox [11,12]. In the semiclassical description, introducing a cutoff to the accessible observables results in the factorization of the two truncated algebras, $\mathcal{A}_\phi^{\text{trunc}}$ and $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$, generated by ϕ_n and $\bar{\phi}_n$, respectively. In context of a black hole, $\mathcal{A}_\phi^{\text{trunc}}$ corresponds to observables outside the black hole, whereas $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ represents the observables inside the black hole that are difficult to construct by a finite product of the operators. During the black hole evaporation, the emitted Hawking radiation, which belongs to $\mathcal{A}_\phi^{\text{trunc}}$, is entangled with the mirror modes in $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$, leading to an increase in the entropy of the Hawking radiation.

When the half of the black hole is evaporated, a point referred to as Page time, the number of the Hawking radiations become sufficiently large that the calculation of the entanglement entropy requires observables beyond the truncated algebra $\mathcal{A}_\phi^{\text{trunc}}$. Adhering to the premise that the Hawking radiation and its mirror modes are independent, by overlooking the constraints in the matrix model even after the Page time, leads to a continuous increase in the entanglement entropy of the Hawking radiation past Page time.

After the Page time, the entanglement entropy calculations must incorporate the constraints in the matrix model, leading to the breakdown of the semi-classical description. Although we expect a decrease in the entanglement entropy following the Page curve [57,58] (see Fig. 4), confirming this decline through methods beyond the semiclassical approaches remains a significant challenge.

In spite of the breakdown of the semiclassical description after the Page time, the *island* conjecture [30–36] has succeeded in reproducing the Page curve surprisingly by using semiclassical calculations. This conjecture proposes that the entanglement wedge of the Hawking radiation, referred to as an “island,” emerges inside the black hole. The emergence of the island inside the black hole, which extremizes the generalized entropy, plays a central role in the decrease of the entanglement entropy after the Page time. Nonetheless, the underlying reason why the

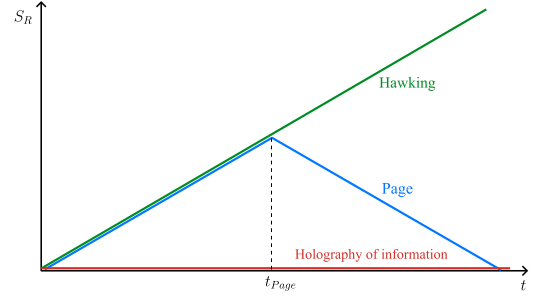


FIG. 4. Entanglement entropy (EE) of Hawking radiation. (Green line) The EE keeps increasing as long as $\mathcal{A}_\phi^{\text{trunc}}$ and $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ are regarded as independent. (Blue line) After the Page time t_{Page} , $\mathcal{A}_\phi^{\text{trunc}}$ and $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ are no longer independent, leading to the reduction in the EE. (Red line) Considering the full algebra $\mathcal{A}_\phi^{\text{full}}$ generated by ϕ , rather than the truncated algebra $\mathcal{A}_\phi^{\text{trunc}}$ of accessible observables, information about mirror operators is consistently present in $\mathcal{A}_\phi^{\text{full}}$, bringing about the vanishing EE throughout the evaporation process.

semiclassical calculations by island managed to reproduce the Page curve has remained elusive.

The emergent factorization in the matrix model elucidates the origin of the island. After Page time, calculating the entanglement entropy of the Hawking radiation necessitates observables associated with large Young tableaux. Some of these Young tableaux can equivalently be viewed as a pair of Young tableaux (λ^L, λ^R) , consisting of a finite number of anti-boxes and boxes, respectively (See Fig. 5). As a result, they can be regarded as observables in $\mathcal{A}_\phi^{\text{trunc}} \otimes \mathcal{A}_{\bar{\phi}}^{\text{trunc}}$, enabling the semiclassical analysis. In other words, certain complicated operators can effectively be considered as two simple operators, one inside the black

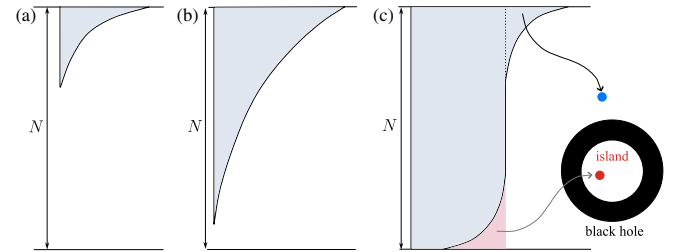


FIG. 5. (a) Before the Page time, the EE calculation is involved with Young tableaux with a finite numbers of boxes, in which the semiclassical description is valid. (b) After the Page time, the number of Hawking radiation becomes large enough to be involved with large Young tableaux. Generally, those large Young tableaux extend beyond the scope of the semiclassical description. (c) Certain large Young tableaux can effectively be represented as a pair of Young tableaux (λ^L, λ^R) , consisting of a finite number of anti-boxes and boxes, respectively. These Young tableaux can be dealt with in the semiclassical framework, wherein the Young tableau λ^L (red) represents the collective excitation in the island inside the black hole while λ^R corresponds to the excitation outside the black hole.

hole and the other outside. Hence, the entanglement wedge of the Hawking radiation includes the truncated algebra, $\mathcal{A}_{\bar{\phi}}^{\text{trunc}} \otimes \mathcal{A}_{\phi}^{\text{trunc}}$. The semiclassical analysis incorporating $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ —the island inside the black hole—can effectively capture the contributions of complicated operators.

VI. HOLOGRAPHY OF INFORMATION

Recently, the holography of information has been proposed, claiming that all information about the black hole interior is already available on the asymptotic boundary [37–44]. Consequently, according to the holography of information, the entanglement entropy of the Hawking radiation would always be 0 throughout the black hole evaporation process. (See Fig. 4.)

The holography of information could be tantamount to asserting that the collective ϕ_n 's can generate full operator algebra $\mathcal{A}_{\phi}$ in the singlet sector of the matrix model, which includes both $\mathcal{A}_{\phi}^{\text{trunc}}$ and $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$, i.e. $\mathcal{A}_{\phi}^{\text{trunc}} \cup \mathcal{A}_{\bar{\phi}}^{\text{trunc}} \subseteq \mathcal{A}_{\phi}^{\text{full}}$. Although one cannot access the observables in the truncated algebra $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ by simple measurements by ϕ_n 's, the information regarding the mirror operators $\bar{\phi}$'s is available in the operator algebra $\mathcal{A}_{\phi}^{\text{full}}$ generated by ϕ 's. Hence, disregarding the measurement, the entanglement entropy of the Hawking radiation remains at zero while the entanglement entropy based on the simple measurement will increase initially.

