

Probing quantum entanglement with generalized parton distributions at the Electron-Ion Collider

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Within the collinear factorization framework based on generalized parton distributions, we calculate the spin density matrix of exclusively produced quark and antiquark pairs $u\bar{u}$, $d\bar{d}$, $s\bar{s}$, $c\bar{c}$, $b\bar{b}$ in electron-proton scattering. The presence of both real and imaginary parts in the scattering amplitudes leads to a rich pattern of entanglement between the quark and the antiquark. We map out kinematical regions where the pairs exhibit entanglement, Bell nonlocality, and nonstabilizerness. We also predict that massive quarks and antiquarks are transversely polarized, similar to the well-known transverse hyperon polarization in unpolarized collisions. In strangeness, charm, and bottom productions, the polarization can reach 50%–80% in certain kinematic regions in the low-energy runs of the Electron-Ion Collider.

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I. INTRODUCTION

Entanglement is a quintessential feature of quantum mechanics that requires the coherent description of a composite quantum system as a whole. While it manifests across vastly different scales in both system size and particle number, historically stringent experimental tests of entanglement have primarily focused on the spin or polarization of elementary particle pairs. A classic example is the entanglement of photon polarizations in the (para)positronium decay $\text{Ps} \rightarrow 2\gamma$ [1]. More definitive tests, including the violation of Bell's inequality [2] or the Clauser-Horne-Shimony-Holt (CHSH) inequality, also involve two-photon final states [3–5]. Photons are advantageous because their polarizations can be measured relatively easily in tabletop or small-scale laboratory experiments.

Entanglement is also ubiquitous among the other elementary particles of the Standard Model that are produced in large-scale collider experiments. However, its direct measurement is much more challenging for a number of reasons. First, many redundant particles are produced in high-energy collisions and they may interfere with the particles of interest. Additionally, if the strong interaction is involved, entanglement generated at the partonic level can be easily washed out by the hadronization process. Moreover, the polarization of final state particles is often difficult to measure. Under these

circumstances, in practice, one can only infer entanglement between a pair of heavy, short-lived particles by measuring the angular distribution of their weak decay products. The first experimental confirmation of entanglement in the top-antitop ($t\bar{t}$) system, as predicted by theory [6–10], has been achieved by the ATLAS and CMS Collaborations at the LHC [11–13] (see also an earlier attempt [14]). Expectations are high [15,16] that similar measurements can be done for other particle species such as W^+W^- [17,18], $\tau^+\tau^-$ [19,20], and $b\bar{b}$ [21,22] pairs, etc., and even light quark-antiquark pairs [23]. Moreover, a violation of the Bell-CHSH inequality, or “Bell nonlocality” could also be detected in these systems [16,24,25], although it adds another layer of practical challenges.¹

In this paper, we continue the discussion of spin-spin entanglement between quark-antiquark ($q\bar{q}$) pairs produced in exclusive deep inelastic scattering (DIS) and ultraperipheral collisions (UPCs) along the lines of [29,30]. Our work is particularly motivated by the future high-luminosity DIS experiment at the Electron-Ion Collider (EIC) [31]. Since an electron is involved in the initial state, DIS offers a cleaner (i.e., less produced particles) environment for studying entanglement than in hadron-hadron collisions. One can realize an even cleaner environment, akin to the final states of e^+e^- annihilation, by focusing on exclusive (diffractive) processes where the target proton does not break up. In [30], the spin density matrix of this process has

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¹The recent controversy (e.g., [26–28]) prompts us to comment that, by Bell nonlocality, we mean certain properties of spin density matrices computed within QCD (see Sec. IV.B). It is not meant to be another test of quantum mechanics or the (non) existence of local hidden variable theories [28].

been computed in the Regge limit where the collision energy is asymptotically large and only the color-singlet gluonic exchange (“Pomeron”) needs to be considered. It has been found that the $q\bar{q}$ pairs are always entangled and always exhibit Bell nonlocality. On the other hand, the EIC runs with variable electron-proton $e + p$ center-of-mass (c.m.) energies in the range $28 < \sqrt{s} < 140$ GeV, which only partially overlaps with the Regge regime. A solid theoretical framework to describe exclusive processes in this window is the generalized parton distribution (GPD) [32,33]. So far, the study of GPDs has been mostly centered around the goal of obtaining the three-dimensional tomographic picture of the nucleon. Here, for the first time, we propose to use GPDs as a tool to probe quantum entanglement at the EIC. We will be particularly interested in whether entanglement and Bell nonlocality are generic features of color-singlet exchanges in QCD or they are emergent phenomena in the high-energy limit. We also evaluate the so-called nonstabilizerness [34,35], a necessary nonclassical resource for achieving quantum advantage in quantum computing.

As a by-product, we find nonvanishing polarization of massive $q\bar{q}$ pairs, even though the colliding particles are unpolarized. This is a kind of single spin asymmetry and is analogous to the transverse polarization of hyperons in unpolarized proton-proton collisions and semi-inclusive DIS [36–39]. We demonstrate that the polarization can be quite sizable, exceeding 60% in certain kinematical regions.

II. EXCLUSIVE DIJET CROSS SECTION

In this section, we review the exclusive dijet cross section calculated in the collinear factorization framework [40]. This sets the stage for the calculation of the associated spin density matrix in the next section. To lowest order, “dijet” simply means an energetic quark-antiquark ($q\bar{q}$) pair. “Exclusive” means that the target proton scatters elastically. Thus, the process of interest is $\gamma^{(*)} + p \rightarrow q + \bar{q} + p'$. This can be thought of as a subprocess in DIS where the virtual photon γ^* with virtuality $Q^2 = -q^2$ is emitted from the incoming electron. In proton-nucleus ($p + A$) UPCs, the photon γ is real ($Q^2 \approx 0$), emitted from the incoming heavy nucleus. We work in a frame where the photon is collinear² $q^\mu = (q^+, -\frac{Q^2}{2q^+}, \mathbf{0})$ and fast moving $q^+ \gg Q$ in the $+x^3$ direction. The incoming and outgoing protons have momenta

$$p^\mu \approx \left(0, (1+\xi)P^-, -\frac{\Delta}{2}\right), \quad p'^\mu \approx \left(0, (1-\xi)P^-, \frac{\Delta}{2}\right). \quad (1)$$

²The light-cone coordinates are defined as $p^\mu = (p^+, p^-, \mathbf{p})$ with $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^3)$. Boldface letters denote two-dimensional vectors $\mathbf{p} = (p^1, p^2)$ and we use $i, j = 1, 2$ for their indices.

We will neglect the proton mass and assume $\Delta \approx 0$ throughout this paper. The latter approximation avoids possible entanglement between the $q\bar{q}$ pair and the proton in momentum space. Moreover, the proton spin does not affect the spin state of the pair because, at high energy, the proton helicity is conserved in near-forward scattering.

In this setup, the produced $q\bar{q}$ pair has momentum

$$\tilde{k}^\mu = \left(zq^+, \frac{k_\perp^2 + m^2}{2zq^+}, \mathbf{k}\right), \quad \tilde{k}'^\mu = \left(\bar{z}q^+, \frac{k_\perp^2 + m^2}{2\bar{z}q^+}, -\mathbf{k}\right), \quad (2)$$

and invariant mass

$$M^2 = (\tilde{k} + \tilde{k}')^2 = \frac{k_\perp^2 + m^2}{z\bar{z}}. \quad (3)$$

$0 < z = \frac{p \cdot \tilde{k}}{p \cdot q} < 1$ is the fraction of the photon energy carried by the quark ($\bar{z} = 1 - z$ for the antiquark). One may regard this process as the $2 \rightarrow 2$ scattering $\gamma^{(*)} + \mathbb{P} \rightarrow q + \bar{q}$, where $\mathbb{P}$ denotes Pomeron³ with momentum

$$\mathbb{P}^\mu = p^\mu - p'^\mu \approx (0, 2\xi P^-, \mathbf{0}). \quad (4)$$

The skewness variable ξ is fixed by momentum conservation in this $2 \rightarrow 2$ scattering as

$$\frac{2\xi}{1+\xi} = \frac{M^2 + Q^2}{W^2 + Q^2} = \frac{M^2 + Q^2}{ys}, \quad (5)$$

with $W^2 = (p + q)^2$ being the $\gamma^* + p$ c.m. energy. On the right-hand side, we introduced the $e + p$ c.m. energy s and the standard DIS variable $y = \frac{q \cdot p}{l \cdot p}$.

We write the unpolarized cross section as

$$\begin{aligned} \frac{d\sigma^{ep}}{dW^2 dQ^2 dz d^2\mathbf{k} d^2\Delta} &= \frac{\alpha_{em}}{\pi s Q^2} \frac{1 + (1-y)^2}{2y} \\ &\times \left(\frac{d\sigma^T}{dz d^2\mathbf{k} d^2\Delta} + \varepsilon \frac{d\sigma^L}{dz d^2\mathbf{k} d^2\Delta} \right) + \dots \end{aligned} \quad (6)$$

The superscripts T and L denote the contributions from the transversely and longitudinally polarized virtual photons and $\varepsilon = \frac{1-y}{1-y+\frac{y^2}{2}}$ is the ratio of their fluxes. The terms omitted in (6) are proportional to $\cos \phi$ or $\cos 2\phi$, where ϕ is the azimuthal angle of $\mathbf{k}$ relative to the lepton scattering plane. These terms arise from interference between different photon polarization states. We neglect them assuming an

³The use of the word Pomeron is due to a lack of better terminology. In the present context, it simply means the color-singlet two-gluon or quark-antiquark exchange in the t channel. In the high-energy (Regge) limit, it is smoothly connected to the Pomeron exchange discussed in [30].

implicit integration over ϕ_k . In UPCs, only the transverse part is relevant

$$\frac{d\sigma^{\text{UPC}}}{dzd^2\mathbf{k}d^2\mathbf{\Delta}} = \int dq^0 \frac{dN}{dq^0} \frac{d\sigma^T}{dzd^2\mathbf{k}d^2\mathbf{\Delta}}, \quad (7)$$

where dN/dq^0 is the well-known equivalent photon flux [41].

$$\begin{aligned} \frac{d\sigma^T}{dzd^2\mathbf{k}d^2\mathbf{\Delta}} &= \frac{\alpha_{em}e_q^2\alpha_s^2}{16\pi^2N_c(1-\xi^2)(k_\perp^2+\mu^2)^4} \\ &\times (k_\perp^2(|2C_F(I_T^{q_1}+I_T^{q_2})-(1-2z)I_T^q|^2+|2C_F(I_T^{q_1}-I_T^{q_2})+I_T^q|^2)+2m^2|I_L^q|^2), \end{aligned} \quad (9)$$

where $\alpha_{em} = \frac{1}{137}$, $\alpha_s = \frac{g^2}{4\pi}$ is the QCD coupling, and $\mu^2 \equiv m^2 + z\bar{z}Q^2$. I 's are convolution integrals with GPDs

$$\begin{aligned} I_L^q &= \int_{-1}^1 dx F_g(x, \xi) \left(\frac{4\xi(1-\beta)(x^2+\xi^2)}{(x^2-\xi^2+i\epsilon)^2} - \frac{2\xi(1-2\beta)}{x^2-\xi^2+i\epsilon} \right), \\ I_L^q &= \int_{-1}^1 dx F_q(x, \xi) \left(\frac{2\xi\bar{z}}{x+\xi-i\epsilon} + \frac{2\xi z}{x-\xi+i\epsilon} \right), \\ I_T^q &= \int_{-1}^1 dx F_g(x, \xi) \left(\frac{2\xi(1-2\beta)(x^2+\xi^2)}{(x^2-\xi^2+i\epsilon)^2} + \frac{4\xi\beta}{x^2-\xi^2+i\epsilon} \right), \\ I_T^{q_1} &= \int_{-1}^1 dx F_q(x, \xi) \left(\frac{2\xi\bar{z}}{x-\xi+i\epsilon} - \frac{2\xi\beta\bar{z}^2}{(1-\beta)(x+\xi-i\epsilon)} + \frac{2\xi\bar{z}^2}{(1-\beta)(x-\xi(1-2\beta)-i\epsilon)} \right), \\ I_T^{q_2} &= \int_{-1}^1 dx F_q(x, \xi) \left(\frac{2\xi\beta z^2}{(1-\beta)(x-\xi+i\epsilon)} - \frac{2\xi z\bar{z}}{x+\xi-i\epsilon} - \frac{2\xi z^2}{(1-\beta)(x+\xi(1-2\beta)+i\epsilon)} \right). \end{aligned} \quad (10)$$

