

Heat kernel of nonminimal second-order operators

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We analyze the spectra of general nonminimal second-order operators. To do this, we derive the local part of the trace of the second Seeley-DeWitt heat kernel coefficient for such operators in a completely model-independent way. Afterwards, we provide three examples to show how our result can be applied in practical scenarios. In particular, we emphasize this discussion when dealing with a toy model of dynamical torsion, which is viewed as a simple instance of higher-spin fields. All our results are compatible with the literature, and we provide a *Mathematica* notebook with the model-independent results that are written in the paper.

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I. INTRODUCTION

The heat kernel formalism provides a powerful method for computing the quantum corrections to the effective action [1]. It has been developed by physicists for dealing with quantum effects on top of classical backgrounds [2–4], while mathematicians later put it on stable grounds and connected it to the index theorem in topology [5–8].

The heat kernel method provides an elegant computational tool for deriving perturbative quantum properties in the presence of background fields in a manifestly covariant fashion. For this reason, it has been widely used in the context of gravitational theories (see, e.g., Refs. [1,9,10]). Among the many phenomenological applications we cite the computation of beta functions [11,12], working out the quantum anomalies [13,14], deriving the quantum corrections to the black hole entropy [15], and analyzing gravitational catalysis [16]. Furthermore, even though this formalism is mostly applied in the 1-loop approximation, it can also be extended to take into account multi-loop diagrams [17–22].

Most of the applications of the heat kernel method have focused on minimal second-order operators, i.e., those whose principal part is given by a Laplace-type operator. This is because in this case one can prove that there exists an asymptotic expansion of the heat kernel for small distances [7], which in turn suggests the introduction of the so-called Seeley-DeWitt bitensors. Such an expansion was found in [3] by comparison with the known flat-space

result of a diffusion process. Then, the coincidence limits of such coefficients can be found in a recursive way [3]. Of particular interest are the first three coefficients $\hat{a}_0$, $\hat{a}_1$, and $\hat{a}_2$, since these dictate the form of the beta functions of marginal and relevant operators in $d = 4$ in the one-loop approximation. In practice, the knowledge of the trace of $\hat{a}_2$ suffices to yield the beta functions of dimensionless coupling constant; for this reason the focus of this paper will be on the derivation of such a trace for a wide class of differential operators.

The most general minimal second-order operators are those that also comprise a derivative interaction. These have been dealt with in [23,24], where the first Seeley-DeWitt coefficients have been derived using a squaring procedure. Furthermore, fourth-order minimal operators have been taken into account in [9,25], and this procedure can be applied to higher-order operators.

On the other hand, if one wants to check the gauge invariance of a given calculation or considers more complicated fields such as torsion, nonmetricity, or higher spins, they will have to deal with nonminimal operators. To derive the heat kernel coefficients of these operators a completely different approach is needed, since there exists no asymptotic expansion when the principal part is not of Laplace type. However, one can rely on a different perturbative expansion as in [9], or they can adopt the so-called nonlocal heat kernel [26]. Recently, such a problem has been reanalyzed in some special examples [27–30]. Of particular interest for the present discussion is the approach of [29], in which a slight modification of the trick introduced in [9] was employed. Indeed, the approach of [29] significantly improves the computational time needed for the derivation of the trace of the $\hat{a}_2$ coefficient. A radically different approach toward the nonminimal operator is given by employing spin-parity decompositions,

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see, e.g., Ref. [26]. Here the idea is to trade nonminimality for higher-order but minimal operators. Such an approach was applied to a dynamical torsion theory in [31], but it will not be pursued here. Finally, nonminimal operators arise also in Proca-like theories, where the principal part is degenerate. In such cases one usually singles out the nonminimal part and derives the heat kernel coefficients by deriving the algebra of the pseudodifferential operators that appear in the Hessian [32–34].

Thus, the aim of this work is to derive the local part of the trace of the second Seeley-DeWitt coefficient for general nonminimal operators in a model-independent way, and to provide some examples to show how the established recipe can be applied in practice. To this end the trick of [9] is applied in the new formulation of [29], and vector bundles are employed to stand for an unspecified target space parametrizing the quantum fields of the theory. On a practical viewpoint the `xAct` packages of *Mathematica* [35–38] have been heavily exploited, out of which `FieldsX` has played a crucial role [39]. Furthermore, the complete form of the model-independent result derived in this paper is also available in the Supplemental Material [40] in a *Mathematica* notebook to provide a handy use of the present findings [41].

The structure of the paper is as follows. First of all, in Sec. II the heat kernel is introduced, and it is related to the trace of the logarithm of a second-order Laplace operator. Next, in Sec. III nonminimal second-order operators are considered in a formal way, singling out the trace log of the principal part in Secs. III A and III B and treating perturbatively the rest in Sec. III C. To turn formal expressions into concrete ones, one has to evaluate functional traces, thus in Sec. IV a method to perform such calculations is presented. In Sec. V the final result for the trace-log of a general nonminimal second-order operator is written down. To this end, some computer-based computational tools and the associated conventions for representing field-space indices are briefly discussed in Sec. VA. Such conventions are employed in Secs. VB and VC to present the resulting modifications to the trace of the second Seeley-DeWitt coefficient $\hat{a}_2$. Section VI provides three examples of applications of the general findings, dealing with a vector field in Sec. VIA, a Kalb-Ramond field in Sec. VIB and a toy model of the torsion in Sec. VIC. The results are summarized and some possible future developments are discussed in Sec. VII. Finally, the Appendixes contain useful commutator identities (Appendix A), they provide all coincidence limits up to dimension-four terms (Appendix B), and they collect the tensor structures that concur to $\text{tr} \hat{a}_2$ that were omitted in Sec. V to avoid overburdening the flow of the main text (Appendix C). Finally, the mostly plus Minkowski signature will be used throughout the paper.

II. THE SCHWINGER-DEWITT TECHNIQUE

Let us consider a minimal second-order differential operator, i.e.,

$$\hat{F}(\hat{\nabla}) = \hat{\square} + \hat{E}, \quad (1)$$

where $\hat{E}$ is an endomorphism. The covariant derivative $\hat{\nabla}_\mu$ is assumed to be compatible with the metric, such that the following commutation relations hold:

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu] \hat{\nabla}_\rho \hat{\phi} = -R^\lambda{}_{\rho\mu\nu} \hat{\nabla}_\lambda \hat{\phi} + \hat{\mathcal{R}}_{\mu\nu} \hat{\nabla}_\rho \hat{\phi}, \quad (2)$$

where $\hat{\mathcal{R}}_{\mu\nu}$ is the generalized curvature acting on field space, and hats stand for field-valued tensors. In practice, an operator like Eq. (1) is usually associated with the Hessian $\hat{\mathcal{H}}$ of a bosonic field theory, or to the square of a fermionic Hessian, through

$$\hat{\mathcal{H}} = \hat{F}(\hat{\nabla}) - m^2 \hat{1}, \quad (3)$$

where $\hat{1}$ is the identity in field space and m is some mass parameter. Then, the Green's function is defined by

$$(\hat{F}(\hat{\nabla}) - m^2) \hat{G}(m^2) = -\hat{1}. \quad (4)$$

By giving an infinitesimal negative imaginary part to the mass, we obtain Feynman's prescription for the Green's function, which allows us to write the latter through the following integral representation over a “proper time” s [3]:

$$\hat{G} = -\frac{\hat{1}}{\hat{F}(\hat{\nabla}) - m^2 + i\epsilon} = i \int_0^\infty ds e^{is(\hat{F}(\hat{\nabla}) - m^2)}. \quad (5)$$

Taking the matrix elements of this equation yields

$$\hat{G}(x, y) = i \int_0^\infty ds \langle x, s | y, 0 \rangle e^{-im^2 s}, \quad (6)$$

where the “transition amplitude” is defined by

$$\langle x, s | y, 0 \rangle \equiv \langle x | e^{i\hat{F}(\hat{\nabla})s} | y \rangle, \quad (7)$$

and it satisfies a Cauchy problem described by the Schrödinger-like equation

$$\begin{aligned} i\partial_s \langle x, s | y, 0 \rangle + \langle x, s | \hat{F}(\hat{\nabla}) | y, 0 \rangle &= 0, \\ \langle x, 0 | y, 0 \rangle &= \delta(x, y). \end{aligned} \quad (8)$$

For notational convenience, we shall denote the “transition amplitude” by

$$\hat{\mathcal{K}}(x, y; s) \equiv \langle x, s | y, 0 \rangle = \langle x | e^{i\hat{F}(\hat{\nabla})s} | y \rangle. \quad (9)$$

In Euclidean signature the differential equation in (8) becomes a heatlike equation, whence $\hat{\mathcal{K}}$ is referred to as the heat kernel. The Cauchy problem (8) is solved by the following ansatz [3]:

$$\hat{\mathcal{K}}(x, y; s) = \frac{i\mathcal{D}^{\frac{1}{2}}(x, y)}{(4\pi is)^\omega} e^{i\frac{\sigma(x, y)}{2s}} \hat{\Lambda}(x, y; s),$$

$$\lim_{x \rightarrow y} \hat{\Lambda}(x, y; 0) = \hat{\mathbb{I}}, \quad (10)$$

which has to be thought of as a modification to the known flat-space solution in absence of background fields [2]. Here and in the following $2\omega = d$, where d is the spacetime dimension. Moreover, $\sigma(x, y)$ is the Synge world function, which is equal to half the square of the geodesic distance between x and y . Thus, it generalizes the flat-space modulus of the difference of the x and y coordinates, and it obeys

$$\sigma = \frac{1}{2} \sigma_\mu \sigma^\mu, \quad \sigma_\mu \equiv \nabla_\mu \sigma. \quad (11)$$

Furthermore, $\mathcal{D}(x, y)$ is the Van Vleck-Morette determinant

$$\mathcal{D}(x, y) = -\det \left(-\frac{\partial^2 \sigma}{\partial x^\mu \partial y^\nu} \right), \quad (12)$$

that describes the divergence or convergence of geodesics emanating from the point x , and it becomes singular whenever we encounter a caustic surface. Such a biscalar density can be rewritten to single out the densities as

$$\mathcal{D}(x, x') = \sqrt{-g(x)} \sqrt{-g(x')} \Delta(x, x'), \quad (13)$$

where $\Delta(x, x')$ is a biscalar which is commonly called the dedensitized Van Vleck determinant. The latter satisfies

$$\Delta^{-1} \nabla_\mu (\sigma^\mu \Delta) = d. \quad (14)$$

Local information is extracted from the previous equations by first differentiating and then taking the coincidence limit. This enables one to approximate these bitensors by expanding them in a covariant Taylor series [9]. The coincidence limits up to mass dimension four are given in Appendix B.

While the Synge world function and the Van Vleck determinant pertain only to the underlying pseudo-Riemannian structure of spacetime, all the information about the operator $\hat{F}$ is encoded in $\hat{\Lambda}$. It is convenient to expand the latter as an infinite series, whose coefficients are the Seeley-DeWitt bitensors $\hat{a}_n$,

$$\hat{\Lambda}(x, x'; s) = \sum_{n=0}^{\infty} (is)^n \hat{a}_n(x, x'). \quad (15)$$

By plugging Eq. (10) into the defining Eq. (9) we find that $\hat{\Lambda}$ satisfies the following differential equation [3]:

$$-i\partial_s \hat{\Lambda} - \frac{i}{s} \sigma^\mu \nabla_\mu \hat{\Lambda} = \Delta^{-1/2} \hat{F} (\Delta^{1/2} \hat{\Lambda}). \quad (16)$$

Such a first-order differential equation is solved by employing the expansion (15) and iteratively deriving local expressions for the Seeley-DeWitt coefficients from the following recurrence relation:

$$n\hat{a}_n + \sigma^\mu \nabla_\mu \hat{a}_n = \Delta^{-1/2} \hat{F} (\Delta^{1/2} \hat{a}_{n-1}); \quad \sigma^\mu \nabla_\mu \hat{a}_0 = 0, \quad a_0(x, x) = \hat{\mathbb{I}}. \quad (17)$$

Thus, once again local information is extracted by repeatedly differentiating it and taking the coincidence limit afterwards. Such limits are given in Appendix B for $\hat{F}(\hat{\nabla}) = \hat{\square} + \hat{E}$.

Let us now connect the heat kernel of the differential operator $\hat{F}(\hat{\nabla})$ to the trace of the logarithm of this operator following Schwinger's presentation [2]. Let $\hat{F}(\hat{\nabla})$ be a differential operator and $\hat{G}(\hat{\nabla})$ its Green's function. Then, for a general functional variation we have

$$\begin{aligned} \delta \text{Tr} \log \hat{F} &= -\text{Tr}(\hat{G} \delta \hat{F}) = -i \text{Tr} \int_0^\infty ds \delta \hat{F} e^{is\hat{F}} \\ &= -\text{Tr} \int_0^\infty \frac{ds}{s} \delta e^{is\hat{F}} = -\delta \left(\text{Tr} \int_0^\infty \frac{ds}{s} e^{is\hat{F}} \right). \end{aligned} \quad (18)$$

Thus, up to an integration constant we have that the trace of the logarithm of $\hat{F}$ can be written in terms of the trace of the heat kernel as

$$\begin{aligned} \text{Tr} \log(\hat{F}(\hat{\nabla})) &= - \int_0^\infty \frac{ds}{s} \text{Tr} e^{is\hat{F}(\hat{\nabla})} \\ &= - \int_0^\infty \frac{ds}{s} \int \sqrt{-g} \text{tr} \hat{\mathcal{K}}(x, x; s). \end{aligned} \quad (19)$$

The last trace is not capitalized since it is to be taken on internal and spacetime indices only. While this equation is valid in general, the asymptotic expansion of the heat kernel given by (10) and (15) is valid only for Laplace-type second-order operators. Thus, when the principal part is not given by $\hat{\square}$ we have to resort to different methods for evaluating the trace log and singling out its divergent parts.

Before focusing on the evaluation of the divergent parts of complicated functional traces, let us make a brief remark. When we are not dealing with scalar fields the Hessian $\mathcal{H}_{ab}$ of a given Lagrange density is not a differential operator in a strict sense, since it does not provide a map to tensors or spinors of the same type. Thus, to make sense out of the tracing procedure in Eq. (19), we need to factor out a “configuration metric,” yielding a true differential operator

$$\mathcal{F}_a{}^b(\hat{\nabla}) = \mathcal{G}^{bc} \mathcal{H}_{ac}(\hat{\nabla}). \quad (20)$$

In practice, such a procedure becomes nontrivial only when the tensor fields that we are integrating out are $O(d)$ reducible, like a symmetric 2-tensor. We shall comment on another type of such a field in the last example in Sec. VIC.

III. NONMINIMAL OPERATORS

In this section we address the main concern of the present work, i.e., a general derivation of the trace of the second Seeley-DeWitt coefficient $\hat{a}_2$ for nonminimal second-order operators. First of all, let us point out that the adjective *nonminimal* has had different interpretation in the literature. For example, in [24] operators involving a derivative interaction are dubbed nonminimal, whereas here we follow the nomenclature of [9], and we refer to nonminimality whenever the principal part of the operator is not just given by $\hat{\square}$. Therefore, our task is to work out the correction to $\text{tr} \hat{a}_2$ in the presence of these nonminimal terms.

The class of operators on which we are focusing is quite broad, and we must make the two following assumptions. First of all, the nonminimality cannot be completely general, in that we require the principal part of the operator to be nondegenerate. Let this operator be written as

$$\hat{F}(\hat{\nabla}) = \hat{D}(\hat{\nabla}) + \hat{Y}(\hat{\nabla}) = \hat{\square} + \hat{N}(\hat{\nabla}) + \hat{Y}(\hat{\nabla}), \quad (21)$$

where $\hat{D}(\hat{\nabla})$ is the principal part, whereas $\hat{Y}$ comprises the derivative interactions and endomorphisms. Thus, the nonminimality is parametrized solely by $\hat{N}$. Let us define the principal symbol of the operator by replacing the covariant derivative $\hat{\nabla}_\mu$ with an arbitrary vector n_μ in $\hat{D}$, i.e.,

$$\hat{D}(n) = \hat{D}^{\mu\nu} n_\mu n_\nu. \quad (22)$$

Then, nondegeneracy means that

$$\det \hat{D}^{\mu\nu} n_\mu n_\nu \neq 0. \quad (23)$$

Furthermore, we also restrict on principal symbols that are parametrized by metric-compatible tensors, i.e.,

$$\hat{\nabla}_\rho \hat{D}^{\mu\nu} = 0. \quad (24)$$

When the background geometry is curved the last assumption merely asserts that the quantum fields that we are integrating out must bear their canonical Weyl weights.

The existence of the asymptotic expansion of the heat kernel Eq. (15) rests upon the hypothesis that the operator is of Laplace type. Clearly, this is not the case when we introduce a nonminimal term $\hat{N}(\hat{\nabla})$ in the principal part. Nevertheless, in this situation we can derive the divergent parts of the effective action employing a trick first

introduced by Barvinsky and Vilkovisky [9], and recently slightly modified by Melichev in [29]. The main observation of [29] that we will employ here is that in absence of multiplicative anomalies [42], our calculation can actually be split in two parts as

$$\text{Tr} \log \hat{F}(\hat{\nabla}) = \text{Tr} \log \hat{D}(\hat{\nabla}) + \text{Tr} \log (1 + \hat{D}^{-1} \hat{Y}). \quad (25)$$

The trick introduced in [9] is to parametrize the principal $\hat{D}(\hat{\nabla})$ as a one parameter family of operators such that

$$\hat{D}(\hat{\nabla}; \zeta)|_{\zeta=1} = \hat{D}(\hat{\nabla}), \quad \hat{D}(\hat{\nabla}; \zeta)|_{\zeta=0} = \hat{\square}, \quad (26)$$

which allows one to perform a perturbative expansion of the first term on the right-hand side of Eq. (25). If the dependence of $\hat{N}$ on ζ is linear we have some important simplifications, which are discussed in the next Sec. III A. However, such an assumption cannot always be made in practice, as we will show in one of our applications in Sec. VI. Thus, the subsequent Sec. III B deals with those scenarios in which the dependence of ζ is the most general one. Finally, in Sec. III C we study the second term on the right-hand side of Eq. (25). Importantly, the latter does not require the introduction of any auxiliary parameter.

A. Trace log of the principal part: The simple case

The trick employed in [9] to evaluate the divergent parts of the trace log of the nonminimal operator relies on the introduction of an auxiliary parameter ζ such as Eq. (26). In the simplest case this parameter appears linearly in $\hat{D}(\hat{\nabla})$, i.e.,

$$\hat{D}(\hat{\nabla}) = (\hat{\square} + \zeta \hat{N})|_{\zeta=1} = \hat{D}(\hat{\nabla}; \zeta)|_{\zeta=1}. \quad (27)$$

Obviously, for $\zeta = 0$ the operator is minimal, and its heat kernel coefficients can be found by means of the standard asymptotic expansion. We consider the inverse of the principal symbol

$$\hat{D}(\hat{\nabla}; \zeta) \hat{G}_0(\hat{\nabla}; \zeta) = \hat{1}, \quad (28)$$

which exists thanks to Eq. (23). Thus, using the fact that $\hat{D}(\hat{\nabla}; \zeta = 0) = \hat{\square}$, we can write the trace of the logarithm of $\hat{D}(\hat{\nabla})$ as

$$\begin{aligned} \text{Tr} \log \hat{D}(\hat{\nabla}) &= \text{Tr} \log \hat{D}(\hat{\nabla}; \zeta)|_{\zeta=1} = \text{Tr} \log \hat{\square} + \int_0^1 d\zeta \frac{d}{d\zeta} \text{Tr} \log \hat{D}(\hat{\nabla}; \zeta) \\ &= \text{Tr} \log \hat{\square} + \int_0^1 d\zeta \text{Tr}(\hat{N} \hat{G}_0(\zeta)) = \text{Tr} \log \hat{\square} + \int_0^1 \frac{d\zeta}{\zeta} \text{Tr}[(\hat{D}(\hat{\nabla}; \zeta) - \hat{\square}) \hat{G}_0(\hat{\nabla}; \zeta)] \\ &= \text{Tr} \log \hat{\square} - \int_0^1 d \log \zeta \text{Tr}(\hat{G}_0(\hat{\nabla}; \zeta) \hat{\square}). \end{aligned} \quad (29)$$

Thus, our new task is to work out a perturbative expansion of $\hat{G}_0(\hat{\nabla}; \zeta)$, which will be used to evaluate the trace of the logarithm of $\hat{D}(\hat{\nabla}; \zeta)$ through the previous equation. The first step in this direction is provided by inverting the principal symbol in momentum space, i.e., solving the following equation for the matrix $\hat{K}(n; \zeta)$:

$$\hat{D}(n; \zeta) \hat{K}(n; \zeta) = n^4 \hat{1}. \quad (30)$$

Then, promoting the vectors n_μ to covariant derivatives defines $\hat{K}(\hat{\nabla}; \zeta)$ by fixing the arbitrary ordering of covariant derivatives. Next, due to the previous equation, we automatically find that

$$\hat{D}(\hat{\nabla}; \zeta) \hat{K}(\hat{\nabla}; \zeta) = \hat{\square}^2 + \hat{M}(\hat{\nabla}; \zeta), \quad (31)$$

where $\hat{M}$ is a differential operator of order two. Then, we compare Eqs. (28) and (31) to find a perturbative expression of $\hat{G}_0(\hat{\nabla}; \zeta)$, which reads

$$\begin{aligned} \hat{G}_0(\hat{\nabla}; \zeta) &= \hat{K} \frac{1}{\hat{\square}^2} - \hat{K} \hat{M} \frac{1}{\hat{\square}^4} + 2\hat{K}[\hat{\square}, \hat{M}] \frac{1}{\hat{\square}^5} \\ &\quad + \hat{K} \hat{M} \hat{M} \frac{1}{\hat{\square}^6} + \dots, \end{aligned} \quad (32)$$

where we use ellipsis to stand for higher-dimensional operators that we are not interested in. To derive this equation we have commuted the inverse box operators to the right using Eqs. (A3) and (A5). Finally, we can insert Eq. (32) into Eq. (29), obtaining the desired relation for the trace log of the principal part,

$$\begin{aligned} \text{Tr log } \hat{D}(\hat{\nabla}) &= \text{Tr log } \hat{\square} - \int_0^1 d \log \zeta \text{Tr}(\hat{G}_0(\zeta) \hat{\square}) \\ &= \text{Tr log } \hat{\square} - \int_0^1 d \log \zeta \text{Tr} \left[\hat{K} \frac{1}{\hat{\square}} - \hat{K} \hat{M} \frac{1}{\hat{\square}^3} \right. \\ &\quad \left. + 2\hat{K}[\hat{\square}, \hat{M}] \frac{1}{\hat{\square}^4} + \hat{K} \hat{M} \hat{M} \frac{1}{\hat{\square}^5} \right], \end{aligned} \quad (33)$$

where the dependence on ζ is understood.