VII. DISCUSSION

In this article, we have investigated the emergent factorization of the Hilbert space in the low energy description of the matrix models and its implications for understanding the black hole information paradox. By analyzing the collective description of a simple matrix model, we have demonstrated how the mirror operator—analogue to a hole in the band theory—emerges as a collective excitation in the low energy sector, thereby providing insights into the black hole complementarity. Furthermore, our study extends to discussing the emergence of the TFD state from a fine-tuned state; elucidating the origin of the island that appears inside black holes; and offering a reinterpretation of the holography of information from the emergent factorization perspective.

Realizing the emergent factorization of the Hilbert space in realistic holographic models for black holes would be crucial. Although the matrix models examined in this work are simplistic enough to elucidate the mechanism behind the emergence of the collective mirror excitations, those models are, in essence, toy models that do not directly describe black holes. Therefore, the investigation on the emergence of the factorization in a concrete holographic matrix model, such as $\mathcal{N} = 4$ SYM, would be highly interesting. In particular, a BPS black hole operator in large N limit could provide a fruitful playground to study the

emergence of the mirror operator. Furthermore, it will be interesting to investigate whether the interior of the black hole and the mirror operator emerge in the semiclassical description of a fuzzball [59,60].

When the truncated “algebras” $\mathcal{A}_{\phi}^{\text{trunc}}$ and $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ are assumed to be closed, the emergent factorization of Hilbert space in the low energy sector leads to the type III_1 von Neumann factor. For example, the mirror observables in $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ can be viewed as the probe discussed in Ref. [61], leading to the identification of the type III_1 factor. Note that we have showed that the mirror operators in $\mathcal{A}_{\bar{\phi}}^{\text{trunc}}$ emerge naturally in the low energy sector, whereas Ref. [61] introduced the mirror operator by hand as a probe. However, the truncated “algebra” is not inherently closed, indicating that it should not be strictly defined as an algebra. Therefore, it becomes imperative to generalize the framework of von Neumann algebra and the Reeh-Schlieder theorem to accommodate a restricted set of accessible observables [62] in the low energy sector in a manner reminiscent of the spirit of the Wilsonian renormalization group.

Previous attempts to tackle the information paradox through semiclassical black hole dynamics have mainly focused on reproducing the correct entanglement across the horizon. In particular, previous works [13] argue that late-time radiation data (typically at null infinity) combined with interior modes can yield a locally vacuumlike horizon, though they emphasize solely on the entanglement structure rather than the full dynamical evolution. Our work is also somewhat limited in this aspect. While we explicitly construct the interior and exterior degrees of freedom, we do not analyze the detailed dynamics (which is studied in some detail in previous works such as Refs. [63–65])—energy eigenstates being labeled by Young diagrams. Instead, we focus our attention on demonstrating how an approximate factorization arises at large N , despite being dependent at finite N . Since our toy model is not meant to describe actual black hole evaporation, a realistic approach would involve large N matrix models with genuine holographic duals such as $\mathcal{N} = 4$ SYM or D1–D5 systems, where emergent factorization may play a central role. Recent developments on fortuitous operators in Refs. [66–68] could further illuminate how such emergent degrees of freedom shape black hole evaporation beyond the entanglement structure. We look forward to investigating the dynamical aspects of these models in future work.

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APPENDIX A: CASIMIR FOR LOW-ENERGY SECTOR OF $SU(N)$ MATRIX MODEL

In this section, we provide an explicit derivation of the decomposition of the energy eigenvalue $E_{SU(N)}(R)$ in the main article for a typical Young tableau in the low energy sector parametrized as $R \equiv (\lambda^L, \lambda^R)$ [See Fig. 1(b) in the main article]. The energy eigenvalue, $SU(N)$ Casimir, is given by

$$E_{SU(N)}(R) = \sum_{j=1}^N r_j (N + r_j - (2j - 1)) - \frac{1}{N} \left(\sum_{j=1}^N r_j \right)^2, \quad (\text{A1})$$

where r_j denotes the number of boxes in the j th row in the Young tableau R . The constituent tableaux λ^R and λ^L are parametrized by their corresponding row lengths $\lambda^R \equiv (\lambda_1^R, \lambda_2^R, \dots, \lambda_p^R)$ and $\lambda^L \equiv (\lambda_1^L, \lambda_2^L, \dots, \lambda_q^L)$, respectively. Thus, the parametrization of the full young tableau R takes the form

$$r_i = \begin{cases} \lambda_1^L + \lambda_i^R & \text{for } 1 \leq i \leq p \\ \lambda_i^L & \text{for } p+1 \leq i \leq N \\ -q\lambda_1^L - \lambda_{N-i+1}^L & \text{for } N-q+1 \leq i \leq N \end{cases}. \quad (\text{A2})$$

In terms of λ^L and λ^R , the energy eigenvalue (A1) can be expressed as

$$\begin{aligned} E_{SU(N)}(R) &= N(N\lambda_1^L + |\lambda^R| - |\lambda^L|) + N\lambda_1^L(\lambda_1^L - N) \\ &\quad + 2\lambda_1^L(|\lambda^R| - |\lambda^L|) + \sum_{j=1}^p \lambda_j^R(\lambda_j^R - 2j + 1) \\ &\quad + \sum_{k=1}^q \lambda_k^L(\lambda_k^L - 2k + 1) + 2N|\lambda^L| \\ &\quad - \frac{1}{N}(N\lambda_1^L + |\lambda^R| - |\lambda^L|)^2, \end{aligned} \quad (\text{A3})$$

where $|\lambda^L|$ and $|\lambda^R|$ denotes the total number of boxes in λ^L and λ^R , i.e., $|\lambda^L| \equiv \sum_{i=1}^p \lambda_i^L$ and $|\lambda^R| \equiv \sum_{i=1}^q \lambda_i^R$. Finally, we obtain

$$E_{SU(N)}(R) = E_{SU(N)}(\lambda^R) + E_{SU(N)}(\lambda^L) + \frac{2}{N}|\lambda^L||\lambda^R|, \quad (\text{A4})$$

leading to the decomposition of the energy eigenvalue of $R = (\lambda^L, \lambda^R)$ as given by Eq. (6) in the main article.

APPENDIX B: EMERGENT FACTORIZED HILBERT SPACE IN SEMICLASSICAL DESCRIPTION OF $U(N)$ MATRIX MODEL

In the operator algebra of the BPS operators, the product of two BPS black hole operators can yield another BPS operator corresponding to a larger BPS black hole. This differs from the operator algebra of the quantum field theory on the curved background, where the background does not take a part in the operator product. To understand the role of the background in the emergent collective excitation, we examine the semiclassical description for collective excitations around a large Young tableau, which is analogous to the giant graviton [69–75] or the LLM geometry [53,76]. Our discussion extends to the emergent factorized Hilbert space in the semiclassical description around the background and its breakdown at finite N , thereby highlighting the constraints of the matrix model as a cutoff in the accessible observables.