$I_L^q, I_T^{q_{1,2}}$, and I_L^g, I_T^g come from the t -channel quark-antiquark and two-gluon exchange diagrams, respectively. In (10), we introduced the variable

$$\beta = \frac{\mu^2}{k_\perp^2 + \mu^2}, \quad (11)$$

often used in diffractive DIS and defined

$$F_{q,g} \equiv H_{q,g} - \frac{\xi^2}{1-\xi^2} E_{q,g}. \quad (12)$$

The quark and gluon GPDs H_q and H_g are normalized such that they reduce to the respective parton distribution functions $H_q(x, 0) = q(x)$ and $H_g(x, 0) = xg(x)$ in the forward limit. On the other hand, the GPDs E_q, E_g have no collinear analogs. The linear combination (12) represents the helicity nonflip amplitude [32]. In the following, we are only concerned with the region $\xi \ll 1$ and, accordingly, neglect $\mathcal{O}(\xi^2)$ terms. The GPD E 's drop out in this approximation.

In [40], $d\sigma^{L/T}$ has been calculated to leading order in collinear factorization using the GPDs. The result reads

$$\frac{d\sigma^L}{dzd^2\mathbf{k}d^2\mathbf{\Delta}} = \frac{\alpha_{em}e_q^2z^2\bar{z}^2Q^2\alpha_s^2}{2\pi^2N_c(1-\xi^2)(k_\perp^2+\mu^2)^4} |I_L^g + 2C_F I_L^q|^2, \quad (8)$$

III. SPIN DENSITY MATRIX

The cross sections (8) and (9) have been obtained by summing over the two spin states of the produced quark and antiquark. In order to compute the spin density matrix, one has to undo this step. Let α, α' be the spin indices of the quark and the antiquark in the amplitude and β, β' be those in the complex-conjugate amplitude. Before setting $\alpha = \beta, \alpha' = \beta'$ and summing over α, α' , the squared scattering amplitude takes the generic form

$$\bar{u}_\alpha(\tilde{k}) \Gamma v_{\alpha'}(\tilde{k}') \bar{v}_{\beta'}(\tilde{k}') \gamma^0 \Gamma^\dagger \gamma^0 u_\beta(\tilde{k}). \quad (13)$$

Up to a prefactor, this is essentially the spin density matrix with matrix indices $\alpha\alpha'\beta\beta'$ in spin space. It describes the quantum state of two “qubits,” representing the spin states of the quark and antiquark with fixed momenta. Clearly, the density matrix is frame dependent because spin and momentum do not transform independently under Lorentz boosts. Results in different frames are related by Wigner rotations. It also depends on the choice of the spinor basis. On the other hand, since Wigner rotations are unitary, physical results such as entanglement and Bell nonlocality are frame independent [15,42]. It is then most

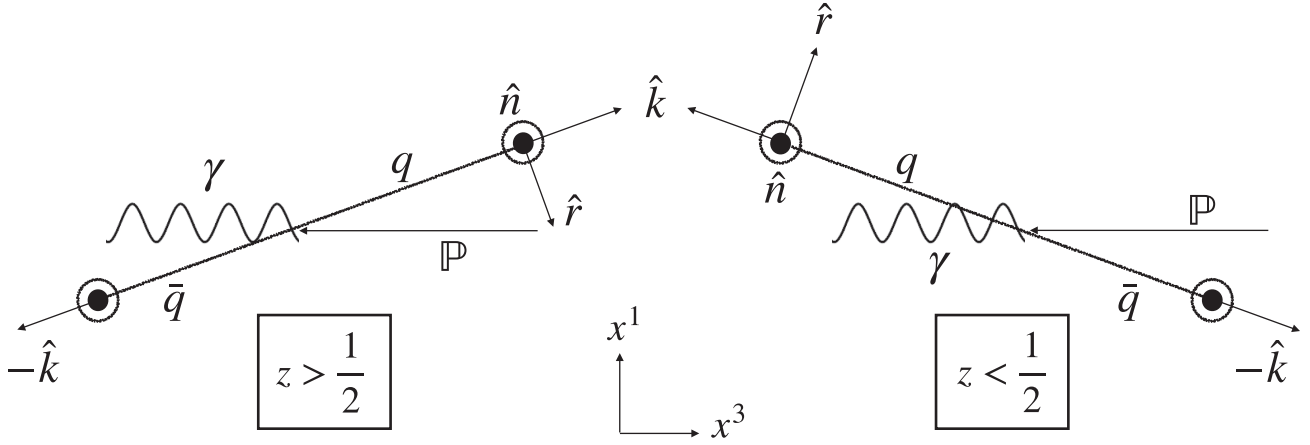


FIG. 1. $q\bar{q}$ production in the photon-Pomeron c.m. frame at $\phi = 0$. Left: $z > \frac{1}{2}$. Right: $z < \frac{1}{2}$.

convenient to calculate spin density matrix using the so-called Jacob-Wick helicity states [43] and in the back-to-back frame of the $q\bar{q}$ pair.

A. Longitudinal photon

In the longitudinal case, (13) takes the form [40]

$$\bar{u}_\alpha(\tilde{k})\gamma^+v_{\alpha'}(\tilde{k}')\bar{v}_{\beta'}(\tilde{k}')\gamma^+u_\beta(\tilde{k}). \quad (14)$$

This is exactly the same as in the Pomeron exchange contribution calculated in [30]. We then immediately see that the spin density matrix C_{ab}^L is the same. It came as a surprise in [30] that C_{ab}^L resulting from the Pomeron exchange is identical to that for the lowest order process $\gamma_L^* + g \rightarrow q + \bar{q}$ calculated in [29]. That is, C_{ab}^L does not depend on whether color-singlet or octet states are exchanged in the t channel, or whether one gluon or multiple gluons are exchanged. We now see that C_{ab}^L remains the same even after including the quark-antiquark exchange channel.

For completeness, and to set up our notation for the transversely polarized case to be discussed later, here we reproduce the result (see the Appendix of [30]). First we go to the photon-Pomeron c.m. frame where the quark and the antiquark have back-to-back momenta [switching to the Cartesian coordinates $k^\mu = (k^0, \vec{k}) = (k^0, \mathbf{k}, k^3)$]

$$k^\mu = \left(\frac{M}{2}, \mathbf{k}, \frac{M}{2}(z - \bar{z})\right) \equiv (k^0, \vec{k}),$$

$$k'^\mu = \left(\frac{M}{2}, -\mathbf{k}, -\frac{M}{2}(z - \bar{z})\right) = (k^0, -\vec{k}), \quad (15)$$

and velocity

$$|\vec{k}| = \frac{M}{2}v, \quad v = \sqrt{1 - \frac{4m^2}{M^2}}. \quad (16)$$

This can be achieved via a Lorentz boost along the x^3 axis $\tilde{k}^\pm = e^{\pm\eta}k^\pm$, $\tilde{k}'^\pm = e^{\pm\eta}k'^\pm$ with

$$e^\eta = q^+ \sqrt{\frac{2z\bar{z}}{k_\perp^2 + m^2}}. \quad (17)$$

In this frame, the quark moves in the direction $\hat{k} = \frac{\vec{k}}{|\vec{k}|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with polar angles

$$\cos\theta = \frac{(z - \bar{z})M}{\sqrt{M^2 - 4m^2}}, \quad \phi = \arg(k^1 + ik^2). \quad (18)$$

Note that $\cos\theta \rightarrow 1$ when $z \rightarrow 1$ and $\cos\theta \rightarrow -1$ when $z \rightarrow 0$. Under this boost, the spinor product transforms as

$$\bar{u}_\alpha(\tilde{k})\gamma^+v_{\alpha'}(\tilde{k}') = e^\eta \bar{u}_\alpha(k)\gamma^+v_{\alpha'}(k'). \quad (19)$$

We then perform a spatial rotation from the $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$ frame to the $(\hat{n}, \hat{r}, \hat{k})$ frame where (see Fig. 1)

$$\hat{n} = \frac{\hat{x}^3 \times \hat{k}}{\sin\theta} = (-\sin\phi, \cos\phi, 0),$$

$$\hat{r} = \hat{k} \times \hat{n} = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta). \quad (20)$$

Under the same rotation, γ^+ transforms as

$$\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \sin\theta\gamma^r + \cos\theta\gamma^k). \quad (21)$$

The spin density matrix is conveniently calculated in the $(\hat{n}, \hat{r}, \hat{k})$ frame with $\hat{k}$ being the spin quantization axis. In this frame, the Jacob-Wick spinors take the form

$$u_\alpha(k) = \begin{pmatrix} \sqrt{k \cdot \sigma} \xi_\alpha \\ \sqrt{k \cdot \bar{\sigma}} \bar{\xi}_\alpha \end{pmatrix},$$

$$v_{\alpha'}(k') = \begin{pmatrix} \sqrt{k' \cdot \sigma} \tilde{\eta}_{-\alpha'} \\ -\sqrt{k' \cdot \bar{\sigma}} \tilde{\bar{\eta}}_{-\alpha'} \end{pmatrix} = \begin{pmatrix} \sqrt{k \cdot \sigma} \tilde{\eta}_{-\alpha'} \\ -\sqrt{k \cdot \bar{\sigma}} \tilde{\bar{\eta}}_{-\alpha'} \end{pmatrix}, \quad (22)$$

with $(\sigma^k = \sigma^3)$ and

$$\sqrt{k \cdot \sigma} = \sqrt{\frac{M}{8}}(\sqrt{1+v}(1-\sigma^k) + \sqrt{1-v}(1+\sigma^k)),$$

$$\sqrt{k \cdot \bar{\sigma}} = \sqrt{\frac{M}{8}}(\sqrt{1-v}(1-\sigma^k) + \sqrt{1+v}(1+\sigma^k)). \quad (23)$$