B. Trace log of the principal part: The general case

Now we want to address the problem of finding an expression for $\text{Tr log } \hat{D}(\hat{\nabla})$ in the most general case. As in the previous simpler case, our starting point is Eq. (26), while the difference is that

$$\frac{d\hat{D}(\hat{\nabla}; \zeta)}{d\zeta} \neq \hat{D}(\hat{\nabla}; \zeta = 1) - \hat{\square}. \quad (34)$$

Therefore, using the trick of [9] we find

$$\begin{aligned} \text{Tr log } \hat{D}(\hat{\nabla}) &= \text{Tr log } \hat{D}(\hat{\nabla}; \zeta)|_{\zeta=1} \\ &= \text{Tr log } \hat{\square} + \int_0^1 d\zeta \frac{d}{d\zeta} \text{Tr log } \hat{D}(\hat{\nabla}; \zeta) \\ &= \text{Tr log } \hat{\square} + \int_0^1 d\zeta \text{Tr} \left(\frac{d\hat{N}(\zeta)}{d\zeta} \hat{G}_0(\zeta) \right), \end{aligned} \quad (35)$$

where we have used the definition of the principal part. The expansion of the inverse $\hat{G}_0$ of $\hat{D}$ Eq. (32) can still be used. Thus, we can rewrite the previous equation as

$$\begin{aligned} \text{Tr log } \hat{D}(\hat{\nabla}) &= \text{Tr log } \hat{\square} + \int_0^1 d\zeta \text{Tr} \left[\frac{d\hat{N}(\zeta)}{d\zeta} \hat{K} \frac{1}{\hat{\square}^2} \right. \\ &\quad \left. - \frac{d\hat{N}(\zeta)}{d\zeta} \hat{K} \hat{M} \frac{1}{\hat{\square}^4} + \frac{d\hat{N}(\zeta)}{d\zeta} 2\hat{K}[\hat{\square}, \hat{M}] \frac{1}{\hat{\square}^5} \right. \\ &\quad \left. + \frac{d\hat{N}(\zeta)}{d\zeta} \hat{K} \hat{M} \hat{M} \frac{1}{\hat{\square}^6} + \dots \right]. \end{aligned} \quad (36)$$

C. Trace log of the rest

Let us now come back to the starting point, which is to derive the divergent parts of the trace log of the operator Eq. (21). Assuming the absence of multiplicative anomalies [42] and using the definition of $\hat{G}(\hat{\nabla})$ (28) we can write

$$\text{Tr log } \hat{F}(\hat{\nabla}) = \text{Tr log } \hat{D}(\hat{\nabla}) + \text{Tr log } (1 + \hat{G}_0 \hat{Y}). \quad (37)$$

The first term on the right-hand side is given by Eq. (33) [or by Eq. (36) in the most general case]; therefore, we now focus on the second one. By using the known Taylor expansion of the logarithm and commuting the $\hat{\square}^{-2}$ operators to the right we find

$$\begin{aligned} \text{Tr log } (1 + \hat{G}_0 \hat{Y}) &= \text{Tr} \left[\hat{G}_0 \hat{Y} - \frac{1}{2} (\hat{G}_0 \hat{Y})^2 + \frac{1}{3} (\hat{G}_0 \hat{Y})^3 - \frac{1}{4} (\hat{G}_0 \hat{Y})^4 + \dots \right] \\ &= \text{Tr} \left[\hat{Y} \hat{K} \frac{\hat{1}}{\hat{\square}^2} - \hat{Y} \hat{K} \hat{M} \frac{\hat{1}}{\hat{\square}^4} + 2\hat{Y} \hat{K}[\hat{\square}, \hat{M}] \frac{\hat{1}}{\hat{\square}^5} - \frac{1}{2} \hat{Y} \hat{K} \hat{Y} \hat{K} \frac{\hat{1}}{\hat{\square}^4} \right. \\ &\quad \left. + \hat{Y} \hat{K}[\hat{\square}, \hat{Y}] \hat{K} \frac{\hat{1}}{\hat{\square}^5} + \hat{Y} \hat{K} \hat{Y}[\hat{\square}, \hat{K}] \frac{\hat{1}}{\hat{\square}^5} - \frac{3}{2} \hat{Y} \hat{K}[\hat{\square}, [\hat{\square}, \hat{Y}]] \hat{K} \frac{\hat{1}}{\hat{\square}^6} + \hat{Y} \hat{K} \hat{M} \hat{Y} \hat{K} \frac{\hat{1}}{\hat{\square}^6} \right. \\ &\quad \left. - 2\hat{Y} \hat{K} \hat{Y} \hat{K}[\hat{\square}, \hat{Y}] \hat{K} \frac{\hat{1}}{\hat{\square}^7} + \frac{1}{3} (\hat{Y} \hat{K})^3 \frac{\hat{1}}{\hat{\square}^6} - \frac{1}{4} (\hat{Y} \hat{K})^4 \frac{\hat{1}}{\hat{\square}^8} + \dots \right]. \end{aligned} \quad (38)$$

Again, our perturbative expansion stops at dimension-four operators, and we also exclude total derivatives. Furthermore, we split the lower derivative part of the operator $\hat{Y}(\hat{\nabla})$ into a derivative interaction and an endomorphism as

$$\hat{Y} = \hat{V} + \hat{E} = \hat{V}^\mu \hat{\nabla}_\mu + \hat{E}. \quad (39)$$

Thus, the contributions encoded in Eq. (38) can be expressed as a series expansion in $\hat{V}$. The affine part of the

latter is given by

$$\begin{aligned} \text{Tr} \log(1 + \hat{G}_0 \hat{Y})|_{V^0} \\ = \text{Tr} \left[\hat{E} \hat{K} \frac{\hat{1}}{\hat{\square}^2} - \hat{E} \hat{K} \hat{M} \frac{\hat{1}}{\hat{\square}^4} - \frac{1}{2} \hat{E} \hat{K} \hat{E} \hat{K} \frac{\hat{1}}{\hat{\square}^4} + \dots \right]. \end{aligned} \quad (40)$$

Next, we consider the terms linear in $\hat{V}$, whose contribution is

$$\begin{aligned} \text{Tr} \log(1 + \hat{G}_0 \hat{Y})|_{V^1} = \text{Tr} \left[\hat{V} \hat{K} \frac{\hat{1}}{\hat{\square}^2} - \hat{V} \hat{K} \hat{M} \frac{\hat{1}}{\hat{\square}^4} + 2 \hat{V} \hat{K} [\hat{\square}, \hat{M}] \frac{\hat{1}}{\hat{\square}^5} - \frac{1}{2} \hat{V} \hat{K} \hat{E} \hat{K} \frac{\hat{1}}{\hat{\square}^4} \right. \\ \left. - \frac{1}{2} \hat{E} \hat{K} \hat{V} \hat{K} \frac{\hat{1}}{\hat{\square}^4} + \hat{E} \hat{K} [\hat{\square}, \hat{V}] \hat{K} \frac{\hat{1}}{\hat{\square}^5} + \hat{V} \hat{K} [\hat{\square}, \hat{E}] \hat{K} \frac{\hat{1}}{\hat{\square}^5} + \dots \right]. \end{aligned} \quad (41)$$

On the other hand, taking into account terms quadratic in $\hat{V}$ yields

$$\begin{aligned} \text{Tr} \log(1 + \hat{G}_0 \hat{Y})|_{V^2} = \text{Tr} \left[-\frac{1}{2} \hat{V} \hat{K} \hat{V} \hat{K} \frac{\hat{1}}{\hat{\square}^4} + \hat{V} \hat{K} [\hat{\square}, \hat{V}] \hat{K} \frac{\hat{1}}{\hat{\square}^5} - \frac{3}{2} \hat{V} \hat{K} [\hat{\square}, [\hat{\square}, \hat{V}]] \hat{K} \frac{\hat{1}}{\hat{\square}^6} \right. \\ \left. + \hat{V} \hat{K} \hat{M} \hat{V} \hat{K} \frac{\hat{1}}{\hat{\square}^6} + \hat{E} \hat{K} \hat{V} \hat{K} \hat{V} \hat{K} \frac{\hat{1}}{\hat{\square}^6} + \dots \right]. \end{aligned} \quad (42)$$

We have listed all terms that depend on the $\hat{M}$ operator and on the endomorphism $\hat{E}$. Cubic terms in this expansion have the following form:

$$\begin{aligned} \text{Tr} \log(1 + \hat{G}_0 \hat{Y})|_{V^3} \\ = \text{Tr} \left[\frac{1}{3} (\hat{V} \hat{K})^3 \frac{\hat{1}}{\hat{\square}^6} - 2 (\hat{V} \hat{K})^2 [\hat{\square}, \hat{V}] \hat{K} \frac{\hat{1}}{\hat{\square}^7} + \dots \right], \end{aligned} \quad (43)$$

whereas quartic ones are given by

$$\text{Tr} \log(1 + \hat{G}_0 \hat{Y})|_{V^4} = \text{Tr} \left[-\frac{1}{4} (\hat{V} \hat{K})^4 \frac{\hat{1}}{\hat{\square}^8} + \dots \right]. \quad (44)$$

After having derived these formal expressions, let us briefly discuss the advantages of following [29] instead of [9]. First of all, in practical applications the integration over ζ is usually computationally expensive; thus, reducing the number of terms to integrate over is more efficient. The second reason is explained by the subsequent observation. Both methods rely on the evaluation of functional traces of the form

$$\text{Tr} \left(\hat{\partial} \frac{\hat{1}}{\hat{\square}^n} \right) = \text{Tr} \left(\hat{\partial}^{\mu_1 \dots \mu_k} \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_k} \frac{\hat{1}}{\hat{\square}^n} \right). \quad (45)$$

When the starting operator involves a derivative interaction $\hat{V}$, the method of [9] requires the evaluation of expressions

with up to 16 covariant derivatives acting on $\hat{\square}^{-10}$. On the other hand, in the present case we must handle expressions with up to 12 derivatives acting on $\hat{\square}^{-8}$. As we will show in the next section, working out the precise form of these functional traces is not particularly involved. Instead, the computational cost increases with the number of derivatives, regardless of the overall mass dimension of a given functional trace. This is due to the appearance of a number of symmetrized metric tensors which is at most equal to half the number of covariant derivatives. In general, handling these kinds of expressions embodies a very expensive part of any such computation. Therefore, the present method significantly improves the computational efficiency of the calculation.

IV. FUNCTIONAL TRACES

In this section we shall deal with the universal functional traces, i.e., objects like

$$\hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_k} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x}, \quad (46)$$

which represent derivative interactions in loop diagrams with n internal lines. Here we are only interested in the logarithmically divergent parts of these diagrams, which will be combined with the explicit form of the differential operators entering Eqs. (33) and (38).

The first step in the evaluation of Eq. (46) is provided by employing a Laplace transform to rewrite the $\hat{\square}^{-n}$ operator as

$$\frac{\hat{1}}{\hat{\square}^n} = \frac{(-i)^n}{(n-1)!} \int_0^\infty ds s^{n-1} e^{is\hat{\square}}. \quad (47)$$

Next, we exploit the asymptotic expansion of the heat kernel Eq. (10) to find a more convenient expression for the exponential that appears in the previous equation. The expansion of $\hat{\Lambda}$ in Eq. (10) reads

$$\hat{\Lambda}(x, y; s) = \hat{\Lambda}_{\hat{\square}}(x, y; s) = \sum_{n \geq 0} (is)^n \hat{a}_n(x, y) \quad (48)$$

where the $\hat{a}_n$ are the Seeley-DeWitt coefficients of the $\hat{\square}$ operator. At this point it is advantageous to modify our covariant derivatives to automatically account for the presence of the exponential of the Synge world function in (10). This is done by defining

$$\hat{\mathfrak{D}}_\mu = e^{-i\frac{g}{2s}} \hat{\nabla}_\mu e^{i\frac{g}{2s}} = \hat{\nabla}_\mu + \frac{i}{2s} \sigma_\mu \hat{1}. \quad (49)$$

Finally, we are able to write a concise expression for the functional trace (46) as the integral over the proper time of some coincidence limit, i.e.,

$$\begin{aligned} & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_k} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x} \\ &= -\frac{(-i)^{n+1}}{(n-1)!} \frac{\sqrt{-g}}{(4\pi i)^\omega} \int_0^\infty \frac{ds}{s^{\omega+1-n}} \\ & \quad \times \left[\hat{\mathfrak{D}}_{\mu_1} \dots \hat{\mathfrak{D}}_{\mu_k} \left(\Delta^{\frac{1}{2}} \hat{\Lambda}_{\hat{\square}}(x, x'; s) \right) \right], \end{aligned} \quad (50)$$

where $[\hat{O}]$ stands for the coincidence limit of the operator $\hat{O}$. The integral in s depends on the spacetime dimension through the ω parameter, and we are only interested in the logarithmically divergent contributions of these diagrams. Furthermore, to obtain tensor structures with the correct mass or background dimensions we need only consider integrals of the form

$$\int_0^\infty ds s^{1-\omega} f(s, \omega) = \frac{1}{2-\omega} f(0, 2) + \dots, \quad (51)$$

where the dots stand for nondivergent terms in dimensional regularization. Following [9] we shall parametrize the pole as

$$\frac{1}{2-\omega} = \log L^2 \quad \text{for } \omega \rightarrow 2. \quad (52)$$

Nevertheless, we will factor out this divergent logarithm at the end of the calculation, thus rewriting our final result as a

modification of the trace of the second Seeley-DeWitt coefficient $\hat{a}_2$, i.e., in a regularization-independent way.

By inspecting of Eqs. (33), (36), and (40)–(44) we can list the functional traces that we need to evaluate:

$$\begin{aligned} & \hat{\nabla}_{\mu_1} \hat{\nabla}_{\mu_2} \frac{\hat{1}}{\hat{\square}} \delta(x, x')|_{x'=x}^{\text{div}}, & \hat{\nabla}_{\mu_1} \hat{\nabla}_{\mu_2} \frac{\hat{1}}{\hat{\square}^2} \delta(x, x')|_{x'=x}^{\text{div}}; \\ & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_4} \frac{\hat{1}}{\hat{\square}^2} \delta(x, x')|_{x'=x}^{\text{div}}, & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_3} \frac{\hat{1}}{\hat{\square}^2} \delta(x, x')|_{x'=x}^{\text{div}}; \\ & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_4} \frac{\hat{1}}{\hat{\square}^3} \delta(x, x')|_{x'=x}^{\text{div}}, & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_6} \frac{\hat{1}}{\hat{\square}^4} \delta(x, x')|_{x'=x}^{\text{div}}; \\ & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_4} \frac{\hat{1}}{\hat{\square}^4} \delta(x, x')|_{x'=x}^{\text{div}}, & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_6} \frac{\hat{1}}{\hat{\square}^5} \delta(x, x')|_{x'=x}^{\text{div}}; \\ & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_8} \frac{\hat{1}}{\hat{\square}^6} \delta(x, x')|_{x'=x}^{\text{div}}, & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_{10}} \frac{\hat{1}}{\hat{\square}^7} \delta(x, x')|_{x'=x}^{\text{div}}. \end{aligned} \quad (53)$$

Instead of simply providing the specific expressions for all these quantities, we prefer to analyze more deeply some general features of functional traces up to dimension-four operators.

Dimensional analysis already suggests that the structure of these objects may be written in an universal way, as we will show below. Let us start by considering $2n-4$ covariant derivatives acting on the n th inverse power of the box operator. By employing Eq. (50) we find

$$\begin{aligned} & \hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_{2n-4}} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x}^{\text{div}} \\ &= \frac{i \log L^2}{16\pi^2} \frac{\sqrt{-g}}{2^{n-2}(n-1)!} g_{\mu_1 \dots \mu_{2n-4}}^{(n-2)} \hat{1}, \end{aligned} \quad (54)$$

where $g_{\mu_1 \dots \mu_{2n-4}}^{(n-2)}$ are completely symmetric tensors defined by

$$\begin{aligned} g^{(0)} &= 1, & g_{\mu\nu}^{(1)} &= g_{\mu\nu}, & g^{(k)} &= 0 \quad \text{for } k < 0; \\ g_{\mu_1 \dots \mu_{2n}}^{(n)} &= \sum_{j=2}^{2n} g_{\mu_1 \mu_j} g_{\mu_2 \dots \mu_{j-1} \mu_{j+1} \dots \mu_{2n}}^{(n-1)}. \end{aligned} \quad (55)$$

We point out that we have extended the definition of these symmetric tensors to negative arguments. This choice is convenient for the present analysis, as it will allow us to write down in a simple fashion all the functional traces that we are interested in.