For this, we consider the semiclassical description for the $U(N)$ matrix model of which the Hamiltonian is defined by

$$H = \text{tr} \left[U \frac{\partial}{\partial U} U \frac{\partial}{\partial U} \right]. \quad (\text{B1})$$

The energy eigenstate of the $U(N)$ matrix model is also written as the $U(N)$ character associated with the Young tableau R [see Eq. (4) in the main article]. The corresponding energy eigenvalue is given by

$$E_{U(N)}(R) = \sum_{j=1}^n r_j [N + r_j - (2j - 1)]. \quad (\text{B2})$$

In the $U(N)$ matrix model, the energy associated with a single box, $\square$, differs from that of a single anti-box, $\bar{\square}$ in contrast to $SU(N)$ case. Furthermore, in the $U(N)$ case, columns that are completely filled with boxes are not trivial. Therefore a reference Young tableau is required to define a new type of conjugation, which gives rise to particle-hole relationship in the low energy sector.

For the semiclassical description of the $U(N)$ matrix model, we introduce a background $N \times M$ Young tableau, $\Lambda^{(0)}$ ($N, M \gg 1$), which plays a similar role to a giant graviton [69–75] or a black hole background in the gravity [see Fig. 6(a)].

$$\Lambda_j^{(0)} \equiv M \quad (j = 1, 2, \dots, N). \quad (\text{B3})$$

Furthermore, we define a $*$ -conjugation of a Young tableau λ with respect to the background $\Lambda^{(0)}$ by subtracting λ from $\Lambda^{(0)}$ (See Fig. 7).

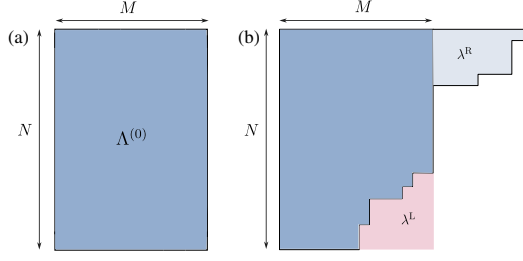


FIG. 6. (a) Background $N \times M$ Young tableau $\Lambda^{(0)}$. (b) A generic Young tableau in the semiclassical description of $U(N)$ matrix model. It is composed of $\mathcal{O}(N^0)$ number of boxes (light blue) and anti-boxes (red) together with background (dark blue).

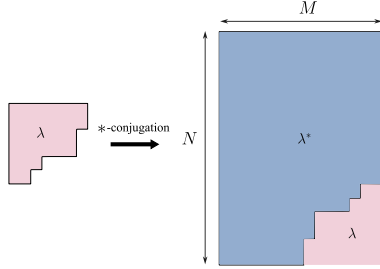


FIG. 7. *-conjugation of Young tableau λ with respect to background Young tableau $\Lambda^{(0)}$.

$$(\lambda^*)_j \equiv M - \lambda_{N+1-j} \quad (j = 1, 2, \dots, N). \quad (\text{B4})$$

This *-conjugation is well defined for Young tableaux that do not exceed the background $\Lambda^{(0)}$, which will be sufficient for the semiclassical description.

In our semiclassical description, we consider fluctuations around the background, $\Lambda^{(0)}$, which form the Hilbert space. A generic Young tableau in the Hilbert space, denoted by (λ^L, λ^R) , is characterized by the configuration where the Young tableau λ^R with a finite number boxes is attached to the background Young tableau $\Lambda^{(0)}$ and the Young tableau λ^L with a finite number boxes is subtracted from $\Lambda^{(0)}$ [see Fig. 6(b)]. The Young tableau λ^L is a collective excitation with respect to the background, and λ^R and λ^L play a role of particle and hole, respectively.

The Hilbert space in the semiclassical description around the background is factorized into two Fock spaces, $\mathcal{H}_\phi \otimes \mathcal{H}_{\bar{\phi}}$, generated by the collective fields, ϕ_n and $\bar{\phi}_n$,

$$\phi_n \equiv \text{tr}(U^n), \quad \bar{\phi}_n \equiv \text{tr}(U^{-n}) \quad (n = 1, 2, 3, \dots). \quad (\text{B5})$$

The emergent factorization of the Hilbert space in the semiclassical description will hold as long as we consider a finite number of boxes for λ^L and λ^R . In this regime, the energy eigenvalue of a state associated with a Young tableau $R = (\lambda^L, \lambda^R)$ is found to be

$$\begin{aligned} E_{U(N)}(R) &= NM^2 + E_{U(N)}(\lambda^R) + E_{U(N)}(\lambda^L) \\ &\quad + 2M(|\lambda^R| - |\lambda^L|) \\ &= N(M^2 + |\lambda^R| + |\lambda^L|) + 2M(|\lambda^R| - |\lambda^L|) \\ &\quad + \sum_j [\lambda_j^R(\lambda_j^R + 1 - 2j) + \lambda_j^L(\lambda_j^L + 1 - 2j)], \end{aligned} \quad (\text{B6})$$

where we will show the explicit derivation in Appendix C. In the limit $N \gg M \gg 1$, this can be indeed viewed as a fluctuation around the background energy NM^2 .

In the semiclassical regime, the collective Hamiltonian that dictates the dynamics of the fluctuations is determined to be

$$\begin{aligned} H &= NM^2 + M \sum_{n=1}^{\infty} n \left(\phi_n \frac{\partial}{\partial \phi_n} - \bar{\phi}_n \frac{\partial}{\partial \bar{\phi}_n} \right) + N \sum_{n=1}^{\infty} n \phi_n \frac{\partial}{\partial \phi_n} \\ &\quad + \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} n \phi_m \phi_{n-m} \frac{\partial}{\partial \phi_n} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n m \phi_{n+m} \frac{\partial}{\partial \phi_n} \frac{\partial}{\partial \phi_m} \\ &\quad + N \sum_{n=1}^{\infty} n \bar{\phi}_n \frac{\partial}{\partial \bar{\phi}_n} + \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} n \bar{\phi}_m \bar{\phi}_{n-m} \frac{\partial}{\partial \bar{\phi}_n} \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n m \bar{\phi}_{n+m} \frac{\partial}{\partial \bar{\phi}_n} \frac{\partial}{\partial \bar{\phi}_m}. \end{aligned} \quad (\text{B7})$$

The energy eigenstate corresponding to a Young tableau $R = (\lambda^L, \lambda^R)$ is expressed as

$$\Psi_{(\lambda^L, \lambda^R)} = P(\lambda^L, \{\bar{\phi}\}) P(\lambda^R, \{\phi\}), \quad (\text{B8})$$

with its eigenvalue yielding the semiclassical spectrum in Eq. (B6). As in the $SU(N)$ case, the state in Eq. (B8) should be interpreted as a direct product rather than a simple product.