Here, $\alpha, \alpha' = \pm$ refers to the helicity (angular momentum projection $\pm \frac{1}{2}$ along the direction of motion). For the antiquark moving in the $-\hat{k}$ direction, we employ the “flipped spinors” $\tilde{\eta}_{-\alpha'} = -i\sigma^2(\eta_{\alpha'})^*$ [44]. Using these basis, we can write the cross section in the form

$$\frac{d\sigma^L}{dz d^2\mathbf{k} d^2\mathbf{\Delta}} = \sum_{\alpha=\beta, \alpha'=\beta'} A^L \xi_\alpha^\dagger \eta_{\alpha'}^\dagger \rho_{\alpha\beta, \alpha'\beta'}^L \xi_\beta \eta_{\beta'}, \quad (24)$$

where

$$\xi_\beta \xi_\beta^\dagger = \frac{1}{2}(\xi_\alpha^\dagger \xi_\beta + (\xi_\alpha^\dagger \sigma^n \xi_\beta) \sigma^n + (\xi_\alpha^\dagger \sigma^r \xi_\beta) \sigma^r + (\xi_\alpha^\dagger \sigma^k \xi_\beta) \sigma^k),$$

$$\tilde{\eta}_{-\alpha'} \tilde{\eta}_{-\alpha'}^\dagger = \frac{1}{2}(\tilde{\eta}_{-\beta'}^\dagger \tilde{\eta}_{-\alpha'} + (\tilde{\eta}_{-\beta'}^\dagger \sigma^n \tilde{\eta}_{-\alpha'}) \sigma^n + (\tilde{\eta}_{-\beta'}^\dagger \sigma^r \tilde{\eta}_{-\alpha'}) \sigma^r + (\tilde{\eta}_{-\beta'}^\dagger \sigma^k \tilde{\eta}_{-\alpha'}) \sigma^k). \quad (28)$$

Finally, we use the formula $\tilde{\eta}_{-\beta'}^\dagger \tilde{\sigma} \tilde{\eta}_{-\alpha'} = -\eta_{\alpha'}^\dagger \tilde{\sigma} \eta_{\beta'}$ to recast the density matrix in the desired form. The result is $B_a^L = \bar{B}_b^L = 0$ and

$$C_{nn}^L = 1, \quad C_{rr}^L = -C_{kk}^L = -\frac{1 - (2 - v^2) \cos^2 \theta}{1 - v^2 \cos^2 \theta} = -\frac{k_\perp^2 - (1 - 2z)^2 m^2}{k_\perp^2 + (1 - 2z)^2 m^2},$$

$$C_{rk}^L = C_{kr}^L = -\frac{\sqrt{1 - v^2} \sin 2\theta}{1 - v^2 \cos^2 \theta} = \frac{2(1 - 2z)k_\perp m}{k_\perp^2 + (1 - 2z)^2 m^2}. \quad (29)$$

The other components are zero. Note that $(C_{rr}^L)^2 + (C_{rk}^L)^2 = 1$.

The matrix C_{ab}^L describes a two-qubit state that is maximally entangled and moreover maximally violates the Bell-CHSH inequality [29]. In particular, for the symmetric pair $z = \frac{1}{2}$ ($\cos \theta = 0$), or in the relativistic limit $k_\perp \rightarrow \infty$, or in the massless case $m = 0$, $C^L = \text{diag}(1, -1, 1)$ represents one of the Bell states

$$\rho^L = |\Phi^+\rangle\langle\Phi^+|, \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle), \quad (30)$$

$$A^L = \frac{\alpha_{em} e_q^2 \alpha_s^2 z^2 \bar{z}^2 Q^2}{2\pi^2 N_c (1 - \xi^2) (k_\perp^2 + \mu^2)^4} |I_L^g + 2C_F I_L^q|^2 \quad (25)$$

is the unpolarized cross section (8). The density matrix ρ^L can be parametrized by

$$\rho^L = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + B_a^L \sigma^a \otimes \mathbb{1} + \bar{B}_b^L \mathbb{1} \otimes \sigma^b + C_{ab}^L \sigma^a \otimes \sigma^b), \quad (a, b = n, r, k), \quad (26)$$

where $\mathbb{1}$ is the unit 2×2 matrix and σ^a are the Pauli matrices. The vectors $B_a^L, \bar{B}_b^L$ represent the polarization of the quark and the antiquark,

$$B_a^L = \text{Tr}[(\sigma^a \otimes \mathbb{1})\rho^L], \quad \bar{B}_b^L = \text{Tr}[(\mathbb{1} \otimes \sigma^b)\rho^L], \quad (27)$$

while the matrix C_{ab}^L describes the correlation between the quark and antiquark spins.

In the actual computation, we evaluate (14) using (22) and the explicit form of the γ matrices $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$. We then exploit the fact that any 2×2 matrix can be expanded in the basis $\{1, \sigma^n, \sigma^r, \sigma^k\}$,

where $\pm$ refers to spin projection $\pm \frac{1}{2}$ along the $\hat{k}$ axis. (This is opposite in sign to helicity for the antiquark moving in the $-\hat{k}$ direction.) On the other hand, in the forward limit $k_\perp \rightarrow 0$ ($\cos \theta = \pm 1$), $C^L = \text{diag}(1, 1, -1)$ represents another Bell state

$$\rho^L = |\Psi^+\rangle\langle\Psi^+|, \quad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle). \quad (31)$$

Moreover, along the line $z = \frac{1}{2} \pm \frac{k_\perp}{2m}$,

$$C^L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mp 1 \\ 0 & \mp 1 & 0 \end{pmatrix}. \quad (32)$$

This represents a state equivalent to the Bell state up to a local phase rotation

$$\rho^L = |\Phi_n^\pm\rangle\langle\Phi_n^\pm|, \quad |\Phi_n^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|++\rangle_n \pm i|--\rangle_n), \quad (33)$$

where the subscript n means that the spin eigenstates are defined with respect to the $\hat{n}$ axis.

B. Transverse photon

We now turn to the transversely polarized case that has a much richer structure from the viewpoint of entanglement. The amplitude is proportional to [40]

$$\begin{aligned} \epsilon^i \bar{u}_\alpha(\tilde{k}) & [-m\gamma^i \gamma^+ I_L^g + k_{\perp j} \gamma^j \gamma^+ (2C_F I_T^{q_1} + \bar{z} I_T^g) + k_{\perp j} \gamma^j \gamma^+ (2C_F I_T^{q_2} - z I_T^g)] v_\alpha(\tilde{k}') \\ & = e^\eta \epsilon^i \bar{u}_\alpha(k) [-m\gamma^i \gamma^+ I_L^g + k_{\perp}^i \gamma^+ X + i\epsilon^{ij} k_{\perp j} \gamma^+ \gamma_5 Y] v_\alpha(k'), \end{aligned} \quad (34)$$

where we defined

$$X = 2C_F(I_T^{q_1} + I_T^{q_2}) + (\bar{z} - z)I_T^g, \quad Y = 2C_F(I_T^{q_1} - I_T^{q_2}) + I_T^g. \quad (35)$$

We used the fact that $\gamma^{i=1,2}$ and γ_5 are invariant under the boost. Since we focus on the ϕ -independent part of the cross section, we may set $\phi = 0$ without losing generality. Squaring (34) and summing over the polarizations of the incoming photon $\epsilon^i \epsilon^{j*} \rightarrow \delta^{ij}$, we find that the cross section is proportional to

$$\begin{aligned} & k_{\perp}^2 |X|^2 \bar{u}_\alpha(k) \gamma^+ v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ u_\beta(k) + k_{\perp}^2 |Y|^2 \bar{u}_\alpha(k) \gamma^+ \gamma_5 v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ \gamma_5 u_\beta(k) \\ & - m^2 |I_L^g|^2 \bar{u}_\alpha(k) \gamma^+ \gamma^i v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ \gamma^i u_\beta(k) \\ & - mk_{\perp} [X I_L^{g*} \bar{u}_\alpha(k) \gamma^+ v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ \gamma^1 u_\beta(k) + X^* I_L^g \bar{u}_\alpha(k) \gamma^1 \gamma^+ v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ u_\beta(k)] \\ & - imk_{\perp} [Y I_L^{g*} \bar{u}_\alpha(k) \gamma^+ \gamma^5 v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ \gamma^2 u_\beta(k) + Y^* I_L^g \bar{u}_\alpha(k) \gamma^+ \gamma^2 v_{\alpha'}(k') \bar{v}_{\beta'}(k') \gamma^+ \gamma^5 u_\beta(k)]. \end{aligned} \quad (36)$$

Similar to [30], we evaluate the spinor products in the $(\hat{n}, \hat{r}, \hat{k})$ frame. The γ matrices in this frame are related to those in the original frame as, at $\phi = 0$,

$$\gamma^1 = -\cos\theta\gamma^r + \sin\theta\gamma^k, \quad \gamma^2 = \gamma^n. \quad (37)$$

We then express the cross section in the form of (24) and (26) with $L \rightarrow T$. The spin-averaged part is as in (9),

$$\begin{aligned} A^T &= \frac{\alpha_{em} e_q^2 \alpha_s^2}{16\pi^2 N_c (1 - \xi^2) (k_{\perp}^2 + \mu^2)^4} \\ &\times (k_{\perp}^2 (|X|^2 + |Y|^2) + 2m^2 |I_L^g|^2). \end{aligned} \quad (38)$$

For the polarization vectors, we find $B_{r,k}^T = \bar{B}_{r,k}^T = 0$, but the n -component turns out to be nonvanishing

$$\begin{aligned} B_n^T &= \frac{-2mk_{\perp} \text{Im}[(X+Y)I_L^{g*}]}{k_{\perp}^2 (|X|^2 + |Y|^2) + 2m^2 |I_L^g|^2}, \\ \bar{B}_n^T &= \frac{-2mk_{\perp} \text{Im}[(X-Y)I_L^{g*}]}{k_{\perp}^2 (|X|^2 + |Y|^2) + 2m^2 |I_L^g|^2}. \end{aligned} \quad (39)$$

This means that the quark and the antiquark are transversely (i.e., normal to the production plane, along the $\hat{n}$ direction in Fig. 1) polarized. It can be readily shown that

$$|B_n^T|, |\bar{B}_n^T| \leq \frac{|\text{Im}[(X \pm Y)I_L^{g*}]|}{\sqrt{2|I_L^{g*}|^2(|X|^2 + |Y|^2)}} \leq \frac{|X \pm Y|}{\sqrt{2(|X|^2 + |Y|^2)}} \leq 1, \quad (40)$$

where in the first inequality we maximized in $k_{\perp}$. The last inequality is saturated when $X = Y$ and $X = -Y$, which occurs at $z = 0$ and $\bar{z} = 0$, respectively. Since the colliding particles $e + p$ or $\gamma + p$ are unpolarized, (39) can be interpreted as single spin asymmetry (SSA), analogous to the transverse polarization of hyperons in unpolarized hadron collisions [36–39].⁴ As is well known, SSA requires a phase, and therefore it cannot be generated in tree-level calculations in parity-conserving theories like QCD. Theoretical descriptions usually invoke higher-twist distribution

⁴The transverse polarization of final state particles can also arise if initial state particles are transversely polarized. See [45] for a recent study at the EIC.

and/or fragmentation functions or their transverse momentum-dependent generalizations [46–51]. However, a phase can also be generated in perturbation theory beyond the tree level [52,53], and this is what happens here. Since the scattering amplitudes (10) are obtained by convoluting GPDs with a hard factor, they are naturally complex. [Notice the $i\epsilon$ terms in (10).] Equation (39) clearly shows that the nonvanishing polarization results from the interference between their real and imaginary parts. In contrast,

in the Regge limit, X , Y , and I_L^g become purely imaginary; see the next subsection. This is why no polarization was found in [30].