The tensor structure of Eq. (54) could have been guessed by dimensional arguments. Indeed, since there are no dimension-one background tensors by applying one more covariant derivative we automatically obtain zero, i.e.,

$$\hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_{2n-3}} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x}^{\text{div}} = 0. \quad (56)$$

Obviously, the situation changes when we further increase the number of covariant derivatives by one. In this case the nontrivial tensor structure turns out to be

$$\hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_{2n-2}} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x}^{\text{div}} = -\frac{i \log L^2}{16\pi^2} \frac{\sqrt{-g}}{2^{n-2}(n-1)!} \left[\frac{1}{12} R g_{\mu_1 \dots \mu_{2n-2}}^{(n-1)} \hat{1} - \sum_{i < j} \left(\frac{1}{6} R_{ij} \hat{1} + \frac{1}{2} \hat{R}_{ij} \right) g_{\mu_1 \dots \hat{i} \dots \hat{j} \dots \mu_{2n-2}}^{(n-2)} + \frac{1}{3} \sum_{i < j < k < l} (R_{kilj} + R_{likj}) g_{\mu_1 \dots \hat{i} \dots \hat{j} \dots \hat{k} \dots \hat{l} \dots \mu_{2n-2}}^{(n-3)} \hat{1} \right]. \quad (57)$$

Notice that we have employed our generalized definition of $g^{(k)} = 0$ for any $k < 0$, see Eq. (55), and we are using the shorthand $\hat{\nabla}_i = \hat{\nabla}_{\mu_i}$. By taking a further covariant derivative we end up with dimension-three terms on the right-hand side, whose precise structure is

$$\hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_{2n-1}} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x}^{\text{div}} = -\frac{i \log L^2}{16\pi^2} \frac{\sqrt{-g}}{2^{n-2}(n-1)!} \left[\frac{1}{12} \sum_i \left(\hat{\nabla}^\alpha \hat{R}_{\mu_i \alpha} + \frac{1}{2} \nabla_{\mu_i} R \right) g_{\mu_1 \dots \hat{i} \dots \mu_{2n-1}}^{(n-1)} - \sum_{i < j < k} \left(\frac{2}{3} \hat{\nabla}_{(i} \hat{R}_{j)k} + \frac{1}{4} \nabla_{(i} R_{jk)} \right) g_{\mu_1 \dots \hat{i} \dots \hat{j} \dots \hat{k} \dots \mu_{2n-1}}^{(n-2)} + \frac{1}{2} \sum_{i < j < k < l < m} (\nabla_i R_{(l|j|m)k} + \nabla_j R_{(l|i|m)k} + \nabla_k R_{(l|i|m)j}) g_{\mu_1 \dots \hat{i} \dots \hat{j} \dots \hat{k} \dots \hat{l} \dots \hat{m} \dots \mu_{2n-1}}^{(n-3)} \right]. \quad (58)$$

Finally, we consider the dimension-four contributions. In such a case many nontrivial tensor structures arise, and the general expression of this type of functional traces is quite complicated. Thus, we choose to write it implicitly in terms of the coincidence limits summarized in the Appendix B. Moreover, we assume that the summed indices never enter the symmetrized metrics. Using this kind of notation we have

$$\hat{\nabla}_{\mu_1} \dots \hat{\nabla}_{\mu_{2n}} \frac{\hat{1}}{\hat{\square}^n} \delta(x, x')|_{x'=x}^{\text{div}} = -\frac{i \log L^2}{16\pi^2} \frac{\sqrt{-g}}{2^{n-2}(n-1)!} \left[\sum_{i_1 < \dots < i_4} g_{\mu_1 \dots \mu_{2n}}^{(n-2)} ([\nabla_{i_1} \dots \nabla_{i_4} \Delta^{1/2}] \hat{1} + [\hat{\nabla}_{i_1} \dots \hat{\nabla}_{i_4} \hat{a}_0]) + \sum_{i_1 < i_2} \sum_{j_1 < j_2} g_{\mu_1 \dots \mu_{2n}}^{(n-2)} [\nabla_{i_1} \nabla_{i_2} \Delta^{1/2}] [\hat{\nabla}_{j_1} \hat{\nabla}_{j_2} \hat{a}_0] + \frac{1}{2} \sum_{i < j} g_{\mu_1 \dots \mu_{2n}}^{(n-1)} ([\hat{\nabla}_i \hat{\nabla}_j \hat{a}_1] + [\nabla_i \nabla_j \Delta^{1/2}] [\hat{a}_1]) - \sum_{i_1 < i_2} \sum_{j_1 < \dots < j_4} g_{\mu_1 \dots \mu_{2n}}^{(n-3)} [\nabla_{j_1} \dots \nabla_{j_4} \sigma] ([\nabla_{i_1} \nabla_{i_2} \Delta^{1/2}] \hat{1} + [\hat{\nabla}_{i_1} \hat{\nabla}_{i_2} \hat{a}_0]) + \frac{1}{2} \sum_{i_1 < \dots < i_4} g_{\mu_1 \dots \mu_{2n}}^{(n-2)} [\nabla_{i_1} \dots \nabla_{i_4} \sigma] [\hat{a}_1] - \sum_{i_1 < \dots < i_6} g_{\mu_1 \dots \mu_{2n}}^{(n-3)} [\nabla_{i_1} \dots \nabla_{i_6} \sigma] \hat{1} - \sum_{i_1 < \dots < i_4} \sum_{j_1 < \dots < j_4} g_{\mu_1 \dots \mu_{2n}}^{(n-4)} [\nabla_{i_1} \dots \nabla_{i_4} \sigma] [\nabla_{j_1} \dots \nabla_{j_4} \sigma] \hat{1} - \frac{1}{4} g_{\mu_1 \dots \mu_{2n}}^{(n)} [\hat{a}_2] \right]. \quad (59)$$

All the functional traces that we have listed in Eq. (53) fall into the three classes that we have just taken into account. Thus, their specific contribution to the derivation of the one-loop divergent terms can be found by specializing these general formulas.

V. FINAL RESULT

In this section we wrap up our previous calculations, and we present the modification to the trace of the second Seeley-DeWitt coefficient due to nonminimal terms. Furthermore, we discuss how known results can be recovered as limiting cases. In Sec. VI and we will show three practical applications of these results.

First of all, by using Eqs. (19), (10), (15), and (52) we can write down the divergent part due to the box operator as

$$\text{Tr} \log \hat{\square}|_{\text{div}} = -\frac{i \log L^2}{16\pi^2} \int \sqrt{-g} \text{tr} \hat{a}_2(\hat{\square}). \quad (60)$$

Thus, to derive the modifications to $\text{tr} \hat{a}_2$ we need to factor out the divergent prefactor, which is the reason why we have already singled out such a numerical factor in the functional traces computed in the previous section.

Before delving into the details of our result, we take on a brief detour to discuss how we have extensively exploited computer algebra to perform most of our calculations.

A. On our computational tools

Combining expressions like Eq. (42) with functional traces like Eq. (57) is generally very lengthy, and numerical mistakes may occur. Therefore, we have opted for performing such a calculation in *Mathematica*, employing the extension `FieldsX` [39] of the `xAct` packages [35–38].

First of all, we have used inner vector bundles to parametrize field space, which is done through the commands `DefVBundleWithMetric` and `DefVBundle`. Two such bundles are needed to derive the coincidence limits of Seeley-DeWitt coefficients and their derivatives, since they are bilocal tensors and covariant derivatives must be performed before letting $y \rightarrow x$.

Covariant derivatives for differentiating vector bundle tensors can be defined using `DefCovD`. This is made compatible with the underlying (psuedo-)Riemannian structure and the vector bundle metric through the options `ExtendedFrom` and `FromMetric`. A drawback of these packages is that, even though we have used the aforementioned definitions and options, covariant differentiation of objects in the vector bundles is still not directly allowed. Problems are due to the automatic canonicalization of tensorial expressions, which is prevented by explicitly specifying the metrics that we have defined as those to be used by `ToCanonical`, i.e., employing the `UseMetricOnVBundle` option of the latter.

Changing the options of `ToCanonical` also reduces the ability of `CollectTensors` to recognize like scalars contractions, which is biggest drawback that we have encountered using `FieldsX`. For example, $\hat{R}^\mu_\nu \hat{R}^\nu_\mu$ and $\hat{R}^{\mu\nu} \hat{R}_{\mu\nu}$ may not be seen to be proportional to each other. A possible way to circumvent this issue is to define additional tensors like $\text{de}\hat{r}V_\alpha{}^\mu$ and to substitute them whenever $\hat{\nabla}_\alpha \hat{V}^\mu$ is present. After making sure that all the covariant derivatives that are present have the form $\hat{\nabla}_\mu = \hat{\mathbb{I}}\nabla_\mu$, we can reset the `UseMetricOnVBundle` option of `ToCanonical`, and employ `CollectTensors` afterward. Finally, by substituting back objects like $\hat{\nabla}_\alpha \hat{V}^\mu$ we recover a simplified expression of the starting tensor quantity.

Due to the generality and complicated nature of our calculation, the final expression of the divergent part of Eq. (25) is still very long. Thus, in order to avoid typos in the transcription, we have chosen to directly print out the resulting scalar contraction from the code using `TeXPrint`, and to attach the code itself to the arXiv version of the paper [41]. This implies that the vector bundle indices are displayed explicitly as

$$\hat{\mathcal{R}}_{\mu\nu}\varphi \equiv \mathcal{R}_{\mu\nu a}{}^{a1}\varphi_{a1}. \quad (61)$$

In an analogous way, we have that

$$\hat{K}^{\mu\nu} \equiv K^{\mu\nu}{}_a{}^{a1}, \quad \hat{V}^\mu \equiv V^\mu{}_a{}^{a1}, \quad \hat{E} \equiv E_a{}^{a1}. \quad (62)$$

To specialize our general result to specific tensor fields, we will then introduce soldering forms like $\theta^{\mu\nu}{}_a$ for a 2-tensor. In this way

$$\varphi_a = \theta^{\mu\nu}{}_a \phi_{\mu\nu}, \quad (63)$$

where $\phi_{\mu\nu}$ is the quantum field whose fluctuations we are taking into account. Then, we have that

$$\theta^{\mu\nu}{}_a \theta_{\rho\sigma}{}^a = \mathbb{1}^{\mu\nu}{}_{\rho\sigma}, \quad (64)$$

where $\mathbb{1}^{\mu\nu}{}_{\rho\sigma}$ is the identity on the (in general reducible) representation of $\mathfrak{gl}(4)$ in which $\phi_{\mu\nu}$ takes values.

B. First part of the result

Having set the notation, we now focus on the first part of our result, i.e., that which is given by the integral over the ζ parameter. Such a contribution to the trace of the second Seeley-DeWitt coefficient is not too complicated when the assumptions of Sec. III A are valid, especially when it is compared to the most general case of Sec. III B. This is because the functional traces needed to evaluate Eq. (33) have fewer tensor structures than those of Eq. (36). For this reason, here we shall only provide the complete result in the first case, deferring the reader to the Appendix C 3 for the general one.

First of all, let us notice that $\hat{M}$ is a second-order differential operator which also contains first-order parts. Thus, in general we have

$$\hat{M} = \hat{M}_2{}^{\mu\nu}{}_a{}^{a1} \hat{\nabla}_\mu \hat{\nabla}_\nu + \hat{M}_3{}^\mu{}_a{}^{a1} \hat{\nabla}_\mu + \text{b.t.}, \quad (65)$$

where the numerical subscripts indicate the mass dimension of the tensors. Moreover, mass-dimension-four terms are always total-derivative endomorphisms, thence we discard them. Finally, $\hat{M}_3$ can only enter the result when considering terms linear in the derivative interaction tensor $\hat{V}$.

The part of Eq. (33) that does not explicitly depend on the Riemann tensor and its contractions is given by

$$\begin{aligned}
\text{tr} \hat{a}_2 \supset & \int_0^1 d \log \zeta \left[\frac{1}{24} \mathcal{R}_{\beta\lambda a2a} \mathcal{R}^{\beta\lambda}_{a1} a^2 K^\alpha_{\alpha}{}^{aa1} - \frac{1}{12} \mathcal{R}_\alpha{}^{\lambda a2}{}_a \mathcal{R}_{\beta\lambda a1 a2} K^{\alpha\beta aa1} - \frac{1}{12} \mathcal{R}_\alpha{}^{\lambda}_{a1} a^2 \mathcal{R}_{\beta\lambda a2a} K^{\alpha\beta aa1} \right. \\
& - \frac{1}{8} \mathcal{R}_{\beta\lambda a2a} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{a1} a^2 - \frac{1}{8} \mathcal{R}_{\beta\lambda a2a} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{aa1} a^2 - \frac{1}{8} \mathcal{R}_{\alpha\beta a2a} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{\lambda a1} a^2 \\
& - \frac{1}{8} \mathcal{R}_{\alpha\lambda a2a} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{\beta a1} a^2 - \frac{1}{8} \mathcal{R}_{\alpha\lambda a2a} K^{\alpha\beta aa1} M_{2\beta}{}^{\lambda}_{a1} a^2 - \frac{1}{8} \mathcal{R}_{\beta\lambda a2a} K^\alpha_{\alpha}{}^{aa1} M_{2\alpha}{}^{\beta\lambda}_{a1} a^2 \\
& - \frac{1}{6} \mathcal{R}_\mu{}^\nu{}_{a2a} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1} a^2 g^{(2)}_{\alpha\beta\lambda\nu} - \frac{1}{6} \mathcal{R}_\lambda{}^\nu{}_{a2a} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1} a^2 g^{(2)}_{\alpha\beta\mu\nu} \\
& \left. + \frac{1}{192} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1} a^2 M_{2\alpha}{}^{\nu\rho}_{a2a} g^{(3)}_{\alpha\beta\lambda\mu\nu\rho} \right]. \quad (66)
\end{aligned}$$

On the other hand, the part that comprises both $\hat{M}_2$ and the Riemann tensor (or its contractions) takes the following form:

$$\begin{aligned}
\text{tr} \hat{a}_2 \supset & \int_0^1 d \log \zeta \left[-\frac{1}{24} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{\beta a1 a} R_{a\lambda} - \frac{1}{24} K^{\alpha\beta aa1} M_{2\beta}{}^{\lambda}_{a1 a} R_{a\lambda} - \frac{1}{24} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{\lambda a1 a} R_{a\beta} \right. \\
& + \frac{1}{12} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1 a} R_\lambda{}^\nu g^{(2)}_{\alpha\beta\mu\nu} + \frac{1}{12} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1 a} R_\mu{}^{\alpha 11} g^{(2)}_{\alpha\beta\lambda\nu} - \frac{1}{24} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{a1 a} R_{\beta\lambda} \\
& - \frac{1}{24} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda}_{aa1 a} R_{\beta\lambda} - \frac{1}{24} K^\alpha_{\alpha}{}^{aa1} M_{2\alpha}{}^{\beta\lambda}_{a1 a} R_{\beta\lambda} + \frac{1}{12} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1 a} R_{\alpha\mu\beta\lambda} \\
& \left. + \frac{1}{12} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1 a} R_{\alpha\lambda\beta\mu} - \frac{1}{6} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1 a} R_\lambda{}^\nu{}_\mu{}^\rho g^{(2)}_{\alpha\beta\nu\rho} + \frac{1}{48} K^{\alpha\beta aa1} M_{2\alpha}{}^{\lambda\mu}_{a1 a} R g^{(2)}_{\alpha\beta\lambda\mu} \right]. \quad (67)
\end{aligned}$$

Finally, the terms that depend neither on $\hat{\mathcal{R}}$ nor on $\hat{M}$ are

$$\begin{aligned}
\text{tr} \hat{a}_2 \supset & \int_0^1 d \log \zeta \left[\frac{1}{45} K^{\alpha\beta a}{}_a R_\alpha{}^\lambda R_{\beta\lambda} - \frac{1}{360} K^\alpha_{\alpha}{}^a{}_a R_{\beta\lambda} R^{\beta\lambda} - \frac{1}{12} \mathcal{R}_{\alpha\beta a1 a} K^{\alpha\beta aa1} R - \frac{1}{36} K^{\alpha\beta a}{}_a R_{\alpha\beta} R \right. \\
& \left. + \frac{1}{144} K^\alpha_{\alpha}{}^a{}_a R^2 - \frac{1}{90} K^{\alpha\beta a}{}_a R_\alpha{}^{\lambda\mu\nu} R_{\beta\lambda\mu\nu} - \frac{1}{90} K^{\alpha\beta a}{}_a R^{\lambda\mu} R_{\alpha\lambda\beta\mu} + \frac{1}{360} K^\alpha_{\alpha}{}^a{}_a R_{\beta\lambda\mu\nu} R^{\beta\lambda\mu\nu} \right]. \quad (68)
\end{aligned}$$

C. Second part of the result

Now we focus on the part of the result that does not depend on the integration over the auxiliary parameter ζ , i.e., Eq. (38). As we have done in Sec. III, we split the final expression as a series in $\hat{V}^\mu \equiv V^\mu{}_a{}^{a1}$.

1. Order $\hat{V}^0$ terms

The contributions that do not depend on $\hat{V}$ come from Eq. (40). They give rise to few scalar contractions when compared to the other ones, and they read

$$\begin{aligned}
\text{tr} \hat{a}_2 \supset & -\frac{1}{6} K^{\alpha\beta aa1} R_{\alpha\beta} E_{a1a} + \frac{1}{12} K^\alpha_{\alpha}{}^{aa1} R E_{a1a} + \frac{1}{24} K^{\alpha\beta aa1} M_{2\alpha}{}^{\mu\nu}_{a1} a^2 E_{a2a} g^{(2)}_{\alpha\beta\mu\nu} \\
& - \frac{1}{2} \mathcal{R}_{\alpha\beta a1 a2} K^{\alpha\beta aa1} E^{a2}{}_a + \frac{1}{48} K^{\alpha\beta aa1} K^{\mu\nu a2 a3} E_{a1 a2} E_{a3 a} g^{(2)}_{\alpha\beta\mu\nu}. \quad (69)
\end{aligned}$$

Thus, if one is interested in deriving the backreaction of some matter fields on the pseudo-Riemannian structure, then, if the assumption of Sec. III A are met, it is sufficient to take into account this last equation, together with Eqs. (66)–(68).

2. Order $\hat{V}^1$ terms

There are many terms which are linear in $\hat{V}$, so we further divide them. Moreover, we choose to write down these terms so that no covariant derivative is acting on $\hat{V}$. The contractions that involve the Ricci tensor and the Ricci scalar are

$$\begin{aligned} \text{tr}\hat{a}_2 \supset & -\frac{1}{12}K^{\alpha\beta aa1}V^\mu{}_{a1a}\nabla_\alpha R_{\beta\mu} + \frac{1}{24}K^{\alpha\beta aa1}V_{\beta a1a}\nabla_\alpha R - \frac{1}{12}K^{\alpha\beta aa1}V^\mu{}_{a1a}\nabla_\beta R_{\alpha\mu} \\ & + \frac{1}{24}K^{\alpha\beta aa1}V_{aa1a}\nabla_\beta R + \frac{1}{24}K^\alpha{}_{\alpha}{}^{aa1}V^\beta{}_{a1a}\nabla_\beta R - \frac{1}{12}K^{\alpha\beta aa1}V^\mu{}_{a1a}\nabla_\mu R_{\alpha\beta}, \end{aligned} \quad (70)$$

while those that are linear in the generalized curvature $\hat{\mathcal{R}}_{\mu\nu}$ are

$$\begin{aligned} \text{tr}\hat{a}_2 \supset & \frac{1}{3}K^{\alpha\beta aa1}V^{\mu a2}{}_a\hat{\nabla}_\alpha\mathcal{R}_{\beta\mu a1a2} - \frac{1}{3}K^{\alpha\beta aa1}V^{\mu a2}{}_a\hat{\nabla}_\mu\mathcal{R}_{\alpha\beta a1a2} + \frac{1}{12}K^{\alpha\beta aa1}V_{\beta}{}^{a2}{}_a\hat{\nabla}^\mu\mathcal{R}_{\alpha\mu a1a2} \\ & + \frac{1}{12}K^{\alpha\beta aa1}V_{\alpha}{}^{a2}{}_a\hat{\nabla}^\mu\mathcal{R}_{\beta\mu a1a2} + \frac{1}{12}K^\alpha{}_{\alpha}{}^{aa1}V^{\beta a2}{}_a\hat{\nabla}^\mu\mathcal{R}_{\beta\mu a1a2}. \end{aligned} \quad (71)$$

Finally, the remaining terms are given in the Appendix C 1.

3. Order $\hat{V}^2$ terms

There is a huge number of tensor structures which are quadratic in $\hat{V}$. For this reason, in the main text we only present those involve the endomorphism $\hat{E}$ and $\hat{M}$. These can be written in the following simple form:

$$\begin{aligned} \text{tr}\hat{a}_2 \supset & -\frac{1}{1920}K^{\alpha\beta aa1}K^{\mu\nu a2a4}K^{\rho\sigma a3a5}V^\lambda{}_{a1a2}V^\tau{}_{a4a3}E_{a5a}g^{(4)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\tau} \\ & -\frac{1}{1920}K^{\alpha\beta aa1}K^{\mu\nu a2a4}M_2{}^{\rho\sigma}{}_{a1}{}^{a3}V^\lambda{}_{a4a}V^\tau{}_{a3a2}g^{(4)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\tau}. \end{aligned} \quad (72)$$

All the other scalar contractions are listed in Sec. C 2 of Appendix C.