In addition, the factor $[\det U]^M$, representing the background, is stripped off in the realization of the state (B8) for the fluctuations. At finite N , the Schur polynomial for the background Young tableau $\Lambda^{(0)}$ is expressed in terms of the determinant operator,

$$P(\Lambda^{(0)}, U) = [\det U]^M. \quad (\text{B9})$$

However, the determinant operator $\det U$ does not belong to the Hilbert space $\mathcal{H}_\phi \otimes \mathcal{H}_{\bar{\phi}}$ since we take into account $\mathcal{O}(N^0)$ numbers of the boxes and anti-boxes for the semiclassical description. Hence the determinant, $\det U$, should be regarded as a c number instead of an operator acting on the Hilbert space.

In the semiclassical description with the realization of the fluctuating state in Eq. (B8), the operator product should be evaluated among ϕ 's and among $\bar{\phi}$'s respectively, not between ϕ and $\bar{\phi}$,

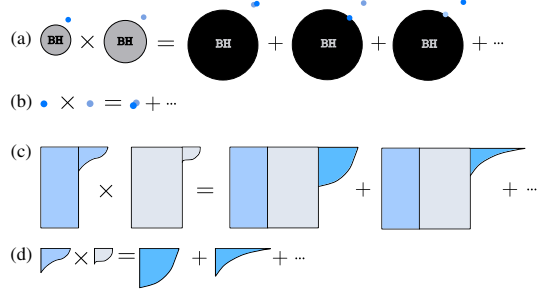


FIG. 8. (a) Product of two BPS black hole operators, representing a black hole with graviton hair, can produce another BPS black hole operator with a large black hole with graviton hair. It is necessary to take into account not only the Operator Product Expansion (OPE) of the gravitons but also the OPE between the graviton and the black hole, etc. (b) In the semiclassical description, the OPE is involved with a graviton while the black hole background does not take part in it. (c) The tensor product of two Young tableaux with background yields a larger background analogous to the OPE of BPS black hole operators. (d) In the semiclassical description, the background Young tableau does not participate in the tensor product.

$$(\lambda_1, \mu_1) \times (\lambda_2, \mu_2) = \sum_{\lambda_3, \mu_3} c_{\lambda_1 \lambda_2}^{\lambda_3} c_{\mu_1 \mu_2}^{\mu_3} (\lambda_3, \mu_3), \quad (\text{B10})$$

where $c_{\lambda\mu}^\nu$ is the tensor product coefficient of $U(N)$ group. The operator product between ϕ and $\bar{\phi}$ is involved with the constraint of the matrix model such as

$$P\left(\begin{array}{c} \square \\ \vdots \\ \square \end{array}\right)_N, \{\phi\} = \frac{1}{N!} \phi_1^N + \dots + \frac{(-1)^{N+1}}{N} \phi_N = \det U, \quad (\text{B11})$$

which is absent in the semiclassical description. In addition, when realizing the fluctuating state as described in Eq. (B8), the background is intact under the operator product. For example, in the operator product

$$\Psi(\cdot, \cdot, \square) \times \Psi(\cdot, \cdot, \square) = \Psi(\cdot, \cdot, \square\square) + \Psi(\cdot, \cdot, \begin{array}{c} \square \\ \square \end{array}), \quad (\text{B12})$$

all Ψ 's in the left-hand and right-hand sides represent fluctuations with respect to same background $\Lambda^{(0)}$. This is analogous to the operator product algebra in the curved background [77–79] where the operator product is analyzed semiclassically in the fixed background geometry.

On the other hand, the background Young tableau $\Lambda^{(0)}$ takes part in the operator product at finite N . This corresponds to the full quantum gravity calculations, in which the background geometry is also treated as an operator. For example, at finite N the product of two operators, which consists of one column corresponding to the background ($M=1$) and a single box (and anti-box, respectively) representing the fluctuation around the background, is given by

Here, the first term on the right-hand side represents the separate product of the two fluctuations while two backgrounds (each one column) combine to create a larger background. The second term indicates the intermingling of ϕ and $\bar{\phi}$. In Fig. 8, we depict an analogous situation in the context of a BPS black hole operator [15–18] in $\mathcal{N}=4$ Yang-Mills theory.

APPENDIX C: CASIMIR FOR SEMICLASSICAL FLUCTUATION AROUND BACKGROUND $\Lambda^{(0)}$ IN $U(N)$ MATRIX MODEL

In this section, we derive the decomposition of the energy eigenvalue $E_{U(N)}(R)$ in Eq. (B6) for a Young tableau $R = (\lambda^L, \lambda^R)$. Recall that for the case of $U(N)$, the Casimir is given by

$$E_{U(N)}(R) = N \sum_{i=1}^N r_i + \sum_{i=1}^N r_i(r_i + 1 - 2i), \quad (\text{C1})$$

where r_i denotes the number of boxes in the j th row of the Young tableau R . As in Appendix A, the Young tableaux λ^L and λ^R are also parametrized by the number of boxes, λ_j^L and λ_j^R .

$$r_i = \begin{cases} M + \lambda_i^R & \text{for } 1 \leq i \leq p \\ M & \text{for } p+1 \leq i \leq N-q \\ M - \lambda_{N-i+1}^L & \text{for } N-q+1 \leq i \leq N \end{cases} \quad (\text{C2})$$

In the above, the row lengths of λ^R are given by $(r_1, r_2, \dots, r_p)$ while the row lengths of λ^L are given by $(s_1, s_2, \dots, s_q)$. Thus, plugging in the parametrization (C2) in (C1) and following an analogous computation as done in the preceding section, we get,

$$\begin{aligned} E_{U(N)}(R) &= N(|\lambda^R| + |\lambda^L|) + NM^2 + 2M(|\lambda^R| - |\lambda^L|) \\ &\quad + \sum_{i=1}^p \lambda_i^R(\lambda_i^R + 1 - 2i) + \sum_{i=1}^q \lambda_i^L(\lambda_i^L + 1 - 2i) \\ &= NM^2 + 2M(|\lambda^R| - |\lambda^L|) + E_{U(N)}(\lambda^R) \\ &\quad + E_{U(N)}(\lambda^L). \end{aligned} \quad (\text{C3})$$