As for the C^T matrix, we obtain

$$C_{nn}^T = \frac{(|X|^2 - |Y|^2)k_\perp^2}{k_\perp^2(|X|^2 + |Y|^2) + 2m^2|I_L^g|^2}, \quad (41)$$

$$C_{rr}^T = \frac{k_\perp^2}{k_\perp^2(|X|^2 + |Y|^2) + 2m^2|I_L^g|^2} \left[-\frac{k_\perp^2 - (1-2z)^2m^2}{k_\perp^2 + (1-2z)^2m^2} |X|^2 + \frac{k_\perp^2 - m^2}{k_\perp^2 + m^2} |Y|^2 \right. \\ \left. + \frac{8z\bar{z}m^4|I_L^g|^2}{(k_\perp^2 + (1-2z)^2m^2)(k_\perp^2 + m^2)} + 4m^2 \left(-\frac{(1-2z)\text{Re}[XI_L^{g*}]}{k_\perp^2 + (1-2z)^2m^2} + \frac{\text{Re}[YI_L^{g*}]}{k_\perp^2 + m^2} \right) \right], \quad (42)$$

$$C_{kk}^T = \frac{1}{k_\perp^2(|X|^2 + |Y|^2) + 2m^2|I_L^g|^2} \left[k_\perp^2 \frac{k_\perp^2 - (1-2z)^2m^2}{k_\perp^2 + (1-2z)^2m^2} |X|^2 + k_\perp^2 \frac{k_\perp^2 - m^2}{k_\perp^2 + m^2} |Y|^2 \right. \\ \left. - \frac{2m^2(k_\perp^4 - (1-2z)^2m^4)|I_L^g|^2}{(k_\perp^2 + (1-2z)^2m^2)(k_\perp^2 + m^2)} + 4m^2k_\perp^2 \left(\frac{(1-2z)\text{Re}[XI_L^{g*}]}{k_\perp^2 + (1-2z)^2m^2} + \frac{\text{Re}[YI_L^{g*}]}{k_\perp^2 + m^2} \right) \right], \quad (43)$$

$$C_{rk}^T = \frac{2k_\perp m}{k_\perp^2(|X|^2 + |Y|^2) + 2m^2|I_L^g|^2} \left[\frac{(1-2z)k_\perp^2|X|^2}{k_\perp^2 + (1-2z)^2m^2} + \frac{k_\perp^2|Y|^2}{k_\perp^2 + m^2} \right. \\ \left. - \frac{2(1-z)m^2(k_\perp^2 + (1-2z)m^2)|I_L^g|^2}{(k_\perp^2 + m^2)(k_\perp^2 + (1-2z)^2m^2)} - \frac{k_\perp^2 - (1-2z)^2m^2}{k_\perp^2 + (1-2z)^2m^2} \text{Re}[XI_L^{g*}] - \frac{k_\perp^2 - m^2}{k_\perp^2 + m^2} \text{Re}[YI_L^{g*}] \right], \quad (44)$$

$$C_{kr}^T = \frac{2k_\perp m}{k_\perp^2(|X|^2 + |Y|^2) + 2m^2|I_L^g|^2} \left[\frac{(1-2z)k_\perp^2|X|^2}{k_\perp^2 + (1-2z)^2m^2} - \frac{k_\perp^2|Y|^2}{k_\perp^2 + m^2} \right. \\ \left. + \frac{2zm^2(k_\perp^2 - (1-2z)m^2)|I_L^g|^2}{(k_\perp^2 + m^2)(k_\perp^2 + (1-2z)^2m^2)} - \frac{k_\perp^2 - (1-2z)^2m^2}{k_\perp^2 + (1-2z)^2m^2} \text{Re}[XI_L^{g*}] + \frac{k_\perp^2 - m^2}{k_\perp^2 + m^2} \text{Re}[YI_L^{g*}] \right]. \quad (45)$$

Unlike in the longitudinal case (29), the C^T matrix depends on the nonperturbative structure of the proton through their dependence on the quark and gluon GPDs. Importantly, the resulting transverse density matrix ρ^T , in general, represents a mixed state $\text{Tr}[(\rho^T)^2] < 1$ [30]. This is due to the combined effect of the transverse photon polarization sum and finite longitudinal momentum transfer $2\xi P^-$ from the proton to the $q\bar{q}$ pair in the form of GPD convolutions ($\Delta \approx 0$ in the present approximation). On the other hand, the proton spin does not affect the entanglement property of the pair because we sum over the initial and final proton spins and there is no spin flip during scattering. The density matrix would take a similar form if the target were a spinless pion, with the pion GPDs replacing the proton GPDs.

One might have expected that $B_n^T(z) = \bar{B}_n^T(z)$ and $C_{rk}^T(z) = C_{kr}^T(z)$ from CP symmetry, as in the case of $q\bar{q} \rightarrow t\bar{t}$ and $gg \rightarrow t\bar{t}$ productions in QCD [8]. However, neither of these relations hold in the present problem since

the initial state $\gamma + \mathbb{P}$ is asymmetric (see also [29]). Instead, the following relations hold exactly

$$B_n^T(z) = -\bar{B}_n^T(\bar{z}), \quad C_{rk}^T(z) = -C_{kr}^T(\bar{z}). \quad (46)$$

[Note that $1-2z \rightarrow -(1-2z)$ under the transformation $z \rightarrow \bar{z} = 1-z$.] For the valence quarks that we treat as massless $m_{u,d} = 0$, this is trivial. For massive quarks $q = s, c, b$, (46) is a consequence of the property $F_q(x, \xi) = -F_q(-x, \xi)$ of the nonvalence quark GPDs, which results in the relations

$$X(z) = -X(\bar{z}), \quad Y(z) = Y(\bar{z}). \quad (47)$$

We may regard (46) as constraints from CP symmetry, since $z \rightarrow \bar{z}$ (or $\cos\theta \rightarrow -\cos\theta$ in the c.m. frame) interchanges the quark and the antiquark.

C. Regge limit

Let us quickly check consistency with the result of [30] obtained in the Regge limit $\xi \rightarrow 0$. In this limit, the quark exchange contributions $I_L^g, I_T^{q_1}, I_T^{q_2}$ are subdominant and neglected altogether. The gluon exchange contributions I_L^g, I_T^g are dominantly imaginary, coming from the $i\epsilon$ prescription in the integrals (10). Keeping only the most singular term in the imaginary part, one finds [40]

$$I_L^g \approx 2\pi i \frac{k_\perp^2 - \mu^2}{k_\perp^2 + \mu^2} H_g(\xi, \xi), \quad I_T^g \approx 2i\pi \frac{-2\mu^2}{k_\perp^2 + \mu^2} H_g(\xi, \xi), \quad (48)$$

where it is assumed that the gluon GPD has a Regge behavior $H_g(\xi, \xi) \sim 1/\xi^\alpha$ with $\alpha \ll 1$. On the other hand, the result of [30] is expressed in terms of the following two integrals:

$$T_1 = \int \frac{d^2 p T(p)}{(\mathbf{p}-\mathbf{k})^2 + \mu^2}, \quad T_2 = -\frac{1}{k_\perp^2} \int d^2 p \frac{\mathbf{p} \cdot \mathbf{k} T(p)}{(\mathbf{p}-\mathbf{k})^2 + \mu^2}, \quad (49)$$

where T is the T -matrix of the $q\bar{q}$ pair (“color dipole”) scattering off the target proton. To make a connection with the collinear GPD calculation, we expand the integrand in powers of the intrinsic transverse momentum $\mathbf{p}$ and use the relation between $T(p)$ and $H_g(\xi, \xi)$ [54]. We find

$$\begin{aligned} T_1 &\approx \frac{k_\perp^2 - \mu^2}{(k_\perp^2 + \mu^2)^3} \int d^2 p \mathbf{p}^2 T(p) \approx \frac{k_\perp^2 - \mu^2}{2(k_\perp^2 + \mu^2)^3} \alpha_s H_g(\xi, \xi), \\ T_2 &\approx -\frac{1}{(k_\perp^2 + \mu^2)^2} \int d^2 p \mathbf{p}^2 T(p) \approx -\frac{1}{2(k_\perp^2 + \mu^2)^2} \alpha_s H_g(\xi, \xi). \end{aligned} \quad (50)$$

This gives the correspondence

$$T_1 \approx \frac{\alpha_s I_L^g}{4\pi i (k_\perp^2 + \mu^2)^2}, \quad T_1 + T_2 \approx \frac{\alpha_s I_T^g}{4\pi i (k_\perp^2 + \mu^2)^2}. \quad (51)$$

Inserting these relations into $A^{T/L}$, and noting that $X \approx (\bar{z} - z) I_T^g, Y \approx I_T^g$, one recovers the dijet cross section originally calculated in the $k_\perp$ factorization framework [55,56], as already noted in [40]. Furthermore, inserting into (41)–(45), we find full agreement with [30]. Note that T_1 linearly vanishes as $k_\perp \rightarrow \mu$ [30]. Consistently, I_L^g has the same behavior in the present approximation.

IV. ENTANGLEMENT, BELL NONLOCALITY, AND NONSTABILIZERNESS

In this section, we discuss three different measures of spin-spin correlation between the produced quark and antiquark: entanglement, Bell nonlocality, and nonstabilizer-ness. The spin density matrix formalism allows one to apply these key concepts in quantum information science to relativistic field theories. Numerical results will be presented in the next section.

A. Entanglement

According to the Peres-Hordecki criterion [57,58], for generic two-qubit systems, the necessary and sufficient condition for separability (i.e., no entanglement) is that the partial transpose of the density matrix

$$\begin{aligned} \rho^{\mathcal{T}} &\equiv \frac{1}{4} (1 \otimes 1^{\mathcal{T}} + B_a \sigma^a \otimes 1^{\mathcal{T}} + \bar{B}_b 1 \otimes (\sigma^b)^{\mathcal{T}} \\ &\quad + C_{ab} \sigma^a \otimes (\sigma^b)^{\mathcal{T}}) \end{aligned} \quad (52)$$

is non-negative definite. (The symbol $\mathcal{T}$ denotes “transpose.”) In the present problem with $\rho = \rho^{L/T}$, it is convenient to formally introduce the spin eigenstates $|\pm\rangle_n = \frac{1}{\sqrt{2}}(|+\rangle_k \pm |-\rangle_k)$ along the direction $\hat{n}$. In the basis $\{|+\rangle_n, |+\rangle_{\bar{n}}, |-\rangle_n, |-\rangle_{\bar{n}}\}$, the (transposed) density matrix is block diagonal [9]

$$\rho^{\mathcal{T}} = \frac{1}{4} \begin{pmatrix} 1 + B_n + \bar{B}_n + C_{nn} & 0 & 0 & C_{rr} + C_{kk} + i(C_{rk} - C_{kr}) \\ 0 & 1 + B_n - \bar{B}_n - C_{nn} & C_{rr} - C_{kk} - i(C_{rk} + C_{kr}) & 0 \\ 0 & C_{rr} - C_{kk} + i(C_{rk} + C_{kr}) & 1 - B_n + \bar{B}_n - C_{nn} & 0 \\ C_{rr} + C_{kk} - i(C_{rk} - C_{kr}) & 0 & 0 & 1 - B_n - \bar{B}_n + C_{nn} \end{pmatrix}. \quad (53)$$

In order for this matrix to be non-negative, the following conditions need to be satisfied:

$$1 \pm C_{nn} \geq 0, \quad (1 \pm C_{nn})^2 - (B_n \pm \bar{B}_n)^2 - (C_{rr} \pm C_{kk})^2 - (C_{rk} \mp C_{kr})^2 \geq 0. \quad (54)$$