4. Order $\hat{V}^3$ terms

The number of independent tensor structures drastically decreases when we move on to consider contributions which are cubic in $\hat{V}$. Indeed, they can be written as

$$\begin{aligned} \text{tr}\hat{a}_2 \supset & -\frac{1}{5760}K^{\alpha\beta a}{}_{a1}K^{\mu\nu}{}_{a2}{}^{a5}K^\rho{}_{\sigma a3a4}V^\lambda{}_{a5a3}V^\tau{}_{a4a}\hat{\nabla}^\rho V^{\kappa a1a2}g^{(4)}{}_{\alpha\beta\mu\nu\sigma\lambda\tau\kappa} \\ & -\frac{1}{2880}K^{\alpha\beta aa1}K^{\mu\nu}{}_{a2}{}^{a5}K^\rho{}_{\sigma a3}{}^{a4}V^\lambda{}_{a1a3}V^\tau{}_{a5a}\hat{\nabla}^\rho V^{\kappa a4a2}g^{(4)}{}_{\alpha\beta\mu\nu\sigma\lambda\tau\kappa} \\ & -\frac{1}{5760}K^{\alpha\beta a}{}_{a1}K^{\mu\nu}{}_{a2}{}^{a5}K^\rho{}_{\sigma}{}^{a3a4}V^\lambda{}_{a5a3}V^\tau{}_{a4a}\hat{\nabla}^\sigma V^{\kappa a1a2}g^{(4)}{}_{\alpha\beta\mu\nu\rho\lambda\tau\kappa} \\ & -\frac{1}{2880}K^{\alpha\beta aa1}K^{\mu\nu}{}_{a2}{}^{a5}K^\rho{}_{\sigma}{}^{a3}{}^{a4}V^\lambda{}_{a1a3}V^\tau{}_{a5a}\hat{\nabla}^\sigma V^{\kappa a4a2}g^{(4)}{}_{\alpha\beta\mu\nu\rho\lambda\tau\kappa} \\ & -\frac{1}{2880}K^{\alpha\beta aa1}K^{\mu\nu a2}{}_{a5}K^{\rho\sigma}{}_{a3}{}^{a4}V^\lambda{}_{a4a}V_{\tau a1a2}\hat{\nabla}^\tau V^{\kappa a5a3}g^{(4)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\kappa} \\ & -\frac{1}{5760}K^{\alpha\beta aa1}K^{\mu\nu a2}{}_{a5}K^{\rho\sigma}{}_{a3}{}^{a4}V^\lambda{}_{a1a2}V_{\tau a4a}\hat{\nabla}^\tau V^{\kappa a5a3}g^{(4)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\kappa} \\ & +\frac{1}{5760}K^{\alpha\beta aa1}K^{\mu\nu a2}{}_{a5}K^{\rho\sigma}{}_{a3}{}^{a4}V^\lambda{}_{a1a2}V^\tau{}_{a4a}\hat{\nabla}^\gamma V^{\kappa a5a3}g^{(5)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\tau\kappa\gamma}. \end{aligned} \quad (73)$$

5. Order $\hat{V}^4$ terms

Finally, terms that are quartic in $\hat{V}$ can be written in the simple fashion

$$\text{tr}\hat{a}_2 \supset \frac{1}{1290240}K^{\alpha\beta aa1}K^{\mu\nu a2a3}K^{\rho\sigma a4a5}K^{\lambda\kappa a6a7}V^\tau{}_{a1a2}V^\gamma{}_{a3a4}V^\eta{}_{a5a6}V^\theta{}_{a7a}g^{(6)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\kappa\tau\eta\theta}. \quad (74)$$

D. Limiting case

Let us take a look at our nonminimal second-order operator (21). If we switch off the nonminimal part $\hat{N}$ the resulting operator is of Laplace type, and it has the derivative interaction which has been considered by Obukhov in [23]. In this limit the first part of our general result (that which is given by the integral over ζ) automatically vanishes, see Sec. VB, while $K^{\mu\nu}{}_a{}^{a1} \rightarrow g^{\mu\nu}\delta_a{}^{a1}$, see Eq. (30). Thus, the expansion performed in Sec. III must yield the same result given in [23] for the trace of the second Seeley-DeWitt coefficient. Indeed, we find complete agreement with [23] up to boundary terms, which is consistent since we have discarded surface terms from the onset.

E. Some words on other computational schemes

Recently, a new computational approach towards non-minimal differential operators was proposed in [34], whose main concern is to derive the low-energy part of the heat kernel in an exact way. It relies on the introduction of spin projectors, which are employed in a different manner with respect to previous studies [31]. Indeed, in flat space these projectors allow one to express the untraced heat kernel as a sum over the spin-parity eigenspaces. When the background geometry is curved the projectors are replaced by quasi-projectors, and an appropriate subtraction procedure is introduced to deal with the nonlocality of them. A systematic perturbative expansion in the background dimension is then carried out on the *full* heat kernel, which is free of IR pathologies.

When some simple operators are considered and a judicious choice of the endomorphism is made the approach of [34] yields an exact expression for the full untraced heat kernel. However, for more complicated differential operators this technique requires to deal with many integrations over proper time of some complicated nested commutators. The final expressions shall still be nonperturbative in the proper time, but they do not allow to implement the final result on a computer to yield an easy-to-use expression. Nevertheless, this result represents a cornerstone in the derivation of infrared properties of 1-loop effective action stemming from nonminimal second-order operators.

On the other hand, the approach that we have followed in this paper was to focus on the traced heat kernel coefficients, and in particular on the second one. By doing this and employing the results of Sec. IV on universal functional traces we were able to provide a model-independent expression for this coefficient. Furthermore, the open-source implementation in *Mathematica* of this Seeley-DeWitt coefficient significantly contributes to the applicability of our results. Finally, no subtraction procedure is needed in this case, since we are only interested in the UV divergent part of the effective action. Therefore, the present paper and [34] provide two complementary results on the heat kernel of nonminimal second-order operators.

VI. APPLICATIONS

In this section we shall present three applications of the results derived in the previous section. Though these examples will only involve bosonic quantum fields, our formulae can also be applied to fermionic higher spins. Concretely, our applications will involve a vector field, an antisymmetric 2-tensor (Kalb-Ramond), and a vector-valued 2-form (like the torsion). However, only in the first case we also take into account derivative interaction, thus fully using the results of Sec. VC. For simplicity none of our examples falls into the most-general case dealt with in Sec. IIIB.

A. Vector field

Let us consider a Lagrangian for a vector field A_μ which comprises a nonminimal kinetic term, as well as cubic and quartic self-interactions. Specifically, we parameterize it as

$$\mathcal{L} = -\frac{1}{2}((\nabla_\mu A_\nu)^2 - \lambda(\nabla_\mu A^\mu)^2) - \frac{b}{2}A^\mu A^\nu R_{\mu\nu} - \frac{c}{2}A^2 R - \frac{f}{3}A^\mu A^\nu \nabla_\mu A_\nu - \frac{g}{4}(A^\mu A_\mu)^2. \quad (75)$$

For example, we may think of this Lagrangian to describe the torsion vector [43–45] or Weyl vectors [46] in a metric-affine geometry. Alternatively, such kind of dynamics may also arise by considering vector fields coupled to scalars as in [47]. The beta functions of higher-derivative gravity due to a nonminimal vector field's quantum fluctuations were also computed in [48]: the author did not consider the cubic interaction, but mixed loops of gravitons and vectors were taken into account. The resulting second-order differential operator which acts on contra-variant vectors A^ν takes the form

$$F^\mu{}_\nu(\nabla) = \square\delta^\mu{}_\nu - \lambda\nabla^\mu\nabla_\nu - \frac{f}{3}(A_\nu\nabla^\mu - A^\mu\nabla_\nu) - bR^\mu{}_\nu - cR\delta^\mu{}_\nu - gA^\mu A_\nu. \quad (76)$$

When dealing with vectors we automatically have up to one nonminimal term, thus we can apply the simplified result of Sec. III A.

In particular, the explicit expressions of the tensors that parameterized all the differential operators are

$$\mathcal{R}_{\mu\nu}{}^\alpha{}_\beta = R^\alpha{}_{\beta\mu\nu}; \quad (77a)$$

$$K^{\alpha\beta\mu}{}_\nu = g^{\alpha\beta}\delta^\mu{}_\nu - \frac{\lambda}{\lambda-1}g^{\alpha\mu}\delta^\beta{}_\nu; \quad (77b)$$

$$M_2^{\alpha\beta\mu}{}_\nu = -\lambda g^{\alpha\mu}R^\beta{}_\nu - \frac{\lambda}{\lambda-1}R^{\alpha\mu}\delta^\beta{}_\nu; \quad (77c)$$

$$M_3^{\alpha\mu}{}_\nu = -\lambda\nabla^\mu R^\alpha{}_\nu - \frac{\lambda}{2}g^{\alpha\mu}\nabla_\nu R; \quad (77d)$$

$$E^\mu{}_\nu = -bR^\mu{}_\nu - cR\delta^\mu{}_\nu - gA^\alpha A_\alpha \delta^\mu{}_\nu - 2gA^\mu A_\nu; \quad (77e)$$

$$V^{\alpha\mu}{}_\nu = \frac{f}{3}(A^\mu \delta^\alpha{}_\nu - g^{\alpha\mu} A_\nu). \quad (77f)$$

In analogy with Eq. (76), we fix the ambiguity in ordering the covariant derivatives of $\hat{K}$ by putting that which acts on

A^ν to the right. The auxiliary integration parameter ζ is introduced by simply letting $\lambda \rightarrow \lambda\zeta$. Then, the final form of the trace of the second Seeley-DeWitt coefficient is found by inserting Eq. (77) into Eqs. (66)–(74) and (C1)–(C4) and computing the ζ integrals. Furthermore, we parametrize the result in terms of $\gamma = \frac{\lambda}{1+\lambda}$. Accordingly, we have that

$$\begin{aligned} \text{tr } \hat{a}_2 = & \left(\frac{5}{1296}(1+\gamma)^2 f^4 - \frac{1}{9}(3+4\gamma+\gamma^2)f^2 g + \frac{1}{4}(24+12\gamma+5\gamma^2)g^2 \right) A_\alpha A^\alpha A_\beta A^\beta \\ & + \frac{1}{432} \left((\gamma^2(-5+4b) - 4\gamma(3+5b) - 4(5+6b))f^2 + 72(\gamma^2(-1+b) + 12b + \gamma(-4+6b))g \right) A^\alpha A^\beta R_{\alpha\beta} \\ & + \frac{1}{360} \left(-8 + 15\gamma(2+\gamma) - 30\gamma(4+\gamma)b + 15(12+6\gamma+\gamma^2)b^2 \right) R_{\alpha\beta} R^{\alpha\beta} \\ & + \left(-\frac{1}{216}(1+\gamma)(-8+6b+36c+\gamma(-5+5b+18c))f^2 + \frac{1}{6}(6(-1+b+6c) + \gamma^2(-2+2b+9c) \right. \\ & \left. + \gamma(-5+3b+18c))g \right) A_\alpha A^\alpha R + \frac{1}{144} (12\gamma(-1+b+2c)(-1+6c) + 8(-1+6c)(-1+3b+6c) \\ & + 3\gamma^2(1+b^2-12c+24c^2+2b(-1+6c)))R^2 - \frac{11}{180} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \\ & + \frac{1}{36} (2-2\gamma b + \gamma^2(-2+2b+9c))f A^\alpha \nabla_\alpha R + \frac{1}{54} (-(1+3\gamma+2\gamma^2)f^3 + 9\gamma(4+5\gamma)fg) A^\alpha A^\beta \nabla_\beta A_\alpha \\ & + \frac{1}{216} (4+4\gamma+7\gamma^2)f^2 \nabla_\alpha A^\alpha \nabla_\beta A^\beta + \frac{1}{432} (-20-20\gamma+\gamma^2)f^2 \nabla_\beta A_\alpha \nabla^\beta A^\alpha. \end{aligned} \quad (78)$$

By setting $\gamma = 0$ we recover the same result that can be found using the squaring procedure outlined in [23].

In [9] a different method is used for dealing with a special case of Eq. (76), i.e., when $c = f = g = 0$ and $b = 1$, which is tailored for gauge theories. In such a special case the dependence on the nonminimal parameter γ identically drops out, which is a consequence of the Ward identity. In the presence of derivative- and self-interactions we observe that choosing $c = 0$ and $b = 1$ no longer suffices to annihilate the γ -dependence of the trace of $\hat{a}_2$. Thus, for any such a vector theory the counterterms stemming from graviton and mixed graviton-vector loops must be taken into account to address the gauge invariance of the effective action, or to prove the presence of a quantum anomaly.

B. Kalb-Ramond field

Let us now consider a Kalb-Ramond field $B_{\mu\nu} = B_{[\mu\nu]}$ [49]. In analogy with the previous example there exists a single divergence for this field, i.e., $\nabla_\mu B^{\mu\nu}$. Accordingly, the Lagrangian only depends on a single nonminimal parameter b , and it can be written as

$$\mathcal{L} = -\frac{1}{2}(\nabla_\alpha B_{\mu\nu} \nabla^\alpha B^{\mu\nu} + b \nabla_\mu B^{\mu\nu} \nabla_\alpha B^\alpha{}_\nu) + \frac{1}{2} B^{\mu\nu} E_{\mu\nu\alpha\beta} B^{\alpha\beta}, \quad (79)$$

where $E_{\mu\nu\alpha\beta}$ is an endomorphism. As a specific example one may consider the Lagrangian that arises by integrating some fermionic matter fields, and one may consider the quantum effective action associated to the Weyl anomaly in the presence of a Kalb-Ramond background as in [50]. By letting $b \rightarrow b\zeta$ we obtain a nonminimal operator whose dependence on the auxiliary parameter ζ is linear, and which reads

$$F^{\mu\nu}{}_{\alpha\beta} = \square \mathbb{1}^{\mu\nu}{}_{\alpha\beta} - b\zeta \nabla^{[\mu} \nabla_{[\beta} \delta^{\nu]}{}_{\alpha]} + E^{\mu\nu}{}_{\alpha\beta}, \quad (80)$$

where $\mathbb{1}^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\nu{}_\alpha \delta^\mu{}_\beta)$ is the identity and the operator is acting on a quantum field bearing contra-variant indices. We first specialize to momentum space and invert the principal symbol, and then we calculate $\hat{M}_2$, finding

$$\begin{aligned} \mathcal{R}_{\mu\nu}{}^{\alpha\beta}{}_{\rho\sigma} = & \frac{1}{2} \delta^\beta{}_\sigma R^\alpha{}_{\rho\mu\nu} - \frac{1}{2} \delta^\beta{}_\rho R^\alpha{}_{\sigma\mu\nu} - \frac{1}{2} \delta^\alpha{}_\sigma R^\beta{}_{\rho\mu\nu} \\ & + \frac{1}{2} \delta^\alpha{}_\rho R^\beta{}_{\sigma\mu\nu}; \end{aligned} \quad (81a)$$

$$\begin{aligned} D^{\alpha\beta\mu\nu}{}_{\rho\sigma} = & -\frac{1}{2} \delta^\mu{}_\sigma \delta^\nu{}_\rho g^{\alpha\beta} + \frac{1}{2} \delta^\mu{}_\rho \delta^\nu{}_\sigma g^{\alpha\beta} - \frac{1}{4} b\zeta \delta^\beta{}_\sigma \delta^\nu{}_\rho g^{\alpha\mu} \\ & + \frac{1}{4} b\zeta \delta^\beta{}_\rho \delta^\nu{}_\sigma g^{\alpha\mu} + \frac{1}{4} b\zeta \delta^\beta{}_\sigma \delta^\mu{}_\rho g^{\alpha\nu} \\ & - \frac{1}{4} b\zeta \delta^\beta{}_\rho \delta^\mu{}_\sigma g^{\alpha\nu}; \end{aligned} \quad (81b)$$

$$K^{\alpha\beta\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\delta^\mu{}_\rho\delta^\nu{}_\sigma g^{\alpha\beta} - \frac{1}{2}\delta^\mu{}_\sigma\delta^\nu{}_\rho g^{\alpha\beta} + \frac{b\zeta}{4+2b\zeta}(\delta^\beta{}_\sigma\delta^\nu{}_\rho g^{\alpha\mu} - \delta^\beta{}_\rho\delta^\nu{}_\sigma g^{\alpha\mu} - \delta^\beta{}_\sigma\delta^\mu{}_\rho g^{\alpha\nu} + \delta^\beta{}_\rho\delta^\mu{}_\sigma g^{\alpha\nu}); \quad (81c)$$

$$M_2^{\alpha\beta\mu\nu}{}_{\rho\sigma} = \frac{1}{8+4b\zeta}[b\zeta(\delta^\beta{}_\sigma(2\delta^\nu{}_\rho R^{\alpha\mu} - 2\delta^\mu{}_\rho R^{\alpha\nu} + b\zeta g^{\alpha\nu} R^\mu{}_\rho - b\zeta g^{\alpha\mu} R^\nu{}_\rho + 4R^{\alpha\mu\nu}{}_\rho - 4R^{\alpha\nu\mu}{}_\rho) + \delta^\beta{}_\rho(-2\delta^\nu{}_\sigma R^{\alpha\mu} + 2\delta^\mu{}_\sigma R^{\alpha\nu} - b\zeta g^{\alpha\nu} R^\mu{}_\sigma + b\zeta g^{\alpha\mu} R^\nu{}_\sigma - 4R^{\alpha\mu\nu}{}_\sigma + 4R^{\alpha\nu\mu}{}_\sigma) + (2+b\zeta)(\delta^\nu{}_\sigma g^{\alpha\mu} R^\beta{}_\rho - \delta^\mu{}_\sigma g^{\alpha\nu} R^\beta{}_\rho - \delta^\nu{}_\rho g^{\alpha\mu} R^\beta{}_\sigma + \delta^\mu{}_\rho g^{\alpha\nu} R^\beta{}_\sigma + 2g^{\alpha\nu} R^{\beta\mu}{}_{\rho\sigma} - 2g^{\alpha\mu} R^{\beta\nu}{}_{\rho\sigma}))]. \quad (81d)$$

As in the previous example, we have decided to sort the covariant derivatives such that the rightmost one generates divergences of $B^{\rho\sigma}$. Both $\hat{K}$ and $\hat{M}_2$ are singular for $b\zeta = -2$, which is exactly the value for which the Lagrangian (79) propagates only the transverse modes. Due to the linear dependence on ζ in $D^{\alpha\beta\mu\nu}{}_{\rho\sigma}$ this example falls into the simplified category of nonminimal operators dealt with in Sec. III A. Thus, to find contribution to the trace of $\hat{a}_2$ due to the nonminimal part of the operator we have to insert the tensors written in the previous Eq. (81) into Eqs. (66)–(68). Performing the ζ integral we find

$$\text{tr } \hat{a}_2 \supset -\frac{b^2}{4(2+b)^2} R_{\alpha\beta} R^{\alpha\beta} + \frac{(-8+b)b}{24(2+b)^2} R^2 + \frac{b^2}{8(2+b)^2} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \quad (82)$$

Next, we consider the remaining part of the trace log of the operator (80), i.e., that which depends on the endomorphism. By using Eq. (69) we obtain the contribution to the trace of the second Seeley-DeWitt coefficient, and the complete form of the trace of this coefficient is given by

$$\begin{aligned} \text{tr } \hat{a}_2 = & \frac{b^2}{12(2+b)^2} E_{\alpha\mu\beta\nu} E^{\alpha\beta\mu\nu} + \frac{24+12b+b^2}{12(2+b)^2} E^{\alpha\beta\mu\nu} E_{\mu\nu\alpha\beta} + \frac{b^2}{12(2+b)^2} E^{\alpha\beta}{}_\alpha{}^\mu E_\mu{}^\nu{}_\beta{}_\nu \\ & - \frac{b(8+b)}{6(2+b)^2} E_\alpha{}^\mu{}_\beta{}_\mu R^{\alpha\beta} - \left(\frac{1}{45} + \frac{b^2}{4(2+b)^2}\right) R_{\alpha\beta} R^{\alpha\beta} + \frac{8+4b+b^2}{12(2+b)^2} E^{\alpha\beta}{}_\alpha{}_\beta R \\ & + \frac{16-8b+7b^2}{72(2+b)^2} R^2 + \frac{b(4+b)}{4(2+b)^2} E^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} + \left(\frac{b^2}{8(2+b)^2} - \frac{13}{90}\right) R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \end{aligned} \quad (83)$$

The gauge theory of a Kalb-Ramond field is written in terms of the square of the field strength $\nabla_{[\mu} B_{\nu\rho]}$. The endomorphism that is obtained by choosing a Lagrangian quadratic in this field strength is

$$E^{\mu\nu}{}_{\alpha\beta} = -\frac{1}{2}\delta_\beta{}^\nu R_\alpha{}^\mu + \frac{1}{2}\delta_\beta{}^\mu R_\alpha{}^\nu + \frac{1}{2}\delta_\alpha{}^\nu R_\beta{}^\mu - \frac{1}{2}\delta_\alpha{}^\mu R_\beta{}^\nu + R_{\alpha\beta}{}^{\mu\nu}. \quad (84)$$

In analogy with what is found for electrodynamics [9], when this specific form of the endomorphism is plugged in Eq. (83) the dependence on the nonminimal parameter b identically drops out, yielding

$$\text{tr } \hat{a}_2 = -\frac{46}{45} R_{\alpha\beta} R^{\alpha\beta} + \frac{2}{9} R^2 + \frac{16}{45} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \quad (85)$$

Thus, the gauge theory of a Kalb-Ramond field on top of a curved pseudo-Riemannian manifold is not anomalous.

C. Torsion field: A simple toy model

Let us now introduce the final example, i.e., a simplified model of dynamical torsion. The latter is a vector-valued two-form $T^\rho{}_{\mu\nu} = T^\rho{}_{[\mu\nu]}$, which is customarily decomposed into its algebraically irreducible components [51]. These are given by a vector, an antisymmetric 3-tensor and a hook-antisymmetric tracefree tensor.

Such an algebraic decomposition has been employed in [29] to derive logarithmically divergent contributions to the effective action due to torsion and graviton fluctuations. Decomposing the torsion in this way may provide a clear algorithm to fathom the possible realization of asymptotic safety in higher-derivative gravity, but it is by no means necessary. Indeed, a second-order operator that comprises all the nine kinetic terms for the torsion can also be written in terms of a principal part parametrized by five free parameters and a configuration-space metric with two unknown coefficients, see Eq. (20) and [9,52]. We argue that the latter approach is more straightforward if one wants

to analyze the most general torsion theory. In this case, one may expect that the simplifying assumption of Sec. III A does not hold, thus using the result of Sec. III B will be necessary.