APPENDIX D: SCHUR POLYNOMIAL IDENTITY FOR *-CONJUGATED YOUNG TABLEAU

Let us consider a $N \times N$ unitary matrix U of which eigenvalues are given by

$$U \sim \text{diag}(x_1, x_2, \dots, x_N). \quad (\text{D1})$$

The Schur polynomial corresponding to a Young tableau with N boxes in one column is given by

$$P\left(\begin{array}{c} \square \\ \vdots \\ \square \end{array}\right)_N, U) = x_1 x_2 \cdots x_N = \det U. \quad (\text{D2})$$

Similarly, the Schur polynomial associated with the Young tableau with $N-1$ boxes in a single column can be written in terms of the determinant $\det U$ and the Schur polynomial of the inverse matrix U^{-1} for one box,

$$\begin{aligned} P\left(\begin{array}{c} \square \\ \vdots \\ \square \end{array}\right)_{N-1}, U) &= \sum_{1 \leq i_1 < i_2 < \cdots < i_{N-1} \leq N} x_{i_1} x_{i_2} \cdots x_{i_{N-1}} \\ &= \det U \sum_{i=1}^N \frac{1}{x_i} = \det U P_1(\square, U^{-1}). \end{aligned} \quad (\text{D3})$$

In general, a Young tableau composed of a single column with $N-n$ boxes can be identified with the Schur polynomial of U^{-1} for n boxes in a single column.

$$\begin{aligned} P_{N-n}\left(\begin{array}{c} \square \\ \vdots \\ \square \end{array}\right)_{N-n}, U) &= \sum_{1 \leq i_1 < i_2 < \cdots < i_{N-n} \leq N} x_{i_1} x_{i_2} \cdots x_{i_{N-n}} \\ &= \det U \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq N} \frac{1}{x_{i_1} x_{i_2} \cdots x_{i_n}} = \det U P_n\left(\begin{array}{c} \square \\ \vdots \\ \square \end{array}\right)_n, U^{-1}). \end{aligned} \quad (\text{D4})$$

For a generic Young tableau λ , the corresponding Schur polynomial can be written as

$$\begin{aligned} P(\lambda, U) &= \frac{\det(x_j^{\lambda_i + N - i})}{\prod_{1 \leq i < j \leq N} (x_i - x_j)} \\ &= \frac{1}{\prod_{1 \leq i < j \leq N} (x_i - x_j)} \begin{vmatrix} x_1^{\lambda_1 + N - 1} & x_2^{\lambda_1 + N - 1} & \cdots & x_N^{\lambda_1 + N - 1} \\ x_1^{\lambda_2 + N - 2} & x_2^{\lambda_2 + N - 2} & \cdots & x_N^{\lambda_2 + N - 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_N} & x_2^{\lambda_N} & \cdots & x_N^{\lambda_N} \end{vmatrix}, \end{aligned} \quad (\text{D5})$$

where λ_j denotes the number of boxes in the j th row of the Young tableau λ .

Denoting the number of boxes in the j th row of the $*$ -conjugated Young tableau λ^* by $\nu_j \equiv (\lambda^*)_j$, we have

$$\lambda_i = M - \nu_{N-i+1}, \quad (\text{D6})$$

where we took the $*$ -conjugation (B4) with respect to a $N \times M$ background Young tableau $\Lambda^{(0)}$. From Eq. (D5), we obtain the relation between two Schur polynomials $P(\lambda, U)$ and $P(\lambda^*, U^{-1})$,

$$\begin{aligned} P(\lambda, U) &= \frac{\prod_i x_i^M}{\prod_{1 \leq i < j \leq N} (x_i^{-1} - x_j^{-1})} \begin{vmatrix} x_1^{-\nu_1 - N + 1} & x_2^{-\nu_1 - N + 1} & \cdots & x_N^{-\nu_1 - N + 1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{-\nu_{N-1} + 1} & x_2^{-\nu_{N-1} + 1} & \cdots & x_N^{-\nu_{N-1} + 1} \\ x_1^{-\nu_N} & x_2^{-\nu_N} & \cdots & x_N^{-\nu_N} \end{vmatrix} \\ &= [\det U]^M P(\lambda^*, U^{-1}). \end{aligned} \quad (\text{D7})$$

This provides the relation between an arbitrary Young tableau and its corresponding $*$ -conjugated one.

APPENDIX E: STATE-DEPENDENT CONSTRUCTION OF MIRROR OPERATOR

In this section, we present the derivation of the state-dependent construction of mirror operators, i.e., Eq. (16) of

the main article. The TFD state Ψ_β that emerges in the low energy sector from a fine-tuned state Ψ_β given by Eq. (11) in the main article, is given by

$$\Psi_\beta = \sum_{\lambda} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\lambda) + \frac{|\lambda|^2}{N})} P(\lambda, \{\bar{\phi}\}) P(\lambda, \{\phi\}), \quad (\text{E1})$$

where λ runs over Young tableaux with a finite number of boxes. The action a Schur polynomial of the mirror operator $\bar{\phi}_n$ corresponding to a $\mathcal{O}(N^0)$ Young tableau ν on the TFD state can be evaluated to be

$$\begin{aligned} P(\nu, \{\bar{\phi}_n\})\Psi_\beta &= \sum_{\lambda} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\lambda) + \frac{|\lambda|^2}{N})} P(\nu, \{\bar{\phi}_n\}) P(\lambda, \{\bar{\phi}\}) P(\lambda, \{\phi\}) \\ &= \sum_{\lambda, \kappa} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\lambda) + \frac{|\lambda|^2}{N})} c_{\nu\lambda}^\kappa P(\kappa, \{\bar{\phi}_n\}) P(\lambda, \{\phi\}), \end{aligned} \quad (\text{E2})$$

where $c_{\nu\lambda}^\kappa$ denotes the tensor product coefficients. Since both the Young tableau λ and ν have $\mathcal{O}(N^0)$ numbers of boxes, the coefficient $c_{\nu\lambda}^\kappa$ does not vanish only when the Young tableau κ in the summation satisfies the condition

$$|\kappa| = |\lambda| + |\nu|. \quad (\text{E3})$$

Recall $\bar{\phi}_n$ corresponds to the mirror creation operator. Hence, let us now consider the action of the annihilation operator on the TFD state. Noting the relation ϕ_n with $\sqrt{n}a_n^\dagger$ and $\frac{\partial}{\partial\phi_n}$ with $\frac{1}{\sqrt{n}}a_n$, we will evaluate the action of $P(\nu, \{n\frac{\partial}{\partial\phi_n}\})$ corresponding to Young tableau ν of $\mathcal{O}(N^0)$ boxes on the TFD state,