Thus, the necessary and sufficient condition for entanglement is

$$\max\{\Delta_1, \Delta_2, |C_{nn}| - 1\} > 0, \quad (55)$$

where

$$\begin{aligned} \Delta_1 &\equiv \sqrt{(C_{rr} - C_{kk})^2 + (C_{rk} + C_{kr})^2 + (B_n - \bar{B}_n)^2 - |1 - C_{nn}|}, \\ \Delta_2 &\equiv \sqrt{(C_{rr} + C_{kk})^2 + (C_{rk} - C_{kr})^2 + (B_n + \bar{B}_n)^2 - |1 + C_{nn}|}. \end{aligned} \quad (56)$$

Note that non-negativity of the density matrix ensured by the condition (55) is not affected by unitary Wigner rotations. This is the physical reason behind the general statement that entanglement is frame independent [42]. In [30], it has been noticed that, in the Regge limit,

$$\Delta_2^{L/T} = -\Delta_1^{L/T} = -2C_{nn}^{L/T}. \quad (57)$$

In the longitudinal case, the $q\bar{q}$ pair is always maximally entangled because $C_{nn}^L = 1$ and $\Delta_1^L = 2 > 0$. In the transverse case where $\Delta_2^T = -2C_{nn}^T \geq 0$, the pair is almost always entangled except when $C_{nn}^T = 0$, which occurs in certain kinematical limits such as $k_\perp = 0$ or $k_\perp = \infty$. Equation (57) is a direct consequence of the following relations obeyed by the components of $C^{L/T}$:

$$(C_{rr})^2 + (C_{rk})^2 + (C_{kr})^2 + (C_{kk})^2 - (C_{nn})^2 - 1 = 0, \quad C_{nn} + C_{rr}C_{kk} - C_{rk}C_{kr} = 0. \quad (58)$$

Away from the Regge limit, (58) continues to hold in the longitudinal case. However, in the transverse case, we find from (41)–(45)

$$\begin{aligned} (C_{rr}^T)^2 + (C_{rk}^T)^2 + (C_{kr}^T)^2 + (C_{kk}^T)^2 - (C_{nn}^T)^2 - 1 &= 8k_\perp^2 m^2 \frac{(\text{Re}[XI_L^{g*}])^2 + (\text{Re}[YI_L^{g*}])^2 - (|X|^2 + |Y|^2)|I_L^g|^2}{((|X|^2 + |Y|^2)k_\perp^2 + 2|I_L^g|^2 m^2)^2}, \\ C_{nn}^T + C_{rr}^T C_{kk}^T - C_{rk}^T C_{kr}^T &= 4k_\perp^2 m^2 \frac{-(\text{Re}[XI_L^{g*}])^2 + (\text{Re}[YI_L^{g*}])^2 + (|X|^2 - |Y|^2)|I_L^g|^2}{((|X|^2 + |Y|^2)k_\perp^2 + 2|I_L^g|^2 m^2)^2}. \end{aligned} \quad (59)$$

In the Regge limit, the right-hand sides vanish because X, Y, I_L^g are purely imaginary. However, in general, X, Y, I_L^g have both real and imaginary parts.⁵ Therefore, in the present calculation, (57) does not hold, and the condition (55) becomes nontrivial to satisfy. We expect to find regions in phase space where the $q\bar{q}$ pair is separable, i.e., not entangled. Interestingly, the right-hand sides of (59) vanish for massless quarks $m = 0$. This means that the $u\bar{u}$ and $d\bar{d}$ pairs, and also the $s\bar{s}$ pair, in practice, are always entangled.

Let us elaborate more on the massless case. The C^T matrix becomes diagonal

$$C_{ab}^T = \begin{pmatrix} \frac{|X|^2 - |Y|^2}{|X|^2 + |Y|^2} & 0 & 0 \\ 0 & -\frac{|X|^2 - |Y|^2}{|X|^2 + |Y|^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (60)$$

The corresponding density matrix can be written in the form

⁵The right-hand sides also vanish if X, Y, I_L^g are all relatively real. Again, this not the case, in general.

$$\rho^T = \frac{|X|^2}{|X|^2 + |Y|^2} |\Phi^+\rangle\langle\Phi^+| + \frac{|Y|^2}{|X|^2 + |Y|^2} |\Phi^-\rangle\langle\Phi^-|. \quad (61)$$

This represents a mixed state between two of the Bell states.

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|++\rangle \pm |--\rangle). \quad (62)$$

Equation (61) is an example of the so-called ‘‘Bell diagonal’’ states [59], which are mixed states comprising only the Bell states

$$\begin{aligned} \rho &= p_1 |\Psi^+\rangle\langle\Psi^+| + p_2 |\Psi^-\rangle \\ &\quad \times \langle\Psi^-| + p_3 |\Phi^+\rangle\langle\Phi^+| + p_4 |\Phi^-\rangle\langle\Phi^-|, \end{aligned} \quad (63)$$

with $\sum_i p_i = 1$. It is known that a Bell diagonal state is separable if $p_i \leq \frac{1}{2}$ for all i . In the present problem, the pair is separable only if $|X| = |Y|$ where $C_{nn}^T = 0$. If either $|X|$ or $|Y|$ (accidentally) vanishes, then one of the Bell states is realized. Note that, in the Regge limit, $X \approx (1 - 2z)I_T^g$ and $Y \approx I_T^g$. Thus, the state is separable when $z = 0, 1$ and becomes the Bell state $|\Phi^-\rangle$ at $z = \frac{1}{2}$. Note that, under

Lorentz transformations, Bell states with sharp momenta remain Bell states up to local unitary transformations.

B. Bell nonlocality

In terms of the correlation matrix C , the Bell-CHSH inequality reads [2,3]

$$\text{Max}_{\{\vec{n}_i\}} |n_1^a C_{ab}(n_2^b + n_4^b) + n_3^a C_{ab}(n_2^b - n_4^b)| \leq 2, \quad (64)$$

where $\vec{n}_i$ ($i = 1, 2, 3, 4$) are unit vectors $|\vec{n}_i| = 1$. Depending on the matrix C_{ab} , quantum two-qubit states can violate this inequality for deliberate choices of $\{\vec{n}_i\}$. When this happens, we say the $q\bar{q}$ pair exhibits Bell nonlocality, and this property is frame independent (see [15] and references therein). The necessary and sufficient

condition for Bell nonlocality is that the largest two of the three eigenvalues $\mu_3 \leq \mu_2 \leq \mu_1$ of the matrix $C^T C$ satisfy [60]

$$1 < \mu_1 + \mu_2 \leq 2. \quad (65)$$

For our problem, the three eigenvalues of

$$C^T C = \begin{pmatrix} C_{nn}^2 & 0 & 0 \\ 0 & C_{rr}^2 + C_{kr}^2 & C_{rr}C_{rk} + C_{kr}C_{kk} \\ 0 & C_{rr}C_{rk} + C_{kr}C_{kk} & C_{kk}^2 + C_{rk}^2 \end{pmatrix} \quad (66)$$

are C_{nn}^2 and

$$\frac{C_{rr}^2 + C_{kr}^2 + C_{kk}^2 + C_{rk}^2 \pm \sqrt{(C_{rr}^2 + C_{kr}^2 - C_{kk}^2 - C_{rk}^2)^2 + 4(C_{rr}C_{rk} + C_{kr}C_{kk})^2}}{2}. \quad (67)$$

In the Regge limit, the identities (58) hold, and the above three eigenvalues reduce to

$$\mu_1 = 1, \quad \mu_2 = \mu_3 = C_{nn}^2. \quad (68)$$

Thus, the inequality is always maximally violated in the longitudinal case $C_{nn}^L = 1$, while in the transverse case it is violated if $C_{nn}^T \neq 0$. Since $C_{nn}^T \leq 0$, this happens to be the same condition as (55) and (57) for nonvanishing entanglement [30]. In other words, Bell nonlocality and entanglement are equivalent. This is remarkable because, in general, the states that exhibit Bell nonlocality form a subset of the states that are entangled [61]. Away from the Regge limit, the Bell-CHSH inequality continues to be maximally violated in the longitudinal case. In the transverse case, it is always violated for the $q = u, d$ quarks because (58) also holds for massless quarks. For massive quarks, there will be kinematical regions where the inequality is not violated. On general grounds, we expect that this region is broader than the region where the pair is separable (not entangled).

C. Nonstabilizerness

Finally, let us consider another measure of entanglement called nonstabilizerness. (See [35] for an introduction friendly to high-energy physicists.) In quantum computing, it is known that entanglement alone does not guarantee quantum advantage [62]. Entangled pairs with nonvanishing nonstabilizerness are crucial for quantum computers to outperform classical ones. Recently, this notion has been imported to the high-energy physics community [35,63,64]. One may ask, for example, how much nonstabilizerness is carried by elementary particle pairs

produced in collider experiments. Of course, at present such a question is mostly of conceptual interest, but it may offer new perspectives on the nature of scattering in quantum field theories. To quantify nonstabilizerness, we adopt the “stabilizer Rényi entropy” [34,35], which takes the following form for the present problem and in the present frame:

$$M_2 = -\ln \left(\frac{1 + B_n^4 + \bar{B}_n^4 + C_{nn}^4 + C_{rr}^4 + C_{kk}^4 + C_{kr}^4 + C_{rk}^4}{1 + B_n^2 + \bar{B}_n^2 + C_{nn}^2 + C_{rr}^2 + C_{kk}^2 + C_{kr}^2 + C_{rk}^2} \right). \quad (69)$$

Note that this quantity depends on the choice of frame and spinor basis [63]. The following results should be understood as such.

Let us study the longitudinal case, which allows for a simple analytical treatment, deferring the transverse case to the numerical section. From (29), we find a compact analytical formula

$$M_2^L = \ln \frac{(k_\perp^2 + m^2(1-2z)^2)^4}{k_\perp^8 + 14k_\perp^4 m^4(1-2z)^4 + m^8(1-2z)^8}. \quad (70)$$

This is plotted in Fig. 2. By construction, M_2 vanishes for the Bell states $|\Phi^+\rangle$ (30), $|\Psi^+\rangle$ (31), and $|\Phi_n^\pm\rangle$ (33) realized along the lines $z = \frac{1}{2}$, $k_\perp = 0$, and $\frac{k_\perp}{m} = |2z - 1|$, respectively. Everywhere else, stabilizer Rényi entropy is nonzero even though the pair is always maximally entangled. M_2^L takes the maximal value

$$M_2^{L,\max} = \ln \frac{4}{3} \approx 0.288 \quad (71)$$

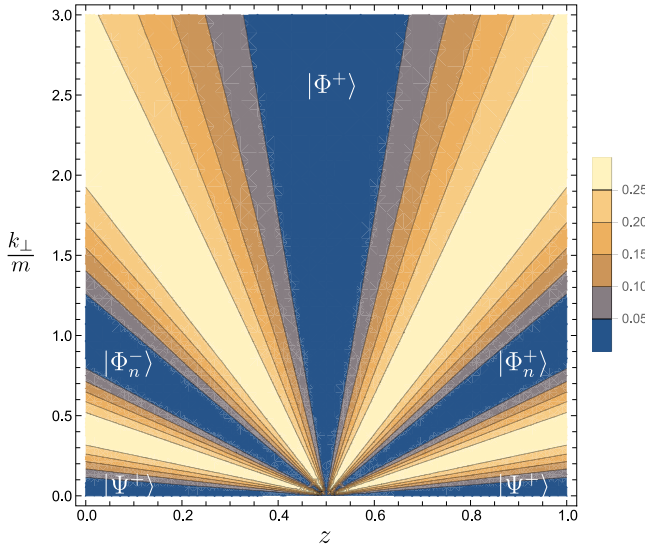


FIG. 2. Stabilizer Rényi entropy M_2^L (70) of a massive quark-antiquark pair produced by a longitudinally polarized virtual photon. M_2^L is suppressed in the blue regions where the pair forms the Bell states (30), (31), and (33).