While a systematic derivation of the phase diagram in the most general metric-torsion theory would settle a common agreement regarding the physical properties of these theories, this is not the aim of the present discussion. Indeed, our aim is to provide a simple example for showing how to treat the presence of nonminimal kinetic terms in torsion theories, and to point out some phenomenological consequences of taking them into account. To this end, we consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}[T^{\rho\mu\nu}\square T^{\rho}_{\mu\nu} + c\nabla^\alpha T^{\rho}_{\alpha\beta}\nabla_\lambda T^{\lambda\beta}_{\rho} - T^{\rho}_{\mu\nu}E^{\mu\nu}_{\rho\lambda}{}^{\alpha\beta}T^{\lambda}_{\alpha\beta}]. \quad (86)$$

This comprises a single nonminimal term, which is parametrized by c , and we shall assume that $-2 < c < 0$. Furthermore, in part of the principal symbol which is proportional to the box operator the indices of the torsion are contracted in the simplest way, thus the configuration metric Eq. (20) is trivial. For the time being we do not assume any specific form of the endomorphism, but we make the following observation. Let us consider one of the possible “field strengths” of the torsion, i.e., $\mathcal{F}^{\rho}_{\lambda\mu\nu} = \nabla_{[\lambda}T^{\rho}_{\mu\nu]}$. This type of gauge approach towards a vector-valued 2-form has been used in flat-space in [53]. Thus, a

gaugelike origin of the previous Lagrangian equation (86) is obtained by considering

$$\mathcal{L} = -\frac{3}{2}\mathcal{F}^{\rho}_{\lambda\mu\nu}\mathcal{F}^{\lambda\mu\nu}_{\rho} \quad (87)$$

in a nonminimal gauge. Then, if we insist to start from the latter Lagrangian, the resulting endomorphism is

$$\begin{aligned} E^{\mu\nu}_{\rho\lambda}{}^{\alpha\beta} &= \frac{1}{2}g^{\beta\nu}g_{\lambda\rho}R^{\alpha\mu} - \frac{1}{2}g^{\beta\mu}g_{\lambda\rho}R^{\alpha\nu} - \frac{1}{2}g^{\alpha\nu}g_{\lambda\rho}R^{\beta\mu} \\ &\quad + \frac{1}{2}g^{\alpha\mu}g_{\lambda\rho}R^{\beta\nu} - g_{\lambda\rho}R^{\alpha\beta\mu\nu} - \frac{1}{2}g^{\beta\nu}R^{\alpha\mu}_{\lambda\rho} \\ &\quad + \frac{1}{2}g^{\beta\mu}R^{\alpha\nu}_{\lambda\rho} + \frac{1}{2}g^{\alpha\nu}R^{\beta\mu}_{\lambda\rho} - \frac{1}{2}g^{\alpha\mu}R^{\beta\nu}_{\lambda\rho}. \end{aligned} \quad (88)$$

Let us go back to the starting Lagrangian equation (86). The second-order differential operator which is associated to it reads as

$$F^{\rho}_{\mu\nu\lambda}{}^{\alpha\beta} = \square\mathbb{1}^{\rho}_{\mu\nu\lambda}{}^{\alpha\beta} + c\delta^{\rho}_{\lambda}\delta^{[\alpha}_{\mu}\nabla^{\beta]}\nabla_{\nu]} - E^{\rho}_{\mu\nu\lambda}{}^{\alpha\beta}, \quad (89)$$

where $\mathbb{1}^{\rho}_{\mu\nu\lambda}{}^{\alpha\beta} = \delta^{\rho}_{\lambda}\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu]}$ is the identity in the space of vector-valued 2-forms. We introduce the auxiliary parameter ζ by letting $c \rightarrow c\zeta$ as in the previous example, and we read off the principal symbol and its inverse in momentum space. The expressions of these tensors and of the generalized curvature are

$$\begin{aligned} \mathcal{R}^{\mu_1\mu_2\lambda}_{\alpha\beta\rho}{}^{\mu\nu} &= \frac{1}{2}\delta^{\nu}_{\beta}\delta^{\lambda}_{\rho}R^{\mu\mu_1\mu_2}_{\alpha} - \frac{1}{2}\delta^{\mu}_{\beta}\delta^{\lambda}_{\rho}R^{\nu\mu_1\mu_2}_{\alpha} - \frac{1}{2}\delta^{\nu}_{\alpha}\delta^{\lambda}_{\rho}R^{\mu\mu_1\mu_2}_{\beta} + \frac{1}{2}\delta^{\mu}_{\alpha}\delta^{\lambda}_{\rho}R^{\nu\mu_1\mu_2}_{\beta} \\ &\quad - \frac{1}{2}\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}R^{\lambda\rho\mu_1\mu_2} + \frac{1}{2}\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}R^{\lambda\rho\mu_1\mu_2}; \end{aligned} \quad (90a)$$

$$D^{\mu_1\mu_2\lambda}_{\alpha\beta\rho}{}^{\mu\nu} = \frac{1}{4}c\zeta\delta^{\lambda}_{\rho}(-\delta^{\nu}_{\alpha}\delta^{\mu_1}_{\beta}g^{\mu_2\nu} + \delta^{\mu}_{\alpha}\delta^{\mu_1}_{\beta}g^{\mu_2\nu} + \delta^{\mu_1}_{\alpha}(\delta^{\nu}_{\beta}g^{\mu_2\nu} - \delta^{\mu}_{\beta}g^{\mu_2\nu})); \quad (90b)$$

$$\begin{aligned} K^{\mu_1\mu_2\lambda}_{\alpha\beta\rho}{}^{\mu\nu} &= \frac{1}{4+2c\zeta}\delta^{\lambda}_{\rho}[\delta^{\nu}_{\alpha}(c\zeta\delta^{\mu_1}_{\beta}g^{\mu_2\nu} - (2+c\zeta)\delta^{\mu}_{\beta}g^{\mu_1\mu_2}) \\ &\quad + c\zeta\delta^{\mu_1}_{\alpha}(-\delta^{\nu}_{\beta}g^{\mu_2\nu} + \delta^{\mu}_{\beta}g^{\mu_2\nu}) + \delta^{\mu}_{\alpha}((2+c\zeta)\delta^{\nu}_{\beta}g^{\mu_1\mu_2} - c\zeta\delta^{\mu_1}_{\beta}g^{\mu_2\nu})]. \end{aligned} \quad (90c)$$

Once again, we have sorted the covariant derivatives to act with divergences on $T^{\rho}_{\mu\nu}$. Clearly, the last equation becomes singular as $c\zeta \rightarrow -2$. Next, we promote momentum space expressions to coordinate-based ones, and we compute the $\hat{M}_2$ differential operator, whose tensor coefficients are given by the equation

$$\begin{aligned} M_2^{\mu_1\mu_2\lambda}_{\alpha\beta\rho}{}^{\mu\nu} &= \frac{c\zeta}{8+4c\zeta}[2\delta^{\mu}_{\beta}\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu_1}_{\alpha} - c\zeta\delta^{\mu_1}_{\alpha}\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu}_{\beta} + 2\delta^{\nu}_{\alpha}\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu_1}_{\beta} \\ &\quad - 2\delta^{\mu}_{\alpha}\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu_1}_{\beta} + c\zeta\delta^{\mu_1}_{\alpha}\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu}_{\beta} - 2\delta^{\mu}_{\alpha}\delta^{\lambda}_{\rho}\delta^{\mu_1}_{\beta}R^{\mu_2\nu} \\ &\quad - c\zeta\delta^{\mu_1}_{\alpha}\delta^{\lambda}_{\rho}\delta^{\mu_2\nu}R^{\mu}_{\beta} + 4\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu}_{\beta}{}^{\mu_1} - 4\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu_1}_{\beta}{}^{\mu} \\ &\quad + 4\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu_1}_{\beta}{}^{\nu} - 4\delta^{\mu}_{\beta}g^{\mu_2\nu}R^{\mu_1\lambda}_{\rho} - 4\delta^{\lambda}_{\rho}g^{\mu_2\nu}R^{\mu}_{\beta}{}^{\mu_1} \\ &\quad + c\zeta\delta^{\mu_1}_{\alpha}g^{\mu_2\nu}R^{\mu\lambda}_{\beta\rho} - 4\delta^{\nu}_{\alpha}g^{\mu_2\nu}R^{\mu_1\lambda}_{\beta\rho} + 4\delta^{\mu}_{\alpha}g^{\mu_2\nu}R^{\mu_1\lambda}_{\beta\rho} \end{aligned}$$

$$\begin{aligned}
& +4\delta_\alpha^{\mu_1}\delta_\rho^\lambda R_\beta^{\mu_2\mu\nu} + 2c\zeta\delta_\alpha^{\mu_1}\delta_\rho^\lambda R_\beta^{\mu_2\mu\nu} - c\zeta\delta_\alpha^{\mu_1}g^{\mu_2\nu}R_\beta^{\lambda\rho} \\
& + \delta_\beta^\nu(\delta_\rho^\lambda(-2g^{\mu_2}R_\alpha^{\mu_1} + (2+c\zeta)\delta_\alpha^{\mu_1}R^{\mu\mu_2}) + 4g^{\mu_2}R_\alpha^{\mu_1\lambda}) \\
& + 2(2+c\zeta)\delta_\alpha^{\mu_1}R_\rho^{\lambda\mu_2} - 2c\zeta\delta_\alpha^{\mu_1}\delta_\beta^{\mu_2}R_\rho^{\lambda\mu\nu} + 4\delta_\alpha^{\mu_1}\delta_\beta^\mu R_\rho^{\lambda\mu_2\nu} \\
& + 2c\zeta\delta_\alpha^{\mu_1}\delta_\beta^\mu R_\rho^{\lambda\mu_2\nu} + \delta_\beta^{\mu_1}(-c\zeta g^{\mu_2\nu}R_\alpha^{\mu\lambda} + \delta_\rho^\lambda(c\zeta g^{\mu_2\nu}R_\alpha^\mu - c\zeta g^{\mu\mu_2}R_\alpha^\nu \\
& -(2+c\zeta)(\delta_\alpha^\nu R^{\mu\mu_2} - \delta_\alpha^\mu R^{\mu_2\nu} + 2R_\alpha^{\mu_2\mu\nu})) + c\zeta g^{\mu\mu_2}R_\alpha^{\nu\lambda} - 4\delta_\alpha^\nu R_\rho^{\lambda\mu_2} \\
& - 2c\zeta\delta_\alpha^\nu R_\rho^{\lambda\mu_2} + 2c\zeta\delta_\alpha^{\mu_2}R_\rho^{\lambda\mu\nu} - 4\delta_\alpha^\mu R_\rho^{\lambda\mu_2\nu} - 2c\zeta\delta_\alpha^\mu R_\rho^{\lambda\mu_2\nu}].
\end{aligned} \tag{91}$$

Now we have all the necessary ingredients to compute the trace of the second Seeley-DeWitt coefficient of the nonminimal operator Eq. (89). First, we focus on the trace log of the nonminimal part of the operator (89). To this end, we insert Eqs. (90) and (91) in the general result, which is given by the sum of Eqs. (66)–(68). By performing the ζ integral we obtain the contribution to the trace of $\hat{a}_2$,

$$\text{tr } \hat{a}_2 \supset -\frac{c^2}{(2+c)^2} R_{\alpha\beta} R^{\alpha\beta} + \frac{(-8+c)c}{6(2+c)^2} R^2 - \frac{(-1+c)c^2}{4(2+c)^2} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \tag{92}$$

Notice the expected singular behavior in the $c \rightarrow -2$ limit. Next, we consider the part of $\text{tr } \hat{a}_2$ which depends on the endomorphism, i.e., Eq. (69). In this case the contribution to the trace of the $\hat{a}_2$ coefficient is

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \frac{c^2}{12(2+c)^2} E^{\alpha\beta\mu\nu\rho\sigma} E_{\nu\beta\rho\alpha\mu\sigma} + \frac{24+12c+c^2}{12(2+c)^2} E^{\alpha\beta\mu\nu\rho\sigma} E_{\nu\rho\sigma\alpha\beta\mu} \\
& + \frac{c^2}{12(2+c)^2} E^{\alpha\beta\mu\nu}{}_\beta{}^\rho E_{\nu\rho}{}^\sigma{}_{\alpha\mu\sigma} - \frac{c(8+c)}{6(2+c)^2} E^\mu{}_\alpha{}^\nu{}_{\mu\beta\nu} R^{\alpha\beta} - \frac{2}{15} R_{\alpha\beta} R^{\alpha\beta} \\
& + \frac{8+4c+c^2}{12(2+c)^2} E^{\alpha\beta\mu}{}_{\alpha\beta\mu} R + \frac{1}{3} R^2 + \frac{c(4+c)}{2(2+c)^2} E^{\alpha\beta\mu\nu}{}_\beta{}^\rho R_{\alpha\nu\mu\rho} - \frac{31}{30} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \\
& - \frac{c(3+2c)}{3(2+c)^2} E^{\alpha\beta\mu}{}_\alpha{}^\nu{}_\rho R_{\beta\mu\nu\rho} + \frac{c(24+11c)}{6(2+c)^2} E^{\alpha\beta\mu}{}_\alpha{}^\nu{}_\rho R_{\beta\nu\mu\rho}.
\end{aligned} \tag{93}$$

Thus, the final result is given by

$$\begin{aligned}
\text{tr } \hat{a}_2 = & \frac{c^2}{12(2+c)^2} E^{\alpha\beta\mu\nu\rho\sigma} E_{\nu\beta\rho\alpha\mu\sigma} + \frac{24+12c+c^2}{12(2+c)^2} E^{\alpha\beta\mu\nu\rho\sigma} E_{\nu\rho\sigma\alpha\beta\mu} \\
& + \frac{c^2}{12(2+c)^2} E^{\alpha\beta\mu\nu}{}_\beta{}^\rho E_{\nu\rho}{}^\sigma{}_{\alpha\mu\sigma} - \frac{c(8+c)}{6(2+c)^2} E^\mu{}_\alpha{}^\nu{}_{\mu\beta\nu} R^{\alpha\beta} \\
& + \frac{8+4c+c^2}{12(2+c)^2} E^{\alpha\beta\mu}{}_{\alpha\beta\mu} R + \frac{c(4+c)}{2(2+c)^2} E^{\alpha\beta\mu\nu}{}_\beta{}^\rho R_{\alpha\nu\mu\rho} \\
& - \frac{c(3+2c)}{3(2+c)^2} E^{\alpha\beta\mu}{}_\alpha{}^\nu{}_\rho R_{\beta\mu\nu\rho} + \frac{c(24+11c)}{6(2+c)^2} E^{\alpha\beta\mu}{}_\alpha{}^\nu{}_\rho R_{\beta\nu\mu\rho} \\
& + \left(\frac{(-8+c)c}{6(2+c)^2} + \frac{1}{3} \right) R^2 - \left(\frac{c^2}{(2+c)^2} + \frac{2}{15} \right) R_{\alpha\beta} R^{\alpha\beta} - \left(\frac{(-1+c)c^2}{4(2+c)^2} + \frac{31}{30} \right) R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}.
\end{aligned} \tag{94}$$

Finally, we specialize the final expression of the trace of the $\hat{a}_2$ coefficient of the differential operator Eq. (89) to the special case of the endomorphism given by Eq. (88). This yields

$$\text{tr } \hat{a}_2 = -\frac{62}{15} R_{\alpha\beta} R^{\alpha\beta} + R^2 + \frac{236+118c-15c^2}{60(2+c)} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \tag{95}$$

Therefore, in strike contrast with the examples of electrodynamics [9] and of the Kalb-Ramond field the dependence on the nonminimal parameter c does not disappear. Accordingly, we cannot treat the Lagrangian equation (87) as describing a gauge theory of a vector-valued 2-form as a consistent quantum field theory on a curved background. Consequently, taking into account the quantum fluctuations of the metric field becomes mandatory in order to understand if this theory generates a quantum anomaly. However, due to the first Bianchi identities, we cannot couple the generalized field strength $\mathcal{F}^\rho{}_{\lambda\mu\nu}$ to

the Riemann tensor, which yields a significantly different behavior with respect to that of the vector theory treated in Sec. VI A.

VII. DISCUSSION AND OUTLOOKS

In the present paper we have derived for the first time a general expression for the local part of the trace of the second Seeley-DeWitt coefficient of a nonminimal operator in $d = 4$. To this end we have employed the trick of [9] for rewriting the trace log of a nonminimal operator as the integral over an auxiliary parameter of a sum of functional traces. However, we have followed [29] in splitting the calculation into two different parts, in such a way that only the first one involves the integration over the auxiliary parameter. As we have discussed at the end of Sec. III such a split significantly improves the computational speed for two reasons. On the one hand, the integrals that have to be computed usually are way less than in the method of [9]; on the other hand, we generate symmetrized metrics $g^{(n)}$ only up to $n = 6$, compared to $n = 8$ in [9]'s method. Thus, we have obtained a reduction of a factor $\frac{1}{13 \times 15}$ in the number of generated tensor structure at order $\hat{V}^4$, and analogous improvements appear at all orders in $\hat{V}$. All of the model-independent formulas for the local part of the trace of the second Seeley-DeWitt coefficient are also attached to this paper in a *Mathematica* notebook [41].

At the end of Sec. V we have discussed the complete compatibility of our result with that of [23] as we switch off the nonminimal terms. Due to the very structure of the trick by [9], the present method allows to access only the trace of the heat kernel coefficients, thus one has to rely on different approaches if he wants to derive the untraced coefficients [26]. Even though we have limited our analysis to the local part of the trace of the second heat kernel coefficient $\hat{a}_2$, in principle the algorithm presented here can also be employed for deriving other traced coefficients as well as the boundary terms.

In Sec. VI we have applied the general result to three simple cases. Our findings are compatible with known results in the literature [9,29,30]. An interesting future development of these applications is to derive the divergent part of the effective action for higher spins, torsion, and nonmetricity fields. This would provide a comprehensive picture of the radiative effects of these quantum fields on the high-energy behavior of higher-derivative gravity. In particular, it would be possible to analyze which parts of the

physical parameter space are compatible with asymptotic safety (see, e.g., Ref. [29] for a recent effort in this direction). Part of this program was carried out in [54], though the author did not consider nonminimal terms for higher spins, while the radiative (in)stability of torsion and nonmetricity theories was studied in [55].

As we have mentioned in the Introduction, there exists another way of deriving the trace log of nonminimal operators within the heat kernel method, which relies on spin-parity decompositions [26]. Such a program was carried out on a maximally symmetric background in [31] integrating out torsion fluctuations, employing the spin-parity decomposition developed in [56]. In general, if the (bosonic) quantum field that we are integrating out has spin s , we will be required to know the precise form of the traced heat kernel coefficients of minimal operators up to order $2 + 2s$. Therefore, while pursuing this route might be preferable when dealing with the torsion or nonmetricity tensors, we argue that it will be less convenient for fields of spin $s \geq 4$.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [41]; embargo periods may apply.

APPENDIX A: COMMUTATOR IDENTITIES

Let us exploit a Laplace transform to write down the inverse of the n th power of $\hat{\square}$, i.e.,

$$\frac{\hat{1}}{\hat{\square}^n} = \frac{(-i)^n}{(n-1)!} \int_0^\infty ds s^{n-1} e^{is\hat{\square}}. \quad (\text{A1})$$

By using this integral representation we can trade the commutation of $\hat{\square}^{-1}$ with the commutation of $\hat{\square}$, thus keeping under control the mass dimension of the operators that we are dealing with. Indeed, the following set of equations holds:

$$\begin{aligned} \frac{\hat{1}}{\hat{\square}} \hat{M} &= -i \int_0^\infty ds e^{is\hat{\square}} \hat{M} = -i \int_0^\infty ds \left(\hat{M} + \sum_{k \geq 1} \frac{(is)^k}{k!} \hat{\square}^k \hat{M} \right) \\ &= -i \int_0^\infty ds \left[\hat{M} + \sum_{k \geq 1} \frac{(is)^k}{k!} \hat{M} \hat{\square}^k + \sum_{k \geq 1} \frac{(is)^k}{k!} k [\hat{\square}, \hat{M}] \hat{\square}^{k-1} + \sum_{k \geq 2} \frac{(is)^k}{k!} \frac{k(k-1)}{2} [\hat{\square}, [\hat{\square}, \hat{M}]] \hat{\square}^{k-2} + \dots \right] \\ &= \hat{M} \frac{\hat{1}}{\hat{\square}} - [\hat{\square}, \hat{M}] \frac{\hat{1}}{\hat{\square}^2} + [\hat{\square}, [\hat{\square}, \hat{M}]] \frac{\hat{1}}{\hat{\square}^3} + \dots, \end{aligned} \quad (\text{A2})$$

where the ellipsis indicates that we are discarding higher-dimensional terms. In doing so we are tacitly assuming that $\hat{M}$ has dimension one, whence triple commutators would at most give rise to boundary terms.