$$\begin{aligned} P\left(\nu, \left\{n\frac{\partial}{\partial\phi_n}\right\}\right)\Psi_\beta &= \sum_{\kappa} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\kappa) + \frac{|\kappa|^2}{N})} P(\kappa, \{\bar{\phi}\}) \\ &\quad \times P\left(\nu, \left\{n\frac{\partial}{\partial\phi_n}\right\}\right) P(\kappa, \{\phi\}), \end{aligned} \quad (\text{E4})$$

where κ runs over the Young tableau of $\mathcal{O}(N^0)$ boxes. To evaluate $P(\nu, \{n\frac{\partial}{\partial\phi_n}\})P(\kappa, \{\phi\})$, we will utilize the product of two Schur polynomials $P(\nu, \{\bar{\phi}\})$ and $P(\kappa, \{\phi\})$ at finite N . At this finite N , ϕ_n and $\bar{\phi}_n$ are not independent, leading to a linear combination of Schur polynomials of ϕ 's as the outcome. Among these resultant Schur polynomials, those that have finite number of Young tableaux correspond to the result of $P(\nu, \{n\frac{\partial}{\partial\phi_n}\})P(\kappa, \{\phi\})$.

$$\begin{aligned} P\left(\nu, \left\{n\frac{\partial}{\partial\phi_n}\right\}\right) P(\kappa, \{\phi\}) \\ = P(\nu, \{\bar{\phi}\}) P(\kappa, \{\phi\})|_{\text{finite number of boxes}}. \end{aligned} \quad (\text{E5})$$

Since the Young tableau ν of anti-boxes is equivalent to the conjugate Young tableau $\bar{\nu}$ of boxes, one may also use the product of $P(\bar{\nu}, \{\phi\})$ and $P(\kappa, \{\phi\})$. Then, using the tensor product of the Young tableau, the product can be decomposed into a linear combination of Schur polynomials.

$$\begin{aligned} P\left(\nu, \left\{n\frac{\partial}{\partial\phi_n}\right\}\right) P(\kappa, \{\phi\}) \\ = P(\bar{\nu}, \{\phi\}) P(\kappa, \{\phi\})|_{\text{finite number of boxes}} \\ = \sum_{\lambda} c_{\bar{\nu}\kappa}^\lambda P(\lambda, \{\phi\})|_{\lambda: \text{finite number of boxes}}, \end{aligned} \quad (\text{E6})$$

where we also select Young tableau λ that has a finite number of boxes. Therefore, one can write the right-hand side of Eq. (E4) as follows.

$$\begin{aligned} P\left(\nu, \left\{n\frac{\partial}{\partial\phi_n}\right\}\right)\Psi_\beta \\ = \sum_{\kappa, \lambda} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\kappa) + \frac{|\kappa|^2}{N})} c_{\bar{\nu}\kappa}^\lambda P(\kappa, \{\bar{\phi}\}) P(\lambda, \{\phi\}), \end{aligned} \quad (\text{E7})$$

where κ and λ run over Young tableaux with finite numbers of boxes. Using the identities of the tensor-product coefficient

$$c_{\bar{\nu}\kappa}^\lambda = c_{\bar{\nu}\lambda}^{\bar{\kappa}} = c_{\nu\lambda}^\kappa, \quad (\text{E8})$$

we finally have

$$\begin{aligned} P\left(\nu, \left\{n\frac{\partial}{\partial\phi_n}\right\}\right)\Psi_\beta \\ = \sum_{\kappa, \lambda} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\kappa) + \frac{|\kappa|^2}{N})} c_{\nu\lambda}^\kappa P(\kappa, \{\bar{\phi}\}) P(\lambda, \{\phi\}). \end{aligned} \quad (\text{E9})$$

For comparison, we can rewrite $P(\nu, \{\bar{\phi}_n\})\Psi_\beta$ as

$$\begin{aligned} P(\nu, \{\bar{\phi}_n\})\Psi_\beta &= \sum_{\lambda, \kappa} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\lambda) + \frac{|\lambda|^2}{N})} c_{\nu\lambda}^\kappa P(\kappa, \{\bar{\phi}_n\}) \\ &\quad \times P(\lambda, \{\phi\}) \\ &= e^{\frac{\beta}{2}(E_{\text{SU}(N)}(\nu) + \frac{|\nu|^2}{N})} \sum_{\lambda, \kappa} e^{-\frac{\beta}{2}(E_{\text{SU}(N)}(\kappa) + \Delta E(\nu, \lambda, \kappa) + \frac{|\kappa|^2}{N})} \\ &\quad \times c_{\nu\lambda}^\kappa P(\kappa, \{\bar{\phi}_n\}) P(\lambda, \{\phi\}). \end{aligned} \quad (\text{E10})$$

Here, we define $\Delta E(\nu, \lambda, \kappa)$ by

$$\Delta E(\nu, \lambda, \kappa) \equiv E_{\text{SU}(N)}(\nu) + E_{\text{SU}(N)}(\lambda) - E_{\text{SU}(N)}(\kappa) + \frac{1}{N}(|\nu|^2 + |\lambda|^2 - |\kappa|^2) \quad (\text{E11})$$

$$\sim \mathcal{O}(N^0), \quad (\text{E12})$$

where we can confirm that it is of order $\mathcal{O}(N^0)$ due to the nonvanishing condition (E3) for the tensor-product coefficient. Consequently, if we ignore $\Delta E(\nu, \lambda, \kappa) \sim \mathcal{O}(N^0)$ in the exponent, we can find the state-dependent construction

of the mirror operator,

$$P(\nu, \{\bar{\phi}_n\})\Psi_\beta \approx e^{\beta(E_{\text{SU}(N)}(\nu))} P\left(\nu, \left\{n \frac{\partial}{\partial \phi_n}\right\}\right) \Psi_\beta. \quad (\text{E13})$$

Strictly speaking, neglecting terms of order $\mathcal{O}(N^0)$ in the exponent would be equivalent to taking the free loop gas limit in Eq. (13) of the main article, where one can immediately obtain the state-dependent construction.