along the following four lines in the $(z, k_\perp)$ plane:

$$\frac{k_\perp}{m} = (\sqrt{2} \pm 1)|2z - 1|, \quad (72)$$

which are clearly visible in Fig. 2. Along these lines, the density matrix takes the form

$$\rho^L = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm \frac{1 \pm i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm \frac{1 \mp i}{\sqrt{2}} & 0 & 0 & 1 \end{pmatrix}, \quad (73)$$

using the same basis as in (53). (The sign combination is arbitrary, except that the 14 and 41 components are complex conjugate to each other.) The result (71) is significantly smaller than the theoretical upper bound for pure two-qubit states [65]

$$M_2^{\max} = \ln \frac{16}{7} \approx 0.827. \quad (74)$$

However, we point out that this limit can be saturated only by states with nonzero polarization vectors, namely, $B, \bar{B} \neq 0$ in the generic parametrization (26). Since $B^L = \bar{B}^L = 0$, the maximal value (74) cannot be expected. A better comparison may be the range of M_2 for the subset of states with vanishing polarization. In the Appendix, we revisit the derivation of [65] by imposing an extra constraint $B = \bar{B} = 0$ and obtain a stronger bound

$$M_2^{\max}|_{B=\bar{B}=0} = \ln \frac{9}{5} \approx 0.588, \quad (75)$$

which is still larger than (71) by a factor of about 2. In the Appendix, we further show that (71) is the maximum value of M_2 if one restricts to an even smaller subset of states. We note that the numbers $\ln \frac{9}{5}$ and $\ln \frac{4}{3}$ have been previously encountered in [63] as the maximal amount of stabilizer Rényi entropy that can be generated from certain initial stabilizer states in QED processes.

V. NUMERICAL RESULTS

In this section we delineate kinematical regions in the parameter space $(z, k_\perp)$ where the $q\bar{q}$ pair exhibits entanglement and Bell nonlocality. We also compute the stabilizer Rényi entropy M_2^T for the transversely polarized photon. We employ the Goloskokov-Kroll (GK) model [66,67] of the quark and gluon GPDs. This model has been fitted to reproduce the exclusive vector meson production data at HERA. At small skewness, the gluon GPD features the Regge behavior

$$H_g(\xi, \xi, \mu_h) \sim \frac{1}{\xi^{0.1+0.06N}}, \quad N = \ln \frac{\mu_h^2}{Q_0^2}, \quad (76)$$

with $Q_0^2 = 4 \text{ GeV}^2$ and $\mu_h = Q$ in electroproduction [66,67]. Thus, the model should produce results consistent with [30] in the Regge limit. Adapting to our process, we use $\mu_h = \max\{Q, m, k_\perp\}$. The quark masses are fixed as $m_u = m_d = 0$ and $m_s = 0.093$, $m_c = 1.27$, and $m_b = 4.18 \text{ GeV}$. For light quark production $q = u, d, s$, we include both the quark and gluon GPD contributions. For heavy quark production $q = c, b$, we include only the gluon GPD contribution (i.e., $I_{L/T}^q = 0$ in this case) since the c, b -quark GPDs are unavailable (and expected to be negligible). Using the GK model, we numerically evaluate the integrals (10) on grid points on a 50×50 lattice in the $(z, k_\perp)$ plane

$$0.1 \leq z \leq 0.9, \quad 1 \text{ GeV} \leq k_\perp \leq k_\perp^{\max} \quad (77)$$

and smoothly interpolate the results. $k_\perp^{\max}$ is chosen appropriately for a given value of the $\gamma + p$ c.m. energy W . The integrals can be evaluated in a numerically stable way. For very small values of ξ , this is achieved by using the contour deformation technique as explained around (43) of [68].

In electroproduction at $Q \neq 0$, the measured cross section (6) is the linear combination of the transverse and longitudinal contributions. Unfortunately, it is difficult to experimentally separate the two contributions at the EIC since typically $\varepsilon \approx 1$ at high-energy colliders (see, however, [69]). This means that the density matrices must be averaged

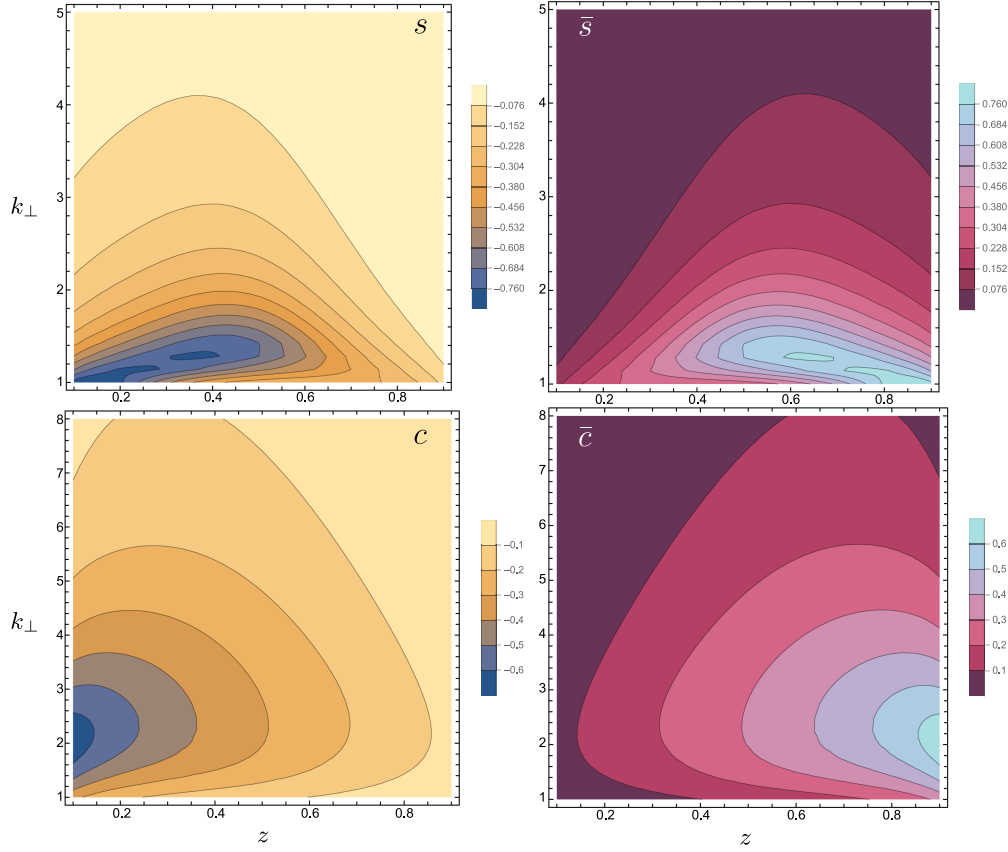


FIG. 3. Upper: polarization of the strange (left) and antistrange (right) quarks in UPC at $W = 30$, $Q = 0$ GeV. Lower: polarization of the charm (left) and anticharm (right) quarks in the same process. Here and in all the plots below, $k_\perp$ in the vertical axis is in units of GeV.

$$A^T \rho^T + \varepsilon A^L \rho^L = \frac{A^T + \varepsilon A^L}{4} (\mathbb{1} \otimes \mathbb{1} + B_n^{\text{DIS}} \sigma^n \otimes \mathbb{1} + \bar{B}_n^{\text{DIS}} \mathbb{1} \otimes \sigma^n + C_{ab}^{\text{DIS}} \sigma^a \otimes \sigma^b), \quad (78)$$

where

$$B_n^{\text{DIS}} = \frac{A^T B_n^T}{A^T + \varepsilon A^L}, \quad \bar{B}_n^{\text{DIS}} = \frac{A^T \bar{B}_n^T}{A^T + \varepsilon A^L}, \quad C_{ab}^{\text{DIS}} = \frac{A^T C_{ab}^T + \varepsilon A^L C_{ab}^L}{A^T + \varepsilon A^L}. \quad (79)$$

$\Delta_{1,2}$, $\mu_{1,2}$, and M_2 defined in the previous section are calculated from these coefficients in the same way. In the following numerical results, we set $\varepsilon = 1$.

A. Polarization

First, we show the results on the polarization of the quark B_n and the antiquark $\bar{B}_n$. As we noted already, B_n and $\bar{B}_n$ are nonvanishing only for massive quarks and for the transversely polarized photon and tend to be suppressed at high c.m. energies. Therefore, it is best to focus on low- W UPC or low- W electroproduction at low Q^2 . In Fig. 3, we plot B_n and $\bar{B}_n$ for the strange (upper panels) and charm

(lower panels) quarks in UPC at $W = 30$ GeV. We find that B_n is negative and $\bar{B}_n$ is positive for all massive quark flavors $q = s, c, b$, consistent with (46). The net polarization along the $\hat{n}$ direction $B_n(z) + \bar{B}_n(z) \propto z - \bar{z}$ vanishes at $z = \frac{1}{2}$. The peak structure in $k_\perp$ is as expected from the analytic formula (39), with the peak magnitude reaching 80%, 60%, and 50% for strange, charm, and bottom quarks, respectively.⁶ These peaks occur when the quark (antiquark) is “soft” $z \ll 1$ ($\bar{z} \ll 1$) in the original frame (2) in which the photon is fast moving and the $q\bar{q}$ pair is produced in the forward rapidity region [cf., the comment below (40)]. Negative B_n means that the quark is polarized in the direction of $-\vec{q} \times \vec{k}_q$, see Fig. 1. Similarly, the antiquark is polarized along the direction $-\vec{q} \times \vec{k}_{\bar{q}}$.

Comments are in order regarding the magnitude of the polarization. In semi-inclusive quark or antiquark production, the imaginary part of the amplitude responsible for nonvanishing polarization is suppressed by a factor of α_s since it requires a loop. Typically, one finds (sub)percent

⁶It should be mentioned that B_n of the strange quark is largest in the low-momentum region $k_\perp \gtrsim 1$ GeV where, strictly speaking, the present perturbative approach is not fully justified in photoproduction $Q = 0$.

level asymmetries in such perturbative calculations [52,53]. However, in exclusive production with color-singlet exchanges in the t channel, the real and imaginary parts of the scattering amplitudes are of the same order in α_s . Consequently, $B_n, \bar{B}_n$ (39) are order unity instead of order α_s . Still, a polarization as large as 50%–80% is quite remarkable. To our knowledge, such a strong polarization of perturbative origin has not been reported in the QCD literature.