Therefore, the commutator of the inverse of $\hat{\square}$ with $\hat{M}$ reads

$$\left[\frac{1}{\hat{\square}}, \hat{M}\right] = -[\hat{\square}, \hat{M}] \frac{1}{\hat{\square}^2} + [\hat{\square}, [\hat{\square}, \hat{M}]] \frac{1}{\hat{\square}^3} + \dots \quad (\text{A3})$$

Finally, using the fact that

$$[AB, C] = A[B, C] + [B, C]A, \quad (\text{A4})$$

we find that

$$\left[\frac{1}{\hat{\square}^2}, \hat{M}\right] = -2[\hat{\square}, \hat{M}] \frac{1}{\hat{\square}^3} + 3[\hat{\square}, [\hat{\square}, \hat{M}]] \frac{1}{\hat{\square}^4} + \dots \quad (\text{A5})$$

APPENDIX B: COINCIDENCE LIMITS

In this section we present all the coincidence limits up to dimension-four operators of the Synge world function, the reduced Van-Vleck determinant, and the Seeley-DeWitt coefficients. As we have done in the main text, we discard surface terms. The Seeley-DeWitt coefficients are those of the minimal operator

$$\hat{F}(\hat{\nabla}) = \hat{\square} + \hat{E}. \quad (\text{B1})$$

1. Covariant derivatives of the Synge world function

We start by considering the Synge world function. All these results are derived by covariant differentiating Eq. (11), sorting the covariant derivatives and taking the coincidence limit. First of all, we have

$$[\sigma] = 0, \quad (\text{B2})$$

which is consistent with the geometrical interpretation of Sec. II. Since σ has dimension of length squared, dimensional analysis tells us that

$$[\nabla_\mu \sigma] = 0. \quad (\text{B3})$$

On the other hand, when acting with two covariant derivatives we find

$$[\nabla_\mu \nabla_\nu \sigma] = g_{\mu\nu}. \quad (\text{B4})$$

Again, since the Synge world function only depends on the background pseudo-Riemannian structure, we have that

$$[\nabla_\mu \nabla_\nu \nabla_\rho \sigma] = 0. \quad (\text{B5})$$

Curvature invariants pop up as soon as we start applying four derivatives. In this case we obtain

$$[\nabla_{\mu_1} \dots \nabla_{\mu_4} \sigma] = -\frac{2}{3} R_{(\mu_3|\mu_1|\mu_4)\mu_2}. \quad (\text{B6})$$

By acting with one more derivative we expect to find expressions proportional to the covariant derivatives of the Riemann tensor. Indeed, the precise form is

$$[\nabla_{\mu_1} \dots \nabla_{\mu_5} \sigma] = -\frac{1}{2} [\nabla_{\mu_1} R_{(\mu_4|\mu_2|\mu_5)\mu_3} + \nabla_{\mu_2} R_{(\mu_4|\mu_1|\mu_5)\mu_3} + \nabla_{\mu_3} R_{(\mu_4|\mu_1|\mu_5)\mu_2}]. \quad (\text{B7})$$

Finally, the most involved result from a computational viewpoint is that whose mass dimension is equal to four. By resorting to computer algebra we find

$$\begin{aligned} [\nabla_{\mu_1} \nabla_{\mu_2} \nabla_{\mu_3} \nabla_{\mu_4} \nabla_{\mu_5} \nabla_{\mu_6} \sigma] = & \frac{1}{45} R_{\mu_1}{}^\alpha{}_{\mu_5\mu_6} R_{\mu_2\alpha\mu_3\mu_4} - \frac{1}{45} R_{\mu_1\mu_5\mu_6}{}^\alpha R_{\mu_2\alpha\mu_3\mu_4} \\ & - \frac{1}{15} R_{\mu_1\mu_6\mu_5}{}^\alpha R_{\mu_2\alpha\mu_3\mu_4} - \frac{1}{45} R_{\mu_1}{}^\alpha{}_{\mu_4\mu_6} R_{\mu_2\alpha\mu_3\mu_5} - \frac{4}{15} R_{\mu_1\mu_4\mu_6}{}^\alpha R_{\mu_2\alpha\mu_3\mu_5} - \frac{1}{45} R_{\mu_1\mu_6\mu_4}{}^\alpha R_{\mu_2\alpha\mu_3\mu_5} \\ & - \frac{1}{45} R_{\mu_1}{}^\alpha{}_{\mu_4\mu_5} R_{\mu_2\alpha\mu_3\mu_6} - \frac{1}{15} R_{\mu_1\mu_4\mu_5}{}^\alpha R_{\mu_2\alpha\mu_3\mu_6} - \frac{1}{45} R_{\mu_1\mu_5\mu_4}{}^\alpha R_{\mu_2\alpha\mu_3\mu_6} - \frac{1}{45} R_{\mu_1}{}^\alpha{}_{\mu_3\mu_6} R_{\mu_2\alpha\mu_4\mu_5} \\ & - \frac{4}{15} R_{\mu_1\mu_3\mu_6}{}^\alpha R_{\mu_2\alpha\mu_4\mu_5} - \frac{1}{45} R_{\mu_1\mu_6\mu_3}{}^\alpha R_{\mu_2\alpha\mu_4\mu_5} - \frac{1}{45} R_{\mu_1}{}^\alpha{}_{\mu_3\mu_5} R_{\mu_2\alpha\mu_4\mu_6} - \frac{1}{15} R_{\mu_1\mu_3\mu_5}{}^\alpha R_{\mu_2\alpha\mu_4\mu_6} \\ & - \frac{1}{45} R_{\mu_1\mu_5\mu_3}{}^\alpha R_{\mu_2\alpha\mu_4\mu_6} + \frac{1}{45} R_{\mu_1}{}^\alpha{}_{\mu_3\mu_4} R_{\mu_2\alpha\mu_5\mu_6} + \frac{4}{15} R_{\mu_1\mu_3\mu_4}{}^\alpha R_{\mu_2\alpha\mu_5\mu_6} + \frac{1}{45} R_{\mu_1\mu_4\mu_3}{}^\alpha R_{\mu_2\alpha\mu_5\mu_6} \\ & + \frac{1}{15} R_{\mu_1}{}^\alpha{}_{\mu_5\mu_6} R_{\mu_2\mu_3\mu_4\alpha} + \frac{2}{15} R_{\mu_1\mu_5\mu_6}{}^\alpha R_{\mu_2\mu_3\mu_4\alpha} - \frac{1}{5} R_{\mu_1\mu_6\mu_5}{}^\alpha R_{\mu_2\mu_3\mu_4\alpha} - \frac{1}{15} R_{\mu_1}{}^\alpha{}_{\mu_4\mu_6} R_{\mu_2\mu_3\mu_5\alpha} \end{aligned}$$

$$\begin{aligned}
& + \frac{2}{15} R_{\mu_1 \mu_6 \mu_4}^\alpha R_{\mu_2 \mu_3 \mu_5 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_4 \mu_5} R_{\mu_2 \mu_3 \mu_6 \alpha} - \frac{1}{5} R_{\mu_1 \mu_4 \mu_5}^\alpha R_{\mu_2 \mu_3 \mu_6 \alpha} + \frac{2}{15} R_{\mu_1 \mu_5 \mu_4}^\alpha R_{\mu_2 \mu_3 \mu_6 \alpha} \\
& + \frac{1}{45} R_{\mu_1}^\alpha R_{\mu_5 \mu_6} R_{\mu_2 \mu_4 \mu_3 \alpha} - \frac{1}{45} R_{\mu_1 \mu_5 \mu_6}^\alpha R_{\mu_2 \mu_4 \mu_3 \alpha} - \frac{4}{15} R_{\mu_1 \mu_6 \mu_5}^\alpha R_{\mu_2 \mu_4 \mu_3 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_3 \mu_6} R_{\mu_2 \mu_4 \mu_5 \alpha} \\
& + \frac{2}{15} R_{\mu_1 \mu_6 \mu_3}^\alpha R_{\mu_2 \mu_4 \mu_5 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_3 \mu_5} R_{\mu_2 \mu_4 \mu_6 \alpha} - \frac{1}{5} R_{\mu_1 \mu_3 \mu_5}^\alpha R_{\mu_2 \mu_4 \mu_6 \alpha} + \frac{2}{15} R_{\mu_1 \mu_5 \mu_3}^\alpha R_{\mu_2 \mu_4 \mu_6 \alpha} \\
& - \frac{1}{45} R_{\mu_1}^\alpha R_{\mu_4 \mu_6} R_{\mu_2 \mu_5 \mu_3 \alpha} - \frac{1}{15} R_{\mu_1 \mu_4 \mu_6}^\alpha R_{\mu_2 \mu_5 \mu_3 \alpha} - \frac{1}{45} R_{\mu_1 \mu_6 \mu_4}^\alpha R_{\mu_2 \mu_5 \mu_3 \alpha} - \frac{1}{45} R_{\mu_1}^\alpha R_{\mu_3 \mu_6} R_{\mu_2 \mu_5 \mu_4 \alpha} \\
& - \frac{1}{15} R_{\mu_1 \mu_3 \mu_6}^\alpha R_{\mu_2 \mu_5 \mu_4 \alpha} - \frac{1}{45} R_{\mu_1 \mu_6 \mu_3}^\alpha R_{\mu_2 \mu_5 \mu_4 \alpha} - \frac{1}{45} R_{\mu_1}^\alpha R_{\mu_3 \mu_4} R_{\mu_2 \mu_5 \mu_6 \alpha} + \frac{1}{3} R_{\mu_1 \mu_3 \mu_4}^\alpha R_{\mu_2 \mu_5 \mu_6 \alpha} \\
& - \frac{1}{45} R_{\mu_1 \mu_4 \mu_3}^\alpha R_{\mu_2 \mu_5 \mu_6 \alpha} - \frac{1}{45} R_{\mu_1}^\alpha R_{\mu_4 \mu_5} R_{\mu_2 \mu_6 \mu_3 \alpha} - \frac{4}{15} R_{\mu_1 \mu_4 \mu_5}^\alpha R_{\mu_2 \mu_6 \mu_3 \alpha} - \frac{1}{45} R_{\mu_1 \mu_5 \mu_4}^\alpha R_{\mu_2 \mu_6 \mu_3 \alpha} \\
& - \frac{1}{45} R_{\mu_1}^\alpha R_{\mu_3 \mu_5} R_{\mu_2 \mu_6 \mu_4 \alpha} - \frac{4}{15} R_{\mu_1 \mu_3 \mu_5}^\alpha R_{\mu_2 \mu_6 \mu_4 \alpha} - \frac{1}{45} R_{\mu_1 \mu_5 \mu_3}^\alpha R_{\mu_2 \mu_6 \mu_4 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_3 \mu_4} R_{\mu_2 \mu_6 \mu_5 \alpha} \\
& + \frac{2}{15} R_{\mu_1 \mu_4 \mu_3}^\alpha R_{\mu_2 \mu_6 \mu_5 \alpha} - \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_6} R_{\mu_3 \mu_4 \mu_5 \alpha} - \frac{2}{15} R_{\mu_1 \mu_2 \mu_6}^\alpha R_{\mu_3 \mu_4 \mu_5 \alpha} + \frac{2}{45} R_{\mu_1 \mu_6 \mu_2}^\alpha R_{\mu_3 \mu_4 \mu_5 \alpha} \\
& - \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_5} R_{\mu_3 \mu_4 \mu_6 \alpha} - \frac{2}{15} R_{\mu_1 \mu_2 \mu_5}^\alpha R_{\mu_3 \mu_4 \mu_6 \alpha} + \frac{2}{45} R_{\mu_1 \mu_5 \mu_2}^\alpha R_{\mu_3 \mu_4 \mu_6 \alpha} + \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_4} R_{\mu_3 \mu_5 \mu_6 \alpha} \\
& + \frac{2}{15} R_{\mu_1 \mu_2 \mu_4}^\alpha R_{\mu_3 \mu_5 \mu_6 \alpha} - \frac{2}{45} R_{\mu_1 \mu_4 \mu_2}^\alpha R_{\mu_3 \mu_5 \mu_6 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_2 \mu_6} R_{\mu_3 \mu_4 \mu_5 \alpha} - \frac{1}{15} R_{\mu_1 \mu_6 \mu_2}^\alpha R_{\mu_3 \mu_4 \mu_5 \alpha} \\
& - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_2 \mu_5} R_{\mu_3 \mu_4 \mu_6 \alpha} - \frac{1}{15} R_{\mu_1 \mu_5 \mu_2}^\alpha R_{\mu_3 \mu_4 \mu_6 \alpha} - \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_6} R_{\mu_3 \mu_5 \mu_4 \alpha} - \frac{2}{15} R_{\mu_1 \mu_2 \mu_6}^\alpha R_{\mu_3 \mu_5 \mu_4 \alpha} \\
& + \frac{2}{45} R_{\mu_1 \mu_6 \mu_2}^\alpha R_{\mu_3 \mu_5 \mu_4 \alpha} - \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_4} R_{\mu_3 \mu_5 \mu_6 \alpha} + \frac{4}{15} R_{\mu_1 \mu_2 \mu_4}^\alpha R_{\mu_3 \mu_5 \mu_6 \alpha} + \frac{2}{45} R_{\mu_1 \mu_4 \mu_2}^\alpha R_{\mu_3 \mu_5 \mu_6 \alpha} \\
& - \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_5} R_{\mu_3 \mu_6 \mu_4 \alpha} - \frac{2}{15} R_{\mu_1 \mu_2 \mu_5}^\alpha R_{\mu_3 \mu_6 \mu_4 \alpha} + \frac{2}{45} R_{\mu_1 \mu_5 \mu_2}^\alpha R_{\mu_3 \mu_6 \mu_4 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_2 \mu_4} R_{\mu_3 \mu_6 \mu_5 \alpha} \\
& - \frac{1}{15} R_{\mu_1 \mu_4 \mu_2}^\alpha R_{\mu_3 \mu_6 \mu_5 \alpha} + \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_3} R_{\mu_4 \mu_5 \mu_6 \alpha} + \frac{2}{15} R_{\mu_1 \mu_2 \mu_3}^\alpha R_{\mu_4 \mu_5 \mu_6 \alpha} - \frac{2}{45} R_{\mu_1 \mu_3 \mu_2}^\alpha R_{\mu_4 \mu_5 \mu_6 \alpha} \\
& - \frac{4}{45} R_{\mu_1}^\alpha R_{\mu_2 \mu_3} R_{\mu_4 \mu_5 \mu_6 \alpha} + \frac{4}{15} R_{\mu_1 \mu_2 \mu_3}^\alpha R_{\mu_4 \mu_5 \mu_6 \alpha} + \frac{2}{45} R_{\mu_1 \mu_3 \mu_2}^\alpha R_{\mu_4 \mu_5 \mu_6 \alpha} - \frac{1}{15} R_{\mu_1}^\alpha R_{\mu_2 \mu_3} R_{\mu_4 \mu_6 \mu_5 \alpha} \\
& - \frac{1}{15} R_{\mu_1 \mu_3 \mu_2}^\alpha R_{\mu_4 \mu_6 \mu_5 \alpha}
\end{aligned} \tag{B8}$$

$$[\nabla_\mu \Delta^{1/2}] = 0. \tag{B10}$$

2. Covariant derivatives of the VanVleck-Morette determinant

Now we turn to the coincidence limits of covariant derivatives acting on the VanVleck determinant. In this case, it is customary to derive such relations acting on the square root of Δ , whose defining equation can be read off from Eq. (14). By looking at Eq. (12) we realize that it has mass dimension zero, while from its geometrical interpretation we expect its coincidence limit to be given by unity, i.e.,

$$[\Delta^{1/2}] = 1. \tag{B9}$$

Thus, dimensional analysis tells us that

When acting with at least two covariant derivatives we find coincidence limits that depend on the curvature invariants. Indeed, the first nontrivial result is

$$[\nabla_\mu \nabla_\nu \Delta^{1/2}] = \frac{1}{6} R_{\mu\nu}. \tag{B11}$$

By applying three derivatives and employing the differential Bianchi identities we find

$$[\nabla_\mu \nabla_\nu \nabla_\rho \Delta^{1/2}] = \frac{1}{12} [\nabla_\rho R_{\mu\nu} + \nabla_\mu R_{\nu\rho} + \nabla_\nu R_{\mu\rho}]. \tag{B12}$$

Finally, for dimension-four terms we make use of computer algebra, and we obtain

$$\begin{aligned}
[\nabla_{\mu_1} \nabla_{\mu_2} \nabla_{\mu_3} \nabla_{\mu_4} \Delta^{1/2}] &= \frac{1}{36} R_{\mu_1 \mu_4} R_{\mu_2 \mu_3} + \frac{1}{36} R_{\mu_1 \mu_3} R_{\mu_2 \mu_4} \\
&+ \frac{1}{36} R_{\mu_1 \mu_2} R_{\mu_3 \mu_4} + \frac{1}{360} R_{\mu_4}^{\alpha} R_{\mu_1 \alpha \mu_2 \mu_3} + \frac{1}{360} R_{\mu_3}^{\alpha} R_{\mu_1 \alpha \mu_2 \mu_4} + \frac{7}{120} R_{\mu_4}^{\alpha} R_{\mu_1 \mu_2 \mu_3 \alpha} \\
&+ \frac{7}{120} R_{\mu_3}^{\alpha} R_{\mu_1 \mu_2 \mu_4 \alpha} + \frac{1}{360} R_{\mu_4}^{\alpha} R_{\mu_1 \mu_3 \mu_2 \alpha} + \frac{1}{180} R_{\mu_2}^{\alpha} R_{\mu_1 \mu_3 \mu_4 \alpha} + \frac{1}{360} R_{\mu_3}^{\alpha} R_{\mu_1 \mu_4 \mu_2 \alpha} \\
&+ \frac{1}{18} R_{\mu_2}^{\alpha} R_{\mu_1 \mu_4 \mu_3 \alpha} + \frac{1}{90} R_{\mu_1}^{\alpha} R_{\mu_4}^{\beta} R_{\mu_2 \alpha \mu_3 \beta} - \frac{1}{20} R_{\mu_1 \mu_4}^{\alpha \beta} R_{\mu_2 \alpha \mu_3 \beta} + \frac{1}{90} R_{\mu_1}^{\alpha} R_{\mu_3}^{\beta} R_{\mu_2 \alpha \mu_4 \beta} \\
&+ \frac{1}{90} R_{\mu_1}^{\alpha} R_{\mu_4}^{\beta} R_{\mu_2 \beta \mu_3 \alpha} + \frac{1}{90} R_{\mu_1}^{\alpha} R_{\mu_3}^{\beta} R_{\mu_2 \beta \mu_4 \alpha} + \frac{1}{40} R_{\mu_1 \mu_4}^{\alpha \beta} R_{\mu_2 \mu_3 \alpha \beta} + \frac{1}{180} R_{\mu_1}^{\alpha} R_{\mu_2 \mu_3 \mu_4 \alpha} \\
&- \frac{1}{20} R_{\mu_1}^{\alpha} R_{\mu_3}^{\beta} R_{\mu_2 \mu_4 \alpha \beta} + \frac{1}{40} R_{\mu_1 \mu_3}^{\alpha \beta} R_{\mu_2 \mu_4 \alpha \beta} + \frac{1}{18} R_{\mu_1}^{\alpha} R_{\mu_2 \mu_4 \mu_3 \alpha} + \frac{1}{90} R_{\mu_1}^{\alpha} R_{\mu_2}^{\beta} R_{\mu_3 \alpha \mu_4 \beta} \\
&+ \frac{1}{90} R_{\mu_1}^{\alpha} R_{\mu_2}^{\beta} R_{\mu_3 \beta \mu_4 \alpha}.
\end{aligned} \tag{B13}$$

3. Covariant derivatives of the Seeley-DeWitt coefficient

After having derived the coincidence limits of the Synge world function and VanVleck-Morette determinant up to dimension-four terms, we focus on the Seeley-DeWitt coefficients. The zeroth coefficient $\hat{a}_0$ is interpreted as a parallel displacement operator [9]. As such, its covariant Taylor expansion only depends on the pseudo-Riemannian structure and on the generalized curvature $\hat{\mathcal{R}}_{\mu\nu}$, and it cannot depend on the specific differential operator that we are dealing with. This geometrical interpretation and the initial conditions of the Cauchy system Eq. (8) yield

$$[\hat{a}_0] = \hat{1}. \tag{B14}$$

Accordingly, dimensional analysis directly tells us that

$$[\hat{\nabla}_{\mu} \hat{a}_0] = 0. \tag{B15}$$

By applying one more derivative and using Eqs. (B3) and (B4) we find the first nontrivial result

$$[\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} \hat{a}_0] = \frac{1}{2} \hat{\mathcal{R}}_{\mu\nu}. \tag{B16}$$

Acting with three derivatives on $\hat{a}_0$ and taking the coincidence limit we find

$$[\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} \hat{\nabla}_{\rho} \hat{a}_0] = \frac{2}{3} \hat{\nabla}_{(\mu} \hat{\mathcal{R}}_{\nu)\rho}. \tag{B17}$$