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- [1] Juan Martin Maldacena, The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
 - [2] Sumit R. Das and Antal Jevicki, String field theory and physical interpretation of $D = 1$ strings, *Mod. Phys. Lett. A* **05**, 1639 (1990).
 - [3] Robert de Mello Koch, Antal Jevicki, Kewang Jin, and Joao P. Rodrigues, $\text{AdS}_4/\text{CFT}_3$ construction from collective fields, *Phys. Rev. D* **83**, 025006 (2011).
 - [4] Antal Jevicki, Kewang Jin, and Qibin Ye, Collective dipole model of AdS/CFT and higher spin gravity, *J. Phys. A* **44**, 465402 (2011).
 - [5] Diptarka Das, Sumit R. Das, Antal Jevicki, and Qibin Ye, Bi-local construction of $\text{Sp}(2N)/\text{dS}$ higher spin correspondence, *J. High Energy Phys.* **01** (2013) 107.
 - [6] Robert de Mello Koch, Antal Jevicki, João P. Rodrigues, and Junggi Yoon, Canonical formulation of $\mathcal{O}(N)$ vector/higher spin correspondence, *J. Phys. A* **48**, 105403 (2015).
 - [7] Robert de Mello Koch, Antal Jevicki, Joao P. Rodrigues, and Junggi Yoon, Holography as a gauge phenomenon in higher spin duality, *J. High Energy Phys.* **01** (2015) 055.
 - [8] Antal Jevicki and Junggi Yoon, Bulk from bi-locals in thermo field CFT, *J. High Energy Phys.* **02** (2016) 090.
 - [9] Antal Jevicki, Kenta Suzuki, and Junggi Yoon, Bi-local holography in the SYK model, *J. High Energy Phys.* **07** (2016) 007.
 - [10] Robert de Mello Koch, Antal Jevicki, Kenta Suzuki, and Junggi Yoon, AdS maps and diagrams of bi-local holography, *J. High Energy Phys.* **03** (2019) 133.
 - [11] S. W. Hawking, Particle creation by black holes, *Commun. Math. Phys.* **43**, 199 (1975); **46**, 206(E) (1976).
 - [12] S. W. Hawking, Breakdown of predictability in gravitational collapse, *Phys. Rev. D* **14**, 2460 (1976).
 - [13] Kyriakos Papadodimas and Suvrat Raju, An infalling observer in AdS/CFT , *J. High Energy Phys.* **10** (2013) 212.
 - [14] Kyriakos Papadodimas and Suvrat Raju, State-dependent bulk-boundary maps and black hole complementarity, *Phys. Rev. D* **89**, 086010 (2014).
 - [15] Chi-Ming Chang and Ying-Hsuan Lin, Words to describe a black hole, *J. High Energy Phys.* **02** (2023) 109.
 - [16] Sunjin Choi, Seok Kim, Eunwoo Lee, Siyul Lee, and Jaemo Park, Towards quantum black hole microstates, *J. High Energy Phys.* **11** (2023) 175.
 - [17] Jaehyeok Choi, Sunjin Choi, Seok Kim, Jehyun Lee, and Siyul Lee, Finite N black hole cohomologies, *J. High Energy Phys.* **12** (2024) 029.
 - [18] Chi-Ming Chang and Ying-Hsuan Lin, Holographic covering and the fortuity of black holes, [arXiv:2402.10129](https://arxiv.org/abs/2402.10129).
 - [19] Gerard 't Hooft, The black hole interpretation of string theory, *Nucl. Phys.* **B335**, 138 (1990).
 - [20] Leonard Susskind, Larus Thorlacius, and John Uglum, The stretched horizon and black hole complementarity, *Phys. Rev. D* **48**, 3743 (1993).
 - [21] Kyriakos Papadodimas and Suvrat Raju, Black hole interior in the holographic correspondence and the information paradox, *Phys. Rev. Lett.* **112**, 051301 (2014).
 - [22] Kyriakos Papadodimas and Suvrat Raju, Remarks on the necessity and implications of state-dependence in the black hole interior, *Phys. Rev. D* **93**, 084049 (2016).
 - [23] Souvik Banerjee, Jan-Willem Bryan, Kyriakos Papadodimas, and Suvrat Raju, A toy model of black hole complementarity, *J. High Energy Phys.* **05** (2016) 004.
 - [24] Jan de Boer, Rik Van Breukelen, Sagar F. Lokhande, Kyriakos Papadodimas, and Erik Verlinde, On the interior geometry of a typical black hole microstate, *J. High Energy Phys.* **05** (2019) 010.
 - [25] Jan De Boer, Rik Van Breukelen, Sagar F. Lokhande, Kyriakos Papadodimas, and Erik Verlinde, Probing typical black hole microstates, *J. High Energy Phys.* **01** (2020) 062.
 - [26] Juan Martin Maldacena, Eternal black holes in anti-de Sitter, *J. High Energy Phys.* **04** (2003) 021.
 - [27] Daniel Harlow, Wormholes, emergent gauge fields, and the weak gravity conjecture, *J. High Energy Phys.* **01** (2016) 122.
 - [28] Monica Guica and Daniel L. Jafferis, On the construction of charged operators inside an eternal black hole, *SciPost Phys.* **3**, 016 (2017).
 - [29] Daniel Harlow and Daniel Jafferis, The factorization problem in Jackiw-Teitelboim gravity, *J. High Energy Phys.* **02** (2020) 177.
 - [30] Ahmed Almheiri, Netta Engelhardt, Donald Marolf, and Henry Maxfield, The entropy of bulk quantum fields and the