It is generally expected that the quark-level polarization is largely retained during fragmentation into heavy baryons $q \rightarrow \Lambda_q, \bar{q} \rightarrow \bar{\Lambda}_q$ [70]. In particular, the process $s \rightarrow \Lambda \rightarrow p + \pi^-$ can be used as a polarimeter of the strange quark. The HERMES Collaboration [38] measured the polarization of Λ and $\bar{\Lambda}$ in semi-inclusive DIS in the photo-production region $Q \approx 0$. They found a positive polarization for Λ but almost zero polarization for $\bar{\Lambda}$. On the other hand, the COMPASS Collaboration [39] found very small values consistent with zero for both Λ and $\bar{\Lambda}$. The contribution from exclusive $s\bar{s}$ production, if any at all in these low-energy experiments, will be significantly diluted by the inclusive background. On the other hand, diffractive events with a rapidity gap are copious at the EIC. By triggering on such events, one should be able to test our prediction in future.

B. Entanglement

The degree of entanglement between the quark and antiquark spins can be quantified by $\Delta_{1,2}$ introduced in (56). First, consider UPCs at the LHC and fix $W = 500$ GeV, $Q = 0$, and $k_\perp^{\max} = 30$ GeV. In Fig. 4, we plot the left-hand side of (55) in the $(z, k_\perp)$ plane for $u\bar{u}$ (left), $c\bar{c}$ (middle), and $b\bar{b}$ (right) pairs, respectively. The results for $d\bar{d}, s\bar{s}$ pairs are similar to the $u\bar{u}$ case. In practice, these are plots of Δ_2^T that dominate in (55). Since W is

typically quite large at the LHC, the results are close to those in the Regge limit calculation in [30]. In particular, the pairs are everywhere entangled [i.e., (55) is always satisfied]. Light quarks are maximally entangled ($\Delta_2^T \approx 2$) along the line $z = \frac{1}{2}$ as discussed around (63). For massive quarks, an almost maximally entangled state

$$|\Psi_n^-\rangle \approx \frac{1}{\sqrt{2}}(|+-\rangle_n - i|-+\rangle_n) \quad (80)$$

is realized at a peak around $z = \frac{1}{2}$ and $k_\perp \approx m$. The peak occurs because $I_g^L \approx 0$ [see the comment below (51)] and hence $C_{nn}^T \approx -1$ in this region. The peak is followed by a dip as $k_\perp$ increases (i.e., entanglement becomes weak) before rising again at higher $k_\perp$ where the mass effect becomes less important. It should be noted that, when $W = 500$ GeV and $k_\perp$ is a few GeV, typically the region $x \sim \xi \sim 10^{-5}$ of the proton wave function is probed. At such small values of x , one may have to switch to the Regge limit calculation [30] including the gluon saturation [71]. The two calculations are qualitatively similar, though. In particular, the peak structure around $k_\perp \sim m$ and $z \sim \frac{1}{2}$ was also found in [30]. It would be interesting to perform a detailed comparison of the spin density matrix with or without gluon saturation and small- x evolution effects.

Next, we consider electroproduction at $W = 30$ and $Q = 10$ GeV. In Fig. 5, we plot the left-hand side of (55) (with $T \rightarrow \text{DIS}$ in the superscript) for $u\bar{u}$ (left), $s\bar{s}$ (middle), $b\bar{b}$ (right) pairs. The $u\bar{u}$ plot is asymmetric under $z \leftrightarrow \bar{z}$. This is because the valence quark integrals $I_L^q, I_T^{q,1,2}$ with $q = u, d$ do not have this symmetry. Actually, the same is true in Fig. 4 (left), but the asymmetry is barely noticeable because at high energy the cross section is dominated by the gluon contribution which is symmetric. Since Q is relatively high, the results reflect a significant

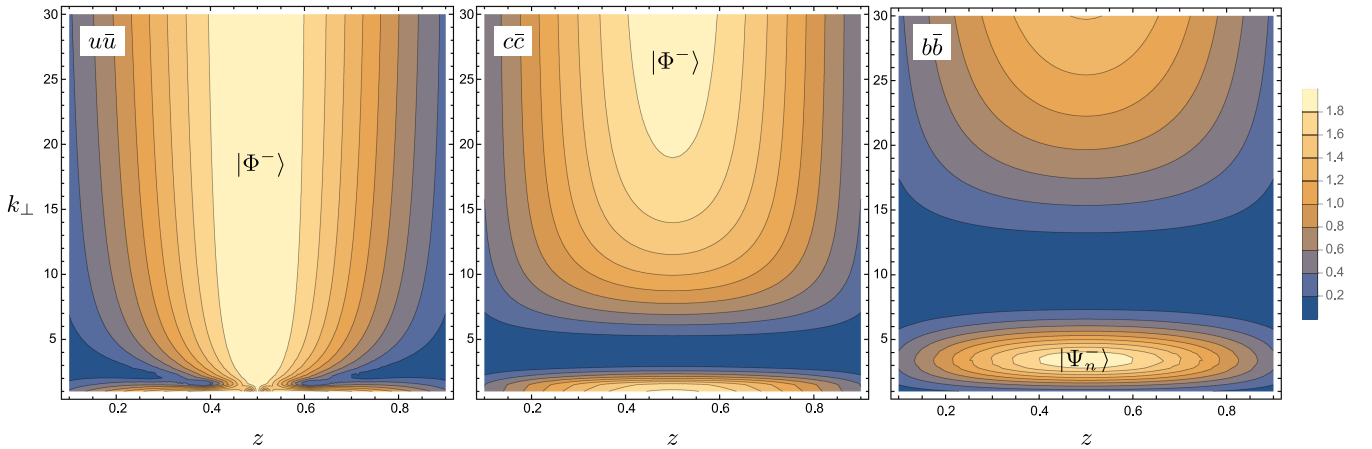


FIG. 4. Left-hand side of (55) in UPC at $W = 500$ GeV, $Q = 0$. Left: $u\bar{u}$ pair. Middle: $c\bar{c}$ pair. Right: $b\bar{b}$ pair. Entanglement is stronger in brighter regions where mostly $\max\{\Delta_1, \Delta_2, |C_{nn} - 1|\} = \Delta_2$. The pair reaches maximally entangled states Φ^- (62) and Ψ_n^- (80) with $\Delta_2 \approx 2$ in the indicated regions.

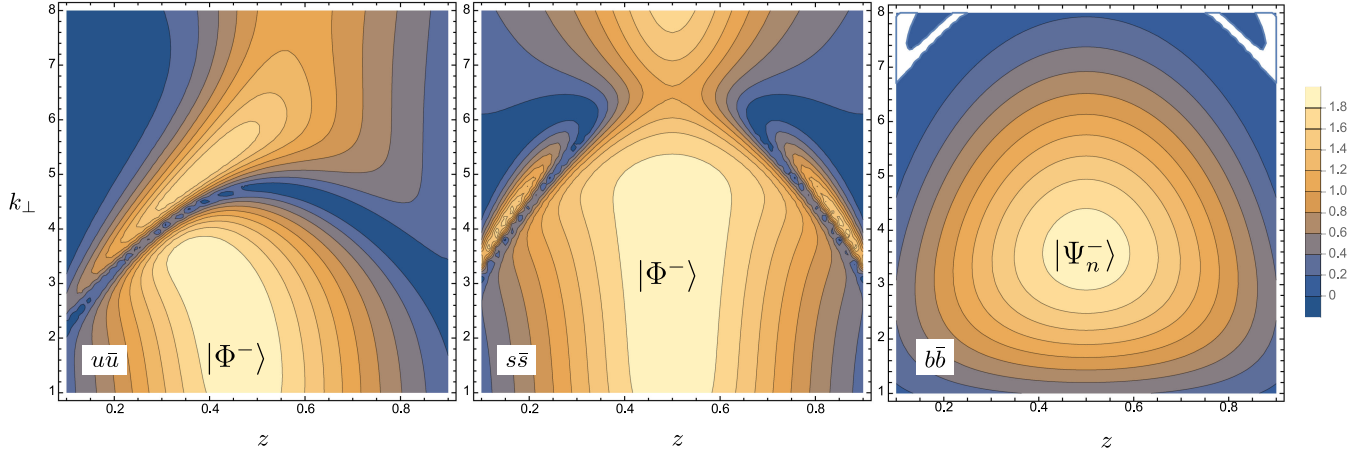


FIG. 5. Left-hand side of (55) in electroproduction at $Q = 10$, $W = 30$ GeV. Left: $u\bar{u}$ pair. Middle: $s\bar{s}$ pair. Right: $b\bar{b}$ pair. The pair is not entangled in the white region.

interference between the transverse and longitudinal photon contributions. In $b\bar{b}$ (and $c\bar{c}$) production, we find narrow white regions where the pair is not entangled (i.e., the pair is separable). We have studied other sets of (W, Q) and found that regions of no entanglement appear only for $c\bar{c}$, $b\bar{b}$ pairs in small patches at most. This means that, even though the conditions (58) do not hold away from the Regge limit, entanglement remains a robust feature of exclusive $q\bar{q}$ production.

C. Bell nonlocality

Next, we delineate the regions where the $q\bar{q}$ pair exhibits Bell nonlocality. In Fig. 6 (left), we plot $\mu_1 + \mu_2 - 1$ [see (65)] for $b\bar{b}$ pairs in the same UPC kinematics as in Fig. 4 (right). We see a large white region where $\mu_1 + \mu_2 - 1 < 0$, meaning that the Bell-CHSH inequality is not violated there. This is in contrast to the finding in [30] that, in the Regge limit, the pair exhibits Bell nonlocality in

the entire kinematical region of $(z, k_\perp)$. It is also in contrast to Fig. 4 (right), which shows that the pair remains entangled even away from the Regge limit. These observations are consistent with the fact that, in general, Bell nonlocality is a sufficient but not necessary condition for entanglement. Since the violation of the Bell-CHSH inequality requires stronger quantum correlations than entanglement, it can be seen in a narrower region of phase space.

In the middle and right panels of Fig. 6, we plot the same quantity for the $s\bar{s}$ and $b\bar{b}$ pairs in electroproduction ($W = 30$, $Q = 10$ GeV) to be directly compared to the middle and right panels in Fig. 5. Again, we find regions where Bell nonlocality is lost, even though the pairs are still entangled.