Finally, by considering the mass-dimension-four terms we have

$$\begin{aligned}
[\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} \hat{\nabla}_{\rho} \hat{\nabla}_{\sigma} \hat{a}_0] &= \frac{1}{8} [\hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}_{\rho\sigma} + \hat{\mathcal{R}}_{\mu\rho} \hat{\mathcal{R}}_{\nu\sigma} + \hat{\mathcal{R}}_{\mu\sigma} \hat{\mathcal{R}}_{\nu\rho} \\
&+ \hat{\mathcal{R}}_{\rho\sigma} \hat{\mathcal{R}}_{\mu\nu} + \hat{\mathcal{R}}_{\nu\sigma} \hat{\mathcal{R}}_{\mu\rho} + \hat{\mathcal{R}}_{\nu\rho} \hat{\mathcal{R}}_{\mu\sigma}] \\
&+ \frac{1}{6} [R^{\lambda}_{(\sigma\rho)\nu} \hat{\mathcal{R}}_{\mu\lambda} + R^{\lambda}_{(\sigma\rho)\mu} \hat{\mathcal{R}}_{\nu\lambda} \\
&+ R^{\lambda}_{(\sigma\nu)\mu} \hat{\mathcal{R}}_{\rho\lambda} + R^{\lambda}_{(\rho\mu)\nu} \hat{\mathcal{R}}_{\sigma\lambda}].
\end{aligned} \tag{B18}$$

Next, we take into account the coincidence limits of $\hat{a}_1$ and its derivatives up to dimension-four operators. By taking the coincidence limit of Eq. (17) with $n = 1$ and using Eq. (B11) we find

$$[\hat{a}_1] = \frac{1}{6} R \hat{1} + \hat{E}. \tag{B19}$$

Dimension-three terms arise by computing the covariant derivative of $\hat{a}_1$ and taking the coincidence limit, and they read

$$[\hat{\nabla}_{\mu} \hat{a}_1] = \frac{1}{12} \nabla_{\mu} R + \frac{1}{2} \hat{\nabla}_{\mu} \hat{E} + \frac{1}{6} \hat{\nabla}^{\lambda} \hat{\mathcal{R}}_{\mu\lambda}. \tag{B20}$$

Eventually, dimension-four terms arise when we act with one more derivative, and they are given by

$$\begin{aligned}
[\hat{\nabla}_{\mu} \hat{\nabla}_{\nu} \hat{a}_1] &= \frac{1}{90} [R^{\alpha\beta} R_{\alpha\mu\beta\nu} + R_{\alpha\beta\gamma\mu} R^{\alpha\beta\gamma}_{\nu} - 2 R_{\mu\alpha} R^{\alpha}_{\nu}] \hat{1} \\
&+ \frac{1}{12} R \hat{\mathcal{R}}_{\mu\nu} + \frac{1}{6} \hat{E} \hat{\mathcal{R}}_{\mu\nu} + \frac{1}{3} \hat{\mathcal{R}}_{\mu\nu} \hat{E} \\
&+ \frac{1}{6} \hat{\mathcal{R}}_{(\mu}^{\alpha} \hat{\mathcal{R}}_{\nu)\alpha},
\end{aligned} \tag{B21}$$

where we have discarded boundary terms.

Lastly, we present the coincidence limit of the $\hat{a}_2$ for the minimal operator with endomorphism $\hat{E}$,

$$[\hat{a}_2] = \frac{1}{180} \left(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - R^{\mu\nu} R_{\mu\nu} + \frac{5}{2} R \right) \hat{1} + \frac{1}{6} \hat{E} R + \frac{1}{2} \hat{E}^2 + \frac{1}{12} \hat{R}^{\mu\nu} \hat{R}_{\mu\nu}. \quad (\text{B22})$$

APPENDIX C: FINAL RESULT: THE REMAINING TERMS

1. Order V^1 terms

The terms that are linear in $\hat{V}$ which were not given in the main text Sec. III C are those that comprise either the endomorphism, or $\hat{M}$. Explicitly, they read

$$\begin{aligned} \text{tr } \hat{a}_2 \supset & \frac{1}{24} K^{\alpha\beta aa1} M_3^\mu{}_{a1}{}^{a3} V^\nu{}_{a3a} g^{(2)}{}_{\alpha\beta\mu\nu} + \frac{1}{24} K^\alpha{}_{\beta a}{}^{a1} V^\mu{}_{a3a} \hat{\nabla}^\alpha M_2^{\nu\rho a1a3} g^{(2)}{}_{\beta\mu\nu\rho} \\ & + \frac{1}{24} K^\alpha{}_{\beta a}{}^{a1} V^\mu{}_{a3a} \hat{\nabla}^\beta M_2^{\nu\rho a1a3} g^{(2)}{}_{\alpha\mu\nu\rho} + \frac{1}{24} K^{\alpha\beta a}{}_{a1} V_{\mu a3a} \hat{\nabla}^\mu M_2^{\nu\rho a1a3} g^{(2)}{}_{\alpha\beta\nu\rho} \\ & - \frac{1}{48} K^{\alpha\beta a}{}_{a1} K_\mu{}^\nu{}_{a3}{}^{a2} V^\rho{}_{a2a} \hat{\nabla}^\mu E^{a1a3} g^{(2)}{}_{\alpha\beta\nu\rho} + \frac{1}{48} K^{\alpha\beta}{}_a{}^{a1} K_\mu{}^\nu{}_{a2}{}^{a3} V^\rho{}_{a1a3} \hat{\nabla}^\mu E^{a2a} g^{(2)}{}_{\alpha\beta\nu\rho} \\ & - \frac{1}{48} K^{\alpha\beta a}{}_{a1} K^\mu{}_{\nu a3}{}^{a2} V^\rho{}_{a2a} \hat{\nabla}^\nu E^{a1a3} g^{(2)}{}_{\alpha\beta\mu\rho} + \frac{1}{48} K^{\alpha\beta}{}_a{}^{a1} K^\mu{}_{\nu a2}{}^{a3} V^\rho{}_{a1a3} \hat{\nabla}^\nu E^{a2a} g^{(2)}{}_{\alpha\beta\mu\rho} \\ & + \frac{1}{48} K^{\alpha\beta a}{}_{a1} K^{\mu\nu}{}_{a3}{}^{a2} V_{\rho a2a} \hat{\nabla}^\rho E^{a1a3} g^{(2)}{}_{\alpha\beta\mu\nu} - \frac{1}{48} K^{\alpha\beta a}{}_{a1} V^\mu{}_{a3a} \hat{\nabla}^\sigma M_2^{\nu\rho a1a3} g^{(3)}{}_{\alpha\beta\mu\nu\rho\sigma} \end{aligned} \quad (\text{C1})$$

2. Order V^2 terms

All the terms that are quadratic in $\hat{V}$ and linear in the Riemann tensor (or its contractions) yield the following contribution to the trace of the second Seeley-DeWitt coefficient:

$$\begin{aligned} \text{tr } \hat{a}_2 \supset & \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\beta\nu\mu\rho} V_{aa3a} V^\rho{}_{a1a2} + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\beta\rho\mu\nu} V_{aa3a} V^\rho{}_{a1a2} \\ & + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\alpha\nu\mu\rho} V_{\beta a3a} V^\rho{}_{a1a2} + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\alpha\rho\mu\nu} V_{\beta a3a} V^\rho{}_{a1a2} \\ & - \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\alpha\mu\beta\nu} V_{\rho a3a} V^\rho{}_{a1a2} - \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\alpha\nu\beta\mu} V_{\rho a3a} V^\rho{}_{a1a2} \\ & + \frac{1}{72} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\beta\mu\nu\rho} V_{aa1a2} V^\rho{}_{a3a} + \frac{1}{72} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\beta\nu\mu\rho} V_{aa1a2} V^\rho{}_{a3a} \\ & + \frac{1}{72} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\alpha\mu\nu\rho} V_{\beta a1a2} V^\rho{}_{a3a} + \frac{1}{72} K^{\alpha\beta aa1} K^{\mu\nu a2a3} R_{\alpha\nu\mu\rho} V_{\beta a1a2} V^\rho{}_{a3a} \\ & + \frac{1}{72} K^{\alpha\beta aa1} K_\beta{}^{\mu a2a3} R_{\alpha\mu\nu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} + \frac{1}{144} K^{\alpha\beta aa1} K^\mu{}_{\beta}{}^{a2a3} R_{\alpha\mu\nu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} \\ & + \frac{1}{72} K^{\alpha\beta aa1} K_\beta{}^{\mu a2a3} R_{\alpha\nu\mu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} + \frac{1}{144} K^{\alpha\beta aa1} K^\mu{}_{\beta}{}^{a2a3} R_{\alpha\nu\mu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} \\ & + \frac{1}{144} K_\alpha{}^{\mu a2a3} K^{\alpha\beta aa1} R_{\beta\mu\nu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} - \frac{1}{144} K^\alpha{}_{\alpha}{}^{aa1} K^{\beta\mu a2a3} R_{\beta\mu\nu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} \\ & + \frac{1}{144} K_\alpha{}^{\mu a2a3} K^{\alpha\beta aa1} R_{\beta\nu\mu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} - \frac{1}{144} K^\alpha{}_{\alpha}{}^{aa1} K^{\beta\mu a2a3} R_{\beta\nu\mu\rho} V^\nu{}_{a1a2} V^\rho{}_{a3a} \\ & + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\alpha\beta} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\mu\nu\rho\sigma} + \frac{1}{288} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\alpha\mu} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\beta\nu\rho\sigma} \\ & + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\alpha\nu} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\beta\mu\rho\sigma} + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\alpha\rho} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\beta\mu\nu\sigma} \\ & + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\alpha\sigma} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\beta\mu\nu\rho} + \frac{1}{288} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\beta\nu} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\alpha\mu\rho\sigma} \\ & + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\beta\rho} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\alpha\mu\nu\sigma} + \frac{1}{144} K^{\alpha\beta aa1} K^{\mu\nu a3a4} R_{\beta\sigma} V^\rho{}_{a1a3} V^\sigma{}_{a4a} g^{(2)}{}_{\alpha\mu\nu\rho} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{288} K^{\alpha\beta aa1} K^{\mu\nu a3 a4} R_{\rho\sigma} V^\rho{}_{a1 a3} V^\sigma{}_{a4 a} g^{(2)}{}_{\alpha\beta\mu\nu} + \frac{1}{96} K^{\alpha\beta aa1} K^{\mu\nu a3 a4} R_{\alpha}{}^{\lambda}{}_{\beta}{}^{\tau} V^\rho{}_{a1 a3} V^\sigma{}_{a4 a} g^{(3)}{}_{\mu\nu\rho\sigma\tau} \\
& - \frac{1}{192} K^{\alpha\beta aa1} K^{\mu\nu a3 a4} R_{\alpha}{}^{\lambda}{}_{\beta}{}^{\tau} V^\rho{}_{a1 a3} V^\sigma{}_{a4 a} g^{(3)}{}_{\beta\mu\nu\rho\sigma\lambda} - \frac{1}{192} K^{\alpha\beta aa1} K^{\mu\nu a3 a4} R_{\beta}{}^{\lambda}{}_{\alpha}{}^{\tau} V^\rho{}_{a1 a3} V^\sigma{}_{a4 a} g^{(3)}{}_{\alpha\mu\nu\rho\sigma\lambda} \\
& - \frac{1}{192} K^{\alpha\beta aa1} K^{\mu\nu a3 a4} R_{\rho}{}^{\lambda}{}_{\alpha}{}^{\tau} V^\rho{}_{a1 a3} V^\sigma{}_{a4 a} g^{(3)}{}_{\alpha\beta\mu\nu\sigma\lambda} - \frac{1}{576} K^{\alpha\beta aa1} K^{\mu\nu a3 a4} R V^\rho{}_{a1 a3} V^\sigma{}_{a4 a} g^{(3)}{}_{\alpha\beta\mu\nu\rho\sigma}. \quad (C2)
\end{aligned}$$

On the other hand, the terms quadratic in $\hat{V}$ and linear in the generalized curvature $\hat{\mathcal{R}}_{\mu\nu}$ are

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & -\frac{1}{96} \mathcal{R}_{\rho\sigma a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\mu\nu} - \frac{1}{96} \mathcal{R}_{\nu\sigma a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\mu\rho} \\
& - \frac{1}{96} \mathcal{R}_{\nu\rho a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\mu\sigma} - \frac{1}{96} \mathcal{R}_{\mu\sigma a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\nu\rho} \\
& - \frac{1}{96} \mathcal{R}_{\mu\rho a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\nu\sigma} + \frac{1}{96} \mathcal{R}_{\mu\nu a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\rho\sigma} \\
& - \frac{1}{96} \mathcal{R}_{\beta\sigma a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\mu\nu\rho} + \frac{1}{96} \mathcal{R}_{\beta\rho a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\mu\nu\sigma} \\
& + \frac{1}{96} \mathcal{R}_{\beta\nu a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\mu\rho\sigma} + \frac{1}{96} \mathcal{R}_{\beta\mu a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\nu\rho\sigma} \\
& - \frac{1}{96} \mathcal{R}_{\alpha\sigma a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\beta\mu\nu\rho} + \frac{1}{96} \mathcal{R}_{\alpha\rho a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\beta\mu\nu\sigma} \\
& + \frac{1}{96} \mathcal{R}_{\alpha\nu a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\beta\mu\rho\sigma} + \frac{1}{96} \mathcal{R}_{\alpha\mu a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\beta\nu\rho\sigma} \\
& + \frac{1}{96} \mathcal{R}_{\alpha\beta a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(2)}{}_{\mu\nu\rho\sigma} + \frac{1}{96} \mathcal{R}_{\mu}{}^{\lambda}{}_{a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(3)}{}_{\alpha\beta\nu\rho\sigma\lambda} \\
& + \frac{1}{96} \mathcal{R}_{\sigma}{}^{\lambda}{}_{a3 a} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(3)}{}_{\alpha\beta\mu\nu\rho\lambda} + \frac{1}{96} \mathcal{R}_{\nu}{}^{\lambda}{}_{a4 a3} K^{\alpha\beta aa1} K^{\mu\nu a2 a4} V^\rho{}_{a1 a2} V^{\sigma a3}{}_a g^{(3)}{}_{\alpha\beta\mu\rho\sigma\lambda}. \quad (C3)
\end{aligned}$$

Finally, the contributions proportional to two covariant derivatives acting on $\hat{V}$ are given by

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \frac{1}{48} K^{\alpha\beta}{}_a{}^{a1} K_{\mu\nu}{}^{a2}{}_{a3} V^\rho{}_{a1 a2} \hat{\nabla}^\mu \hat{\nabla}^\nu V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\rho\sigma} + \frac{1}{48} K^{\alpha\beta}{}_a{}^{a1} K_{\mu}{}^{\nu a2}{}_{a3} V_{\rho a1 a2} \hat{\nabla}^\rho \hat{\nabla}^\mu V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\nu\sigma} \\
& + \frac{1}{48} K^{\alpha\beta}{}_a{}^{a1} K_{\nu}{}^{\mu a2}{}_{a3} V_{\rho a1 a2} \hat{\nabla}^\rho \hat{\nabla}^\nu V^{\sigma a3}{}_a g^{(2)}{}_{\alpha\beta\mu\sigma} - \frac{1}{96} K^{\alpha\beta}{}_a{}^{a1} K_{\mu}{}^{\nu a2}{}_{a3} V^\rho{}_{a1 a2} \hat{\nabla}^\mu \hat{\nabla}^\lambda V^{\sigma a3}{}_a g^{(3)}{}_{\alpha\beta\nu\rho\sigma\lambda} \\
& - \frac{1}{96} K^{\alpha\beta}{}_a{}^{a1} K_{\nu}{}^{\mu a2}{}_{a3} V^\rho{}_{a1 a2} \hat{\nabla}^\nu \hat{\nabla}^\lambda V^{\sigma a3}{}_a g^{(3)}{}_{\alpha\beta\mu\rho\sigma\lambda} - \frac{1}{96} K^{\alpha\beta}{}_a{}^{a1} K_{\mu\nu}{}^{a2}{}_{a3} V_{\rho a3 a} \hat{\nabla}^\rho \hat{\nabla}^\lambda V^{\sigma a1 a2} g^{(3)}{}_{\alpha\beta\mu\nu\sigma\lambda} \\
& - \frac{1}{192} K^{\alpha\beta}{}_a{}^{a1} K_{\mu\nu}{}^{a2}{}_{a3 a} V^\rho{}_{a3 a} \hat{\nabla}^\lambda \hat{\nabla}_\lambda V^{\sigma a1 a2} g^{(3)}{}_{\alpha\beta\mu\nu\rho\sigma} + \frac{1}{320} K^{\alpha\beta}{}_a{}^{a1} K_{\mu\nu}{}^{a2}{}_{a3 a} \hat{\nabla}^\tau \hat{\nabla}^\lambda V^{\sigma a1 a2} g^{(4)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\tau}. \quad (C4)
\end{aligned}$$

3. Integrand: The general case

Here we present the full result for the modification of the local part of $\text{tr } \hat{a}_2$ due to the principal part of the operator in the most general case, whose derivation has been outlined in Sec. III B. Since many independent scalar contractions are present, we split this result into multiple pieces. Furthermore, we denote the ζ derivative of the nonminimal symbol $\frac{d\hat{N}}{d\zeta}$ simply by $d\hat{N}$ for notational convenience.

We start by taking into account the terms that involve the generalized curvature $\hat{\mathcal{R}}_{\mu\nu}$. The contribution which is quadratic in the latter is given by

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \left[\frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\beta}{}^{\nu a2}{}_a \mathcal{R}_{\mu\nu a3 a2} K_{\alpha}{}^{\mu}{}_{a1}{}^{a3} + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\beta}{}^{\nu}{}_{a3}{}^{a2} \mathcal{R}_{\mu\nu a2 a} K_{\alpha}{}^{\mu}{}_{a1}{}^{a3} \right. \\
& \left. + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha}{}^{\nu a2}{}_a \mathcal{R}_{\mu\nu a3 a2} K_{\beta}{}^{\mu}{}_{a1}{}^{a3} + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha}{}^{\nu}{}_{a3}{}^{a2} \mathcal{R}_{\mu\nu a2 a} K_{\beta}{}^{\mu}{}_{a1}{}^{a3} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{24} dN^\alpha_{\alpha}{}^{aa1} \mathcal{R}_\beta{}^{\nu a2}{}_a \mathcal{R}_{\mu\nu a3a2} K^{\beta\mu}{}_{a1}{}^{a3} + \frac{1}{24} dN^\alpha_{\alpha}{}^{aa1} \mathcal{R}_\beta{}^{\nu}{}_{a3}{}^{a2} \mathcal{R}_{\mu\nu a2a} K^{\beta\mu}{}_{a1}{}^{a3} \\
& + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_\beta{}^{\nu a2}{}_a \mathcal{R}_{\mu\nu a3a2} K^\mu{}_{aa1}{}^{a3} + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_\beta{}^{\nu}{}_{a3}{}^{a2} \mathcal{R}_{\mu\nu a2a} K^\mu{}_{aa1}{}^{a3} \\
& + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_\alpha{}^{\nu a2}{}_a \mathcal{R}_{\mu\nu a3a2} K^\mu{}_{\beta a1}{}^{a3} + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_\alpha{}^{\nu}{}_{a3}{}^{a2} \mathcal{R}_{\mu\nu a2a} K^\mu{}_{\beta a1}{}^{a3} \\
& + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_\alpha{}^{\nu a2}{}_a \mathcal{R}_{\beta\nu a3a2} K^\mu{}_{\mu a1}{}^{a3} + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_\alpha{}^{\nu}{}_{a3}{}^{a2} \mathcal{R}_{\beta\nu a2a} K^\mu{}_{\mu a1}{}^{a3} \\
& - \frac{1}{8} dN^{\alpha\beta aa1} \mathcal{R}_{a\nu}{}^{a2}{}_a \mathcal{R}_{\beta\mu a3a2} K^{\mu\nu}{}_{a1}{}^{a3} - \frac{1}{8} dN^{\alpha\beta aa1} \mathcal{R}_{a\nu a3}{}^{a2} \mathcal{R}_{\beta\mu a2a} K^{\mu\nu}{}_{a1}{}^{a3} \\
& - \frac{1}{8} dN^{\alpha\beta aa1} \mathcal{R}_{a\mu}{}^{a2}{}_a \mathcal{R}_{\beta\nu a3a2} K^{\mu\nu}{}_{a1}{}^{a3} - \frac{1}{8} dN^{\alpha\beta aa1} \mathcal{R}_{a\mu a3}{}^{a2} \mathcal{R}_{\beta\nu a2a} K^{\mu\nu}{}_{a1}{}^{a3} \\
& - \frac{1}{8} dN^{\alpha\beta aa1} \mathcal{R}_{a\beta}{}^{a2}{}_a \mathcal{R}_{\mu\nu a3a2} K^{\mu\nu}{}_{a1}{}^{a3} - \frac{1}{8} dN^{\alpha\beta aa1} \mathcal{R}_{a\beta a3}{}^{a2} \mathcal{R}_{\mu\nu a2a} K^{\mu\nu}{}_{a1}{}^{a3} \\
& - \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\rho\sigma a2a} \mathcal{R}^{\rho\sigma}{}_{a3}{}^{a2} K^{\mu\nu}{}_{a1}{}^{a3} g^{(2)}{}_{\alpha\beta\mu\nu} \Big]. \tag{C5}
\end{aligned}$$