- entanglement wedge of an evaporating black hole, *J. High Energy Phys.* **12** (2019) 063.
- [31] Ahmed Almheiri, Raghu Mahajan, Juan Maldacena, and Ying Zhao, The Page curve of Hawking radiation from semiclassical geometry, *J. High Energy Phys.* **03** (2020) 149.
- [32] Ahmed Almheiri, Raghu Mahajan, and Juan Maldacena, Islands outside the horizon, [arXiv:1910.11077](#).
- [33] Geoff Penington, Stephen H. Shenker, Douglas Stanford, and Zhenbin Yang, Replica wormholes and the black hole interior, *J. High Energy Phys.* **03** (2022) 205.
- [34] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini, Replica wormholes and the entropy of Hawking radiation, *J. High Energy Phys.* **05** (2020) 013.
- [35] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini, The entropy of Hawking radiation, *Rev. Mod. Phys.* **93**, 035002 (2021).
- [36] Dongsu Bak, Chanju Kim, Sang-Heon Yi, and Junggi Yoon, Unitarity of entanglement and islands in two-sided Janus black holes, *J. High Energy Phys.* **01** (2021) 155.
- [37] Alok Laddha, Siddharth G. Prabhu, Suvrat Raju, and Pushkal Shrivastava, The holographic nature of null infinity, *SciPost Phys.* **10**, 041 (2021).
- [38] Chandramouli Chowdhury, Olga Papadoulaki, and Suvrat Raju, A physical protocol for observers near the boundary to obtain bulk information in quantum gravity, *SciPost Phys.* **10**, 106 (2021).
- [39] Suvrat Raju, Lessons from the information paradox, *Phys. Rep.* **943**, 1 (2022).
- [40] Chandramouli Chowdhury, Victor Godet, Olga Papadoulaki, and Suvrat Raju, Holography from the Wheeler-DeWitt equation, *J. High Energy Phys.* **03** (2022) 019.
- [41] Suvrat Raju, Failure of the split property in gravity and the information paradox, *Classical Quantum Gravity* **39**, 064002 (2022).
- [42] Tuneer Chakraborty, Joydeep Chakravarty, Victor Godet, Priyadarshi Paul, and Suvrat Raju, Holography of information in de Sitter space, *J. High Energy Phys.* **12** (2023) 120.
- [43] Joydeep Chakravarty, Diksha Jain, and Akhil Sivakumar, Holography of information in massive gravity using Dirac brackets, *J. High Energy Phys.* **06** (2023) 109.
- [44] Robert de Mello Koch and Garreth Kemp, Holography of information in AdS/CFT, *J. High Energy Phys.* **12** (2022) 095.
- [45] John B. Kogut and Leonard Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, *Phys. Rev. D* **11**, 395 (1975).
- [46] A. Jevicki and B. Sakita, Loop space representation and the large N behavior of the one plaquette Kogut-Susskind Hamiltonian, *Phys. Rev. D* **22**, 467 (1980).
- [47] A. Jevicki and B. Sakita, Collective field approach to the large N limit: Euclidean field theories, *Nucl. Phys.* **B185**, 89 (1981).
- [48] Antal Jevicki, Nonperturbative collective field theory, *Nucl. Phys.* **B376**, 75 (1992).
- [49] Jean Avan and Antal Jevicki, Collective field theory of the matrix vector models, *Nucl. Phys.* **B469**, 287 (1996).
- [50] Antal Jevicki and Junggi Yoon, Field theory of primaries in W_N minimal models, *J. High Energy Phys.* **11** (2013) 060.
- [51] Antal Jevicki and Junggi Yoon, S_N orbifolds and string interactions, *J. Phys. A* **49**, 205401 (2016).
- [52] Masao Nomura, A soluble nonlinear bose field as a dynamical manifestation of symmetric group characters and young diagrams, *Phys. Lett.* **117A**, 289 (1986).
- [53] Robert de Mello Koch, Geometries from Young diagrams, *J. High Energy Phys.* **11** (2008) 061.
- [54] Yasushi Takahashi and Hiroomi Umezawa, Thermo field dynamics, *Int. J. Mod. Phys. B* **10**, 1755 (1996).
- [55] Y. Takahashi and H. Umezawa, Thermo field dynamics, *Int. J. Mod. Phys. B* **10**, 1755 (1996).
- [56] W. Israel, Thermo field dynamics of black holes, *Phys. Lett.* **57A**, 107 (1976).
- [57] Don N. Page, Average entropy of a subsystem, *Phys. Rev. Lett.* **71**, 1291 (1993).
- [58] Don N. Page, Information in black hole radiation, *Phys. Rev. Lett.* **71**, 3743 (1993).
- [59] Samir D. Mathur, The Fuzzball proposal for black holes: An elementary review, *Fortschr. Phys.* **53**, 793 (2005).
- [60] Iosif Bena and Nicholas P. Warner, Black holes, black rings and their microstates, *Lect. Notes Phys.* **755**, 1 (2008).
- [61] Elliott Gesteau and Leonardo Santilli, Explicit large N von Neumann algebras from matrix models, *Adv. Theor. Math. Phys.* **28**, 2245 (2024).
- [62] Sudip Ghosh and Suvrat Raju, Quantum information measures for restricted sets of observables, *Phys. Rev. D* **98**, 046005 (2018).
- [63] Steven B. Giddings and William M. Nelson, Quantum emission from two-dimensional black holes, *Phys. Rev. D* **46**, 2486 (1992).
- [64] Samir D. Mathur, The quantum structure of black holes, *Classical Quantum Gravity* **23**, R115 (2006).
- [65] Steven B. Giddings and Julie Perkins, Quantum evolution of the Hawking state for black holes, *Phys. Rev. D* **106**, 065011 (2022).
- [66] Chi-Ming Chang, Witten index of BMN matrix quantum mechanics, [arXiv:2404.18442](#).
- [67] Chi-Ming Chang, Yiming Chen, Bik Soon Sia, and Zhenbin Yang, Fortuity in SYK models, *J. High Energy Phys.* **08** (2025) 003.
- [68] Robert de Mello Koch, Minkyoo Kim, Seok Kim, Jehyun Lee, and Siyul Lee, Brane-fused black hole operators, *J. High Energy Phys.* **07** (2025) 216.
- [69] John McGreevy, Leonard Susskind, and Nicolaos Toumbas, Invasion of the giant gravitons from Anti-de Sitter space, *J. High Energy Phys.* **06** (2000) 008.
- [70] Marcus T. Grisaru, Robert C. Myers, and Oyvind Tafjord, SUSY and goliath, *J. High Energy Phys.* **08** (2000) 040.
- [71] Akikazu Hashimoto, Shinji Hirano, and N. Itzhaki, Large branes in AdS and their field theory dual, *J. High Energy Phys.* **08** (2000) 051.
- [72] Sumit R. Das, Antal Jevicki, and Samir D. Mathur, Giant gravitons, BPS bounds and noncommutativity, *Phys. Rev. D* **63**, 044001 (2001).
- [73] Sumit R. Das, Antal Jevicki, and Samir D. Mathur, Vibration modes of giant gravitons, *Phys. Rev. D* **63**, 024013 (2001).
- [74] Steve Corley, Antal Jevicki, and Sanjaye Ramgoolam, Exact correlators of giant gravitons from dual $N = 4$ SYM theory, *Adv. Theor. Math. Phys.* **5**, 809 (2002).

- [75] Vijay Balasubramanian, Micha Berkooz, Asad Naqvi, and Matthew J. Strassler, Giant gravitons in conformal field theory, *J. High Energy Phys.* **04** (2002) 034.
- [76] Hai Lin, Oleg Lunin, and Juan Martin Maldacena, Bubbling AdS space and 1/2 BPS geometries, *J. High Energy Phys.* **10** (2004) 025.
- [77] Samuel Leutheusser and Hong Liu, Causal connectability between quantum systems and the black hole interior in holographic duality, *Phys. Rev. D* **108**, 086019 (2023).
- [78] S. Leutheusser and H. Liu, Emergent times in holographic duality, *Phys. Rev. D* **108**, 086020 (2023).
- [79] Edward Witten, Why does quantum field theory in curved spacetime make sense? And what happens to the algebra of observables in the thermodynamic limit?, in *Dialogues Between Physics and Mathematics*, edited by M. L. Ge and Y. H. He (Springer, Cham, 2021), [10.1007/978-3-031-17523-7_11](https://doi.org/10.1007/978-3-031-17523-7_11).