D. Stabilizer Rényi entropy

Finally, in Figs. 7 and 8, we plot the stabilizer Rényi entropy M_2 (69) in UPC and electroproduction,

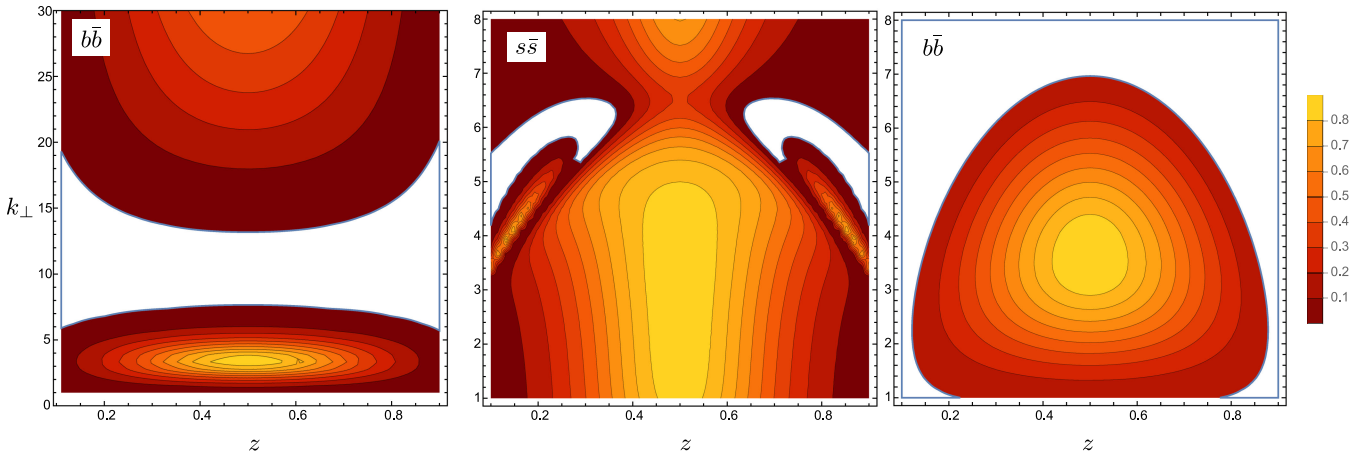
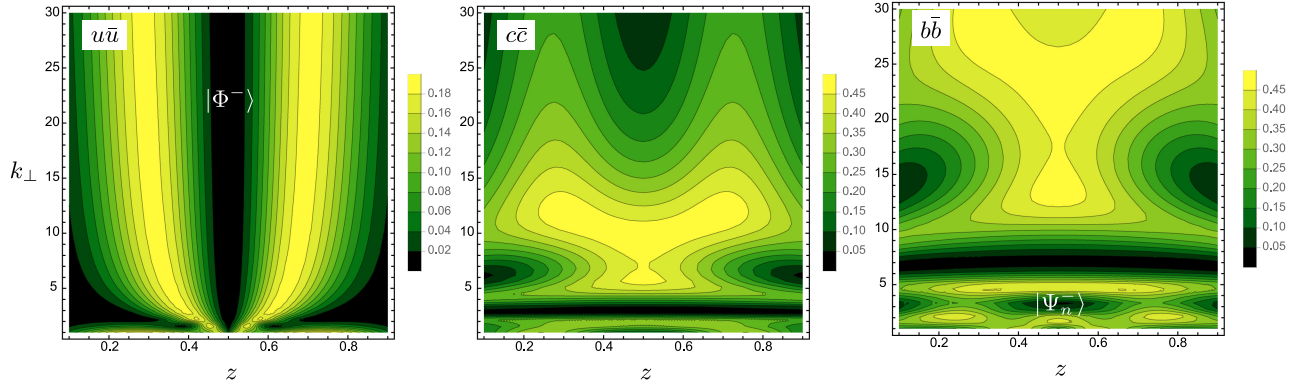
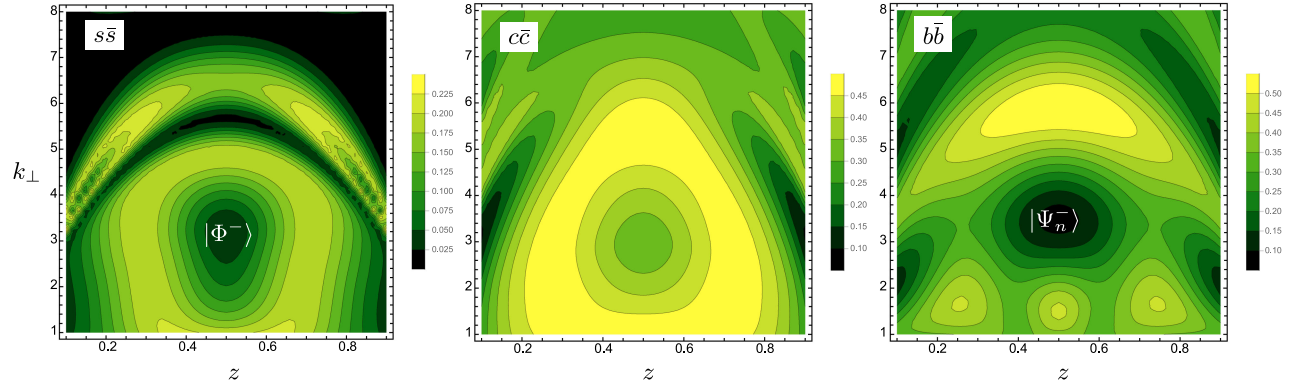


FIG. 6. Plot of $\mu_1 + \mu_2 - 1$, see (65). The Bell-CHSH inequality is not violated in the white regions. Left: $b\bar{b}$ production in UPC at $W = 500$ GeV, $Q = 0$, compare with Fig. 4 (right). Middle: $s\bar{s}$ production in electroproduction, compare with Fig. 5 (middle). Right: $b\bar{b}$ pair production in electroproduction, compare with Fig. 5 (right).


 FIG. 7. Stabilizer Rényi entropy (69) in UPC ($W = 500$ GeV) for $u\bar{u}$ (left), $c\bar{c}$ (middle), and $b\bar{b}$ (right) pairs.

 FIG. 8. Stabilizer Rényi entropy (69) in electroproduction ($W = 30$, $Q = 10$ GeV) for $s\bar{s}$ (left), $c\bar{c}$ (middle), and $b\bar{b}$ (right) pairs.

respectively, for different quark flavors. As expected, M_2 is suppressed when entanglement is nearly maximal; but, other than this general statement, it is difficult to anticipate in which kinematical regions M_2 is enhanced. We, however, see that M_2 tends to be larger for heavier quarks, with the maximal value around 0.58 for the $b\bar{b}$ pair in electroproduction with $W = 30$, $Q = 3$ GeV, see Fig. 8 (right). Curiously, in the entire parameter space we have explored, M_2 is always smaller than (75), although, in principle, it can exceed this bound because $B_n, \bar{B}_n \neq 0$.

VI. CONCLUSIONS

In this paper, we have analyzed exclusive quark-antiquark production in electroproduction and UPCs from the viewpoint of quantum information science (QIS). This is a generalization of the previous work in the high-energy (Regge) limit [30] to more practical c.m. energies at the EIC where collinear factorization with GPDs is an appropriate theoretical tool. The scattering amplitudes computed in the collinear factorization framework are complex, in contrast to being purely real in the one-gluon exchange approximation [29] and purely imaginary in the Regge limit. An immediate consequence is that massive $q\bar{q}$ pairs are transversely polarized without the usual α_s suppression

in inclusive production. In low-energy UPC or electroproduction, the polarization can easily exceed 50% in certain kinematical regions. Moreover, we have observed rich patterns of entanglement, Bell nonlocality, and stabilizer Renyi entropy due to the interference between the real and imaginary parts of the scattering amplitudes, as well as between the transverse and longitudinal photons. Together with the EIC's ability to vary W and Q over an unprecedentedly wide kinematical range, this makes DIS a particularly fascinating arena for exploring QIS in collider physics.

We have focused on the polarization and entanglement of the produced $q\bar{q}$ pairs to leading order in perturbation theory. In future, the impact of the next-to-leading order corrections should be investigated (cf. [72]). Moreover, other types of GPDs such as the polarized GPDs $\tilde{H}_{q,g}$ can contribute and become important at low energy. Most importantly, whether the entanglement properties discussed in this paper survive as observables in realistic experiments is a nontrivial problem that requires separate theoretical investigations. For massive quarks s, c, b (or antiquarks $\bar{s}, \bar{c}, \bar{b}$), the canonical method is to tag events where the quarks fragment into heavy baryons $\Lambda, \Lambda_c, \Lambda_b$ (or antibaryons $\bar{\Lambda}, \bar{\Lambda}_c, \bar{\Lambda}_b$) [70]. Their subsequent weak decays

such as $\Lambda \rightarrow p + \pi^-$ allow one to reconstruct the polarization vectors $B_a, \bar{B}_a$ [36,37] and the spin density matrix C_{ab} [22,73]. For light quarks, methods outlined in [23] may be applicable to DIS. In both cases, detailed simulations with a realistic detector setup at the EIC are necessary [74,75].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: MAXIMAL STABILIZER R NYI ENTROPY WITHOUT POLARIZATION

In this appendix, we determine the upper limit of the stabilizer R nyi entropy [34] for generic two-qubit systems with vanishing polarization. Following [65], we parametrize pure two-qubit states as

$$|\psi\rangle = c_1|++\rangle + c_2|+-\rangle + c_3| - + \rangle + c_4|--\rangle, \quad (\text{A1})$$

$$\begin{aligned} c_1 &= \sin\theta_1 \sin\theta_2 e^{i\phi_1}, & c_2 &= \sin\theta_1 \cos\theta_2 e^{i\phi_2}, \\ c_3 &= \cos\theta_1 \sin\theta_3 e^{i\phi_3}, & c_4 &= \cos\theta_1 \cos\theta_3. \end{aligned} \quad (\text{A2})$$

Without loss of generality, we may take $0 \leq \theta_{1,2,3} \leq \frac{\pi}{2}$ and $0 \leq \phi_{1,2,3} \leq 2\pi$. The polarization of the first qubit can be read off from the reduced density matrix traced over the second qubit

$$\text{Tr}_2 |\psi\rangle\langle\psi| = \begin{pmatrix} |c_1|^2 + |c_2|^2 & c_1 c_3^* + c_2 c_4^* \\ c_1^* c_3 + c_2^* c_4 & |c_3|^2 + |c_4|^2 \end{pmatrix} = \frac{1 + \vec{B}_1 \cdot \vec{\sigma}}{2}. \quad (\text{A3})$$

If we require that $\vec{B}_1 = 0$, then

$$|c_1|^2 + |c_2|^2 = |c_3|^2 + |c_4|^2 = \frac{1}{2}, \quad c_1 c_3^* + c_2 c_4^* = 0. \quad (\text{A4})$$

Similarly, if we require that $\vec{B}_2 = 0$,

$$|c_1|^2 + |c_3|^2 = |c_2|^2 + |c_4|^2 = \frac{1}{2}, \quad c_1 c_2^* + c_3 c_4^* = 0. \quad (\text{A5})$$

Substituting (A2) into (A4), we get

$$\sin\theta_1^2 = \cos\theta_1^2 \quad \sin\theta_2 \sin\theta_3 e^{i(\phi_1 - \phi_3)} + \cos\theta_2 \cos\theta_3 e^{i\phi_2} = 0. \quad (\text{A6})$$

This means that

$$\theta_1 = \frac{\pi}{4} \quad \phi_1 - \phi_3 = \phi_2 \pm \pi, \quad \theta_3 = \frac{\pi}{2} - \theta_2. \quad (\text{A7})$$

It is easy to check that, when (A7) is satisfied, (A5) is automatically satisfied. The maximal value of M_2 for systems with $\vec{B}_1 = \vec{B}_2 = 0$ is therefore obtained by minimizing (16) of [65] with the constraints (A7). There are many degenerate minima. For example, we find

$$\cos\theta_2 = \sqrt{\frac{1}{3}}, \quad \phi_1 = \frac{\pi}{2}, \quad \phi_2 = \frac{5}{4}\pi, \quad \phi_3 = \frac{\pi}{4}, \quad (\text{A8})$$

to a good numerical accuracy. The corresponding value of M_2 is given by (75).

Now let us further require that $\theta_2 = \frac{\pi}{2}$, and accordingly, $\theta_3 = 0$. (Alternatively, one may set $\theta_2 = 0$ and $\theta_3 = \frac{\pi}{2}$.) The density matrix becomes

$$\rho^L = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & e^{i\phi_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\phi_1} & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A9})$$

Minimizing (16) of [65] under this additional constraint, we find the extremal condition $\cos(4\phi_1) = -1$, or equivalently,

$$\phi_1 = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}. \quad (\text{A10})$$

At these points, M_2 takes the maximal value that coincides with (71). The four values of ϕ_1 in (A10) correspond to the four branches (73).

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