Next, we consider the terms that are linear in both the generalized curvature and the Riemann tensor (or its contractions). These are

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \Big[-\frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_{\mu\nu a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\alpha\beta} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\nu a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\alpha\mu} \\
& - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\mu a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{a\nu} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_{a\nu a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\beta\mu} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\mu a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\beta\nu} \\
& - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\beta a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\mu\nu} + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\mu a2a} K^\mu{}_{\alpha}{}^{a1}{}^{a2} R + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\mu a2a} K^\mu{}_{\beta}{}^{a1}{}^{a2} R \\
& + \frac{1}{24} dN^\alpha_{\alpha}{}^{aa1} \mathcal{R}_{\beta\mu a2a} K^{\beta\mu}{}_{a1}{}^{a2} R + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\mu a2a} K^\mu{}_{aa1}{}^{a2} R + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\mu a2a} K^\mu{}_{\beta a1}{}^{a2} R \\
& + \frac{1}{24} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\beta a2a} K^\mu{}_{\mu a1}{}^{a2} R + \frac{1}{6} dN^{\alpha\beta aa1} \mathcal{R}_\nu{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\alpha\beta\mu\rho} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\mu{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\alpha\beta\nu\rho} \\
& - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\nu{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\alpha\mu\beta\rho} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\beta{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\alpha\mu\nu\rho} \\
& - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\mu{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{a\nu\beta\rho} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\beta{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{a\nu\mu\rho} \\
& - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\alpha{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\beta\mu\nu\rho} - \frac{1}{12} dN^{\alpha\beta aa1} \mathcal{R}_\alpha{}^\rho{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a2} R_{\beta\nu\mu\rho} \Big]. \tag{C6}
\end{aligned}$$

Finally, we present the part of the modification to the trace of $\hat{a}_2$ which depends on the dimension-two tensor $\hat{M}_2$ and $\hat{\mathcal{R}}_{\mu\nu}$, which read

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \Big[\frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\rho\sigma a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\beta\mu\nu} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\nu\sigma a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\beta\mu\rho} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\nu\rho a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\beta\mu\sigma} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\mu\sigma a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\beta\nu\rho} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\mu\rho a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\beta\nu\sigma} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\mu\nu a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\beta\rho\sigma} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\sigma a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\mu\nu\rho} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\rho a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\mu\nu\sigma} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\nu a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\mu\rho\sigma} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\beta\mu a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2{}^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\alpha\nu\rho\sigma} \Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\sigma a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\beta\mu\nu\rho} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\rho a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\beta\mu\nu\sigma} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\nu a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\beta\mu\rho\sigma} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\mu a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\beta\nu\rho\sigma} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\alpha\beta a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(2)}{}_{\mu\nu\rho\sigma} + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\sigma}{}^{\lambda}{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(3)}{}_{\alpha\beta\mu\nu\rho\lambda} \\
& + \frac{1}{48} dN^{\alpha\beta aa1} \mathcal{R}_{\rho}{}^{\lambda}{}_{a2a} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} g^{(3)}{}_{\alpha\beta\mu\nu\sigma\lambda} \Big]. \tag{C7}
\end{aligned}$$

The part of the result that depends quadratically on the dimension-two tensor $\hat{M}_2$ has the following simple form:

$$\text{tr } \hat{a}_2 \supset -\frac{1}{1920} \int_0^1 d\zeta dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a3} M_2^{\rho\sigma}{}_{a3}{}^{a2} M_2^{\lambda\tau}{}_{a2a} g^{(4)}{}_{\alpha\beta\mu\nu\rho\sigma\lambda\tau}. \tag{C8}$$

On the other hand, the number of terms that involve both $\hat{M}_2$ and the Riemann tensor (or its traces) is huge. Thus, we further split them into two parts, the first of which features only the Ricci tensor and scalar, i.e.,

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \Bigg[\frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\alpha\beta} g^{(2)}{}_{\mu\nu\rho\sigma} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\alpha\mu} g^{(2)}{}_{\beta\nu\rho\sigma} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\alpha\nu} g^{(2)}{}_{\beta\mu\rho\sigma} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\alpha\rho} g^{(2)}{}_{\beta\mu\nu\sigma} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\alpha\sigma} g^{(2)}{}_{\beta\mu\nu\rho} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\beta\mu} g^{(2)}{}_{\alpha\nu\rho\sigma} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\beta\nu} g^{(2)}{}_{\alpha\mu\rho\sigma} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\beta\rho} g^{(2)}{}_{\alpha\mu\nu\sigma} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\beta\sigma} g^{(2)}{}_{\alpha\mu\nu\rho} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\mu\nu} g^{(2)}{}_{\alpha\beta\rho\sigma} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\mu\rho} g^{(2)}{}_{\alpha\beta\nu\sigma} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\mu\sigma} g^{(2)}{}_{\alpha\beta\nu\rho} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\nu\rho} g^{(2)}{}_{\alpha\beta\mu\sigma} + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\nu\sigma} g^{(2)}{}_{\alpha\beta\mu\rho} \\
& + \frac{1}{144} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\rho\sigma} g^{(2)}{}_{\alpha\beta\mu\nu} - \frac{1}{96} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\rho}{}^{\lambda} g^{(3)}{}_{\alpha\beta\mu\nu\sigma\lambda} \\
& - \frac{1}{96} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R_{\sigma}{}^{\lambda} g^{(3)}{}_{\alpha\beta\mu\nu\rho\lambda} - \frac{1}{288} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho\sigma}{}_{a2a} R g^{(3)}{}_{\alpha\beta\mu\nu\rho\sigma} \Bigg], \tag{C9}
\end{aligned}$$

while those that involve the Riemann tensor explicitly are given by

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \Bigg[-\frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{\rho a2a} R_{\alpha\mu\beta\nu} - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{2\nu a2a} R_{\alpha\mu\beta\rho} \\
& - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{\nu a2a} R_{\alpha\mu\beta\rho} - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{\rho a2a} R_{\alpha\nu\beta\mu} \\
& - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{a2a} R_{\alpha\nu\beta\rho} - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu}{}_{\mu a1}{}^{a2} M_2^{\nu\rho}{}_{a2a} R_{\alpha\nu\beta\rho} \\
& - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{\mu a2a} R_{\alpha\nu\beta\rho} - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{a2a} R_{\alpha\nu\mu\rho} \\
& - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu}{}_{\beta a1}{}^{a2} M_2^{\nu\rho}{}_{a2a} R_{\alpha\nu\mu\rho} - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu}{}_{\beta a1}{}^{a2} M_2^{\nu\rho}{}_{a2a} R_{\alpha\nu\mu\rho} \\
& - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{\beta a2a} R_{\alpha\nu\mu\rho} - \frac{1}{72} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1}{}^{a2} M_2^{\rho}{}_{2\nu a2a} R_{\alpha\rho\beta\mu} \Bigg]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2\nu a2a}{}^{\rho}R_{\alpha\rho\beta\mu}-\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2\mu}{}^{\rho}{}_{a2a}R_{\alpha\rho\beta\nu} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{\mu a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\alpha\rho\beta\nu}-\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2}{}^{\rho}{}_{\mu a2a}R_{\alpha\rho\beta\nu} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2\beta}{}^{\rho}{}_{a2a}R_{\alpha\rho\mu\nu}-\frac{1}{72}dN^{\alpha\beta aa1}K_{\beta}{}^{\mu}{}_{a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\alpha\rho\mu\nu} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{\beta a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\alpha\rho\mu\nu}-\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2}{}^{\rho}{}_{\beta a2a}R_{\alpha\rho\mu\nu} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2\alpha}{}^{\rho}{}_{a2a}R_{\beta\nu\mu\rho}-\frac{1}{72}dN^{\alpha\beta aa1}K_{\alpha}{}^{\mu}{}_{a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\beta\nu\mu\rho} \\
& -\frac{1}{72}dN^{\alpha}{}_{\alpha}{}^{aa1}K^{\beta\mu}{}_{a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\beta\nu\mu\rho}-\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{aa1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\beta\nu\mu\rho} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2}{}^{\rho}{}_{aa2a}R_{\beta\nu\mu\rho}-\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2\alpha}{}^{\rho}{}_{a2a}R_{\beta\rho\mu\nu} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K_{\alpha}{}^{\mu}{}_{a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\beta\rho\mu\nu}-\frac{1}{72}dN^{\alpha}{}_{\alpha}{}^{aa1}K^{\beta\mu}{}_{a1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\beta\rho\mu\nu} \\
& -\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{aa1}{}^{a2}M_{2}{}^{\nu\rho}{}_{a2a}R_{\beta\rho\mu\nu}-\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2}{}^{\rho}{}_{aa2a}R_{\beta\rho\mu\nu} \\
& +\frac{1}{48}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1}{}^{a2}M_{2}{}^{\rho\sigma}{}_{a2a}R_{\rho}{}^{\lambda}{}_{\sigma}{}^{\tau}g^{(3)}_{\alpha\beta\mu\lambda\tau}\Big]. \tag{C10}
\end{aligned}$$

We are left only with those terms that depend quadratically on the pseudo-Riemannian curvatures. Since many tensor structures arise, we further split them in two. Thus, we first write down those terms that do not explicitly depend on the Riemann tensor,

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \Big[-\frac{1}{36}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha\nu}R_{\beta\mu}-\frac{1}{36}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha\mu}R_{\beta\nu} \\
& -\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu}{}_{\mu a1a}R_{\alpha}{}^{\nu}R_{\beta\nu}-\frac{1}{36}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha\beta}R_{\mu\nu}-\frac{1}{90}dN^{\alpha\beta aa1}K_{\beta}{}^{\mu}{}_{a1a}R_{\alpha}{}^{\nu}R_{\mu\nu} \\
& -\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu}{}_{\beta a1a}R_{\alpha}{}^{\nu}R_{\mu\nu}-\frac{1}{90}dN^{\alpha\beta aa1}K_{\alpha}{}^{\mu}{}_{a1a}R_{\beta}{}^{\nu}R_{\mu\nu}-\frac{1}{90}dN^{\alpha}{}_{\alpha}{}^{aa1}K^{\beta\mu}{}_{a1a}R_{\beta}{}^{\nu}R_{\mu\nu} \\
& -\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu}{}_{aa1a}R_{\beta}{}^{\nu}R_{\mu\nu}+\frac{1}{720}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\rho\sigma}R^{\rho\sigma}g^{(2)}_{\alpha\beta\mu\nu}+\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{\mu a1a}R_{\alpha\beta}R \\
& +\frac{1}{72}dN^{\alpha\beta aa1}K_{\beta}{}^{\mu}{}_{a1a}R_{\alpha\mu}R+\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{\beta a1a}R_{\alpha\mu}R+\frac{1}{72}dN^{\alpha\beta aa1}K_{\alpha}{}^{\mu}{}_{a1a}R_{\beta\mu}R \\
& +\frac{1}{72}dN^{\alpha}{}_{\alpha}{}^{aa1}K^{\beta\mu}{}_{a1a}R_{\beta\mu}R+\frac{1}{72}dN^{\alpha\beta aa1}K^{\mu}{}_{aa1a}R_{\beta\mu}R-\frac{1}{288}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R^2g^{(2)}_{\alpha\beta\mu\nu}\Big]. \tag{C11}
\end{aligned}$$

On the other hand, the terms that only depend on the Riemann tensor are

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \Big[-\frac{1}{40}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha\nu}{}^{\rho\sigma}R_{\beta\mu\rho\sigma}-\frac{1}{40}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha\mu}{}^{\rho\sigma}R_{\beta\nu\rho\sigma} \\
& +\frac{1}{180}dN^{\alpha\beta aa1}K^{\mu}{}_{\mu a1a}R_{\alpha}{}^{\nu\rho\sigma}R_{\beta\nu\rho\sigma}+\frac{1}{20}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\mu}{}^{\sigma}R_{\beta\nu\rho\sigma}+\frac{1}{20}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha\nu}{}^{\rho\sigma}R_{\beta\rho\mu\sigma} \\
& -\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\nu}{}^{\sigma}R_{\beta\rho\mu\sigma}-\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\mu}{}^{\sigma}R_{\beta\rho\nu\sigma}-\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\nu}{}^{\sigma}R_{\beta\sigma\mu\rho} \\
& -\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\mu}{}^{\sigma}R_{\beta\sigma\nu\rho}+\frac{1}{180}dN^{\alpha\beta aa1}K_{\beta}{}^{\mu}{}_{a1a}R_{\alpha}{}^{\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\frac{1}{180}dN^{\alpha\beta aa1}K^{\mu}{}_{\beta a1a}R_{\alpha}{}^{\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\
& +\frac{1}{180}dN^{\alpha\beta aa1}K_{\alpha}{}^{\mu}{}_{a1a}R_{\beta}{}^{\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\frac{1}{180}dN^{\alpha}{}_{\alpha}{}^{aa1}K^{\beta\mu}{}_{a1a}R_{\beta}{}^{\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\frac{1}{180}dN^{\alpha\beta aa1}K^{\mu}{}_{aa1a}R_{\beta}{}^{\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\
& -\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma}R_{\mu\rho\nu\sigma}-\frac{1}{90}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma}R_{\mu\sigma\nu\rho}-\frac{1}{720}dN^{\alpha\beta aa1}K^{\mu\nu}{}_{a1a}R_{\rho\sigma\lambda\tau}R^{\rho\sigma\lambda\tau}g^{(2)}_{\alpha\beta\mu\nu}\Big]. \tag{C12}
\end{aligned}$$

Finally, the terms that display both the Riemann tensor and its contractions are

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \left[-\frac{7}{120} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\nu{}^\rho R_{\alpha\beta\mu\rho} - \frac{7}{120} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\mu{}^\rho R_{\alpha\beta\nu\rho} \right. \\
& - \frac{1}{36} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R R_{\alpha\mu\beta\nu} - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\nu{}^\rho R_{\alpha\mu\beta\rho} - \frac{1}{180} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\beta{}^\rho R_{\alpha\mu\nu\rho} \\
& - \frac{1}{36} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R R_{\alpha\nu\beta\mu} - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\mu{}^\rho R_{\alpha\nu\beta\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\mu a1a} R^{\nu\rho} R_{\alpha\nu\beta\rho} \\
& - \frac{1}{18} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\beta{}^\rho R_{\alpha\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K_{\beta}{}^\mu{}_{a1a} R^{\nu\rho} R_{\alpha\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\beta a1a} R^{\nu\rho} R_{\alpha\nu\mu\rho} \\
& - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\nu{}^\rho R_{\alpha\rho\beta\mu} - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\mu{}^\rho R_{\alpha\rho\beta\nu} - \frac{1}{180} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho R_{\beta\mu\nu\rho} \\
& - \frac{1}{18} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho R_{\beta\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K_\alpha{}^\mu{}_{a1a} R^{\nu\rho} R_{\beta\nu\mu\rho} + \frac{1}{180} dN^\alpha{}_{\alpha}{}^{aa1} K^{\beta\mu}{}_{a1a} R^{\nu\rho} R_{\beta\nu\mu\rho} \\
& \left. + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\alpha a1a} R^{\nu\rho} R_{\beta\nu\mu\rho} \right]. \tag{C13}
\end{aligned}$$

$$\begin{aligned}
\text{tr } \hat{a}_2 \supset & \int_0^1 d\zeta \left[-\frac{1}{36} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\alpha\nu} R_{\beta\mu} - \frac{1}{36} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\alpha\mu} R_{\beta\nu} \right. \\
& - \frac{1}{90} dN^{\alpha\beta aa1} K^\mu{}_{\mu a1a} R_\alpha{}^\nu R_{\beta\nu} - \frac{1}{36} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\alpha\beta} R_{\mu\nu} - \frac{1}{90} dN^{\alpha\beta aa1} K_{\beta}{}^\mu{}_{a1a} R_\alpha{}^\nu R_{\mu\nu} \\
& - \frac{1}{90} dN^{\alpha\beta aa1} K^\mu{}_{\beta a1a} R_\alpha{}^\nu R_{\mu\nu} - \frac{1}{90} dN^{\alpha\beta aa1} K_\alpha{}^\mu{}_{a1a} R_\beta{}^\nu R_{\mu\nu} - \frac{1}{90} dN^\alpha{}_{\alpha}{}^{aa1} K^{\beta\mu}{}_{a1a} R_\beta{}^\nu R_{\mu\nu} \\
& - \frac{1}{90} dN^{\alpha\beta aa1} K^\mu{}_{\alpha a1a} R_\beta{}^\nu R_{\mu\nu} + \frac{1}{720} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\rho\sigma} R^{\rho\sigma} g^{(2)}{}_{\alpha\beta\mu\nu} - \frac{1}{288} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R^2 g^{(2)}{}_{\alpha\beta\mu\nu} \\
& - \frac{7}{120} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\nu{}^\rho R_{\alpha\beta\mu\rho} - \frac{7}{120} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\mu{}^\rho R_{\alpha\beta\nu\rho} - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\nu{}^\rho R_{\alpha\mu\beta\rho} \\
& - \frac{1}{180} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\beta{}^\rho R_{\alpha\mu\nu\rho} - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\mu{}^\rho R_{\alpha\nu\beta\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\mu a1a} R^{\nu\rho} R_{\alpha\nu\beta\rho} \\
& - \frac{1}{18} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\beta{}^\rho R_{\alpha\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K_{\beta}{}^\mu{}_{a1a} R^{\nu\rho} R_{\alpha\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\beta a1a} R^{\nu\rho} R_{\alpha\nu\mu\rho} \\
& - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\nu{}^\rho R_{\alpha\rho\beta\mu} - \frac{1}{360} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\mu{}^\rho R_{\alpha\rho\beta\nu} - \frac{1}{180} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho R_{\beta\mu\nu\rho} \\
& - \frac{1}{40} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\alpha\nu}{}^\rho R_{\beta\mu\rho\sigma} - \frac{1}{18} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho R_{\beta\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K_\alpha{}^\mu{}_{a1a} R^{\nu\rho} R_{\beta\nu\mu\rho} \\
& + \frac{1}{180} dN^\alpha{}_{\alpha}{}^{aa1} K^{\beta\mu}{}_{a1a} R^{\nu\rho} R_{\beta\nu\mu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\alpha a1a} R^{\nu\rho} R_{\beta\nu\mu\rho} - \frac{1}{40} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\alpha\mu}{}^\rho R_{\beta\nu\rho\sigma} \\
& + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\mu a1a} R_\alpha{}^\nu{}^\rho R_{\beta\nu\rho\sigma} + \frac{1}{20} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\beta\nu\rho\sigma} + \frac{1}{20} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\alpha\nu}{}^\rho{}^\sigma R_{\beta\rho\mu\sigma} \\
& - \frac{1}{90} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\beta\rho\mu\sigma} - \frac{1}{90} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\beta\rho\nu\sigma} - \frac{1}{90} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\beta\sigma\mu\rho} \\
& - \frac{1}{90} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\beta\sigma\nu\rho} + \frac{1}{180} dN^{\alpha\beta aa1} K_{\beta}{}^\mu{}_{a1a} R_\alpha{}^\nu{}^\rho R_{\mu\nu\rho\sigma} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\beta a1a} R_\alpha{}^\nu{}^\rho R_{\mu\nu\rho\sigma} \\
& + \frac{1}{180} dN^{\alpha\beta aa1} K_\alpha{}^\mu{}_{a1a} R_\beta{}^\nu{}^\rho R_{\mu\nu\rho\sigma} + \frac{1}{180} dN^\alpha{}_{\alpha}{}^{aa1} K^{\beta\mu}{}_{a1a} R_\beta{}^\nu{}^\rho R_{\mu\nu\rho\sigma} + \frac{1}{180} dN^{\alpha\beta aa1} K^\mu{}_{\alpha a1a} R_\beta{}^\nu{}^\rho R_{\mu\nu\rho\sigma} \\
& - \frac{1}{90} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\mu\nu\rho\sigma} - \frac{1}{90} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_\alpha{}^\rho{}^\sigma R_{\mu\sigma\nu\rho} \\
& \left. - \frac{1}{720} dN^{\alpha\beta aa1} K^{\mu\nu}{}_{a1a} R_{\rho\sigma\lambda\tau} R^{\rho\sigma\lambda\tau} g^{(2)}{}_{\alpha\beta\mu\nu} \right]. \tag{C14}
\end{aligned}$$

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